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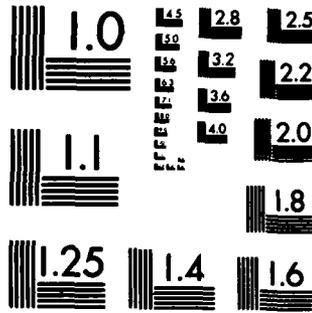
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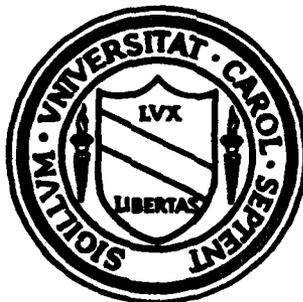
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Department of Statistics  
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Chapel Hill, North Carolina

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ON A PROBLEM CONCERNING SPACINGS

Shihong Cheng

TECHNICAL REPORT #27

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$$\lim_n \frac{2nM_n}{\log n} = 1 \text{ a.s.}$$

which refines a result of Chow, Geman and Wu.

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ON A PROBLEM CONCERNING SPACINGS

Shihong Cheng  
Peking University  
and  
University of North Carolina

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## 1. Introduction

Let  $\{U_n, n=1,2,\dots\}$  be an i.i.d. sequence uniformly distributed on  $[0,1]$ , and  $U_1^{(n)} \leq \dots \leq U_n^{(n)}$  be the order statistics of  $U_1, \dots, U_n$ . The random variables  $S_i^{(n+1)} = U_i^{(n)} - U_{i-1}^{(n)}$ ,  $i=1, \dots, n+1$  are called the spacings divided by  $U_1, \dots, U_n$ , where  $U_0^{(n)} \triangleq 0$ ,  $U_{n+1}^{(n)} \triangleq 1$ . The maximum of spacings plays an important part in nonparametric problems. Its exact and asymptotic behavior has been studied by many authors (See Darling [4], Pyke [8], Slud [9], Devroye [5] and so on). Write  $W_i^{(n+1)} = S_i^{(n+1)} \wedge S_{i+1}^{(n+1)}$ ,  $i=1, \dots, n$ . The behavior of  $M_{n+1} = \max_{1 \leq i \leq n} W_i^{(n+1)}$  is important in cross-validated kernel density estimation. (See Chow, Geman and Wu [3] and Marron [7]). In this paper we give the exact distribution of  $M_n$  in section 2. In section 3 we discuss the behavior of a certain distribution function, which will be called the Fibonacci distribution function. Chow, Geman and Wu [3] have shown that there exists a constant  $C$  such that

$$(1.1) \quad P(nM_n / \log n \leq C \text{ i.o.}) = 0,$$

where "i.o." means infinitely often. In section 4, we will refine (1.1) by showing that

$$(1.2) \quad P(\lim_n 2nM_n / \log n = 1) = 1.$$

The limiting distribution of  $M_n$  is also discussed in this section.

## 2. The exact distribution of $M_n$ .

Let  $Y_1, \dots, Y_n$  be  $n$  random variables whose joint distribution function is  $F(y_1, \dots, y_n)$ . We call  $Y_1, \dots, Y_n$  exchangeable random variables if  $F(y_{i_1}, \dots, y_{i_n}) = F(y_1, \dots, y_n)$  for any permutation  $\{i_1, \dots, i_n\}$  of  $\{1, \dots, n\}$ . Given  $y \in \mathbb{R}$ , let

$$A_{j_1 \dots j_k} = \{Y_j > y, j \in \{j_1, \dots, j_k\}; Y_j \leq y, j \in \overline{\{j_1, \dots, j_k\}}, 1 \leq j_1 < \dots < j_k \leq n\}.$$

If  $Y_1, \dots, Y_n$  are exchangeable, the probabilities of the events  $A_{j_1 \dots j_k}$  will be the

same. Denote

$$(2.1) \quad F^{(k)}(y) = P(A_{j_1 \dots j_k}), \quad 1 \leq j_1 < \dots < j_k \leq n$$

and  $F^{(0)}(y) = F(y_1, \dots, y_n)$ .

