MULTI-ECHELON REPAIRABLE ITEM PROVISIONING
IN A TIME-VARYING ENVIRONMENT USING
THE RANDOMIZATION TECHNIQUE

by
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SCHOOL OF ENGINEERING AND APPLIED SCIENCE
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**Title:** Multi-echelon repairable item provisioning in a time-varying environment using the randomization technique

**Abstract:**
Multi-echelon repairable item provisioning systems are considered under a time-varying environment. Such conditions could arise, for example, in a military context where a shift from peacetime operation to wartime operation takes place; or, in a civilian setting where a public transit system decides to increase its hours of operation or frequency of service.

Exact Markovian models, incorporating a finite population of repairable
components and limited repair capacity (nonample service), are treated, with transient solutions obtained using the randomization technique.
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Exact Markovian models, incorporating a finite population of repairable components and limited repair capacity (nonample service), are treated, with transient solutions obtained using the randomization technique.
1. INTRODUCTION

This paper develops models of the transient behavior of Markovian repairable item provisioning systems. A multi-echelon structure of repair and resupply is of concern. It is desired to develop analytical solution techniques for exact models of systems with finite numbers of items and finite repair capacities (i.e., a finite number of repair channels at each repair facility). This is in contrast to Dyna-METRIC [see Hillestad (1981) and Hillestad and Carrillo (1980)], which is an approximate model assuming an infinite calling population of items and ample repair capacities.

Most multi-echelon repairable item work has concentrated on steady state solutions and revolves around the METRIC type of model, which assumes an infinite population of items which can fail and an infinite repair capacity, so that no queue ever forms at the repair
Recently, the METRIC type of model has been extended to provide transient solutions for a time varying environment (Hillestad and Hillestad and Carrillo, *op. cit.*), and is called Dyna-METRIC.

Exact models for finite item populations and repair capacities have been studied by Gross, Miller, and Soland (1981), but for steady state solutions only. This paper presents results on transient solutions for such problems.

For convenience in describing the various multi-echelon systems studied, the following classification scheme is adopted. A system is described by seven symbols, in the format \( (n_1, n_2, n_3/n_4, n_5/n_6, n_7) \). Table I provides the definitions of \( n_1 \) through \( n_7 \).

As an example of the classification scheme, consider a military supply system with five bases, base and depot repair, base and depot

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n_1 )</td>
<td>Number of bases ((1,2,...))</td>
</tr>
<tr>
<td>( n_2 )</td>
<td>Number of levels of repair ((1,2,...))</td>
</tr>
<tr>
<td>( n_3 )</td>
<td>Number of levels of supply ((1,2,...))</td>
</tr>
<tr>
<td>( n_4 )</td>
<td>Number of item types ((1,2,...))</td>
</tr>
<tr>
<td>( n_5 )</td>
<td>Number of levels of indenture ((0,1,...))</td>
</tr>
<tr>
<td>( n_6 )</td>
<td>Size of item population ((f \text{ or } \infty))</td>
</tr>
<tr>
<td>( n_7 )</td>
<td>Capacity of repair facilities ((f \text{ or } \infty))</td>
</tr>
</tbody>
</table>
spares, two types of items with no indenture, finite calling population size (finite numbers of items), and finite repair capacities. This system would be represented as (5,2,2/2,0/f,f). The same system with infinite calling population and infinite repair capacity (Dyna-METRIC assumptions) would be denoted (5,2,2/2,0/inf,inf). The most general system descriptor would be (b,r,s/k,j/f,f); that is, a system with b bases, r levels of repair, s levels of supply, k item types, j levels of indenture, and finite calling populations and repair capacities. To describe fully any system, it would still be necessary actually to draw the network structure; nevertheless, this classification scheme will be most useful in delineating the systems under discussion.

Our goal is to present exact models of Markovian repairable item provisioning systems which can be solved numerically for exact transient state probabilities and other performance measures. We are also concerned with the practical problem of algorithmic implementation on a computer. It is possible to solve nontrivial, fairly complex, systems using the algorithmic approach of "randomization." We have developed highly efficient algorithms for two cases [(1,1,1/1,0/f,f) and (2,2,2/1,0/f,f)], and discuss generalizations of these situations.

The approach we use is a two stage procedure. The first stage is a method of modeling called SERT and the second is an algorithm based on the randomization technique [the reader is referred to Gross and Miller (1982) for a full methodological discussion of these]. A good discussion of the advantages of the randomization procedure as a numerical analysis tool is found in Grassmann (1977).
2. \textit{SERT Modeling Approach}

Since we assume a finite Markovian system (all failure times and repair times are independent exponential random variables), the system can be fully described by its infinitesimal generator matrix

$$Q \equiv \lim_{\Delta t \to 0} \frac{[P(\Delta t) - I]}{\Delta t}$$

where \( P(\Delta t) = \{p_{ij}(\Delta t)\}; \ p_{ij}(\Delta t) \equiv \Pr(\text{system is in state } j \text{ at time } t + \Delta t \mid \text{ in state } i \text{ at time } t) \). The matrix \( Q \) is finite, say \( N \times N \), since we are assuming a finite number of items in the system. For complex systems, \( N \) can be quite large (this will be seen later), so it is necessary to have algorithms as efficient as possible, both in running times and storage demands made on the computer.

It is not necessary to store this \( N \times N \) \( Q \) matrix if we use the \textit{SERT} modeling approach. Briefly [for more detail see Gross and Miller, \textit{op. cit.}], we must describe the \textit{State space}, the \textit{types of Events}, the \textit{transition Rates} (the off-diagonal nonzero elements of the \( Q \) matrix), and the \textit{Target states}, that is, the state to which the system goes when a given type of event occurs.

Given a state space \( S \) of size \( N \) with \( s \) denoting a given state of the system \((s \in S)\), and an event space \( E \) of size \( E \) with \( e_j \) denoting an event of type \( j \) \((e_j \in E)\), it is necessary to consider a rate vector and a target state vector for each \( e_j \), which we denote by \( r^j \) and \( t^j \), respectively. The dimensions of \( r^j \) and \( t^j \) are \( 1 \times N \). Since in these models \( E << N \), it is much more efficient with respect to computer storage requirements to work only with the \( r^j \) and \( t^j \) vectors, rather than the \( N \times N \) \( Q \) matrix, as the former requires consideration of only \( 2 \cdot E \cdot N \) elements rather than \( N^2 \).
The SERT modeling procedure will be illustrated on the (1,1,1/1,0/f,f) and (2,2,2/1,0/f,f) systems to follow.

