THE PMC SYSTEM LEVEL FAULT MODEL:
MAXIMALITY PROPERTIES OF
THE IMPLIED FAULTY SETS

G. G. L. Meyer

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Electrical Engineering and Computer Science Department
The Johns Hopkins University
Baltimore, Maryland 21218

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ABSTRACT

In this paper, we examine the implied faulty sets in the case of the PMC system level fault model. We show that those sets possess a maximality property whenever the system is one-step \( \tau \)-diagnosable, no two modules test each other and the number of faulty modules is no larger than \( \tau \). In addition, we propose a syndrome-decoding algorithm based on that maximality property.
INTRODUCTION

Since its introduction in 1967, the PMC system level fault model proposed by Preparata, Metze and Chien \(^9\) has been the subject of much attention. Conditions that insure one-step \(r\)-diagnosability have been proposed in \([1,3]\) and \([9]\), and decoding algorithms have been proposed in \([2]\) and in \([4]-[8]\).

One of the major stumbling blocks for the synthesis of decoding algorithms for the PMC model is the absence of known useful properties that result from the assumptions of the model. Thus, the existing algorithms depend either on strong assumptions on the structure of the testing interconnection \([5,7]\), on unproven conjectures \([2]\), or on searches with the associated drawback — namely, backtracking \([4]\) — or they are insured to work only when few faults are present \([6,8]\).

It seems reasonable to assume that if a PMC model is one-step \(r\)-diagnosable, then well-chosen quantities exist that exhibit useful properties. In previous work \([5]-[8]\), we used the concept of the implied faulty set to analyze the PMC model. The implied faulty set of a module is simply the set of all the modules in the system that may be deduced to be faulty under the assumption that the module is non-faulty. The usefulness of this concept has been demonstrated in \([5]\) and \([7]\) for the case of \(D_3,r\) interconnection structures, and in \([6]\) and \([8]\) for the case in which no two modules test each other, and the number of faulty modules is small.

In this paper, we show that the implied faulty sets of one-step \(r\)-diagnosable PMC models in which no two modules test each other satisfy a principle of optimality: the module that corresponds to a maximal implied faulty set is always faulty.
THE PMC SYSTEM LEVEL FAULT MODEL

Consider a system $S$ of $n$ modules $U_0, U_1, \ldots, U_{n-1}$ and a testing inter-
connection design $TID = \{(i, j) | U_i \text{ tests } U_j \}$. It is assumed that when $(i, j)$
is in $TID$, the test outcome $a_{ij}$ of $U_i$ testing $U_j$ is $a_{ij} = 0$ if $U_i$ believes $U_j$ to
be nonfaulty, and $a_{ij} = 1$ if $U_i$ believes $U_j$ to be faulty. A complete set of
test outcomes, i.e., an outcome $a_{ij}$ for each $(i, j)$ in $TID$ is called a syndrome.
The diagnosis problem consists in partitioning $S$ into the set $G_S$ of non-faulty
modules and the set $F_S$ of faulty modules from the knowledge of one of the
possible corresponding syndromes. In this paper, we assume that the only
faults that may occur are solid, and that the test-fault relationship satisfies at
least the Preparata-Metze-Chien assumption given below.

_Hypothesis 1:_  
(i) If $(i, j)$ is in $TID$ and $U_i$ is nonfaulty, then $a_{ij} = 0$ implies that $U_j$ is non-
faulty, and $a_{ij} = 1$ implies that $U_j$ is faulty;
(ii) If $(i, j)$ is in $TID$ and $U_i$ is faulty, then $U_j$ may be nonfaulty or faulty
regardless of the value of $a_{ij}$.

Given $(G_S, F_S)$, Hypothesis 1 implies that only a subset of all possible syn-
dromes may occur. Determining all possible syndromes that correspond to a
given partition $(G_S, F_S)$ of $S$ is not difficult. On the other hand, the problem
we address in this paper — that is, given a syndrome produced by a partition
$(G_S, F_S)$ of $S$, find $(G_S, F_S)$ — is much more difficult to solve.

