



Characteristic Boundary Conditions for ARO-1

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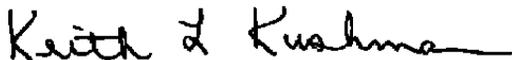
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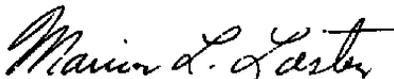
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PREFACE

The work reported herein was conducted by the Arnold Engineering Development Center (AEDC), Air Force Systems Command (AFSC). The Air Force project manager was Dr. Keith Kushman, AEDC/DOT. The results of the research were jointly obtained by Sverdrup Technology, Inc./AEDC Group, operating contractor for Propulsion Testing, and Calspan Field Services, Inc./AEDC Division, operating contractor for the Aerospace Flight Dynamics effort at the AEDC, AFSC, Arnold Air Force Station, Tennessee, under AEDC Project Number D205PW (Calspan Project Number P32A-C7). The manuscript was submitted for publication on November 22, 1982.

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1.0 INTRODUCTION

The computer program designated ARO-1 (Ref. 1) is an effective tool for the solution of problems definable in terms of the three-dimensional, unsteady Euler equations. Unprecedented flexibility and applicability of the code to a broad spectrum of aerodynamic flows was achieved through use of a finite volume approach in Cartesian coordinates. The explicit MacCormack algorithm expressed in fine-tuned, CRAY-vectorized coding yielded a very fast solver. Unfortunately, these specific attributes which make the code so useful have delayed correction of an error in the boundary condition routine.

Proper specification of boundary conditions is a difficult problem in computational fluid dynamics, increasingly so with added dimensions and dependent variables. There are actually two fundamental problems associated with boundary conditions: (1) over- or under-specification of boundary information can create an unstable numerical system, and (2) incorrect specification will yield invalid results, often without warning. Cline (Ref. 2), among others, has demonstrated that the theory of characteristics can be used to deduce what information is defined in the interior flow and thus what information remains to be specified at the boundary. In principle, this solves the first problem. The second problem requires solution through experience and induction.

Application of characteristics theory to ARO-1 is not straightforward because of the Cartesian frame of reference. This report presents the derivation of characteristic relations in generalized coordinates and discusses the specialization necessary for inclusion in ARO-1.

2.0 COORDINATE SYSTEMS AND NOTATION

The body-fitted computational grid is presumed given in Cartesian coordinates (x^1 , x^2 , x^3), referenced as the unbarred system. A new system, termed barred, is defined locally at a boundary point as sketched in Fig. 1. The basis vectors in the linear barred system are constructed in the following manner. Choose \bar{e}_1 to be directed from a boundary point towards the associated interior point such that

$$\bar{e}_1 \cdot \bar{n} = 1 \quad (1)$$

where \bar{n} is the inward unit normal to the boundary. Select \bar{e}_2 as

$$\bar{e}_2 = \frac{\bar{e}_1 \times \bar{n}}{|\bar{e}_1 \times \bar{n}|} \quad (2)$$

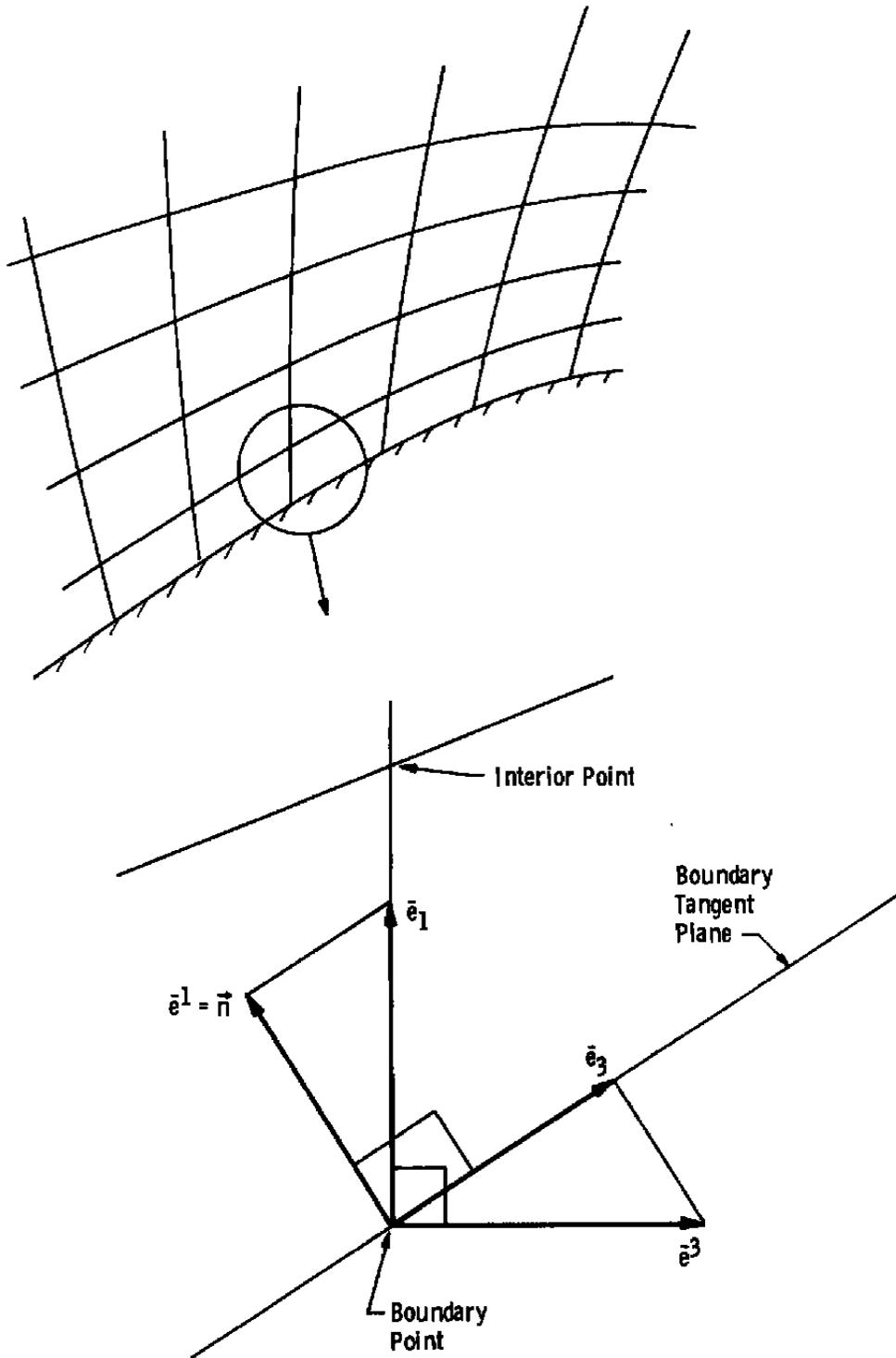


Figure 1. The barred basis vectors.

and should $\bar{e}_1 \times \bar{n} = 0$, then select \bar{e}_2 as a unit vector in an arbitrary direction perpendicular to \bar{n} . Finally, designate \bar{e}_3 orthogonal to \bar{n} and \bar{e}_2 :

$$\bar{e}_3 = \bar{n} \times \bar{e}_2 \quad (3)$$

The barred system is linear but not necessarily Cartesian. In general, \bar{e}_1 is not orthogonal to \bar{e}_3 . The \bar{x}^1 axis is directed through the interior point, and the \bar{x}^2 and \bar{x}^3 axes lie in the tangent plane.