3. RANDOMIZATION COMPUTATIONAL TECHNIQUE

Transient solutions to Markovian queues require solving the set of differential equations

\[
\pi'(t) = \pi(t)Q
\]

where \( \pi(t) \) is the vector of transient probabilities, i.e., \( \pi(t) = \{\pi_j(t)\} \), where \( \pi_j(t) = \Pr\{\text{system is in state } j \text{ at time } t\} \) and \( \pi'(t) \) is the vector of derivatives with respect to time, i.e., \( \pi'(t) = \{\pi'_j(t)\} \).

Many numerical techniques can be used to solve these linear, first order differential equations; for example, numerical integration [see Maron (1982)]. We choose instead a technique referred to as randomization [see Gross and Miller, op. cit., or Grassmann (1977)], which is ideally suited to the SERT modeling of Markovian systems.

Randomization is based upon the ability to transform the continuous parameter Markov process analysis to an analysis of a discrete parameter Markov chain (MC) whose transition times are generated by a Poisson process.

Consider the continuous parameter process we are modeling. It remains in a given state \( s \) until one of many (\( E \) possible) events occurs, which then changes its state to, say, \( s' \) (if it is an event of type \( e_j \), \( s' = t^j_s \)). Since all times are exponential, the time to the next event is the minimum of exponentials which is also exponential. Hence, if we look at the process only at transition times, it is a discrete parameter Markov chain, whose holding (transition) times are exponentially distributed.
Since the mean of the holding time in state $s$ may depend on $s$, we do not quite have a Poisson process generating these transitions. To get around this, we denote the mean of the minimum holding time over all states (this always exists since $S$ is finite) as $1/\Lambda$ and consider a Poisson process with rate $\Lambda$ as a generator of the transitions. Because this generates transitions at a greater rate than desired (since $\Lambda$ is the maximum of the state-dependent rates), we must thin the process to model the actual state-dependent holding times by adjusting the discrete parameter MC transition probability matrix. Without the adjustment, the embedded discrete parameter MC has transition probability matrix $P = \{p_{ij}\}$, $p_{ij} = q_{ij} / \sum_{j \neq i} q_{ij}$. The parameter $\Lambda$ is $\max_{i,j} q_{ij}$. Adjusting the $P$ matrix for the thinning operation gives the transition probability matrix $\tilde{P} = \{\tilde{p}_{ij}\}$, where

$$\tilde{p}_{ij} = \frac{q_{ij}}{\sum_{j \neq i} q_{ij}} \cdot \frac{1}{\Lambda} = \frac{q_{ij}}{\Lambda}.$$ 

Thus, the computing formula for $\pi(t) = \{\pi_s(t)\}$ is

$$\pi_s(t) = \sum_{n=0}^{\infty} \sum_{i=0}^{N} \phi_s(n) \frac{e^{-\Lambda t} (\Lambda t)^n}{n!},$$

(1)

where $\phi_s(n)$ is the probability that the system is in state $s$ after $n$ transitions generated by the Poisson ($\Lambda$) process, and $e^{-\Lambda t} (\Lambda t)^n/n!$ is the probability that there are $n$ transitions of the Poisson ($\Lambda$) process in time $(0,t)$. The $\phi_s(n)$ can be determined using $\tilde{P}$ and $\pi_i(0)$ in the standard MC way; that is,

$$\phi(n) = \pi(0) (\tilde{P})^n$$

$$= \phi(n-1) \tilde{P},$$

(2)
where \( \pi(0) \) is the starting state probability vector. The infinite sum in (1) must be truncated at some appropriate point (see Gross and Miller, op. cit.) which can be set to guarantee a bound on the error.

Using the target and rate vectors from the SERT procedure, we can compute the \( \hat{\phi}(n) \) vectors recursively in a manner more efficient than using (2) by the following algorithm:

(i) \( \hat{\phi}(0) = \pi(0) \)

(ii) \( \hat{\phi}(n + 1) \) is computed from \( \hat{\phi}(n) \) as follows:

\[
(3) \quad \phi_s(n + 1) = \phi_s(n) \cdot \left[ \begin{array}{c} \frac{E}{j} \frac{r^i_j}{r_s} \\ 1 - \frac{1}{\lambda} \end{array} \right]
\]

then

(b) for \( j = 1, 2, \ldots, E \), and \( s \in S \), add

\[
\phi_s(n) \cdot (r^j_s / \lambda) \text{ to } \hat{\phi}_j(n + 1).
\]

What this algorithm does is to operate simultaneously on all components of \( \hat{\phi}(n + 1) \) using the components of \( \hat{\phi}(n) \) and the transition probabilities from \( \hat{\theta} \). Note that \( (r^j_s / \lambda) \) is \( \hat{\theta}^{i_j} \); that is, the probability of going from state \( s \) to state \( t^i_s, t^j_s \) (the target state that event \( j \) causes the system to switch to when it is in state \( s \)) given an occurrence of the Poisson \((\lambda)\) process. The term

\[
(1 - \sum_{j=1}^{E} r^j_s / \lambda)
\]

is the probability of ignoring an occurrence of the Poisson \((\lambda)\) process, which is called thinning and which leaves the system in state \( j \) (a null event).

This algorithm is very efficient for sparse \( Q \) matrices, which is the situation we generally have for repairable item systems, since it avoids the zero multiplication that would result from using the matrix multiplication of (2).
4. SIMPLE NETWORK MODEL: \((1,1,1/1,0/f,f)\) SYSTEM

The simplest case of a repairable item system is the simple machine repair model shown in Figure 1. The situation modeled has a population consisting of \(M\) items desired to be operational at all times and \(Y\) spares to support the system. There are \(c\) repair channels, so that a maximum of \(c\) items can be undergoing repair simultaneously. If more than \(c\) items require repair, a queue forms at the repair facility. Failure and repair times are exponentially distributed random variables with the mean time to failure of any item denoted by \(1/\lambda\) and the mean time to repair an item denoted by \(1/\mu\); i.e., \(\lambda\) and \(\mu\) are the failure rate and repair rate, respectively. The total number of items in the system is \(M + Y = N\) and if \(s\) items are in repair, then \(N - s\) items are at the operating node, so that when \(s > Y\), the population is operating at less than the desired strength of \(M\).

