Stabilizing Effect of Gas Conductivity Evolution on the Resistive Sausage Mode of a Propagating Beam

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**STABILIZING EFFECT OF GAS CONDUCTIVITY EVOLUTION ON THE RESISTIVE SAUSAGE MODE OF A PROPAGATING BEAM**

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**ABSTRACT:**
Previous theoretical work has shown that a highly current neutralized charged particle beam propagating in a pre-ionized plasma channel of fixed conductivity is subject to a resistive sausage instability. We show that the instability is stabilized, for the case of beam propagation into initially ionized gas, when the effect of beam-collisional ionization on the gas conductivity is modeled fully self-consistently.
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1. Introduction

A self-pinched electron or ion beam propagating in gas will excite a substantial return current if the gas is pre-ionized, if it is rapidly ionized collisionally by the beam head, or if avalanche breakdown is driven by inductive electric fields at the beam head. One may well expect highly current neutralized beams to be subject to a variety of instabilities excited by the repulsive force between beam and return current, and several model calculations have reached this conclusion with regard to beams propagating in resistive plasma. The situation differs sharply from that of a non-current-neutralized beam in a resistive plasma, for which the hose mode is the only unstable mode.

Under conditions where they are unstable, the axisymmetric beam modes appear to be particularly dangerous to propagation, because they are almost inevitably excited at large amplitude. Unless the beam emittance is perfectly matched at injection, the beam will oscillate in radius, and in particular, the violent pinchdown associated with the process of nose expansion and erosion can be expected to excite some radial oscillations. Non-axisymmetric modes, such as hose and filamentation, must grow out of initially low-level noise if the accelerator produces a high-quality beam, and thus must e-fold many more times before they pose a threat to beam integrity.

We classify the linearized normal modes of a beam by an azimuthal mode number \( n \) and a radial mode number \( n \). For a mode \((n,m)\), all perturbed quantities \( \psi \) are of the form

\[
\psi(r,\theta,z,t) = \exp(\imath m \theta - \imath \Omega t) \hat{\psi}(r,t),
\]

where \( \xi = ct - z \) is the position in the beam measured back from the beam head. Roughly speaking, \( n \) is the number of oscillations of \( \hat{\psi}(r,t) \) as \( r \) varies

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from zero to the outside of the beam, for fixed $\zeta$. The $m = 0$ (axisymmetric) modes are sometimes referred to collectively as sausage modes, but we shall reserve this term for the $m = 0, n = 1$ mode, which corresponds roughly to self-similar radial oscillation or "breathing" of the beam, and we shall refer to the next axisymmetric mode, $m = 0, n = 2$, as the axisymmetric hollowing mode. A recent theoretical study by Uhm and Lampe, based on the simplifying assumptions of a fixed flat-topped radial profile of plasma conductivity and a flat-topped beam radial profile, predicted sausage instability when the ratio of plasma return current to beam current $I_p/I_b > 0.50$ and axisymmetric hollowing instability when $I_p/I_b > 0.38$. Lee extended the theory of the sausage mode to arbitrary beam profiles and arbitrary (but fixed) conductivity profiles, and also predicted sausage instability when $I_p/I_b$ exceeds a threshold whose exact value depends on the profile. For similar Bennett profiles of beam current and conductivity, the threshold is 0.69; when $I_p/I_b > 0.75$, instability at $\Omega = 0$ is predicted. However the sausage instability has not been observed, as far as we are aware, for beams propagating in neutral gas, although $I_p/I_b$ often exceeds any of these instability thresholds.

In this paper, we present a more complete linearized theory of the sausage mode of a relativistic electron beam which begins from Lee's formulation but includes a self-consistent treatment of the plasma conductivity, including collisional ionization of the gas by the perturbed beam. We find that the conductivity channel perturbs in such a way as to follow the distortions of the beam. This inhibits the separation of beam and plasma current, which is the cause of the instability, and consequently leads to a more stringent instability condition, which is never satisfied for beams propagating in high density initially unionized gas. (The situation at low
gas density, \( \leq 50 \) torr depending on conditions, is discussed later in this introduction.) This instability condition is [see Eq. (41) in Sec. 2]

\[
\frac{I_p}{I_{eff}} > \frac{(2\lambda + \sqrt{3})^2}{\lambda + 1} H,
\]

where \( I_{eff} \) is a radially-averaged net current which determines the mean pinch force,

\[
\lambda \equiv \frac{k I_b}{2c} \equiv \frac{d}{dt} \left( \frac{\sigma(r=0, \xi) a^2}{2c} \right)
\]

is a measure of the rate of change of conductivity \( \sigma \) due to beam-molecule collisions, \( k \) is a coefficient depending on the ionization coefficient and mobility of the particular gas, and \( H \) is a factor of order unity.