From Eqs. (1) through (3) it follows that

$$\bar{e}_1 \cdot \bar{e}_2 \times \bar{e}_3 = 1 \quad (4)$$

and the reciprocal basis vectors are given by

$$\bar{e}^1 = \bar{e}_2 \times \bar{e}_3 = \bar{n} \quad (5)$$

$$\bar{e}^2 = \bar{e}_3 \times \bar{e}_1 = \bar{e}_2 \quad (6)$$

$$\bar{e}^3 = \bar{e}_1 \times \bar{e}_2 \quad (7)$$

It should be noted that \bar{e}^1 , \bar{e}_1 , \bar{e}^3 , and \bar{e}_3 lie in the same plane; that \bar{e}^1 , \bar{e}_2 , and \bar{e}_3 are orthonormal; that \bar{e}_1 and \bar{e}^3 are of the same magnitude; and that

$$\bar{e}_j \cdot \bar{e}^k = \delta_j^k \quad (8)$$

Since both the barred and unbarred systems are linear, the transformation between them is expressible as a matrix of constants,

$$M_j^k = \frac{\partial x^k}{\partial \bar{x}^j} \quad (9)$$

consistent with

$$\bar{e}_j = M_j^k e_k \quad (10)$$

where M_j^k is the k th component of \bar{e}_j when expressed in terms of the unbarred basis vectors. The inverse transformation

$$W_j^k = \frac{\partial \bar{x}^k}{\partial x^j} \quad (11)$$

is obtained consistent with

$$M_j^\alpha W_\alpha^k = \delta_j^k \quad (12)$$

The metric tensors are given by

$$\bar{g}_{jk} = \bar{e}_j \cdot \bar{e}_k \quad (13)$$

$$\bar{g}^{jk} = \bar{e}^j \cdot \bar{e}^k \quad (14)$$

where

$$\bar{g}_{j\alpha} \bar{g}^{\alpha k} = \delta_j^k \quad (15)$$

Note that the definition of the basic vectors implies that $\bar{g}^{11} = 1$. Contravariant and covariant components are related by

$$\bar{u}_j = \bar{g}_{jk} \bar{u}^k \quad (16)$$

and vectors can be expressed in either fashion; for example,

$$\bar{q} = \bar{u}^j \bar{e}_j = \bar{u}_j \bar{e}^j \quad (17)$$

The magnitude of the velocity vector is given by

$$q = \sqrt{\bar{g}^{jk} \bar{u}_j \bar{u}_k} = \sqrt{\bar{g}_{jk} \bar{u}^j \bar{u}^k} = \sqrt{\bar{u}_j \bar{u}^j} \quad (18)$$

For subsequent reference, frequently used expressions and their associated inverses are grouped below:

$$M_j^k = \frac{\partial x^k}{\partial \bar{x}^j} \quad W_j^k = \frac{\partial \bar{x}^k}{\partial x^j} \quad (19)$$

$$\bar{e}_j = M_j^k e_k \quad e_j = W_j^k \bar{e}_k \quad (20)$$

$$e^k = M_j^k \bar{e}^j \quad \bar{e}^k = W_j^k e^j \quad (21)$$

$$x^k - x_0^k = M_j^k \bar{x}^j \quad \bar{x}^k = W_j^k (x^j - x_0^j) \quad (22)$$

$$u^k = M_j^k \bar{u}^j \quad \bar{u}^k = W_j^k u^j \quad (23)$$

$$\bar{u}_j = M_j^k u_k \quad u_j = W_j^k \bar{u}_k \quad (24)$$

The "o" subscripts in Eq. (22) refer to the specific boundary point which is the origin of the barred system. Since the unbarred system is Cartesian, $e_j = e^j$. Scalars, such as q and p , are the same in both systems and are not barred.

Since the barred system is linear, the metric tensors are constant throughout space. Thus, the Christoffel symbols are zero, and covariant differentiation is simply partial differentiation, such as

$$\bar{u}^j{}_{,k} = \frac{\partial \bar{u}^j}{\partial \bar{x}^k} \quad (25)$$

In spite of the barred system's having been completely defined here, actual implementation of characteristic boundary conditions does not require computation of the second and third basis vectors or components.

3.0 CHARACTERISTICS OF THE EULER EQUATIONS

The Euler equations can be written as

$$\bar{G}_t + \bar{F}^k{}_{,k} = 0 \quad (26)$$

where

$$\bar{G} = \begin{bmatrix} \rho \\ \rho \bar{u}^i \\ E \end{bmatrix} \quad (27)$$

and

$$\bar{F}^k = \begin{bmatrix} \rho \bar{u}^k \\ \rho \bar{u}^i \bar{u}^k + \bar{g}^{ik} p \\ (E + p) \bar{u}^k \end{bmatrix} \quad (28)$$

and i and k assume values 1, 2, and 3. The equation of state is taken to be

$$p = (\gamma - 1) (E - \rho q^2/2) \quad (29)$$

These equations can be put in the form

$$\bar{A} \bar{Q}_t + \bar{B}^k \bar{Q}_{\bar{x}^k} = 0 \quad (30)$$

where

$$\bar{Q} = \begin{bmatrix} \rho \\ \bar{u}^j \\ p \end{bmatrix} \quad (31)$$

$$\bar{A} = \begin{bmatrix} 1 & 0 & 0 \\ \bar{u}^i & \delta_j^i \rho & 0 \\ \frac{1}{2} q^2 & \rho \bar{u}_j & \frac{1}{\gamma - 1} \end{bmatrix} \quad (32)$$

and

$$\bar{B}^k = \begin{bmatrix} \bar{u}^k & \delta_j^k \rho & 0 \\ \bar{u}^i \bar{u}^k & \delta_j^i \rho \bar{u}^k + \delta_j^k \rho \bar{u}^i & \bar{g}^{ik} \\ \frac{1}{2} q^2 \bar{u}^k & \delta_j^k (E + p) + \rho \bar{u}_j \bar{u}^k & \frac{\gamma}{\gamma - 1} \bar{u}^k \end{bmatrix} \quad (33)$$

with i and j assuming values 1, 2, and 3. Equation (30) is further rearranged to yield

$$\bar{A} \bar{Q}_t + \bar{B} \bar{Q}_{\bar{x}^1} = \bar{C} \quad (34)$$

where

$$\bar{B} = \bar{B}^1 \quad (35)$$

and

$$\bar{C} = -\bar{B}^2 \bar{Q}_{\bar{x}^2} - \bar{B}^3 \bar{Q}_{\bar{x}^3} \quad (36)$$

Thus, the derivatives with respect to t and \bar{x}^1 are kept on the left-hand side, defining a reference plane that can be easily treated by the theory of characteristics; everything else is transposed to the right-hand side.