Not all the partitions of $S$ into nonfaulty and faulty modules may explain a
given syndrome. A partition $(G, F)$ of $S$ is consistent with a given syndrome if
and only if the assumption that all the modules in $G$ are nonfaulty and all the
modules in $F$ are faulty is consistent with the syndrome. The partition $(G_S, F_S)$
is obviously consistent, but unfortunately, many partitions usually exist that are
consistent with any given syndrome. Thus, given a syndrome, one cannot
identify the faulty modules without additional assumptions.

If we assume that the *a priori* probability that a set of modules \( F \) is faulty
is inversely proportional to the cardinality \( |F| \) of \( F \), then it is reasonable to
look for the consistent partitions of \( S \) that are most likely to occur: namely, the
consistent partitions of \( S \) in which \( |F| \) is minimal. Such partitions, called
*minimal consistent partitions* are the solutions to the following discrete minimization
problem.

**Problem 1:** Given a syndrome, find a consistent partition \((G_#, F_#)\) of \( S \) such
that \( |F_#| \leq |F| \) for all the partitions \((G, F)\) of \( S \) that are consistent with the
syndrome.

If the number of faulty modules does not exceed \( \tau \), at least one consistent
partition, namely \((G_5, F_5)\), exists such that \( |F_5| \leq \tau \). If only one such parti-
tion exists, then finding the partition \((G_5, F_5)\) reduces to solving Problem 1
whenever \( |F_5| \leq \tau \). Thus, in the context of our paper, one-step \( \tau \)-
diagnosability [9] reduces to:

**Definition 1:** A system \( S \) is one-step \( \tau \)-diagnosable if and only if whenever a
consistent partition \((G, F)\) exists such that \( |F| \leq \tau \), that partition is the
unique solution to Problem 1.

**IMPLIED NON-FAULTY AND FAULTY SETS**

We have reduced the problem of identifying the partition \((G_5, F_5)\) to that
of solving a discrete minimization problem, Problem 1. Our approach to this
problem depends on the concepts of implied non-faulty and faulty sets.
Definition 2: The implied non-faulty set $M(U_i)$ of a module $U_i$ (with respect to a given syndrome) is the set of all the modules in $S$ that may be deduced to be non-faulty under the assumption that $U_i$ is non-faulty.

Definition 3: The implied faulty set $L(U_i)$ of a module $U_i$ (with respect to a given syndrome) is the set of all the modules in $S$ that may be deduced to be faulty under the assumption that $U_i$ is non-faulty.

If the module $U_i$ is in $G_S$, then $M(U_i)$ is a subset of $G_S$, $L(U_i)$ is a subset of $F_S$, and therefore, $M(U_i)$ and $L(U_i)$ are disjoint. Thus, if $M(U_i)$ and $L(U_i)$ are not disjoint, we may conclude that $U_i$ is in $F_S$. Let $F_0$ and $G_0$ be the sets defined by

$$F_0 = \{ U_i \in S \mid M(U_i) \cap L(U_i) \neq \phi \},$$

and

$$G_0 = S - F_0.$$

The set $F_0$ is a subset of $F_S$, provided that the basic assumption on the fault-test relationship, namely Hypothesis 1, holds. The fact that $F_0$ is a subset of $F_S$ does not depend on any assumption concerning the maximum number of faulty modules, nor on assumptions concerning the structure of the testing interconnection network. The set $F_0$ is not difficult to obtain. Given a system $S$ and a syndrome, we may compute $F_0$ and consider the reduced system $S_0$ obtained by deleting $F_0$ from $S$ and the corresponding reduced syndrome obtained by deleting all the test links between $G_0$ and $F_0$ from the original syndrome. Note that if $S$ is one-step $r$-diagnosable, then $S_0$ is one-step $(r-|F_0|)$-diagnosable.