The inequality (1) can easily be satisfied for beams propagating into pre-ionized gas. The instability predicted under those conditions typically has a smaller growth rate than the hose mode, but could dominate if the sausage mode is initiated at larger amplitude, as it normally will be in a well-prepared beam.

For beams injected into neutral gas, a very brief burst of avalanche ionization at the beam head typically has a strong influence on the degree of current neutralization. This effect is tacitly included in our theory, since \( I_{eff} / I_b \) is treated as a free parameter. However our analytic theory treats only beam-collisional ionization, and not avalanche, in the beam body where the instability grows. This is usually a good approximation, except in low density gas (5 - 50 torr depending on beam current density and gas type). We have found some cases in low density gas where avalanche at the beam head is so strong that the instability condition (1) is satisfied, but in all of these
cases noted to date. Avalanche should also be included in modeling in the beam body. Heuristic considerations indicate that the neglected avalanche term in the beam body would enhance stability, by further perturbing the conductivity channel so as to follow the beam distortions. We have tested this idea by performing a few simulations with the axisymmetric beam envelope code VIPER-O, which includes avalanche everywhere and permits sausage-like oscillations to develop (but does not permit any higher axisymmetric perturbations). Sausage instability did not occur in these cases, even though condition (1) was satisfied. Further study of this low-density regime is needed, however.

Particle simulations at several laboratories\textsuperscript{8-10} have recently observed strong axisymmetric instabilities under a variety of conditions that are compatible with the instability condition

\[ \frac{I_p}{I_b} \geq 0.50 \]  

(2)

but are incompatible with condition (1). We have shown by means of a simulation analysis that these instabilities involve the hollowing mode, not the sausage mode, and are triggered by a complex set of circumstances with other requirements in addition to (2). These results are reported in a separate paper.\textsuperscript{11}

The outline of this paper is as follows. We introduce our model and list its assumptions in Sec. 2. Our analytic calculation of the sausage mode dispersion relation is presented in Sec. 3. Our conclusions, with regard to beam propagation in initially neutral gas, pre-ionized gas, or in a channel of fixed conductivity profile, are discussed in Sec. 4.
2. **Formalism and Assumptions**

In this section we develop a fully analytic theory of the sausage mode of a relativistic electron beam that includes the modifications of the channel conductivity that result from beam-collisional ionization treated self-consistently with the sausaging of the beam. We find that the conductivity channel tends to follow the sausage distortion of the beam; as a result, spatial separation of the plasma return current density $J_p$ from the beam current density $J_b$ is reduced, and the mode is found to be much more stable than it would be in a fixed conductivity channel.

In order to carry out the analysis in simple form, we make a number of simplifying assumptions, most of them having wide validity. Four of these assumptions have been widely employed in beam propagation theory. They are that the background gas can be regarded as an immobile medium with a scalar conductivity $\sigma(x,t)$, that the beam is highly relativistic ($\gamma > 1$) and paraxial ($v_\perp \ll v_z$ for all electrons), and that therefore $v_z = c$.

We consider only instability growth in the region of the beam which we shall call the beam "body", behind the pinch point but forward of the beam tail where recombination limits the conductivity. This is the region where violent axisymmetric instability has been observed in simulations, and where theory indicates that instabilities should grow most rapidly. Here the conductivity $\sigma$ is large enough to insure space charge neutrality in the vicinity of the beam, and Maxwell's equations reduce to Ampere's law,

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial A(r, \xi, z)}{\partial r} \right) - \frac{4\pi \sigma(r, \xi, z)}{c} \frac{\partial A}{\partial \xi} = - \frac{4\pi}{c} J_b(r, \xi, z)$$

(3)

for an axisymmetric beam, where $A(r, \xi, z)$ is the axial component of the vector
potential. We can expect the beam body in equilibrium to have a Bennett radial profile,

\[ J_{bo}(r, \zeta, z) = \frac{J_{bo}}{1 + \frac{r^2}{a_o^2}}. \]  

(Other profiles could be treated.) In the beam body, the Bennett radius \( a_o \) typically decreases slowly with \( \zeta \) and increases very slowly (erosion and Nordsieck expansion\(^1\),\(^6\)) with \( z \); we use the approximation that \( a_o \) is completely independent of both \( z \) and \( \zeta \) in equilibrium. Since avalanche is normally unimportant in the beam body except for beams propagating in low density gas, e.g. \( P < 50 \text{ torr} \) for a typical induction linac beam with \( I_b = 10 \text{ kA} \), \( a_o = 0.5 \text{ cm} \), and since recombination is by definition unimportant in the beam body, we assume that the conductivity evolution is determined by direct collisional ionization of the gas by the beam,

\[ \frac{2}{\beta \zeta} \sigma(r, \zeta, z) = \kappa J_b. \]  