Let $\bar{\lambda}$ and \bar{T} be an eigenvalue and eigenvector defined by

$$(\bar{\lambda} \bar{A}^* - \bar{B}^*) \bar{T} = 0 \quad (37)$$

Then, multiplying Eq. (34) by \bar{T}^* and using Eq. (37), one obtains

$$\bar{T}^* \bar{A} (\bar{Q}_t + \bar{\lambda} \bar{Q}_{\bar{x}^1}) = \bar{T}^* \bar{C} \quad (38)$$

Along the characteristic direction given by

$$\begin{aligned} d\bar{x}^1 &= \bar{\lambda} dt \\ d\bar{x}^2 &= 0 \\ d\bar{x}^3 &= 0 \end{aligned} \quad (39)$$

the total time derivative becomes

$$\frac{d\bar{Q}}{dt} = \bar{Q}_t + \bar{\lambda} \bar{Q}_{x_1} \quad (40)$$

and Eq. (38) becomes

$$\bar{T}^* \bar{A} \frac{d\bar{Q}}{dt} = \bar{T}^* \bar{C} \quad (41)$$

Equation (41) is termed the compatibility equation and gives a relation among the total time derivatives in a characteristic direction.

Equation (37) has five eigenvalues, only three being distinct since one of them is triply repeated. Appearing in the solution of Eq. (37) is the speed of sound, given by

$$a = \sqrt{\gamma p / \rho} \quad (42)$$

The eigenvalues and eigenvectors of Eq. (37) are as follows:

$$\bar{\lambda}_1 = \bar{u}^1 - a \quad (43)$$

$$\bar{T}_1^* = (\gamma - 1) \left(\frac{1}{2} q^2 + \frac{a}{\gamma - 1} \bar{u}^1, -\bar{u}_i - \delta_i^1 \frac{a}{\gamma - 1}, 1 \right) \quad (44)$$

$$\bar{\lambda}_2 = \bar{u}^1 + a \quad (45)$$

$$\bar{T}_2^* = (\gamma - 1) \left(\frac{1}{2} q^2 - \frac{a}{\gamma - 1} \bar{u}^1, -\bar{u}_i + \delta_i^1 \frac{a}{\gamma - 1}, 1 \right) \quad (46)$$

$$\bar{\lambda}_3 = \bar{u}^1 \quad (47)$$

$$\bar{T}_3^* = (\gamma - 1) \left(\frac{1}{2} q^2 - \frac{a^2}{\gamma - 1}, -\bar{u}_i, 1 \right) \quad (48)$$

$$\bar{\lambda}_4 = \bar{u}^1 \quad (49)$$

$$\bar{T}_4^* = \frac{1}{\rho} \left(-\bar{u}_2, \bar{g}_{i2}, 0 \right) \quad (50)$$

$$\bar{\lambda}_5 = \bar{u}^1 \quad (51)$$

$$\bar{T}_3^* = \frac{1}{\rho} \left(-\bar{u}_3, \bar{g}_{i3}, 0 \right) \quad (52)$$

4.0 ANALYTIC SIMPLIFICATIONS

The left-hand side of Eq. (41) involves total time derivatives in a characteristic direction, whereas the right-hand side involves spatial partial derivatives. The two sides will be simplified separately. Two of the factors on the left-hand side can be multiplied to yield

$$\bar{A} \frac{d\bar{Q}}{dt} = \begin{bmatrix} \frac{d\rho}{dt} \\ \frac{d}{dt} (\rho\bar{u}^i) \\ \frac{dE}{dt} \end{bmatrix} \quad (53)$$

The left-hand sides corresponding to the five eigenvectors are

$$\bar{T}_1^* \bar{A} \frac{d\bar{Q}}{dt} = \frac{d\rho}{dt} - a\rho \frac{d\bar{u}^1}{dt} \quad (54)$$

$$\bar{T}_2^* \bar{A} \frac{d\bar{Q}}{dt} = \frac{d\rho}{dt} + a\rho \frac{d\bar{u}^1}{dt} \quad (55)$$

$$\bar{T}_3^* \bar{A} \frac{d\bar{Q}}{dt} = \frac{d\rho}{dt} - a^2 \frac{d\rho}{dt} \quad (56)$$

$$\bar{T}_4^* \bar{A} \frac{d\bar{Q}}{dt} = \frac{d\bar{u}_2}{dt} \quad (57)$$

$$\bar{T}_5^* \bar{A} \frac{d\bar{Q}}{dt} = \frac{d\bar{u}_3}{dt} \quad (58)$$

Now the right-hand side, Eq. (36), can be written as

$$\bar{C} = - (\delta_{\beta}^{\alpha} - M_1^{\alpha} W_{\beta}^1) M_k^{\beta} W_{\alpha}^j B^k \bar{Q}_{\bar{x}j} \quad (59)$$

Using Eqs. (11), (23), and (31), one obtains

$$W_{\alpha}^j \bar{Q}_{\bar{x}^j} = \begin{bmatrix} \frac{\partial \rho}{\partial x^{\alpha}} \\ W_m^i \frac{\partial u^m}{\partial x^{\alpha}} \\ \frac{\partial p}{\partial x^{\alpha}} \end{bmatrix} \quad (60)$$

Another pair of products can be reduced using Eqs. (12), (14), (21), (23), (24), and (33) to yield

$$M_k^{\beta} \bar{B}^k = \begin{bmatrix} u^{\beta} & M_j^{\beta} \rho & 0 \\ W_n^i u^n u^{\beta} & W_k^i M_j^n \rho u^{\beta} + M_j^{\beta} W_n^i \rho u^n & W_n^i g^{n\beta} \\ \frac{1}{2} q^2 u^{\beta} & M_j^{\beta} (E+p) + M_j^n \rho u_n u^{\beta} & \frac{\gamma}{\gamma-1} u^{\beta} \end{bmatrix} \quad (61)$$