Lemma 2.1

Suppose that  $Y_1, \dots, Y_n$  are exchangeable. Then

$$(2.2) \quad P(\max_{1 \leq i \leq n-1} (Y_i \wedge Y_{i+1}) \leq y) = \sum_{k=0}^{[(n+1)/2]} \binom{n-k+1}{k} F^{(k)}(y),$$

where  $F^{(k)}(y)$  is defined by (2.1) and  $[x]$  is the integer part of  $x$ .

Proof. The event  $\{\max_{1 \leq i \leq n-1} (Y_i \wedge Y_{i+1}) \leq y\}$  means that there is no index  $i$  such that  $\{Y_i > y\}$  and  $\{Y_{i+1} > y\}$  happen simultaneously. Hence this event is the union of  $E_k$ ,  $k=0, 1, \dots, [(n+1)/2]$ , where  $E_k$  is the event "there are  $k$  integers  $1 \leq j_1 < \dots < j_k \leq n$  which do not contain two consecutive integers, such that the event  $A_{j_1 \dots j_k}$  happens." We obtain

$$P(\max_{1 \leq i \leq n-1} (Y_i \wedge Y_{i+1}) \leq y) = \sum_{k=0}^{[(n+1)/2]} P(E_k)$$

since  $\{E_k\}$  are disjoint events. Since  $Y_1, \dots, Y_n$  are exchangeable, it follows that

$$P(E_k) = \binom{n-k+1}{k} F^{(k)}(y),$$

where  $\binom{n-k+1}{k}$  is the number of  $k$ -element subsets that can be selected from the set  $\{1, \dots, n\}$  and that do not contain two consecutive integers (see [1], Chapter 3).

We first find the exact distribution of  $M_n$ . Define

$$(x)_+ = \begin{cases} x & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

We have

Theorem 2.2

$$(2.3) \quad P(M_n \leq x) = \sum_{k=0}^{[(n+1)/2]} \binom{n-k+1}{k} \sum_{t=0}^{n-k} (-1)^t \binom{n-k}{t} \{[1-(k+t)x]_+\}^{n-1}$$

Proof. It is known that

$$P(S_1^{(n)} > x_1, \dots, S_n^{(n)} > x_n) = \left[ \left( 1 - \sum_{i=1}^n x_i \right)_+ \right]^{n-1},$$

where  $x_1, \dots, x_n$  are nonnegative numbers (see Devroye [5]). Hence the spacings

$S_1^{(n)}, \dots, S_n^{(n)}$  are exchangeable, and Lemma 2.1 can be used in this case. Notice that

$$\begin{aligned} & P(S_1^{(n)} > x, \dots, S_k^{(n)} > x, S_{k+1}^{(n)} \leq x, \dots, S_n^{(n)} \leq x) \\ &= P(S_1^{(n)} > x, \dots, S_k^{(n)} > x) - \sum_{t=1}^{n-k} (-1)^{t-1} \sum_{k+1 \leq j_1 < \dots < j_t \leq n} P(S_1^{(n)} > x, \dots, S_k^{(n)} > x, S_{j_1}^{(n)} > x, \dots, S_{j_t}^{(n)} > x) \\ &= \sum_{t=0}^{n-k} (-1)^t \binom{n-k}{t} \left[ 1 - (k+t)x \right]_+^{n-1}, \end{aligned}$$

so that the theorem is proved.