For modeling purposes, the operating node of the two-node network pictured in Figure 1 can be considered to be an \(M\) channel queue, so

\[\text{M Operating Units} \quad \text{Failed Units} \left[\lambda \text{ Min}(M,N-s)\right]\]

\[\text{Y Spares} \quad \text{Repaired Units} \left[\mu \text{ Min}(s,c)\right]\]

\[\text{Repair Facility} \quad c \text{ channels} \quad s\]

Figure 1.--\((1,1,1/1,0/f,f)\) system.
that if there are more than $M$ units at this node ($s < Y$), the queue represents on-hand spares. If no queue is present and some "servers" are idle at the operating node ($s > Y$), then the population is operating at degraded strength (as mentioned above).

Utilizing the SERT methodology, the state of the system $s$ can be considered to be the number of items in or awaiting repair (in re-supply). The size of $S$ is $N = M + Y + 1$, since $s = 0, 1, 2, \ldots, M + Y$.

There are only two types of events ($E = 2$). The first type of event is a failure. The rate vector for failures is

$$r^f_s = \begin{cases} M \lambda & 0 \leq s \leq Y \\ (M + Y - s) \lambda & Y < s < N \end{cases}$$

and the target state vector is

$$t^f_s = \begin{cases} s + 1 & s = 0, 1, \ldots, N - 1 \\ N & s = N \end{cases}$$

since a failure increases the number in resupply by one, except when $N$ are in resupply. In this case a failure cannot occur, but we show $t^f_N$ as $N$; that is, a "mythical" failure does not change the state.

The second type of event is a service completion. The $s$th component of the rate vector $r^r_s$ is

$$r^r_s = \begin{cases} s \mu & 0 \leq s < c \\ c \mu & c \leq s \leq N \end{cases}$$

and the corresponding $s$th component of the $t^r_s$ vector is

$$t^r_s = \begin{cases} s - 1 & 1 \leq s \leq N \\ 0 & s = 0 \end{cases}.$$
We can now apply the SERT multiplication algorithm (3), which reduces to

(i) $\phi(0) = \pi(0)$

(ii) (a) For $s = 0,1,...,N$, set

$$\phi_s(n + 1) = \phi_s(n) \cdot \left( 1 - \frac{r_s^f + r_s}{\Lambda} \right),$$

then

(b₁) For $s = 0,1,...,N$, add

$$\phi_s(n) \cdot \frac{r_s^f}{\Lambda} \text{ to } \phi_s(n + 1)$$

(b₂) For $s = 0,1,...,N$, add

$$\phi_s(n) \cdot \frac{r_s}{\Lambda} \text{ to } \phi_s(n + 1).$$

From $\phi(n)$, we can obtain $\pi(t)$ rather easily using equation (1), and use $\pi(t)$ to obtain system performance measures, such as system availability, which we define as the probability that the desired number of components ($N$) are operating at time $t$; that is,

$$A(t) = \sum_{s=0}^{N} \pi_s(t)$$

Various other measures such as expected backorder level, expected number of units operating, etc., can also be readily calculated from $\pi(t)$.

Computational efficiency for system performance measures can be gained by using $\phi(n)$ and then converting to continuous time (see Gross and Miller, op. cit.), for example, defining
we have

\[ A_n = \sum_{s=0}^Y \phi_s(n), \]

\[ A(t) = \sum_{n=0}^T A_n \frac{e^{-At}(At)^n}{n!}. \]

This algorithm has been coded as an interactive FORTRAN program called REPTRAN1. Sample input and output \([A(t) \text{ vs. } t]\) are shown in Figures 2 and 3, respectively. Also shown is the equivalent \((1,1,1/1,0/\infty,\infty)\) calculations—the Dyna-METRIC model. This has also been coded up as part of the REPTRAN1 program.

Finite source, finite repair models \((-,-,-/,-,-/f,f)\) are referred to as "closed queuing network" models, for these types of models can indeed be viewed as closed queuing networks \([\text{the } (1,1,1/1,0/\infty,\infty) \text{ of REPTRAN1 is the simplest closed network, namely, a two-stage cyclic queue.}]\)

5. SPECIAL CASE: \((2,2,2/1,0/f,f)\)

We now present an implementation of the techniques discussed previously to the computation of transient probabilities and availabilities of a \((2,2,2/1,0/f,f)\) system. (It has been coded as a FORTRAN program called REPTRAN2.) The system is shown in Figure 4. Items of one type move around a network with six nodes, namely, operational at Base 1 (BU1), in or awaiting repair at Base 1 (BR1), operational at Base 2 (BU2), in or awaiting repair at Base 2 (BR2), operational (ready spares) at depot (DU), and in or awaiting repair at depot (DR). The description of the system will follow the approach used earlier. The number of operating machines
$ RUN REPTRAN1

THIS IS REPTRAN1 PROGRAM

DO YOU WANT A HARDCOPY (Y OR N)?
N
DO YOU WANT TO PRINT ALL OUTPUT (Y OR N)?
Y
Type: Initial time (assume zero), final time, time increment
0,15,1
NUMBER OF TIME POINTS = 16  NUMBER OF TIME PERIODS = 15
Case number: 1
DO YOU WANT TO RUN A CLOSED QUEUEING NETWORK OR A DYNAMETRIC MODEL
TYPE 1 OR 2 ACCORDINGLY
1
Type: M, C, Y, EPSILON
3,2,2,.001
Type number of lambda values to be used
2
Type lambda values
.2,.3
Type times of shift of lambdas (start w/ 0)
0,6
Type number of mu values to be used
2
Type mu values
.5,.75
Type times of shift of mu (start w/ 0)
0,10
Do you want to type in an initial prob. vector? (Type Y or N)
If you type N, program assumes (1,0,0,...,0) as initial prob. vector.
N
DO YOU WANT TO RUN ANOTHER CASE?
Y
Case number: 2
DO YOU WANT TO RUN A CLOSED QUEUEING NETWORK OR A DYNAMETRIC MODEL
TYPE 1 OR 2 ACCORDINGLY
2
Type LAMBDA1, LAMBDA2 AND THE TIME AT WHICH THE CHANGE OF LAMBDA OCCURS
.6,.9,.6
Type MU1, MU2, MU3
.5,.5,.75
Type the times where the mus change values

Figure 2.—Sample input for REPTRAN1: Closed queueing network and Dyna-METRIC runs.
Figure 3: Sample output for BRPMAN (1) and Dyna-MIRC (2) runs.