Let $\bar{\tau}_i$ be the components of \bar{T}

$$\bar{T} = (\bar{\tau}_i) \quad (62)$$

then, from Eqs. (12), (18), (29), (59), (60), and (61)

$$\begin{aligned} \bar{T}^{\alpha} \bar{C} &= -(\delta_{\beta}^{\alpha} - M_1^{\alpha} W_{\beta}^1) \left\{ \bar{\tau}_1 \frac{\partial}{\partial x^{\alpha}} (\rho u^{\beta}) \right. \\ &\quad + \bar{\tau}_{i+1} W_n^i \frac{\partial}{\partial x^{\alpha}} (\rho u^n u^{\beta} + g^{n\beta} p) \\ &\quad \left. + \bar{\tau}_5 \frac{\partial}{\partial x^{\alpha}} [(E+p)u^{\beta}] \right\} \end{aligned} \quad (63)$$

Define

$$\hat{T}^{\alpha} = [\bar{\tau}_1, \bar{\tau}_{i+1} W_n^i, \bar{\tau}_5] \quad (64)$$

then Eq. (63) can be written

$$\bar{T}^{\alpha} \bar{C} = -(\delta_{\beta}^{\alpha} - M_1^{\alpha} W_{\beta}^1) \hat{T}^{\alpha} F^{\beta, \alpha} \quad (65)$$

The significant simplification here is that the local coordinate system dependence is removed and the equations are expressed in a global Cartesian system consistent with ARO-1. The

overbars are essentially gone. Listed below are the five \hat{T} vectors for each of the corresponding characteristic directions.

$$\hat{T}_1^* = (\gamma - 1) \left[\frac{1}{2} q^2 + \frac{a}{\gamma-1} \bar{u}^1, -u_i - W_i^1 \frac{a}{\gamma-1}, 1 \right] \quad (66)$$

$$\hat{T}_2^* = (\gamma - 1) \left[\frac{1}{2} q^2 - \frac{a}{\gamma-1} \bar{u}^1, -u_i + W_i^1 \frac{a}{\gamma-1}, 1 \right] \quad (67)$$

$$\hat{T}_3^* = (\gamma - 1) \left[\frac{1}{2} q^2 - \frac{a^2}{\gamma-1}, -u_i, 1 \right] \quad (68)$$

$$\hat{T}_4^* = \frac{1}{\rho} M_2^k \left[-u_k, g_{ki}, 0 \right] \quad (69)$$

$$\hat{T}_5^* = \frac{1}{\rho} M_3^k \left[-u_k, g_{ki}, 0 \right] \quad (70)$$

The first three compatibility equations do not involve \bar{u}_2 or \bar{u}_3 . It is desirable to eliminate these components from the fourth and fifth equations as well. Combining Eqs. (24), (41), (57), (58), (65), (69), and (70), the last two compatibility equations can be written

$$M_s^k \frac{du_k}{dt} = -(\delta_j^s - M_j^\alpha W_\beta^1) \frac{1}{\rho} M_s^k (-u_k, g_{ki}, 0) F_{,\alpha}^\beta \quad (71)$$

with $s = 2$ and 3 . Define

$$\hat{T}_{5-k}^* = \frac{1}{\rho} (-u_k, g_{ki}, 0) \quad (72)$$

for $k = 1, 2$, and 3 ; then Eq. (71) can be written

$$M_s^k \frac{du_k}{dt} = M_s^k (\bar{T}_{5+k}^* \bar{C}) \quad (73)$$

From differentiation of Eq. (23) one obtains

$$W_k^1 \frac{du^k}{dt} = \frac{d\bar{u}^1}{dt} \quad (74)$$

Consider the system of three equations, Eq. (74) and Eq. (73) with $s = 2$ and 3 . Note that since the unbarred system is Cartesian, $u_k = u^k$, and the system can be written

$$\begin{bmatrix} W_1^1 & W_2^1 & W_3^1 \\ M_2^1 & M_2^2 & M_2^3 \\ M_3^1 & M_3^2 & M_3^3 \end{bmatrix} \begin{bmatrix} \frac{du^1}{dt} \\ \frac{du^2}{dt} \\ \frac{du^3}{dt} \end{bmatrix} = \begin{bmatrix} \frac{d\bar{u}^1}{dt} \\ M_2^1(\bar{T}_6^* \bar{C}) + M_2^2(\bar{T}_7^* \bar{C}) + M_2^3(\bar{T}_8^* \bar{C}) \\ M_3^1(\bar{T}_6^* \bar{C}) + M_3^2(\bar{T}_7^* \bar{C}) + M_3^3(\bar{T}_8^* \bar{C}) \end{bmatrix} \quad (75)$$

The rows of the coefficient matrix are the components with respect to the unbarred coordinates of \bar{e}_1 , \bar{e}_2 , and \bar{e}_3 , which are orthonormal. Therefore, the coefficient matrix is orthogonal and its inverse is its transpose. Thus, one can directly solve Eq. (75) to obtain

$$\frac{du^k}{dt} = W_k^1 \frac{d\bar{u}^1}{dt} + \bar{T}_{3+k}^* \bar{C} - W_k^1 \sum_{\alpha=1}^3 W_\alpha^1 (\bar{T}_{3+\alpha}^* \bar{C}) \quad (76)$$

The fourth and fifth compatibility equations can therefore be replaced with Eq. (76) with $k = 1, 2$, and 3 , which does not involve \bar{u}_2 or \bar{u}_3 or the components of \bar{e}_2 or \bar{e}_3 . This represents a significant reduction in computational burden. Knowledge of the tangent-plane variable dependence is not required for implementing characteristic boundary conditions out of the plane.

5.0 FINITE-DIFFERENCE APPROXIMATIONS

All of the right-hand sides of the compatibility equations involve the quantity

$$\hat{C} = -(\delta_\beta^\alpha - M_\beta^\alpha W_\beta^1) F^{\beta, \alpha} \quad (77)$$