Clearly, if Hypothesis 1 is satisfied and if $L(U_i) \cap G_0 = \phi$ for every
module $U_i$ in $G_0$, then the partition $(G_0, F_0)$ is a solution to Problem 1. Thus, Definition 1 implies that, in some cases, the set $F_S$ of faulty modules is the set $F_0$.

**Lemma 1:** If Hypothesis 1 is satisfied, if $S$ is one-step $r$-diagnosable, if $L(U_i) \cap G_0 = \emptyset$ for every module $U_i$ in $G_0$, and if $|F_0| \leq r$, then $F_S = F_0$.

**MAXIMALITY OF THE IMPLIED FAULTY SETS**

The set $F_0$ may be computed as soon as the implied non-faulty and faulty sets have been obtained, and we know that every module in $F_0$ is faulty. We have no reason to believe that $F_0$ contains all the faulty modules. We could use the fact [8] that $F = L(G)$, where

$$L(G) = \{ U_j \mid U_j \in L(U_i), U_i \in G \}$$

whenever $(G,F)$ is a minimal consistent partition, to search for the minimal consistent partitions, but this search may be tedious. We will now present a property of the implied faulty sets that holds when the Hakimi-Amin [3] sufficient conditions for one-step $r$-diagnosability are satisfied.

**Hypothesis 2 (Hakimi-Amin):**
(i) every module is tested by at least $r$ other modules;
(ii) no two modules test each other.

We know from [3] that Problem 1 possesses a unique solution whenever Hypotheses 1 and 2 are satisfied and the number of faulty modules is not greater than $r$. In such a case, the implied faulty sets possess a property that greatly simplifies the task of decoding syndromes.

**Theorem 1:** If Hypotheses 1 and 2 are satisfied, and if $1 \leq |F_0| \leq r$, then at
least one module \( U_i \) in \( S \) exists such that either \( M(U_i) \cap L(U_i) \neq \emptyset \), or \( |L(U_i)| \geq r+1 \), or both.

**Proof:** See Appendix.

We may use Theorem 1 to exhibit a maximality property of the implied faulty sets. If \( F_S \) is non-empty and if \( F_0 \) is empty, then at least one module \( U_i \) exists so that \( |L(U_i)| \geq r+1 \); thus, the modules \( U_j \) in \( S \) that maximize \( |L(U_j)| \) are faulty.

**Corollary 1:** If Hypotheses 1 and 2 are satisfied, if \( 1 \leq |F_S| \leq r \) and if \( M(U_i) \cap L(U_i) = \emptyset \) for every module in \( S \), then the modules \( U_j \) that maximize \( |L(U_j)| \) are in \( F_S \).

Corollary 1 may be used recursively to generate the set \( F_S \) of faulty modules in \( S \), and thus, when Hypotheses 1 and 2 are satisfied, an iterative "greedy-type" algorithm will produce the set of faulty modules, provided that the set \( F_0 \) is first identified.

**Algorithm 1:**

Step 0: Let \( F_0 = \{ U_i \in S \mid M(U_i) \cap L(U_i) \neq \emptyset \} \), and let \( k = 0 \).

Step 1: Let \( h_k = \max \{ |L(U_i) \cap (S - F_k)| \mid U_i \in S - F_k \} \).

Step 2: If \( h_k = 0 \), let \( F_k = F_k \) and stop; otherwise, go to Step 3.

Step 3: Let \( H_k = \{ U_i \in S - F_k \mid |L(U_i) \cap (S - F_k)| = h_k \} \).

Step 4: Let \( F_{k+1} = F_k \cup H_k \).

Step 5: Let \( k = k+1 \), and go to Step 1.