Closed queuing network

AVAILABILITY VS. TIME

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The number of repairable items at Base 1 equals the maximum of the desired number, MS1, and the number of machines at node BU1 (#BU1); similarly for Base 2.* The number of spares available at the depot spares pool is the number at node DU (#DU). The number of busy repair channels at Base 1 equals the minimum of the number of repair channels, BCl, and the number of items at node BR1 (#BR1); similarly for Base 2 and the depot. Thus, the system parameters (failure and repair rates) and the number of machines at each node give us total information about the system. Knowledge of how machines move around the network completes the description of the system. The only point which must yet be specified in detail is the assignment rule for filling backorders to the depot.

Consider the situation where the number of machines at node DR (#DR) is greater than the spares allocated to the depot, DS. In this case there will be no machines at node DU and a backorder level of #DR - DS at the depot.

*Quantities with "#" preceding them are system state variables; quantities without # are either node designators, event descriptors, or preset parameter values.
The question arises concerning the allocation of the next machine to complete repair at the depot: To which base should it be given? We must define an allocation function based on the state of the system. For now, let us define the state of the system as the number of machines at each node:

\[ s = (\#BU1, \#BR1, \#BU2, \#BR2, \#DR, \#DU) \]

The number of backorders at the depot from Bases 1 and 2, respectively, are

\[ \#BD1 = BS1 - (\#BU1 + \#BR1) \]
\[ \#BD2 = BS2 - (\#BU2 + \#BR2) \]

where BS1 and BS2 are the numbers of spares allocated to Bases 1 and 2, respectively. Possible allocation functions are

\[
\text{ALL1}(s) = \begin{cases} 
1 & \text{if } \#BD1 > \#BD2 \\
2 & \text{if } \#BD2 > \#BD1
\end{cases}
\]

or

\[
\text{ALL2}(s) = \begin{cases} 
1 & \text{if } \#BD1/BS1 > \#BD2/BS2 \\
2 & \text{if } \#BD1/BS1 > \#BD2/BS2
\end{cases}
\]

In the program REPTRAN2 we use a randomized generalization of the second allocation function: Let PROB.ALL(*) be a function that equals the probability of assigning the repaired machine to Base 1. The assignment rule is given in Table II. The weights W1 and W2 are supplied by the user and reflect his

**TABLE II**

ASSIGNMENT RULE FOR SENDING A REPAIRED MACHINE FROM DEPOT IN BACKORDER SITUATION

<table>
<thead>
<tr>
<th>Condition</th>
<th>Probability of Assignment to Base 1</th>
<th>Probability of Assignment to Base 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>W1 * #BD1 &gt; W2 * #BD2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>W1 * #BD1 = W2 * #BD2</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>W1 * #BD1 &lt; W2 * #BD2</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

*Because it is a closed system, we shall see later that s can be characterized by fewer state variables.*
strategy for favoring one base over the other when both have backordered units.

The description of the system is now complete and we proceed to describe the Markov model of the system using the SERT approach. This requires (i) identifying the state space (as a vector), (ii) defining the event set, and for each event (iii) computing the vector of transition rates, and (iv) computing the vector of target states. The randomization algorithm can then be used to compute transient probabilities to any user-specified accuracy (ε). The events are given in Table III; once the state space is described as a one-dimensional vector, steps (iii) and (iv) are straightforward.

In general, the state space appears to have six dimensions, but because of one-for-one ordering and conservation of the total number of items in the system, the state space actually has a lower dimension.

TABLE III

THE EVENTS THAT ACCOUNT FOR ALL THE STATE CHANGES OF A (2,2,2/1,0/f,f) SYSTEM WITH A SINGLE TYPE OF ITEM

<table>
<thead>
<tr>
<th>Name</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>B1R</td>
<td>Repair completed at Base 1</td>
</tr>
<tr>
<td>B2R</td>
<td>Repair completed at Base 2</td>
</tr>
<tr>
<td>F1B</td>
<td>Failure at Base 1 (base repairable)</td>
</tr>
<tr>
<td>F2B</td>
<td>Failure at Base 2 (base repairable)</td>
</tr>
<tr>
<td>F1D</td>
<td>Failure at Base 1 (depot repairable)</td>
</tr>
<tr>
<td>F2D</td>
<td>Failure at Base 2 (depot repairable)</td>
</tr>
<tr>
<td>DR1</td>
<td>Repair completed at depot and sent to depot spares pool if not backorder situation; otherwise sent to Base 1</td>
</tr>
<tr>
<td>DR2</td>
<td>Repair completed at depot while backorder situation and sent to Base 2</td>
</tr>
</tbody>
</table>
The description of the state space breaks into two situations: no depot spares available, and some depot spares available,

\[ S = S_0 \cup S_+ \]

where

- \( S_0 \) = states with depleted depot spares pool
- \( S_+ \) = states with nondepleted depot spares pool.