For purposes of constructing finite differences of Eq. (77), three pairs of points are chosen. Let $d_{\alpha\beta}^k$ be the k 'th component with respect to the unbarred system of a vector from the boundary point to one of the chosen points, α indicating the pair (1, 2, or 3), and β indicating which point in the pair (1 or 2). Let \vec{d}_α be the vector from the first point of the α -pair to the second, and let d_α^k be its components with respect to the unbarred system; then

$$d_\alpha^k = d_{\alpha 2}^k - d_{\alpha 1}^k \quad (78)$$

The three pairs of points are chosen so that the vector \vec{d}_1 is in the same direction as \bar{e}_1 and the vectors \vec{d}_1 , \vec{d}_2 , and \vec{d}_3 are not coplanar. Since \vec{d}_1 and \bar{e}_1 are parallel, let c be their constant of proportionality; then by Eq. (9),

$$d_1^k = c M_1^k \quad (79)$$

Multiplying by W_k^1 yields

$$c = d_1^k W_k^1 \quad (80)$$

The finite-difference formula is derived from

$$F_{\alpha\beta}^j = F_o^j + F_{o,\ell}^j d_{\alpha\beta}^\ell + O(2) \quad (81)$$

where the $\alpha\beta$ subscript on F^j indicates its evaluation at that point, and the o subscript indicates its value at the boundary point. It is assumed that all the chosen points are in a neighborhood of the boundary point where the second-order terms are negligible. Define

$$F_\alpha^j = F_{\alpha 2}^j - F_{\alpha 1}^j \quad (82)$$

Neglecting second-order terms, Eqs. (78), (81), and (82) yield

$$F_\alpha^j = F_{o,\ell}^j d_\alpha^\ell \quad (83)$$

Multiplying by D_j^α (defined below), one obtains

$$F_\alpha^j D_j^\alpha = F_{o,\ell}^j d_\alpha^\ell D_j^\alpha \quad (84)$$

Since it is desired to obtain an approximation to Eq. (77), D_j^α is defined to assure that

$$D_j^\alpha d_\alpha^\ell = \delta_j^\ell - M_1^\ell W_j^1 \quad (85)$$

Selecting the three pairs of points such that the \bar{d}_α^k vectors are not coplanar ensures the existence of the inverse, \tilde{d}_ℓ^k , such that

$$d_\alpha^\ell \bar{d}_\ell^k = \delta_\alpha^k \quad (86)$$

Multiplication of Eq. (85) by \tilde{d}_ℓ^k and using Eqs. (79) and (80) yields

$$D_j^k = \tilde{d}_j^k - \frac{W_j^1 \delta_1^k}{d_1^\alpha W_\alpha^1} \quad (87)$$

By using Eqs. (84) and (85), an approximation to Eq. (77) is finally given as

$$\hat{C} = -F_\alpha^j D_j^\alpha \quad (88)$$

This approximation is valid in a small region, limited to the neighborhood where the second-order terms of Eq. (81) are negligible.

The procedure for computing \hat{C} is summarized as follows: (1) calculate d_j^k and their inverse from Eqs. (78) and (86); (2) evaluate D_j^k from Eq. (87) in which W_α^1 are the components of \bar{n} expressed in the unbarred system; (3) compute F_b^i with Eq. (82), in which Eq. (28) is employed in the unbarred sense; thence (4) \hat{C} is given by Eq. (88).

6.0 BOUNDARY CONDITION METHODOLOGY

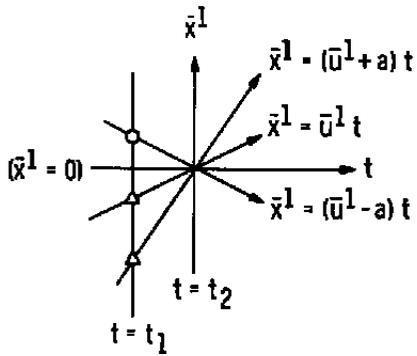
The solution is known at time $t = t_1$, and it is desired to update the variables on the boundary ($\bar{x}^1 = 0$) at $t = t_2$. Three characteristic lines are constructed in the (t, \bar{x}^1) plane intersecting the boundary as sketched in Fig. 2. The analysis can be divided into five cases:

- a. subsonic inflow
- b. supersonic inflow
- c. subsonic outflow
- d. supersonic outflow
- e. solid wall

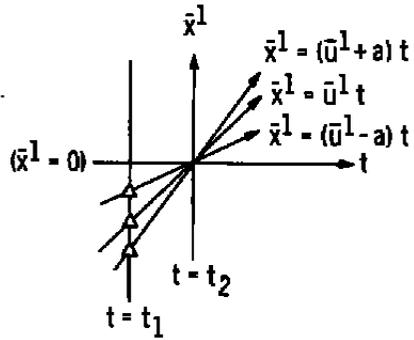
For boundary condition considerations, the terms subsonic and supersonic refer to the normal component of the velocity. At the solid wall the normal component is zero, and only one case is needed. In Fig. 2, the flow is known at the points marked with circles since these points are inside the computational region at $t = t_1$. The time step

$$\Delta t = t_2 - t_1 \quad (89)$$

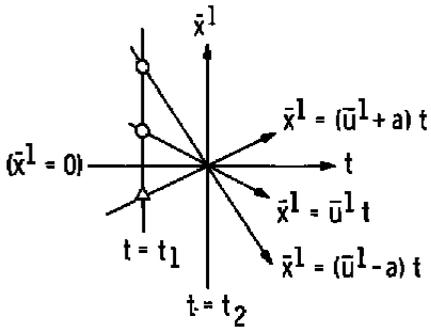
as determined by the Courant-Friedrich-Lewy stability criterion in ARO-1 is such that the circled points are on the \bar{x}^1 axis between the boundary point and the associated interior point. Therefore, the values at points designated by circles can be obtained by interpolation rather than extrapolation. The flow is undefined at the points marked with triangles since these points are outside the computational region at $t = t_1$; thus the compatibility equations along the characteristic lines passing through the triangles cannot be used and must be replaced with user-specified boundary conditions. For supersonic inflow, none of the characteristic lines can be used, which means that at such boundary points all flow variables must be specified. For supersonic outflow, all of the characteristic lines are used, which means that nothing can be specified; the flow at the boundary must be a result of the flow calculations. The remaining three cases will be treated separately.



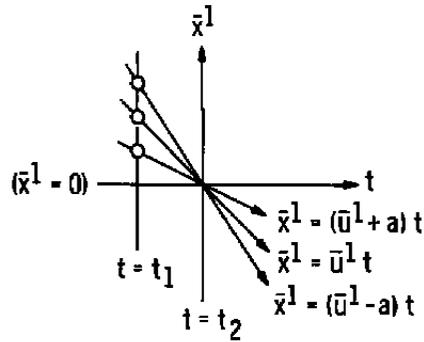
a. Subsonic inflow



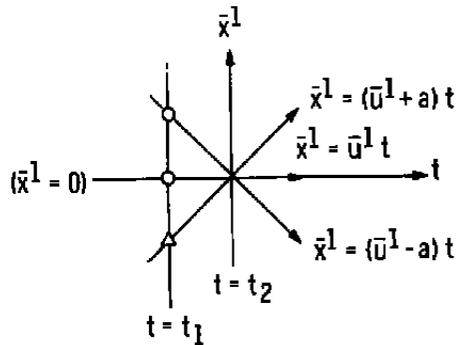
b. Supersonic inflow



c. Subsonic outflow



d. Supersonic outflow



e. Solid wall ($\bar{u}^1 = 0$)

Figure 2. Types of boundary conditions.