The fact that \( S \) contains a finite number of modules implies that Algorithm 1 terminates after a finite number of iterations. Using Lemma 1, Theorem 1 and Corollary 1, we may then obtain the following result.
Theorem 2: If Hypotheses 1 and 2 are satisfied, and if $|F_3| < \tau$, the set $F_A$ generated by Algorithm 1 is equal to the set of faulty modules $F_3$.

APPENDIX: Proof of Theorem 1

Our proof of Theorem 1 is similar to the one used by Hakimi and Amin in [3]. First, we assume that the result of the theorem does not hold; we then partition the system $S$ and using that partition, we exhibit two inequalities which taken together, lead to a contradiction. Thus, in this appendix we shall assume that we have a PMC system level model in which:

(A1) every module is tested by at least $\tau$ other modules,

(A2) no two modules test each other,

(A3) the number of faulty modules $|F_3|$ satisfies $1 \leq |F_3| \leq \tau$,

(A4) $M(U_i) \cap L(U_i) = \emptyset$ for every module $U_i$ in $S$, and

(A5) $|L(U_i)| \leq \tau$ for every module $U_i$ in $S$.

Let $U_*$ be a faulty module in $S$ such that for every faulty module $U_i$ in $S$,

$$|M(U_*) \cap F_3| \geq |M(U_i) \cap F_3|.$$  \hspace{1cm} (1)

We now partition our system $S$ into five subsets: $V_1, V_2, V_3, V_4$ and $V_5$.

Let $V_1 = M(U_*) \cap F_3$. Thus, $V_1$ consists of all the modules in $F_3$ that must be nonfaulty if $U_*$ is assumed to be nonfaulty and $U_*$ is in $V_1$.

Let $V_2$ be the subset of $S$ that consists of all the modules in $L(U_*)$ that are actually faulty, i.e.,

$$V_2 = L(U_*) \cap F_3.$$

Let $V_3$ be the set of all faulty modules that are not in $V_1$ or $V_2$, i.e.,

$$V_3 = F_3 - (V_1 \cup V_2).$$
Let \( V_4 \) be the subset of \( S \) that consists of all the modules in the implied faulty set of \( U_s \) that are actually nonfaulty, i.e.,
\[
V_4 = L(U_s) \cap G_s.
\]

Let \( V_5 \) be the set of all nonfaulty modules that are not in \( L(U_s) \), i.e.,
\[
V_5 = G_s - V_4.
\]

Clearly, the sets \( V_1, V_2, V_3 \) form a partition for \( F_5 \), and the sets \( V_4 \) and \( V_5 \) form a partition for \( G_s \).

For \( i = 1, 2, 3, 4, 5 \) and \( j = 1, 2, 3, 4, 5 \), let \( E_{i,j} \) be the set of testing links from \( V_i \) into \( V_j \), let \( v_i \) denote the cardinality of the partition block \( V_i \), let \( e_{i,j}^0 \) denote the cardinality of the set of 0-links from \( V_i \) to \( V_j \), let \( e_{i,j}^1 \) denote the cardinality of the sets of 1-links from \( V_i \) to \( V_j \), and let \( e_{i,j} = e_{i,j}^0 + e_{i,j}^1 \), i.e., \( e_{i,j} \) is the cardinality of the set of testing links \( E_{i,j} \).

The definition of the partition blocks \( V_i \) implies that the number and type of testing links between blocks may not be arbitrary.

**Lemma 2:** The testing links sets \( E_{i,j} \) satisfy:

(i) \( e_{1,3} - e_{5,1} - e_{5,4} = 0 \);
(ii) \( E_{1,1}, E_{1,5}, E_{3,1}, E_{4,4}, E_{4,5} \) and \( E_{5,5} \) consists only of 0-links;
(iii) \( E_{1,2}, E_{1,4}, E_{3,2}, E_{3,4}, E_{4,1}, E_{4,2}, E_{4,3}, E_{5,2} \) and \( E_{5,3} \) consist only of 1-links;
(iv) \( E_{2,1}, E_{2,2}, E_{2,3}, E_{2,4}, E_{2,5}, E_{3,3} \) and \( E_{3,5} \) consist of both 0-links and 1-links.