First consider \( S_0 \). In this case, it is possible to describe the state of the system with four numbers:

\[ (#BU1, #BD1, #BU2, #BD2) \]

It is known that \( #DU = 0 \) and the remaining machines are at node DR. The feasible states of \( S_0 \) are subject to two constraints:

\[ #BU1 + #BD1 \leq BS1 \]
\[ #BU2 + #BD2 \leq BS2 \]

and thus \( S_0 \) is a Cartesian product,

\[ S_0 = T_1 \times T_2 \]

where

\[ T_1 = \{ (#BD1, #BU1): #BD1 + #BU1 \leq BS1 \} \]
\[ T_2 = \{ (#BD2, #BU2): #BD2 + #BU2 \leq BS2 \} \]

These sets are shown in Figure 5. (The notation \( T \) is used because the spaces are triangular.) Note that the number of points in \( T_1 \) and \( T_2 \) are

\[ |T_1| = \frac{(BS1+1)(BS1+2)}{2} \]
\[ |T_2| = \frac{(BS2+1)(BS2+2)}{2} \]

respectively, and the number of states in \( S_0 \) is the product

\[ |S_0| = \frac{(BS1+1)(BS1+2)(BS2+1)(BS2+2)}{4} \]
Figure 5.—Examples of state space to describe individual bases; in the case BS1 = 4 and BS2 = 5.

Now let us consider the states where the spares pool at the depot is not empty, $S_+$. In this case the state of the system can be described by three numbers:

$$(\#BU1, \#BU2, \#DU)$$

The constraints on these are

$$\#BU1 \leq BS1$$
$$\#BU2 \leq BS2$$
$$1 \leq \#DU \leq DS$$

We can condition on the value of $\#DU$ to get $S_+$ into the form

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\[ S_+ \cup S_1 \cup S_2 \cup \ldots \cup S_{DS} \]

where \( S_i \) consists of states with exactly \( i \) machines in the depot spares pool; such a set is depicted in Figure 6. Note that each \( S_i \) is a rectangle and

\[ |S_i| = (BS_1 + 1)(BS_2 + 1) \]

Thus

\[ |S_+| = (BS_1 + 1)(BS_2 + 1)DS \]

and the total number of states is

\[ |S| = \frac{(BS_1+1)(BS_1+2)(BS_2+1)(BS_2+2)}{4} + (BS_1 + 1)(BS_2 + 1)DS \]

Examples of state space sizes are given in Table IV.

Although in principle we can apply the randomization algorithm to a system whose state space is described in this complex multidimensional form, we choose to work with one-dimensional state spaces. We made this choice for two reasons. First, one general implementation of the randomization algorithm will work on all systems after they are put into

\[
\begin{array}{cccccc}
\#UB1 & 2 & 3 & 4 & 5 \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\#UB2 & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
4 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

Figure 6.—The state space describing status of bases given that depot spares are available; BS1 = 4 and BS2 = 5.
TABLE IV

SIZE OF THE STATE SPACES OF (2,2,2/1,0/1,2) SYSTEM
FOR SELECTED VALUES OF BS1, BS2, AND DS

| BS1 | BS2 | DS | |S_0| + |S_+| - |S| |
|-----|-----|----|---|---|---|---|
| 2   | 2   | 2  | 36 | 18 | 54 |
| 4   | 4   | 2  | 225| 50 | 275|
| 6   | 6   | 2  | 784| 98 | 882|
| 8   | 8   | 2  | 2025| 162| 2187|
| 10  | 10  | 2  | 4356| 242| 4598|
| 12  | 12  | 2  | 8281| 338| 8619|
| 18  | 18  | 2  | 36100| 722| 36822|
| 24  | 24  | 2  | 105625| 1250| 106875|

one-dimensional form, and it seems to decrease the problems in verifying
the program. Second, and even more important, working directly with
vectors rather than structures such as

\[ S = T_1 \times T_2 \cup_{i=1}^{DS} S_i \]

will speed up the algorithm by a significant factor because it is much
faster to "fetch" and "store" an element of a one-dimensional vector
than it is to "fetch" and "store" an element of a complicated multi-
dimensional array. We estimate that this speeds up REPTRAN2 by a factor
of four or more.

Thus, let us consider putting the elements of

\[ S = T_1 \times T_2 \cup_{i=1}^{DS} S_i \]
into a one-dimensional form. First consider $T_1$; this triangular shaped region can be transformed into a linear space $L_1$ by placing the rows end to end,

$L_1 = \{00, 01, 02, 03, 04, 10, 11, 12, 13, 20, 21, 22, 30, 31, 40\}$

Doing the same thing for $T_2$ yields $L_2$; thus,

$$S_0 = T_1 \times T_2 \equiv L_1 \times L_2$$

becomes a two-dimensional set which can be linearized again by placing the rows end to end as shown graphically in Figure 7. The first $|S_0|$ elements of our one-dimensional vector will correspond to the states in $S_0$ in linear order as presented in Table IV.

The remaining $|S_+|$ elements in the one-dimensional vector will correspond to the states in $S_+$ arranged linearly as follows:

$$S_+ = \bigcup_{i=1}^{DS} S_i$$

where $S_i$ is depicted in Figure 6. Each $S_i$ is put into one-dimensional form by putting rows end to end (as in Figure 7). Finally, these DS

$$S_0 = L_1 \times L_2$$

00 01 02 03 04 10 11 12 13 20 21 22 30 31 40

0000, 0001, ..., 0050, 0100, 0101, ..., 0150, ..., 4000, 4001, ..., 4050

Linear version

Figure 7.--Depiction of linearization of the Cartesian product being transformed into a one-dimensional space.
one-dimensional $S_1$'s are put end to end starting with $S_1$ and ending with $S_{DS}$. (Thus the last state in this one-dimensional listing is (BS1, BS2, DS), which corresponds to BS1 units "up" at Base 1 ($\#BU1 = BS1$), BS2 units "up" at Base 2, and DS units in the spares pool at the depot, i.e., a perfect system with no failed units. The first state listed in the one-dimensional version of the state space is (0, 0, 0, 0), which corresponds to BS1 units in repair at Base 1, BS2 units in repair at Base 2, and DS units in repair at the depot.)

This completes the description of the state space of this $(2,2,2/1,0/f,f)$ system as a one-dimensional vector. It is used by the program REPTRAN2. A major part of REPTRAN2 is subroutines that take the parameters and rates and compute one-dimensional vectors of transition rates and one-dimensional vectors of target states for each event in Table III. It then uses the randomization algorithm logic to compute transient state probabilities and availabilities as described in Section 2.