6.1 SUBSONIC INFLOW

For subsonic inflow (Fig. 2a), the flow transmits information to the boundary by way of only the first characteristic line

$$\bar{x}^1 = (\bar{u}^1 - a)t \quad (90)$$

with compatibility equation

$$\frac{dp}{dt} - aq \frac{d\bar{u}^1}{dt} = \hat{T}_1 \hat{C} \quad (91)$$

This equation supplies only one of the five relations needed to specify the boundary conditions. For subsonic inflow, the stagnation pressure and temperature are usually known, and the assumption of isentropic flow is generally reasonable. The remaining two conditions are a constraint on the direction of the flow

$$u^j = q\ell^j \quad (92)$$

where the direction cosines, ℓ^j , are specified at the boundary; alternately, a zero derivative could be specified which requires the flow direction at the boundary to be the same as at the associated interior point.

Reference 3 gives isentropic relations for pressure-velocity dependence as

$$p = p_T \left[1 - \frac{\gamma - 1}{2} \left(\frac{q}{a_T} \right)^2 \right]^{\gamma/(\gamma-1)} \quad (93)$$

and sound speed-velocity as

$$a = a_T \left[1 - \frac{\gamma - 1}{2} \left(\frac{q}{a_T} \right)^2 \right]^{1/2} \quad (94)$$

Rigorous evaluation of these expressions would require expensive iteration at each boundary point to achieve consistency among the variables. However, linearization in time avoids that iteration and yields consistency in the following fashion. Differentiating Eq. (93) and simplifying with Eqs. (94) and (42), one obtains

$$\Delta p = \frac{p}{p_T} \Delta p_T - \rho q \Delta q + \frac{1}{2} \rho q^2 \frac{\Delta a_T^2}{a_T^2} \quad (95)$$

Finite differencing Eq. (91) gives

$$\Delta p - a \rho \Delta \bar{u}^1 = (\hat{T}_1^* \hat{C}) \Delta t \tag{96}$$

Using Eqs. (23) and (92), differentiation yields

$$\Delta \bar{u}^1 = W_\alpha^1 \ell^\alpha \Delta q + q W_\alpha^1 \Delta \ell^\alpha \tag{97}$$

Combining Eqs. (95), (96), and (97) results in

$$\rho \left[q + a W_\alpha^1 \ell^\alpha \right] \Delta q = \frac{p}{p_T} \Delta p_T + \frac{1}{2} \rho q^2 \frac{\Delta a_T^2}{a_T^2} - a \rho q W_\alpha^1 \Delta \ell^\alpha - (T_1^* C) \Delta t \tag{98}$$

As shown in Fig. 3, the point designated I1 is the interior point that was used to define the barred coordinate system, evaluated at time t_1 . All finite differences in Eqs. (95) through (98) are taken between the points O2, the boundary at time t_2 , and C1, the characteristic point in the interior at time t_1 . Equation (91) is valid only along this characteristic direction.

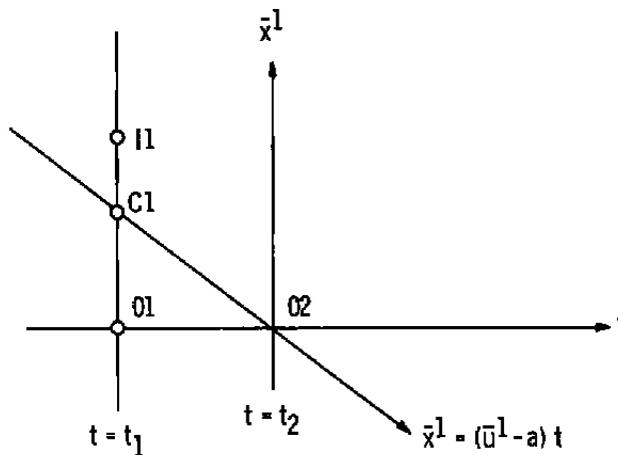


Figure 3. Nomenclature for the subsonic inflow case.

In summary, the procedure for updating the dependent variables Q at the subsonic inflow boundary point O2 is as follows: from Eq. (23),

$$\bar{u}_{01}^1 = W_\alpha^1 u_{01}^\alpha \tag{99}$$

where W_α^1 are the components of \bar{n} in the unbarred Cartesian coordinate system.

From Fig. 3, to first-order accuracy, the characteristic slope at O2 is approximated by that at O1 to yield

$$\bar{x}_{C1}^1 = -\Delta t(\bar{u}_{01}^1 - a_{01}) \quad (100)$$

From Eq. (22)

$$\bar{x}_{i1}^1 = W_\alpha^1 (x_{i1}^\alpha - x_{01}^\alpha) \quad (101)$$

and interpolation yields the dependent variables at the characteristic point, C1, as

$$Q_{C1} = Q_{01} + \frac{\bar{x}_{C1}^1}{\bar{x}_{i1}^1} (Q_{i1} - Q_{01}) \quad (102)$$

The local Mach number

$$M_{C1} = \frac{q_{C1}}{a_{C1}} \quad (103)$$

is used to evaluate total conditions from the isentropic relations given in Ref. 3.

$$a_{TC1} = a_{C1} \left[1 + \frac{\gamma - 1}{2} (M_{C1})^2 \right]^{1/2} \quad (104)$$

$$p_{TC1} = p_{C1} \left[1 + \frac{\gamma - 1}{2} (M_{C1})^2 \right]^{\gamma/(\gamma - 1)} \quad (105)$$

Since total conditions are prescribed at the boundary (possibly as a function of time), the differences required in Eq. (98) are given by

$$\Delta a_T = a_{T02} - a_{TC1} \quad (106)$$

$$\Delta p_T = p_{T02} - p_{TC1} \quad (107)$$

Direction cosines are computed at C1 using

$$l_{C1}^\alpha = \frac{u_{C1}^\alpha}{q_{C1}} \quad (108)$$

If the flow direction is specified at the boundary (possibly as a function of time), then

$$\Delta l^\alpha = l_{02}^\alpha - l_{C1}^\alpha \quad (109)$$

or, alternatively, if zero derivatives are specified, then

$$\Delta l^\alpha = 0 \quad (110)$$

and

$$f_{02}^\alpha = f_{C1}^\alpha \quad (111)$$

The velocity difference, Δq , along the characteristic is then computed from Eq. (98) and the velocity on the boundary is given by

$$q_{02} = q_{C1} + \Delta q \quad (112)$$

with the Cartesian components expressed as

$$u_{02}^\alpha = q_{02} f_{02}^\alpha \quad (113)$$

State variables, p_{02} and a_{02} , are computed using the isentropic relations, Eqs. (93) and (94), of Ref. 3 and density evaluated from