(However the effect of avalanche in the beam head is tacitly included as an initial condition, i.e. the values of \( \sigma \) and \( I_p/I_b \) specified at the front of the beam body segment included in the calculation, \( \zeta = \zeta_o \), depend implicitly on avalanche in the beam head, \( \zeta < \zeta_o \).) We neglect the fairly weak dependence of \( \sigma \) on plasma temperature (through the plasma electron collision frequency \( v_m \)) and thus treat \( \kappa \) as a constant. A reasonable estimate of \( \kappa \), based on the typical value\(^{12} \) \( v_m = 2 \times 10^{12} \text{ sec}^{-1} \) for air at standard density and \( 0.5 \text{ eV} \leq T_e \leq 1 \text{ eV} \), is

\[ \kappa = 8.8 \times 10^{-4} \text{ cm/statcoul}. \]
To facilitate the analysis, we also make the following additional assumptions. The equilibrium electric field $E_{z0}(r,\zeta,z)$, which has a weak radial dependence\textsuperscript{1,6}

$$E_{z0}(r,\zeta,z) = \ln \left( \frac{1 + b^2/a_0^2}{1 + r^2/a_0^2} \right), \tag{6a}$$

is regarded as independent of $r$,

$$E_{z0}(r,\zeta,z) = E_{z0}(0,\zeta,z). \tag{6b}$$

In (6a) $b$ is a large radius where charge neutrality fails. Approximation (6b) is common to all previous stability analyses. We note elsewhere that the breakdown of this approximation plays a key role in the destabilization of the hollowing mode,\textsuperscript{11} but we believe that the approximation is acceptable in analysis of the sausage mode.

As a result of Eqs. (4) - (6), the equilibrium conductivity and plasma current density, $\sigma_0(r,\zeta)$ and $J_{po}(r,\zeta)$, both have Bennett profiles of radius $a_0$. This turns out to be very helpful to the analysis: it allows us to reduce a problem in $r,\zeta$ and $z$ to the form of ordinary differential equations in $\zeta$ and $z$ only. For related problems of interest, the $r$-dependence cannot be eliminated. For example, in the beam "tail" where beam-collisional ionization is balanced by recombination, $\sigma(r) = [J_b(r)]^{1/2}$, a broader profile than $J_b(r)$. As a result, much of the plasma current flows outside the beam and has no destabilising effect on the beam, which essentially guarantees that the sausage mode will be stable. On the other hand, if the channel is fully ionized, as is the case typically for applications to ion-beam inertial fusion, $\sigma(r) = T_e^{3/2} = [J_b(r)]^{3/2}$ for Spitzer conductivity and no effective
heat loss mechanism. Thus $\sigma(r)$ is narrower than $J_b(r)$, causing the plasma current to peak on axis, further destabilizing all beam modes.

We assume for convenience that $J_{bo}$ and $J_{po}$ are also independent of $\zeta$. (The analysis could be carried out without these assumptions, but not in closed form.) The justification for the latter is that the decay length $\tau_o$ of $J_{bo}$ is long compared to the decay length $\tau_1$ of current perturbations,

$$\tau_0 = \frac{2\pi \sigma a^2}{c} \ln \frac{b}{a_0} \quad (7a)$$

$$\tau_1 = \frac{\pi \sigma a^2}{2c} \quad (7b)$$

$\tau_1$ also characterizes the instability growth length.

We consider only beam perturbations in the form of self-similar radial expansion and contraction, i.e. the perturbed beam is of the form

$$J_b(r, \zeta, z) = \frac{I_b}{\pi a^2(\zeta, z)} \frac{1}{[1 + r^2/a^2(\zeta, z)]^2} \quad (8)$$

which is a reasonable but not exact representation of the sausage mode. The variation of the root-mean-square beam radius $a(\zeta, z)$ is calculated from the Lee-Cooper envelope equation,

$$\frac{\partial^2 a^2}{\partial z^2} = \frac{\epsilon^2}{a^2} - \frac{U^2}{a} \quad (9)$$

where $U^2$ is a measure of the average pinch force,

$$U^2 = \frac{2I_b}{I_A} \int_0^\infty r^2 \frac{J_b(r)}{I_b} dr \int_0^{2\pi} [J_b(r) + J_p(r)] \frac{2\pi r}{I_b} dr \quad (10)$$
$I_A = 17\gamma \text{ kA}$ is the Alfvén current, $J_p$ is the plasma current density$^7$, and $\epsilon$ is the emittance, defined as $\bar{a} \langle \theta \rangle$, where $\langle \theta \rangle$ is the root-mean-square beam electron velocity angle.