The program REPTRAN2 can also compute transient probabilities for $(2,2,2/1,0/f,f)$ systems whose underlying rates (failure and repair rates) change at discrete points in time (on a lattice time scale)—up to five changes are accommodated for each underlying rate.

We note that by setting certain parameters to special values, REPTRAN2 can handle the corresponding special cases. If $DS = 0$, we lose the second level of supply and get a $(2,2,1/1,0/f,f)$ system. If $FB1 = 1$ then all failures at Base 1 are base repairable and Base 1 does not interact with the depot: thus Base 1 is a $(1,1,1/1,0/f,f)$ system and Base 2 and the depot form a $(1,2,2/1,0/f,f)$ system, or $(1,2,1/1,0/f,f)$ if $DS = 0$. 
6. COMPUTER RUNS

The REPTRAN2 program has been run for various \((2,2,2/1,0/f,f)\) systems as described in Section 5. The program executes in quite short times for systems with large state spaces. It also compares the probabilities computed by the Dyna-METRIC model for corresponding \((-,-,-/-\infty,\infty)\) systems, which are also a part of the REPTRAN2 code. [For the Dyna-METRIC developments, details for these corresponding systems, see Gross, Kioussis, and Miller (1982).] Furthermore, it is reassuring that when REPTRAN2 was run on a system whose parameters gave rise to a \((1,1,1/1,0/f,f)\) subsystem (as discussed in Section 4), the program gave the same answers as REPTRAN1.

REPTRAN2 is interactive and the user has the option of running either the Dyna-METRIC model or the closed queuing network model for a \((2,2,2/1,0/\ast,\ast)\) system. Availability is defined as the probability that the number of machines "up" at a base meets or exceeds the number desired to be operating, and is the primary measure outputted.

We ran a few sample cases to get an idea of how well this program performs (with reference to time) in computing exact transient solutions. The sample cases we ran are described in Table V. For each case, all the necessary parameters (which the user must supply) are listed. The failure rates and repair rates shift at the times indicated for each case.

For example, in Case 1a: At time \(t_0 = 0\), the failure rate of a machine at Base 1 is \(\lambda_1 = 0.4\); then at time \(t_1 = 6\), this rate changes to \(\lambda_1 = 0.6\). At time \(t_0 = 0\) the repair rate of a single repair channel at Base 1 is \(\mu_1 = 0.5\); then at time \(t_1 = 10\), this rate changes to \(\mu_1 = 0.75\). Similar changes occur for the failure and repair rates at Base 2 and the repair rate at the depot.
### TABLE V
PARAMETER VALUES OF THE SAMPLE CASES SOLVED USING REPTRAN2

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>BSi</td>
<td>Allocation of total stock to Base i (operating machines plus spares), i = 1, 2</td>
</tr>
<tr>
<td>MSi</td>
<td>Desired number of working machines at Base i</td>
</tr>
<tr>
<td>BCi</td>
<td>Number of repair channels in repair shop at Base i</td>
</tr>
<tr>
<td>FBI</td>
<td>Proportion of items failing at Base i, base repairable</td>
</tr>
<tr>
<td>t₀</td>
<td>Time zero</td>
</tr>
<tr>
<td>t₀λᵢ</td>
<td>Time of shift in mean failure rate, Base i</td>
</tr>
<tr>
<td>t₀μᵢ</td>
<td>Time of shift in mean repair rate, Base i</td>
</tr>
<tr>
<td>λᵢ(₀)</td>
<td>Initial mean failure rate, Base i items</td>
</tr>
<tr>
<td>λᵢ(λᵢ)</td>
<td>Shifted mean failure rate, Base i items</td>
</tr>
<tr>
<td>μᵢ(₀)</td>
<td>Initial mean repair rate, Base i items</td>
</tr>
<tr>
<td>μᵢ(μᵢ)</td>
<td>Shifted mean repair rate, Base i items</td>
</tr>
<tr>
<td>Wi</td>
<td>Weighting factor for filling depot backorders for Base i</td>
</tr>
<tr>
<td>DS</td>
<td>Number of depot spares</td>
</tr>
<tr>
<td>DC</td>
<td>Number of depot repair channels</td>
</tr>
<tr>
<td>μ₀(₀)</td>
<td>Initial mean depot repair rate</td>
</tr>
<tr>
<td>t₀D</td>
<td>Time of shift in mean depot repair rate</td>
</tr>
<tr>
<td>μ₀(₀)</td>
<td>Shifted mean depot repair rate</td>
</tr>
<tr>
<td>ε</td>
<td>Error tolerance</td>
</tr>
</tbody>
</table>
Table V--continued

B. REPTRAN2 Run Description

<table>
<thead>
<tr>
<th>BS1</th>
<th>MS1</th>
<th>BC1</th>
<th>FB1</th>
<th>λ₁(t₀)</th>
<th>λ₁(t₁)</th>
<th>u₁(t₀)</th>
<th>u₁(t₁)</th>
</tr>
</thead>
<tbody>
<tr>
<td>BS2</td>
<td>MS2</td>
<td>BC2</td>
<td>FB2</td>
<td>λ₂(t₀)</td>
<td>λ₂(t₁)</td>
<td>u₂(t₀)</td>
<td>u₂(t₁)</td>
</tr>
<tr>
<td>DS</td>
<td>DC</td>
<td>W1</td>
<td>W2</td>
<td>μ₆(t₀)</td>
<td>μ₆(t₁)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ε</td>
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<td></td>
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Cases

la

<p>| | | | | | | | |</p>
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<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>2</td>
<td>.7</td>
<td>.4(0)</td>
<td>.6(6)</td>
<td>.5(0)</td>
<td>.75(10)</td>
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<tr>
<td>5</td>
<td>3</td>
<td>2</td>
<td>.5</td>
<td>.4(0)</td>
<td>.6(8)</td>
<td>.6(0)</td>
<td>.9(12)</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>.4</td>
<td>.6</td>
<td>.3(0)</td>
<td>.45(11)</td>
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<td></td>
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.001

lb

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<td>4</td>
<td>2</td>
<td>2</td>
<td>.7</td>
<td>.4(0)</td>
<td>.6(6)</td>
<td>.5(0)</td>
<td>.75(10)</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>2</td>
<td>.5</td>
<td>.4(0)</td>
<td>.6(8)</td>
<td>.6(0)</td>
<td>.9(12)</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>.5</td>
<td>.5</td>
<td>.3(0)</td>
<td>.45(11)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