**Proof:** Let \( U_i \) be in \( V_1 \), and let \( U_j \) be in \( S \). If a 0-link from \( U_i \) to \( U_j \) exists, then \( U_j \) must be in \( V_1 \cup V_5 \), and if there is a 1-link from \( U_i \) to \( U_j \), then \( U_j \) must be in \( L(U_s) \) and hence, in \( V_2 \cup V_4 \). We may then conclude that \( E_{1,1} \) and \( E_{1,5} \) consist of only 0-links, that \( E_{1,2} \) and \( E_{1,4} \) consist of only 1-links, and
that $e_{1,3} = 0$.

Let $U_i$ be in $V_3$, and let $U_j$ be in $S$. If a 0-link from $U_i$ to $U_j$ exists, then $U_j$ must be in $V_1 \cup V_3 \cup V_5$; if there is a 1-link from $U_i$ to $U_j$, then $U_j$ must be in $V_2 \cup V_3 \cup V_4 \cup V_5$. We may then conclude that $E_{3,1}$ consists of only 0-links, and that $E_{3,2}$ and $E_{3,4}$ consist of only 1-links.

Let $U_i$ be in $V_4$ and let $U_j$ be in $S$. By construction, $U_i$ is non faulty and $U_j$ is faulty whenever $U_j$ is in $V_1 \cup V_2 \cup V_3$; $U_j$ is non faulty whenever $U_j$ is in $V_4 \cup V_5$. We may then conclude that $E_{4,1}$, $E_{4,2}$ and $E_{4,3}$ consist of only 1-links, and that $E_{4,4}$ and $E_{4,5}$ consist of only 0-links.

We cannot have any testing links from $V_5$ to $V_1$ or $V_4$, because whenever a non faulty module tests a module in either $V_1$ or $V_4$, it must be in $L(U_i)$, thus $e_{5,1} = e_{5,4} = 0$. By construction, a module $U_i$ in $V_5$ is non faulty and thus, $E_{5,2}$ and $E_{5,3}$ consist of only 1-links, and $E_{5,5}$ consists of only 0-links. □

Every module is tested by at least $\tau$ other modules, and therefore,

$$e_{1,1} + e_{2,1} + e_{3,1} + e_{4,1} \geq \tau v_1, \quad (2)$$

and

$$e_{1,4} + e_{2,4} + e_{3,4} + e_{4,4} \geq \tau v_4. \quad (3)$$

No two modules test each other, and thus

$$e_{1,1} \leq v_1 (v_1 - 1)/2, \quad (4)$$

$$e_{4,4} \leq v_4 (v_4 - 1)/2, \quad (5)$$

$$e_{1,4} + e_{4,1} \leq v_1 v_4, \quad (6)$$

and

$$e_{2,4} \leq v_2 v_4. \quad (7)$$
The fact that $|L(U_j)| \leq r$ for every module $U_j$ in $S$ implies that the number of 1-links from a partition block $V_i$ to $S$ cannot be larger than the number of 0-links from $S$ to $V_i$, and therefore,

$$e_1^1 + e_2^1 + e_3^1 + e_4^1 \leq e_2^0,$$  \hspace{1cm} (8)

and

$$e_{3,4} \leq e_{2,3}^0 + e_{3,3}^0.$$  \hspace{1cm} (9)

The maximality of the module $U_*$ on which the basic partition is based implies that no faulty module may find more than $v_1-1$ faulty modules non faulty, and thus,

$$e_2^0 + e_2^0 + e_2^0 \leq v_2(v_1-1),$$  \hspace{1cm} (10)

and

$$e_{3,1} + e_{3,3}^0 \leq v_3(v_1-1).$$  \hspace{1cm} (11)