The principal effect omitted by the model (8) - (10) is sausage oscillation damping due to phase mixing among beam electrons of different betatron frequency.$^{15,16}$ This effect is included phenomenologically by adding a damping term in the form derived by Lee and Yu$^{17}$,

$$\frac{\partial^2 \epsilon^2}{\partial z^2} = -\left( \frac{2 \alpha^2 \epsilon^2}{I_A} \right) \frac{\partial^2 \epsilon^2}{\partial z^2}. \tag{11}$$

The damping constant $\alpha$ is sensitive to the beam profile.$^{13}$ Lee$^5,17$ estimates $\alpha = 0.7$ for use with a Bennett profile.

We perform a linearized perturbation calculation in which small perturbations are added to the equilibrium profiles of $J_b$, $J_p$, $\sigma$ and $A_z$ described above. Perturbed quantities are calculated from the linearized forms of Eqs. (3), (5), (9), (11), and Ohm's law,

$$J_p = \sigma E_z. \tag{12}$$

Because of the assumed radial dependences of the equilibrium and perturbations, the equations reduce to coupled ordinary differential equations.

We note particularly that both the equilibrium and the perturbed conductivity are calculated self-consistently from Eq. (5); this is the new feature of the present calculation and leads to enhanced mode stability.
3. Calculation

The equilibrium described in Sec. 2 is specified by

\[ J_{bo} = \tilde{J}_{bo} (1 + r^2/a_0^2)^{-2}, \quad (13a) \]

\[ J_{po} = \tilde{J}_{po} (1 + r^2/a_0^2)^{-2}, \quad (13b) \]

\[ \sigma = \kappa J_{bo} \zeta (1 + r^2/a_0^2)^{-2}, \quad (13c) \]

\[ \Lambda_o = \tilde{\Lambda}_o (\zeta) \ln \frac{1 + r^2/a_0^2}{1 + b^2/a_0^2} = - \tilde{\Lambda}_o (\zeta) \ln \frac{b^2}{a_0^2}, \quad (13d) \]

\[ E_{oz} = - \frac{d\Lambda_o}{d\zeta} = \frac{d\tilde{\Lambda}_o}{d\zeta} \ln \frac{b^2}{a_0^2}. \quad (13e) \]

Equations (3), (9) and (10) reduce to the equilibrium relations

\[ \tilde{\Lambda}_o + \tau_o \frac{d\tilde{\Lambda}_o}{d\zeta} = - \frac{\pi a_0^2}{c} J_{bo}, \quad (14) \]

\[ \varepsilon_o^2/a_o^2 = \tilde{u}_o^2 = \frac{I_{\text{eff}}}{I_A}. \quad (15) \]

where

\[ I_{\text{eff}} \equiv I_b - I_p \quad (16) \]

for the present case of beam and plasma currents with the same profile.7

The front of the beam segment of interest is taken to be at \( \zeta = \zeta_0 \); the value of \( \zeta_0 \) may be chosen so that Eq. (13c) gives the conductivity desired as an initial condition in \( \zeta \).
The perturbed parts of $J_b$, $J_p$, $\sigma$ and $\Lambda$, designated
$\delta J_b$, $\delta J_p$, $\delta \sigma$ and $\delta \Lambda$, are treated as small quantities. To first order Eqs. (3),
(5), (9), (11) and (12) become

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) \delta A - \frac{4 \pi \sigma_0}{c} \frac{\partial \delta A}{\partial \zeta} - \frac{4 \pi}{c} \frac{\partial \delta A}{\partial \zeta} - \frac{4 \pi}{c} \delta J_b, $$

(17a)

$$\frac{\partial \delta \sigma}{\partial \zeta} = \kappa \delta J_b, $$

(17b)

$$\delta J = \sigma_0 \frac{\partial \delta A}{\partial \zeta} - \frac{3 \Lambda_0}{\partial \zeta} \delta \sigma, $$

(17c)

and

$$\frac{\partial^2 \delta \sigma}{\partial z^2} = - \frac{2U_0^2}{a_0^2} \delta \sigma - \frac{2U_0}{a_0} \frac{\partial \delta \sigma}{\partial z} - \frac{\delta U^2}{a_0^2}, $$

(18)

where we have used (11) and (15) to simplify (9). These equations admit the
solution

$$\delta J_b = - \tilde{J}_b(\zeta, z) \frac{1 - r^2/a_0^2}{(1 + r^2/a_0^2)^3}, $$

(19a)

$$\delta J_p = - \tilde{J}_p(\zeta, z) \frac{1 - r^2/a_0^2}{(1 + r^2/a_0^2)^3}, $$

(19b)

$$\delta \sigma = - \tilde{\sigma}(\zeta, z) \frac{1 - r^2/a_0^2}{(1 + r^2/a_0^2)^3}, $$