### 3. The Fibonacci distribution function.

Let  $X_n \sim G(x)$ ,  $n=1,2,\dots$  be an i.i.d. sequence and  $Z_n = \max_{1 \leq i \leq n-1} (X_i \wedge X_{i+1})$ . From (2.2) we have

Lemma 3.1

$$(3.1) \quad P(Z_n \leq x) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \binom{n-k+1}{k} G^{n-k}(x) [1-G(x)]^k.$$

Now define the Fibonacci distribution function by

$$(3.2) \quad F_n^*(x) = \begin{cases} 0 & x < 0 \\ \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \binom{n-k+1}{k} x^{n-k} (1-x)^k & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}.$$

The name Fibonacci was chosen because

$$F_0 \stackrel{\Delta}{=} 0, \quad F_1 = F_2 = 1, \quad F_{n+2} = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \binom{n-k+1}{k}, \quad n=1,2,\dots$$

is the sequence of Fibonacci numbers. By using the generating function method for finding the values of Fibonacci numbers, the sum  $g_n \stackrel{\Delta}{=} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \alpha^k$  can be found as follows.

Lemma 3.2

For any  $n=0,1,2,\dots$ ,

$$(3.3) \quad g_n = \{[1+(1+2\alpha)/\beta][(1+\beta)/2]^n + [1-(1+2\alpha)/\beta][(1-\beta)/2]^n\}/2 ,$$

where  $\beta = (1+4\alpha)^{1/2}$ .

Proof. For convenience, let  $\binom{n}{k} = 0$  if  $k < 0$  or  $n < k$ . Then it follows that

$$\binom{n-k+1}{k} = \binom{n-k}{k-1} + \binom{n-k}{k} , \quad k=0,1,\dots, [(n+1)/2] ,$$

and therefore that

$$g_{n+1} = \alpha g_{n-1} + g_n , \quad n=1,2,\dots .$$

Hence we obtain

$$P(x) - [xP(x) + \alpha x^2 P(x)] = 1 + \alpha x , \quad \text{i.e.}$$

$$P(x) = (1+\alpha x)/(1-x-\alpha x^2) ,$$

where  $P(x) \triangleq \sum_{n=0}^{\infty} g_n x^n$  is the generating function of the sequence  $\{g_n\}$ . Expand  $P(x) = (1+\alpha x)/(1-x-\alpha x^2)$  into a power series:

$$\begin{aligned} P(x) &= \{[1+(1+2\alpha)/\beta]/[1-x(1+\beta)/2] + [1-(1+2\alpha)/\beta]/[1-x(1-\beta)/2]\}/2 \\ &= \sum_{n=0}^{\infty} \{[1+(1+2\alpha)/\beta][(1+\beta)/2]^n + [1-(1+2\alpha)/\beta][(1-\beta)/2]^n\} x^n / 2 . \end{aligned}$$

Comparing the above series with the definition of  $P(x)$ , we have (3.3), to complete the proof.

Now the Fibonacci d.f. can be represented as

$$(3.4) \quad F_n^*(x) = x^n g_{n+1}((1-x)/x) , \quad 0 < x < 1 ,$$

where  $g_{n+1}((1-x)/x)$  is the value of  $g_{n+1}$  at  $\alpha = (1-x)/x$ . We discuss the asymptotic behavior of the Fibonacci distribution function as follows.

Theorem 3.3

If  $x_n \in (0,1)$ ,  $n=1,2,\dots$  is a sequence such that, as  $n \rightarrow \infty$ ,

$$(3.5) \quad ny_n^3 \rightarrow 0 ,$$

where  $y_n = 1 - x_n$ , then we have

$$(3.6) \quad F_n^*(X_n) = [1+0(y_n)+0(ny_n^3)] \exp[-ny_n^2] .$$

Proof. Write

$$(3.7) \quad F_n^*(x_n) = u_n(x_n) [1 - v_n(x_n)/u_n(x_n)] / (2x_n) ,$$

where

$$u_n(x_n) = [1 + (1+2y_n/x_n)/(1+4y_n/x_n)^{1/2}] \{x_n [1 + (1+4y_n/x_n)^{1/2}] / 2\}^{n+1}$$

$$v_n(x_n) = [1 - (1+2y_n/x_n)/(1+4y_n/x_n)^{1/2}] \{x_n [1 - (1+4y_n/x_n)^{1/2}] / 2\}^{n+1} .$$