$$\rho_{02} = \frac{\gamma p_{02}}{(a_{02})^2} \quad (114)$$

6.2 SUBSONIC OUTFLOW

For subsonic outflow (Fig. 4), the interior flow transmits information via the first characteristic line

$$\bar{x}^1 = (\bar{u}^1 - a)t \quad (115)$$

with compatibility equation

$$\frac{dp}{dt} - a \frac{d\bar{u}^1}{dt} = \hat{T}_1^* \hat{C} \quad (116)$$

and along the coincident third, fourth, and fifth characteristic lines

$$\bar{x}^1 = \bar{u}^1 t \quad (117)$$

with compatibility equations

$$\frac{dp}{dt} - (a)^2 \frac{dq}{dt} = \hat{T}_3^* \hat{C} \quad (118)$$

and from Eq. (76), written as

$$\frac{du^k}{dt} = W_k^1 \frac{d\bar{u}^1}{dt} + \hat{T}_{5+k}^* \hat{C} - W_k^1 \sum_{\alpha=1}^3 W_\alpha^1 \hat{T}_{5+\alpha}^* \hat{C} \quad (119)$$

with $k = 1, 2,$ and 3 . Although Eq. (119) provides three equations, considering the dependence of \bar{u}^1 [see Eqs. (73) and (74)], they are of rank two; therefore, one further condition remains to be specified. It will be taken as a specified static pressure, p . For practical applications, pressure is specified at one isolated boundary point and zero gradient in the outflow direction applied at all other boundary points.

Referring to Fig. 4, the solution is known at time t_1 at the points 01 and I1 and, through interpolation, at the characteristic points B1 and C1. The procedure for updating the dependent variables Q at the subsonic outflow boundary point 02 is, from Eqs. (22) and (23):

$$\bar{x}_{I1}^1 = W_\alpha^1 (x_{I1}^\alpha - x_{01}^\alpha) \quad (120)$$

$$\bar{u}_{01}^1 = W_\alpha^1 u_{01}^\alpha \quad (121)$$

where W_α^1 are the components of \bar{n} in the unbarred system. From Fig. 4, with first-order accuracy, the characteristic slopes are evaluated at 01 to yield

$$\bar{x}_{B1}^1 = -\Delta t \bar{u}_{01}^1 \quad (122)$$

$$\bar{x}_{C1}^1 = -\Delta t (\bar{u}_{01}^1 - a_{01}) \quad (123)$$

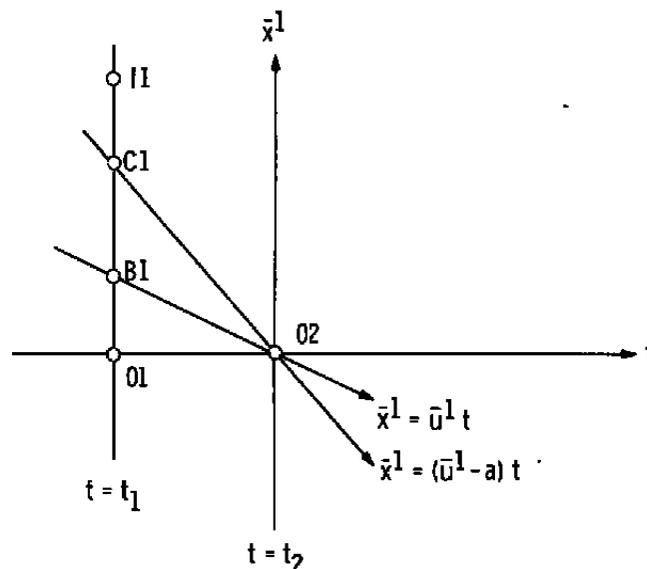


Figure 4. Nomenclature for the subsonic outflow case.

Interpolation yields

$$Q_{B1} = Q_{01} + \frac{\bar{x}_{B1}^1}{\bar{x}_{11}^1} (Q_{11} - Q_{01}) \quad (124)$$

$$Q_{C1} = Q_{01} + \frac{\bar{x}_{C1}^1}{\bar{x}_{11}^1} (Q_{11} - Q_{01}) \quad (125)$$

and from Eq. (23)

$$\bar{u}_{B1}^1 = W_{\alpha}^1 u_{B1}^{\alpha} \quad (126)$$

$$\bar{u}_{C1}^1 = W_{\alpha}^1 u_{C1}^{\alpha} \quad (127)$$

A finite-difference representation of Eq. (116) is written as

$$(p_{02} - p_{C1}) - a_{C1} \rho_{C1} (\bar{u}_{02}^1 - \bar{u}_{C1}^1) = \Delta t [\hat{T}_1^* \hat{C}]_{C1} \quad (128)$$

which yields \bar{u}_{02}^1 where $p_{02} = p_{11}$ except at one select point where p_{02} is user-specified.

A finite-difference representation of Eq. (118) is written as

$$(p_{02} - p_{B1}) - (a_{B1})^2 (\rho_{02} - \rho_{B1}) = \Delta t [\hat{T}_3^* \hat{C}]_{B1} \quad (129)$$

which yields ρ_{02} . Finally, differencing of Eq. (119) can be constructed as

$$(u_{02}^{\alpha} - u_{B1}^{\alpha}) = W_{\alpha}^1 (\bar{u}_{02}^1 - \bar{u}_{B1}^1) + \Delta t \left[\hat{T}_{5+\alpha}^* \hat{C} - W_{\alpha}^1 \sum_{k=1}^3 W_k^1 \hat{T}_{5+k}^* \hat{C} \right]_{B1} \quad (130)$$

which is solved to yield u_{02}^{α} which then completes calculation of Q_{02} at the subsonic outflow boundary.

6.3 SOLID WALL BOUNDARY

Information from the flow is transmitted to a solid wall boundary (see Fig. 2d) via the same characteristic lines as in the subsonic outflow case, Eqs. (115) through (119). The additional condition needed to determine the flow at the wall is

$$\bar{q} \cdot \bar{n} = 0 \quad (131)$$

which can be analyzed

$$\bar{q} \cdot \bar{n} = \bar{u}^\alpha \bar{e}_\alpha \cdot \bar{e}^1 = \bar{u}^\alpha \delta_\alpha^1 = 0 \quad (132)$$

resulting in

$$\bar{u}^1 = 0 \quad (133)$$

Referring to Fig. 5, the procedure for computing Q at 02 is [from Eq. (22)]:

$$\bar{x}_{I1}^1 = W_\alpha^1 (x_{I1}^\alpha - x_{01}^\alpha) \quad (134)$$

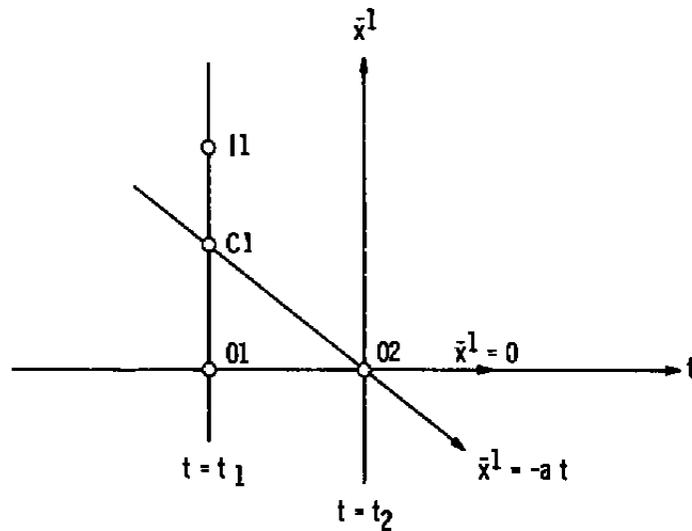


Figure 5. Nomenclature for solid wall boundary conditions.

From Fig. 5, assuming the characteristic slope at 02 is approximately that at 01 (first-order accuracy),

$$\bar{x}_{C1}^1 = \Delta t a_{01} \quad (135)$$

then interpolation yields

$$Q_{C1} = Q_{01} + \frac{\bar{x}_{C1}^1}{\bar{x}_{I1}^1} (Q_{I1} - Q_{01}) \quad (136)$$

and from Eq. (23)

$$\bar{u}_{C1}^1 = W_\alpha^1 u_{C1}^\alpha \quad (137)$$

A finite-difference representation of Eq. (116) is written as

$$(p_{02} - p_{C1}) - a_{C1} \rho_{C1} (\bar{u}_{02}^1 - \bar{u}_{C1}^1) = \Delta t [\hat{T}_1^* \hat{C}]_{C1} \quad (138)$$

to allow calculation of p_{02} where $\bar{u}_{02}^1 = 0$ from Eq. (133). The density at point 02 is obtained from a finite-difference representation of Eq. (118) in the form

$$(p_{02} - p_{01}) - (a_{01})^2 (\rho_{02} - \rho_{01}) = \Delta t [\hat{T}_3^* \hat{C}]_{01} \quad (139)$$

and the Cartesian velocity components evaluated from

$$u_{02}^\alpha = u_{01}^\alpha + \Delta t \left[\hat{T}_{3+\alpha}^* \hat{C} - w_\alpha^1 \sum_{k=1}^3 w_k^1 \hat{T}_{3+k}^* \hat{C} \right]_{01} \quad (140)$$

It should be recognized that Eq. (131) can be relaxed to permit a specified surface-normal velocity to allow imposition of an auxiliary wall boundary condition. For example, viscous boundary-layer effects on the inviscid Euler solution can be included using the concepts developed in Ref. 4.

7.0 CONCLUDING REMARKS

Characteristic boundary conditions as described herein have been implemented in the three-dimensional, time-dependent Euler solver known as ARO-1. Experience with the modified code to date indicates improved accuracy of the solutions, particularly for internal duct flow computations with subsonic inflow/outflow conditions. However, the accuracy improvement relative to the solid wall boundary conditions using zero normal pressure gradient is significantly less than expected. Vorticity is generated at curved surfaces and manifested as total pressure losses which are convected streamwise. Characteristic boundary conditions reduce the magnitude of these losses by only a few percent. Computational grid refinement appears to be necessary to minimize the total pressure loss, but, since ARO-1 is an explicit code, the associated increase in computer time is generally prohibitive.

The ARO-1 code is routinely used to enhance the ground test capabilities of the wind tunnels and engine test facilities at AEDC. The assumption of zero normal pressure gradient as a wall boundary condition has proven to yield sufficiently accurate solutions for most engineering applications. Whenever increased precision is deemed necessary, both grid refinement and characteristic wall boundary conditions must be employed.

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NOMENCLATURE

\bar{A}	See Eq. (32)
a	Speed of sound, Eq. (42)
\bar{B}	See Eq. (35)
\bar{B}^k	See Eq. (33)
\hat{C}	See Eq. (77)
\bar{C}	See Eq. (36)
c	Proportionality constant between \bar{d}_1 and \bar{e}_1
D_j^k	See Eq. (87)
\bar{d}_α	Vector from the first point of the α -pair to the second point
d_α^k	k th component of \bar{d}_α ; see Eq. (78)

$d_{\alpha\beta}^k$	kth component of a vector from the boundary point to the β -point of the α -pair
\bar{d}_{α}^k	The inverse of d_{α}^k , Eq. (86)
E	Specific energy
$\bar{e}_1, \bar{e}_2, \text{ and } \bar{e}_3$	The basis vectors for the barred system, Eqs. (1) through (3)
$\bar{e}^1, \bar{e}^2, \text{ and } \bar{e}^3$	The reciprocal base vectors for the barred system, Eqs. (5) through (7)
\bar{F}^k	See Eq. (28)
F_{δ}^k	The value of F^k at the boundary point
F_{α}^k	See Eq. (82)
$F_{\alpha\beta}^k$	The value of F^k at the β -point of the α -pair
\bar{G}	See Eq. (27)
\bar{g}_{jk} and \bar{g}^{jk}	Metric tensors, Eqs. (13) and (14)
ℓ^{α}	Direction cosines of \bar{q} , Eq. (92)
M	Mach number, Eq. (103)
M_j^k	Transformation between barred and unbarred systems, Eq. (9)
\bar{n}	Unit normal at the given boundary point
p	Static pressure, Eq. (29)
\bar{Q}	See Eq. (31)
q	Flow speed, Eq. (18)

\bar{q}	Velocity vector
\bar{T}	An eigenvector
\bar{T}_k	The kth eigenvector
\hat{T}	See Eq. (64)
\hat{T}_k	See Eqs. (66) through (70), and Eq. (72)
t	Time
\bar{u}_j and \bar{u}^j	Covariant and contravariant components with respect to the barred system of \bar{q}
W_j^k	Transformation between the barred and unbarred systems, Eq. (11)
\bar{x}^j	Barred coordinates
α	Dummy repeated index
γ	Ratio of specific heats
Δ	Finite difference
δ_j^k	Kronecker delta
λ	An eigenvalue
λ_k	The kth eigenvalue
ρ	Density
\bar{r}_i	Components of \bar{T} , Eq. (62)

SUBSCRIPTS

01,02, B1, C1, and I1 Points shown in Figs. 3, 4, and 5

- T Total conditions
- o Given boundary point

SUPERSCRIPTS

- * An asterisk indicates the transpose
- ($\bar{\quad}$) Overbar indicates association with the local boundary coordinate system