(19c)

$$\delta \Lambda = - \tilde{\Lambda}(\zeta, z) \frac{1 - r^2/a_0^2}{1 + r^2/a_0^2}, $$

(19d)

which corresponds to self-similar oscillations, since
\[
\frac{1 - r^2/a_0^2}{(1 + r^2/a_0^2)^3} = -\frac{4}{c^3} \frac{d}{da} \left[ \frac{1}{a^2} \frac{1}{(1 + r^2/a^2)^2} \right] \Delta a.
\]  
(20)

Equations (17) and (18) then reduce to the ordinary differential equations

\[
\begin{align*}
\left(1 + \frac{a^2}{2c} \frac{J_{bo}}{Ja} \right) \Delta a = & \frac{\Delta a}{2c} \frac{J}{Ja} + \frac{\Delta a}{2c} \frac{d\Delta a}{dc} \ln \left( \frac{b_0^2}{a_0^2} \right) \Delta \sigma, \\
\frac{\partial \sigma}{\partial c} = & \sigma \frac{\\\Delta a}{\Delta c}, \\
J_p = & -\sigma \frac{\partial \Delta a}{\partial c} - \frac{d\Delta a}{dc} \Delta \sigma, \\
\frac{\partial^2 J_b}{\partial z^2} = & -\frac{2U_0^2}{a_0^2} J_b - \alpha \frac{J_b}{a_0} \frac{\partial J_b}{\partial z} - \frac{\delta U^2}{a_0^2}.
\end{align*}
\]  
(21a)
(21b)
(21c)
(21d)

We address the linearized envelope equation (21d) first. Since \(I_b\) and \(I_p\) are unchanged by self-similar expansion, only the cross terms between \(J_p\) and \(J_b\) in Eq. (10) contribute to the perturbed pinch force \(\delta U^2\). Recalling our convention that \(I_p\) and \(I_b\) are both positive, we find

\[
\frac{\delta U^2}{U_0^2} = \frac{1}{3J_{bo}} \left( -\frac{I_p}{I_{eff}} J_b + \frac{I_b}{I_{eff}} J_p \right),
\]  
(22)

and the perturbed envelope equation (21d) then reduces to

\[
\frac{\partial^2 J_b}{\partial (z/\lambda_B)^2} = -\alpha \frac{\partial J_b}{\partial (z/\lambda_B)} - \left[ 2 - \frac{2}{3} \frac{I_p}{I_{eff}} \right] J_b + \frac{2}{3} \frac{I_b}{I_{eff}} J_p,
\]  
(23)

where we have defined an average betatron wavelength

\[
\lambda_B = \frac{a_0}{U_0}.
\]  
(24)
The quantity\(^{13} \bar{a}_o\) appears only through \(\lambda_b\).

In order to close the analysis, we must express \(\bar{J}_p\) in terms of \(\bar{J}_b\) through Eqs. (21a-c). First we simplify Eq. (21a) by using the approximation that \(I_p\) as well as \(I_b\) are independent of \(\zeta\), so that

\[
\frac{\bar{J}_p}{\bar{J}_b} = -\frac{I_p}{I_b} = \text{const}
\]  

and from Eqs. (12) and (13a)

\[
\bar{J}_p = \bar{J}_b \zeta \frac{d\bar{a}_o}{d\zeta} \ln \frac{b^2}{a_0^2}.
\]  

(The analysis could be carried out without these approximations, but would then require numerical solution of a complicated ordinary differential equation, rather than yielding the solutions we shall find in closed form.)

Thus Eq. (21a) can be written as

\[
\left(\frac{2c}{a_0^2} + \kappa \bar{J}_b \zeta \frac{2}{2\zeta} \right) \bar{\lambda} = \bar{J}_b - \frac{I_p}{\pi I_b} \bar{\sigma}.
\]  

Next we rewrite Eq. (21b) as

\[
\left(1 + \zeta \frac{2}{2\zeta} \right) \bar{\sigma} = \kappa \bar{J}_b,
\]  

and use (28) to eliminate \(\bar{\sigma}\) from (27), which yields an expression for \(\bar{\lambda}\) in terms of \(\bar{J}_b\),

\[
\left(1 + \zeta \frac{2}{2\zeta} \right) \left(-\frac{2c}{a_0^2} + \kappa \bar{J}_b \zeta \frac{2}{2\zeta} \right) \bar{\lambda} = \left(1 + \zeta \frac{2}{2\zeta} - \frac{I_p}{I_b} \right) \bar{J}_b.
\]  

We also use Eqs. (13), (19) and (26) in (17c), to obtain
\[
\tilde{J}_p = \frac{1}{c^{2}b} \tilde{\sigma} - \kappa \tilde{J}_{bo} \zeta \frac{\tilde{a}}{\tilde{c}}.
\]  

(30)

and using (28) again in (30) gives

\[
(1 + \zeta \frac{\partial}{\partial \zeta}) \tilde{J}_p = \frac{1}{b} \tilde{J}_b - \kappa \tilde{J}_{bo} (1 + \zeta \frac{\partial}{\partial \zeta}) \zeta \frac{\tilde{a}}{\tilde{c}}.
\]  

(31)

Equations (29) and (31) together specify \( \tilde{J}_p \) as a linear function of \( \tilde{J}_b \).