Noticing that

$$\begin{aligned} & (1+2y_n/x_n)/(1+4y_n/x_n)^{1/2} \\ &= (1+2y_n/x_n)(1-2y_n/x_n+0(y_n^2)) = 1 + 0(y_n^2) , \end{aligned}$$

we know  $v_n(x_n)/u_n(x_n) = 0(y_n^2)$ , and therefore

$$1 - v_n(x_n)/u_n(x_n) = 1 + 0(y_n^2) .$$

Hence to prove (3.6), we need only to show that

$$u_n(x_n)/(2x_n) = (1+0(y_n))(1+0(ny_n^3)) \exp(-ny_n^2)$$

(See (3.7)). This follows from

$$\begin{aligned} u_n(x_n)/(2x_n) &= (1+0(y_n^2)) \{x_n [1 + (1+4y_n/x_n)^{1/2}] / 2\}^n [1 + (1+4y_n/x_n)^{1/2}] / 2 \\ &= (1+0(y_n)) \{x_n [1 + (1+2y_n/x_n - 2y_n^2/x_n + 0(y_n^3))] / 2\}^n \\ &= (1+0(y_n)) (1 - y_n^2 + 0(y_n^3))^n \\ &= (1+0(y_n)) \exp\{-ny_n^2 + 0(ny_n^3)\} \\ &= (1+0(y_n))(1+0(ny_n^3)) \exp\{-ny_n^2\} , \end{aligned}$$

completing the proof of theorem 3.3.

4. The asymptotic behavior of  $M_n$

Unless otherwise stated,  $X_n, n=1,2,\dots$  will be an i.i.d. exponential sequence in this section.

Lemma 4.1

Let  $T_n = \sum_{i=1}^n X_i$ . The spacings  $(S_1^{(n)}, \dots, S_n^{(n)})$  are distributed as  $(X_1/T_n, \dots, X_n/T_n)$ .

Proof. See Pyke [8].

Lemma 4.2

For all  $x>0$ , the following inequalities hold:

(4.1)  $P(T_n/n-1>x) \leq \exp[-nx^2(1-x)/2]$

(4.2)  $P(T_n/n-1<-x) \leq \exp(-nx^2/2)$  .

Proof. See Devroye [5].

Lemma 4.3

Let  $\{\xi_n\}, \{\eta_n\}$  be r.v. sequences. If there exist  $a_n>0, b_n$  such that

$$P(\xi_n \leq a_n x + b_n) \xrightarrow{P} \Psi(x)$$

$$\eta_n/n \xrightarrow{P} 1 \text{ and } (\eta_n/n-1)b_n/a_n \xrightarrow{P} 0 ,$$

then

$$P(\xi_n/\eta_n \leq (a_n x + b_n)/\eta_n) \xrightarrow{P} \Psi(x) .$$

Proof. The sequences  $\{(\xi_n - b_n)/a_n\}$  and  $\{(\xi_n - b_n \eta_n/n)/a_n\}$  have the same limiting distribution since  $(\eta_n/n-1)b_n/a_n \rightarrow 0$  in probability. The sequences  $\{(\xi_n - b_n \eta_n/n)/a_n\}$  and  $\{(n\xi_n/\eta_n - b_n)/a_n\}$  have the same limiting distribution since  $\eta_n/n \rightarrow 1$  in probability. Hence lemma 4.3 is proved.

Theorem 4.4

For any  $x \in R$

$$(4.3) \quad \lim_n P(M_n \leq x/2n + \log n/2n) = \exp[-\exp(-x)] .$$

Proof. By using lemma 4.1, (4.3) is equivalent to

$$(4.4) \quad \lim_n P(Z_n/T_n \leq x/2n + \log n/2n) = \exp[-\exp(-x)] .$$

By using lemma 4.2, it can be shown that

$$(T_n/n - 1) \log n \xrightarrow{P} 0 .$$

Hence by using lemma 4.3, (4.4) holds if we show that

$$(4.5) \quad \lim_n P(Z_n \leq (x + \log n)/2) = \exp[-\exp(-x)] .$$

From (3.1) and (3.2), we have

$$P(Z_n \leq (x + \log n)/2) = F_n^*(x_n) ,$$

where  $x_n = 1 - \exp[-(x + \log n)/2]$ . It is easy to check that  $ny_n^3 \rightarrow 0$ ,  $y_n \rightarrow 0$ ,  $ny_n^2 \rightarrow \exp(-x)$ .