.001

2a

<p>| | | | | | | | |</p>
<table>
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<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>2</td>
<td>.7</td>
<td>.2(0)</td>
<td>.4(6)</td>
<td>.5(0)</td>
<td>.75(10)</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>2</td>
<td>.5</td>
<td>.1(0)</td>
<td>.15(8)</td>
<td>.4(0)</td>
<td>.6(12)</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>.4</td>
<td>.6</td>
<td>.3(0)</td>
<td>.45(11)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

.001
Table V--continued

<p>| | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2b</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>.7</td>
<td>.2(0)</td>
<td>.3(6)</td>
<td>.5(0)</td>
<td>.75(10)</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>.5</td>
<td>.1(0)</td>
<td>.15(8)</td>
<td>.4(0)</td>
<td>.6(12)</td>
</tr>
</tbody>
</table>
|    | 2 | 3 | .4| .6| .3(0)| .45(11)|   |   |.
| .001|   |   |   |   |   |   |   |   |
| 3  |   |   |   |   |   |   |   |   |
|    | 4 | 2 | 2 | .7| .2(0)| .3(6)| .5(0)| .75(10)|
|    | 5 | 3 | 2 | .5| .1(0)| .15(8)| .4(0)| .6(12)|
|    | 2 | 2 | .4| .6| .3(0)| .45(11)|   |   |.
| .00001|   |   |   |   |   |   |   |   |
| 4  |   |   |   |   |   |   |   |   |
|    | 8 | 4 | 4 | .7| .4(0)| .6(6)| .5(0)| .75(10)|
|    | 10| 6 | 4 | .5| .4(0)| .6(8)| .6(0)| .9(12)|
|    | 4 | 4 | .5| .5| .3(0)| .45(11)|   |   |.
| .001|   |   |   |   |   |   |   |   |
| 5  |   |   |   |   |   |   |   |   |
|    | 18|14 | 2 | .6667| .2(0)| .3(6)| 1.0(0)| 1.5(10)|
|    | 13|10 | 2 | .6667| .143(0)| .2143(6)| 1.0(6)| 1.5(10)|
|    | 3 | 4 | .5| .5(0)| .75(10)|   |   |   |.

The seven cases in Table V were solved exactly over 16 time points \{0,1,...,15\}. The CPU execution times are shown in Table VI, as well as the size of the state space of the model. Also shown is the set of time points at which rate changes (failure or repair) occurred.
TABLE VI
SIZE OF STATE SPACE AND RUNNING TIMES FOR REPTRAN2 SOLUTION OF VARIOUS (2,2,2/1,0/f,f) SYSTEMS

<table>
<thead>
<tr>
<th>System Number</th>
<th>Size of State Space</th>
<th>Homogeneous Time Interval</th>
<th>CPU (sec.)a</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a</td>
<td>375</td>
<td>0, 6, 8, 10, 11, 12</td>
<td>10.18</td>
</tr>
<tr>
<td>1b</td>
<td>375</td>
<td>0, 6, 8, 10, 11, 10</td>
<td>10.48</td>
</tr>
<tr>
<td>2a</td>
<td>375</td>
<td>0, 6, 8, 10, 11, 12</td>
<td>8.07</td>
</tr>
<tr>
<td>2b</td>
<td>375</td>
<td>0, 6, 8, 10, 11, 12</td>
<td>8.87</td>
</tr>
<tr>
<td>3</td>
<td>375</td>
<td>0, 6, 8, 10, 11, 12</td>
<td>9.04</td>
</tr>
<tr>
<td>4</td>
<td>3366</td>
<td>0, 6, 8, 10, 11, 12</td>
<td>190.25</td>
</tr>
<tr>
<td>5</td>
<td>20748</td>
<td>0, 6, 10</td>
<td>1393.73</td>
</tr>
</tbody>
</table>

aRunning time is approximately proportional to size of state space x number of event types x number of occurrences of Poisson process.

The seven cases in Table VI were also solved using the DynaMETRIC approximate model (2,2,2/1,0/∞,∞). For this comparison it was necessary to select failure arrival rates at each base. We chose an arrival rate of M1 * λ1 at the Base 1 repair shop and M2 * λ2 at the Base 2 repair shop. For systems operating at moderate to high availabilities, these should be approximately correct.

We show the computer input and output for case 5 given in Table V as Figures 8 through 12. The interactive input is shown, first modelled as an exact queuing network (2,2,2/1,0/f,f) and then modelled as a DynaMETRIC model (2,2,2/1,0/∞,∞). Both models were solved for availabilities and then superimposed on three plots. The exact availabilities are
TYPE: INITIAL TIME (assume zero), FINAL TIME, TIME INCREMENT 0,15,1
NUMBER OF TIME POINTS = 16  NUMBER OF TIME PERIODS = 15

CASE NUMBER : 1
DO YOU WANT TO RUN A CLOSED QUEUEING NETWORK OR A DYNAMETRIC MODEL
TYPE 1 OR 2 ACCORDINGLY
1

TYPE BS1, MS1, BC1, AND FB1
18, 14, 2, .6667
BS1 : 18  MS1 : 14  BC1 : 2  FB1 : 0.6667

TYPE BS2, MS2, BC2, AND FB2
13, 10, 2, .6667
BS2 : 13  MS2 : 10  BC2 : 2  FB2 : 0.6667

TYPE DS, DC, U1, U2
3, 4, .5, .5
DS : 3  DC : 4  U1 : 0.5000  U2 : 0.5000

TYPE NUMBER OF FAILURE RATES TO BE USED IN BASE 1
2
TYPE THE FAILURE RATES
-2, -3
TYPE THE TIME WHERE THOSE CHANGES IN RATES OCCUR (START WITH 0)
0, 6
TYPE NUMBER OF REPAIR RATES TO BE USED IN BASE 1
2