We observe that (23), (29) and (31) constitute a closed system of linear equations with constant coefficients if \( \frac{\partial}{\partial z} \) and \( \frac{\partial}{\partial \ln \zeta} \) are regarded as the basic operators. Thus it is natural to use \( z \) and \( \ln \zeta \) as the independent variables. A complete solution of the problem with perturbation initial conditions at \( \zeta = \zeta_0 \) would require a Laplace transform analysis, which is beyond the scope of this paper, but the equations admit a free mode solution in the form

\[
f(\zeta,z) = \hat{f} \exp \left[ -i\Omega z \frac{\lambda}{\beta} - i\omega \ln \left( \zeta/\zeta_0 \right) \right]
\]

\[
= \hat{f} e^{-i\Omega z \frac{\lambda}{\beta}} \left( \frac{\zeta}{\zeta_0} \right)^{\frac{\zeta}{\zeta_0}} \left[ \cos\left( \omega \ln \frac{\zeta}{\zeta_0} \right) - i \sin\left( \omega \ln \frac{\zeta}{\zeta_0} \right) \right],
\]  

(32)

where \( f(\zeta,z) \) is any perturbed quantity. The \( z \)-dependence of the mode is in the usual exponential form, but in \( \zeta \) the mode shows power law growth or decay and oscillation with steadily increasing wavelength, due to the non-uniformity of the equilibrium in \( \zeta \), i.e. the linear increase of \( \sigma_o(r,\zeta) \) with \( \zeta \).

Using (32) in (29) and (31) yields the required relation between \( \tilde{J}_p \) and \( \tilde{J}_b \),

\[
\tilde{J}_p = \left[ -\frac{I_p/I_b}{1 - i\omega} + \frac{i\omega(1 - i\omega - I_p/I_b)}{(1 - i\omega)(1 - i\omega + \frac{2\kappa \omega}{\kappa a_o} - i\omega)} \right] \tilde{J}_b.
\]  

(33)
We also use (32) to reduce the envelope equation (23) to algebraic form, and use (33) to eliminate \( \tilde{J}_p \). After some algebra, but no approximations, the envelope equation reduces to the dispersion relation

\[
- \Omega^2 - i\omega = -2
\]

\[
+ \frac{2}{3} \frac{1}{1 + \lambda^2 \omega^2} [ \omega^2 \left( \frac{1 + \lambda}{1 + \omega^2} \frac{I_p}{I_{\text{eff}}} - \lambda^2 \right) + i\omega \left( \lambda - \frac{1 - \lambda \omega^2}{1 + \omega^2} \frac{I_p}{I_{\text{eff}}} \right)], \quad (34)
\]

where

\[
\lambda \equiv \frac{d\tau_p}{dz} = \frac{\omega I_B}{2c}, \quad (35a)
\]

\( \tau_p \) from Eq. (7b) is the characteristic decay time of the perturbed plasma current, and \( \alpha \) is the phenomenological damping constant from Eq. (11).

If we use Eq. (5b) for \( \kappa \), we can write Eq. (35a) in the convenient form

\[
\lambda = 0.044 \left( \frac{I_B}{1 \text{ kA}} \right). \quad (35b)
\]
4. Results

The dispersion relation (34) can be solved for either $\Omega$ or $\bar{\omega}$ as a function of the other. The mode is unstable if $\Omega_1 > 0$ for real $\bar{\omega}$ or if $\bar{\omega}_1 > 0$ for real $\Omega$. We consider the instability condition $\Omega_1 > 0$ for real $\bar{\omega}$. Rewriting (34) in the schematic form

$$-\Omega^2 - i\omega = F_1(\lambda, I_p/I_{\text{eff}}, \bar{\omega}) + IF_2(\lambda, I_p/I_{\text{eff}}, \bar{\omega}),$$

where $F_1$ and $F_2$ are real, this condition is

$$F_1 > -F_2^2/\alpha^2.$$  

(36)

For $\alpha > 0$ (no phase mix damping included in the formalism) all modes are unstable; as might be expected the conditions for instability become more restrictive as $\alpha$ increases, but instability can occur even in the limit $\alpha = 0$ if $F_1 > 0$, i.e.