Hence (4.5) follows from (3.6), completing the proof of (4.3).

Remark 1. Using the methods of section 3, we can show that Gnedenko's theorems (See [6]) for the i.i.d. case will still be valid for the sequence  $\{X_i \wedge X_{i+1}\}$  where  $\{X_i\}$  is i.i.d. but not necessarily exponential. The same statement may also follow from Watson [10].

Remark 2. It is easy to show, from theorem 4.4, that

$$(4.6) \quad 2nM_n/\log n \xrightarrow{P} 1 .$$

Now we turn our attention to proving (1.2). It is easy to see that (1.2) is equivalent to the following equations:

$$(4.7) \quad P(\limsup_n 2nM_n/\log n \leq 1) = 1 ,$$

$$(4.8) \quad P(\liminf_n 2nM_n/\log n \geq 1) = 1 .$$

But (4.7) and (4.8) are equivalent to the statement that for any  $\delta > 0$ ,

$$(4.9) \quad P(2nM_n/\log n > 1 + \delta \text{ i.o.}) = 0 ,$$

$$(4.10) \quad P(2nM_n/\log n \leq 1-\delta \text{ i.o.}) = 0.$$

Lemma 4.5

Equation (4.10) holds.

Proof. Let  $A_n = \{2nM_n/\log n \leq 1-\delta\}$ ,  $n=1,2,\dots$ . By the Borel-Cantelli lemma, to prove (4.10), it is sufficient to show that

$$(4.10) \quad \sum_{n=1}^{\infty} P(A_n) < \infty.$$

Using lemma 3.1 and 3.2, we have

$$\begin{aligned} (4.11) \quad P(A_n) &= P(Z_n \leq (1-\delta)T_n \log n / (2n)) \\ &\leq P(Z_n \leq (1-\delta)(1+\epsilon_n) \log n / 2) + P(T_n/n > 1+\epsilon_n) \\ &\leq P(Z_n \leq (1-\delta/2) \log n / 2) + \exp(-n\epsilon_n^2/4), \end{aligned}$$

where  $\epsilon_n = 2n^{-(1-\delta/2)/2}$  and therefore  $(1-\delta)(1+\epsilon_n) \leq 1 - \delta/2$ ,  $1-\epsilon_n > 1/2$  if  $n$  is sufficiently large. By letting  $x_n = 1 - \exp[-(1-\delta/2) \log n / 2]$ , it is easy to check that  $ny_n^3 \rightarrow 0$ ,  $ny_n^2 = n^{\delta/2}$ , and therefore theorem 3.3 can be used for this case. Hence

$$\begin{aligned} &P(Z_n \leq (1-\delta/2) \log n / 2) \\ &= (1+o(1)) \exp(-n^{\delta/2}). \end{aligned}$$

Now from (4.11), we have

$$P(A_n) \leq C \exp(-n^{\delta/2})$$

where  $C$  is a constant. Then (4.10) follows since  $\sum_{n=1}^{\infty} \exp(-n^{\epsilon})$  converges for any  $\epsilon > 0$ , completing the proof of lemma 4.5.

To prove (4.9) we need the following lemma, which is stronger than the Borel-Cantelli lemma.

Lemma 4.6

Let  $\{A_n\}$  be a sequence of events with  $\lim_{n \rightarrow \infty} P(A_n) = 0$ . If either  $\sum_{n=1}^{\infty} P(A_n^C A_{n+1}) < \infty$  or  $\sum_{n=1}^{\infty} P(A_n A_{n+1}^C) < \infty$ , then  $P(A_n \text{ i.o.}) = 0$ .

Proof. See Barndorff-Neilsen [2].

Lemma 4.7

Equation (4.9) holds.