TYPE THE REPAIR RATES
1, 1.5
TYPE THE TIME WHERE THOSE CHANGES IN RATES OCCUR (START WITH 0)
0, 10
TYPE NUMBER OF FAILURE RATES TO BE USED IN BASE 2
2

TYPE THE FAILURE RATES (BASE 2)
.143, .2145
TYPE THE TIME WHERE THOSE CHANGES IN RATES OCCUR (START WITH 0)
0, 6
TYPE NUMBER OF REPAIR RATES TO BE USED IN BASE 2
2

TYPE THE REPAIR RATES (BASE 2)
1, 1.5
TYPE THE TIME WHERE THOSE CHANGES IN RATES OCCUR (START WITH 0)
0, 10
TYPE NUMBER OF REPAIR RATES TO BE USED IN DEPOT
2

TYPE THE REPAIR RATES (DEPOT)
.5, .75
TYPE THE TIME WHERE THOSE CHANGES IN RATES OCCUR (START WITH 0)
0, 10
TYPE THE DESIRED MAGNITUDE OF THE ERROR (EPSILON)
.001

Figure 8.—Sample input for REPTRAN2 case 5, closed queuing network.
DO YOU WANT TO RUN ANOTHER CASE? (Y or N)

Y
CASE NUMBER: 2
DO YOU WANT TO RUN A CLOSED QUEUEING NETWORK OR A DYNAMETRIC MODEL
TYPE 1 OR 2 ACCORDINGLY
2
TYPE THE TIME AT WHICH THE FAILURE RATE AND THE REPAIR RATE CHANGE AT BASE
6, 10
TYPE THE TWO FAILURE RATES FIRST AND THEN THE TWO REPAIR RATES
2.8, 4.2, 1.0, 1.5
TYPE THE STOCK LEVEL AT BASE 1 AND THE PROPORTION OF THE FAILED ITEMS GO:
BASE REPAIR 1
4, .6667
TYPE THE TIME AT WHICH THE FAILURE RATE AND THE REPAIR RATE CHANGE AT BASE
6, 10
TYPE THE TWO FAILURE RATES FIRST AND THEN THE TWO REPAIR RATES (BASE 2)
1.43, 2.143, 1.0, 1.5
TYPE THE STOCK LEVEL AT BASE 2 AND THE PROPORTION OF THE FAILED ITEMS GO:
BASE REPAIR 2
3, .6667
TYPE THE TIME AT WHICH THE REPAIR RATE CHANGES AT THE DEPOT
10
TYPE THE TWO REPAIR RATES (FOR DEPOT)
.5,.75
TYPE THE STOCK LEVEL AT THE DEPOT
3

Figure 9.—Sample input for REPTRAN2 case 5, Dyna-METRIC.

plotted using the symbol "1" and the approximate Dyna-METRIC availabilities are plotted using the symbol "2." For time points where the symbol "1" fails to appear, it coincides with the "2." "Availability 1" is availability at Base 1, "Availability 2" is availability at Base 2, and "Availability 3" is simultaneous availability at both Base 1 and Base 2.
<table>
<thead>
<tr>
<th>TIME</th>
<th>0.0</th>
<th>1.0</th>
<th>2.0</th>
<th>3.0</th>
<th>4.0</th>
<th>5.0</th>
<th>6.0</th>
<th>7.0</th>
<th>8.0</th>
<th>9.0</th>
<th>10.0</th>
<th>11.0</th>
<th>12.0</th>
<th>13.0</th>
<th>14.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>AVAILABILITY 1</td>
<td>1.0</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>AVAILABILITY 1 VS. TIME</td>
<td>0.0</td>
<td>0.4</td>
<td>0.6</td>
<td>0.8</td>
<td>1.0</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
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<td>2</td>
</tr>
</tbody>
</table>

Figure 10.—Sample output for REPTRAN2 case 5,
1: closed queuing network,
2: Dyna-METRIC,
availabilities at Base 1.
<table>
<thead>
<tr>
<th>TIME</th>
<th>0.0</th>
<th>1.0</th>
<th>2.0</th>
<th>3.0</th>
<th>4.0</th>
<th>5.0</th>
<th>6.0</th>
<th>7.0</th>
<th>8.0</th>
<th>9.0</th>
<th>10.0</th>
<th>11.0</th>
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<th>13.0</th>
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<td>0.5</td>
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**AVAILABILITY 2 VS. TIME**

Figure 11.—Sample output for REPTAN2 case 5,
1: closed queueing network,
2: Dyna-METRIC,
availabilities at Base 2.
Figure 12.—Sample output for REPTRAN2 case 5,
1: closed queuing network,
2: Dyna-METRIC,
simultaneous availabilities at
both bases.
7. CONCLUSIONS

The SERT modelling technique and randomization computing algorithm [methodology developed in Gross and Miller (1982)] has been applied and implemented for computing transient performance measures of multi-echelon repairable item inventory systems. We have shown that it is feasible to compute exact probabilities for systems with large state spaces (20,000 or more states). Furthermore, for the types of systems under consideration, we believe that significantly larger cases are feasible using a truncated state space approach; that is, lumping the vast number of very low probability states together as one.

For example, in a \((l,l,l/l,0/f,f)\) case with 31 states \((M = 20, Y = 10)\), we found that the probabilities of \(s\) units in resupply for \(s \geq 15\) was zero to at least three significance figures. Lumping states \(s = 15, 16, \ldots, 31\) together would reduce the problem from 31 states to 16, a savings of almost 50%. We estimate in the 20,000 state space example that such a procedure would easily cut the number of states in half.

We have used the SERT modelling technique on a \((b,2,2/k,1/f,f)\) system [see Gross, Kioussis, and Miller (1982)]. This has not been coded, but for moderate \(b\) and \(k\), using the truncated state space approach, development of an efficient code should be feasible. Conceptually, of course, the most general \((b,r,s/k,j/f,f)\) system could be modelled using SERT; the problem, of course, is the state space size for cases other than those with very small values of \(b, r, s, k,\) and \(j\). The truncated state space approach offers the most promise for treating these models.
ACKNOWLEDGMENT

The authors thank Mr. L. Kioussis for his outstanding programming efforts, that resulted in the REPTRAN1 and REPTRAN2 codes.

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