$$\frac{1 + \lambda}{1 + \omega^2} \frac{I_p}{I_{\text{eff}}} > 4\lambda^2 + \frac{3}{\omega^2}.$$  

(38)

There are modes $\bar{\omega}$ that satisfy condition (38) if and only if

$$\frac{I_p}{I_{\text{eff}}} > \frac{(2\lambda + \sqrt{3})^2}{\lambda + 1},$$

(39)

and the range of unstable modes is given by

$$c_1 - c_2 < \Omega^2 < c_1 + c_2.$$  

(40a)
where

\[ C_1 = \frac{1}{8\lambda^2} \left[ (1 + \lambda) \frac{I_p}{I_{\text{eff}}} - 4\lambda^2 - 3 \right], \quad (40a) \]

\[ C_2 = \frac{1}{8\lambda^2} \left[ \left( (1 + \lambda) \frac{I_p}{I_{\text{eff}}} - 4\lambda^2 - 3 \right)^2 - 48\lambda^2 \right]^{1/2}. \quad (40c) \]

If condition (39) is satisfied, there are very firm model-independent grounds for expecting strong instability, independent of the damping coefficient \( \alpha \). In fact, if one solves for \( \bar{w}(\Omega) \) it is seen that instability can occur \( \Omega > 0 \) in this case even with \( \Omega = 0 \), i.e. mere time-independent beam non-uniformities can grow unstably as one moves back in the beam. If the weaker condition (37) is satisfied, instability is still predicted, but of an oscillatory and somewhat weaker form, somewhat dependent on the model and the value of \( \alpha \). Lee uses \( \alpha = 0.7 \), corresponding to a Bennett profile truncated at three to four Bennett radii. We can write the critical value of \( I_p/I_{\text{eff}} \) for instability, from Eq. (37), in the form of a correction to (39),

\[ \frac{I_p}{I_{\text{eff}}} > \frac{(2\lambda + \sqrt{3})^2}{\lambda + 1} H(\alpha = 0.7, \lambda). \quad (41) \]

Numerical evaluation shows that \( H(\alpha = 0.7, \lambda) \) is quite close to unity, varying only from 0.75 for \( \lambda = 0 \) to 0.81 for \( \lambda = \), so (39) is a reasonably accurate instability criterion even for finite \( \alpha \).

Under conditions where avalanche is unimportant even at the pinch point, e.g. for a typical induction line beam with \( I_p \leq 10 \text{ kA} \), \( \epsilon_0 \geq 0.3 \text{ cm} \), in air at density close to or above ambient, the instability condition (41) is never satisfied. In these cases, Sharp and Lampe have shown that \( I_p/I_{\text{eff}} \) reaches a peak value of...
\[
\frac{I_p}{I_{\text{eff}}} = 3\lambda \text{ to } 4\lambda \quad (42a)
\]

at the pinch point, and then falls off to a fairly constant value

\[
\frac{I_p}{I_{\text{eff}}} \sim (1 \pm 0.3)\lambda \quad (42b)
\]

a few centimeters further back in \( \zeta \). Equation (42b) is incompatible with condition (41).

If avalanche is important near the pinch point, \( \frac{I_p}{I_{\text{eff}}} \) is increased. Nevertheless a survey performed with the simulation code SIM1011, which does include avalanche, indicates that over a very wide range of beam parameters in air the instability condition (41) is not satisfied over a long enough stretch of beam to permit effective mode growth. [In some cases (41) is satisfied for a region of only a few centimeters about the pinch point.] For example, beams with \( I_b = 10 \text{ kA} \) and radius \( \gtrsim 0.5 \text{ cm} \) are predicted to be sausage-stable in air at densities above 50 torr, and beams with \( I_b \) up to at least 100 kA are predicted to be sausage-stable in air at ambient density. We have found, however, that the instability condition (41) is satisfied in some low-density regimes where avalanche is so strong near the pinch point that \( \frac{I_{\text{eff}}}{I_b} \) is small, e.g. in air at 10 torr \( \frac{I_{\text{eff}}}{I_b} = 0.08 \) for a 10 kA beam with radius 0.5 cm and current risetime 0.33 nsec. This low density regime around 10 torr is of interest, since considerable experimental effort has been devoted to it in past20 and present experiments.18

However in this regime avalanche often dominates the conductivity physics even well behind the pinch point. Thus the conductivity model (5) used in the present theory is seriously incomplete. Inclusion of avalanche self-consistently in the model of the beam body probably would be stabilizing.
since it further reduces the spatial separation of the plasma current from the beam current. To date, we have been unable to treat avalanche self-consistently in an analytic theory, but we are presently using the axisymmetric beam envelope code VIPER-0 to simulate sausage evolution in this regime. In limited studies to date we have found no case for which the sausage mode is unstable for beam injection into neutral gas.\textsuperscript{11}

Recent particle simulations\textsuperscript{8-10} have shown very strong axisymmetric instabilities in some regimes where (41) predicts stability. We present a detailed simulation analysis of these instabilities in a companion paper.\textsuperscript{11} We find that the unstable mode is the \((m = 0, n = 2)\) hollowing mode, not the \((m = 0, n = 1)\) sausage mode.