Proof. Let  $u_n = (1+\delta)\log n/2$ ,  $\tilde{u}_n = (1+\delta)\log n/(2n)$  and  $A_n = \{M_n > \tilde{u}_n\}$ . By using lemma 4.6, to prove (4.9) we need only to show

$$(4.12) \quad \sum_{n=1}^{\infty} P(A_n A_{n+1}^c) < \infty$$

Writing  $E_i^{(n)}$  for the set  $\{S_i^{(n)} \wedge S_{i+1}^{(n)} > \tilde{u}_n, S_j^{(n)} \wedge S_{j+1}^{(n)} \leq \tilde{u}_{n+1}, j \neq i\}$  and noticing that  $\{\tilde{u}_n\}$  is nonincreasing, we have

$$\begin{aligned} P(A_n A_{n+1}^c) &= P(M_n > \tilde{u}_n, M_{n+1} \leq \tilde{u}_{n+1}) \\ &= \sum_{i=1}^{n-1} P(E_i^{(n)} \cap \{U_i^{(n-1)} - \tilde{u}_{n+1} \leq U_n \leq U_i^{(n-1)} + \tilde{u}_{n+1}\}) \\ &= \sum_{i=1}^{n-1} \int_{E_i^{(n)}} P(U_i^{(n-1)} - \tilde{u}_{n+1} \leq U_n \leq U_i^{(n-1)} + \tilde{u}_{n+1} | U_1, \dots, U_{n-1}) dP \\ &\leq 2\tilde{u}_n \sum_{i=1}^{n-1} P(E_i^{(n)}) \\ &\leq 2\tilde{u}_n P(M_n > \tilde{u}_n) \\ &= 2\tilde{u}_n P(Z_n > u_n T_n/n) \\ &\leq 2\tilde{u}_n P(Z_n > (1-\epsilon_n)u_n) + 2\tilde{u}_n P(T_n/n < 1-\epsilon_n) \end{aligned}$$

for any  $\epsilon_n > 0$ . Furthermore, using lemma 4.2, we obtain that

$$\tilde{u}_n P(T_n/n < 1-\epsilon_n) \leq (1+\delta)(\log n) \exp(-n\epsilon_n^2/2)/n,$$

and therefore can choose  $\epsilon_n \rightarrow 0$  such that

$$\sum_{n=1}^{\infty} \tilde{u}_n P(T_n/n < 1-\epsilon_n) < \infty.$$

Hence to show (4.12), it is sufficient to prove

$$(4.13) \quad \sum_{n=1}^{\infty} \tilde{u}_n P(Z_n > (1+\delta/2) \log n / 2) < \infty$$

since  $(1-\epsilon_n)(1+\delta) > 1 - \delta/2$  and therefore

$$P(Z_n < (1-\epsilon_n)u_n) \leq P(Z_n > (1+\delta/2) \log n / 2)$$

for large enough  $n$ . Let  $x_n = 1 - \exp[-(1+\delta/2) \log n / 2] = 1 - n^{-(1+\delta/2)/2}$ ,  $y_n = 1 - x_n$ .

Then we have  $ny_n^3 = o(y_n)$  and therefore (3.6) can be rewritten as

$$F_n^*(x_n) = [1 + o(y_n)] \exp(-ny_n^2).$$

Now (4.13) follows from the fact that

$$\begin{aligned} P(Z_n > (1+\delta/2) \log n / 2) &= 1 - F_n^*(x_n) \\ &\leq C_1 [1 - \exp(-n^{-\delta/2})] + C_2 n^{-(1+\delta/2)/2} \\ &\leq C_3 n^{-\delta/2} + C_2 n^{-(1+\delta/2)/2} \end{aligned}$$

( $C_1, C_2, C_3$  are constants) and the fact that the series  $\sum_{n=1}^{\infty} \log n / n^{1+\epsilon}$  converges for any  $\epsilon > 0$ . This completes the proof of lemma 4.7.

Combining lemma 4.5 and 4.1, we obtain

Theorem 4.8

Equation (1.2) holds.

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