Equations (2) and (4) imply

$$e_{4,1} \geq v_1 - v_1(v_1-1)/2 - e_2^1 - e_3^1.$$  \hspace{1cm} (12)

Equations (3), (5) and (7) imply

$$e_{4,4} \geq v_4 - v_2v_4 - e_{3,4}^0 - v_4(v_4-1)/2.$$  \hspace{1cm} (13)

Thus, using (6), we obtain

$$X + Y \geq 0$$  \hspace{1cm} (14)

where

$$X = v_1v_4 - v_1 + v_1(v_1-1)/2 - v_1 - v_4(v_4-1)/2 + v_2v_4.$$  \hspace{1cm} (15)

and

$$Y = e_2^1 + e_{3,1} + e_{3,4}^0.$$  \hspace{1cm} (16)
Using (9), we obtain
\[ Y \leq e_{2,1} + e_{3,1} + e_{2,3}^0 + e_{3,3}^0. \]  
(17)
and using (11), we find that
\[ Y \leq e_{2,1} + e_{2,3}^0 + v_3(v_1-1). \]  
(18)
Inequalities (8) and (10) yield
\[ e_{2,1}^1 + e_{2,2}^1 + e_{2,3}^1 + e_{2,4}^1 + e_{2,1}^0 + e_{2,2}^0 + e_{2,3}^0 \leq e_{2,2}^0 + v_2(v_1-1), \]  
(19)
thus
\[ e_{2,1}^0 + e_{2,1}^1 + e_{2,3}^0 \leq v_2(v_1-1), \]  
(20)
and we may conclude that
\[ e_{2,1} + e_{2,3}^0 \leq v_2(v_1-1). \]  
(21)
Using (18) and (21), we obtain
\[ Y \leq v_3(v_1-1) + v_2(v_1-1). \]  
(22)
The number of faulty modules is at most \( r \), i.e.,
\[ v_1 + v_2 + v_3 \leq r, \]  
(23)
and thus
\[ v_1(v_1-1)/2 + v_2(v_1-1) + v_3(v_1-1) \leq r(v_1-1) - v_1(v_1-1)/2. \]  
(24)
The fact that \( |L(U_i)| \leq r \) for every module \( U_i \) in \( S \) implies that \( v_2 + v_4 \leq r \), and thus
\[ v_2 \leq r - v_4. \]  
(25)
Using (14), (15), (16), (22), (24) and (25), we obtain
\[ v_1v_4 - vv_1 + r(v_1-1) - v_1(v_1-1)/2 - rv_4 + v_4(v_4-1)/2 + (r-v_4)v_4 \geq 0. \]  
(26)
Equation (26) may be rewritten as

\[
(v_1 - v_4)/2 - (v_1 - v_4)^2/2 - \tau \geq 0.
\]  

(27)

It is easy to verify that

\[
(v_1 - v_4)/2 - (v_1 - v_4)^2/2 \leq 0.125
\]  

(28)

for all values of \( v_1 \) and \( v_4 \), and thus

\[
0.125 - \tau \geq 0.
\]  

(29)

Equation (29) implies that \( \tau \) must be equal to 0, and this contradicts (A3).

We have shown that the five basic assumptions given at the beginning of the appendix lead to a contradiction and thus may not hold. We can therefore conclude that if a PMMC model satisfies (A1), (A2) and (A3), then (A4) and (A5) may not hold simultaneously; that is, if (A1), (A2) and (A3) hold, then at least one module \( U_i \) in \( S \) exists so that either (i) \( M(U_i) \cap L(U_i) \neq \phi \), or (ii) \( |L(U_i)| \geq \tau + 1 \), or both.
REFERENCES


In this paper, we examine the implied faulty sets in the case of the PIC system level fault model. We show that those sets possess a maximality property whenever: the system is one-step ε-diagnosable, no two modules test each other and the number of faulty modules is no larger than ε. In addition, we propose a syndrome-decoding algorithm based on that maximality property.