For beams propagating into a pre-ionized channel, \(I_p/I_{\text{eff}}\) can be arbitrarily large, and the instability conditions (41) or even (39) can be satisfied. The limit of a fixed pre-ionized channel, i.e. the case in which beam-induced conductivity augmentation is negligible compared to the pre-existing conductivity, has been considered previously by Lee.\textsuperscript{5} In our formalism, this is the limit \(\lambda \rightarrow 0, 0 \text{ fixed}, \lambda \bar{\omega}\text{(}\Omega\text{)} \text{ fixed}, \bar{\omega}\text{(}\Omega\text{)} = \zeta\) restricted to a range \(\zeta_0 < \zeta < \zeta_0 + \Delta \zeta\), where \(\Delta \zeta \ll 1\) and \(\tilde{\sigma}_0(\zeta)\) has the essentially constant value \(\tilde{J}_{bo} / \zeta_0\). Since \(\zeta\) is proportional to \(\lambda\), we note also that \(\zeta_0 + \lambda\) is the limit. The dependence of any perturbed quantity \(f\) on \(\zeta\), from Eq. (32), then reduces to

\[
f(\zeta) = \hat{f} \exp(-i\omega \zeta / \zeta_0) \approx \hat{f} \exp(-i\omega(\zeta - \zeta_0)),
\]

i.e. exponential growth and sinusoidal oscillation with complex frequency

\[
\omega = \bar{\omega}/\zeta_0.
\]
Our dispersion relation (34) reduces to one previously derived by Lee:

\[-a^2 - i\omega = -2 + \frac{2}{3} \left( \frac{I_p}{I_{\text{eff}}} + \frac{I_b}{I_{\text{eff}}} \right) \frac{i\omega}{1 - i\omega}. \]  

(45)

Our condition (39) for instability to occur and not be stabilized by any value of the damping coefficient \( \alpha \) reduces to

\[ \frac{I_p}{I_{\text{eff}}} > 3, \]  

(46)

in agreement with Ref. 5.

In conclusion, then, we have determined the mode structure, Eq. (32), for instability growth in the beam body, and have calculated a dispersion relation, Eq. (34), and instability conditions, (39) and (41), for the sausage mode. We find that the sausage mode instability condition is not satisfied for beams injected into neutral gas under conditions that satisfy our assumptions. However the instability conditions are usually satisfied for beams injected into a pre-ionized gas channel with the same profile as the beam (and with \( 4\pi \sigma \sigma_0 > c \)), because of the substantial current neutralization under those conditions. There is a region of low neutral gas density where our theory would predict instability but where the theory is itself inapplicable because avalanche is strong and persistent. Although sausage stability properties are not well understood in this regime, early indications are that sausage instability does not occur.

To our knowledge, sausage instability has not been observed experimentally for beams injected into neutral gas, in agreement with our conclusions.
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References

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4. H. S. Uhm and M. Lampe, Phys. Fluids 25, 1444 (1982). The threshold values quoted for \( I_p/I_b \) are for instability with \( \Omega > 0 \), where the flat-top beam model is applicable. The model is artificially unstable for \( \Omega \) near the betatron frequency \( \Omega_b \), due to the lack of spread in \( \Omega_b \).
7. Our convention is that \( I_p \) and \( I_b \) are both positive, although the beam and plasma currents flow in opposite directions in all cases considered in this paper. However our current densities are given their actual signs, i.e. \( J_b \) and \( J_p \) have opposite signs.

13. For the Bennett profile the root-mean-square radius is infinite, as has been noted frequently. There are a number of other peculiarities associated with the Bennett profile, e.g. the damping coefficient \( a \) defined in Eq. (11) also is infinite for this profile. These singularities are due to the relatively slow fall-off of the Bennett density profile at large radii. In practice, the Bennett profile is an accurate representation of the beam density out to two or three Bennett radii under many conditions, but the beam density falls off more rapidly at larger radii. The value of \( \bar{a} \) or \( a \) depends on this outer radius only logarithmically, and \( \bar{a}/a \) can be regarded as a fixed ratio, typically 1.0 to 1.5, during the sausage evolution. The instability thresholds are independent of the Bennett cut-off, while instability growth rates in \( z \) depend weakly on it through a mean betatron wavelength which is proportional to \( \bar{a} \).


21. The VIPER-O simulations were performed by Dr. Richard Hubbard.