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Approximation by
Smooth Bivariate Splines
On a Three-Direction Mesh

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Let $S := \mathbb{W}_{k,\Delta}^\rho$ be the space of bivariate piecewise polynomial functions in $C^\rho$, of degree $\leq k$, on the mesh $\Delta$ obtained from a uniform square mesh by drawing in the same diagonal in each square.

de Boor and Höllig have given the following upper bound

$$m < m(k) := \min\{2(k-\rho), k+1\}$$

for the approximation order $m$ of $S$.

In this paper, the lower bound

$$m > m(k) - 2$$

is demonstrated. This result is close to de Boor and Höllig's conjecture that $m$ never differs from $m(k)$ by more than 1.

Incidentally, the approximation order of $\mathbb{W}_{4,\Delta}^1$ is shown to be 4.

AMS (MOS) Subject Classifications: 41A15, 41A63, 41A25.

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SIGNIFICANCE AND EXPLANATION

Univariate splines have been proved quite useful in practice. However, if one wants to fit a surface, or solve a partial differential equation numerically, one would naturally think of using multivariate splines. Here splines still mean piecewise polynomial functions. In this respect, a basic question is to ascertain, for a given mesh \( \mathcal{A} \) and a family \( S \) of splines on \( \mathcal{A} \), what its optimal approximation order is. This question is challenging even for a regular triangular mesh \( \mathcal{A} \), as soon as one demands that the approximating functions have a certain amount of smoothness. The report records a step toward answering the above question.
1. Introduction

In this paper we study approximation order of smooth bivariate splines on a three-direction mesh. The work in this respect was initiated by de Boor and DeVore [BD] and de Boor and Höllig [BH 1,2,3]. Here we follow them and introduce some notations. Let

\[ \Delta := \bigcup \{ x \in \mathbb{R}^2 ; x(1) = n, x(2) = n, \text{ or } x(2) - x(1) = n \} \ . \]

Namely, the mesh \( \Delta \) is obtained from a uniform square mesh by drawing in the same diagonal in each square. Let

\[ S := W^p_{k, \Delta} := W^p_{k, \Delta} \cap C^p \]

be the space of bivariate \( pp \) (piecewise polynomial) functions in \( C^p \), of total degree \( \leq k \), on the mesh \( \Delta \). Also, by \( W^p_k \) we denote the space of polynomials of total degree \( \leq k \). We are interested in the approximation order of \( S \). The approximation order of \( S \) is, by definition, the integer \( m \) for which the following holds: For all sufficiently smooth function \( f \),

\[ \text{dist}(f, S_h) = O(h^m) \]

while, for some \( C^\infty \)-function \( f \),

\[ \text{dist}(f, S_h) \neq o(h^m) \ . \]

Here, the scale \((S_h)\) of approximating spaces is generated from \( S \) by simple scaling,

\[ S_h := \sigma_h(S) \]

with

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\[(a,f)(x) := f(x/h), \text{ all } f, x, h.\]

de Boor and DeVore have given the following lower bound for \(m\) (see [BD]):

\[m > \rho + 2 \text{ in case } \rho < \rho(k) := \lfloor (2k-2)/3 \rfloor.\]

In contrast, \(S\) has approximation order \(0\) for \(\rho > \rho(k)\).

An upper bound for \(m\) has been obtained by de Boor and Höllig (see [BH 3; Theorem 3]):

\[m < m(k) := \min\{2(k-\rho), k+1\}.\]

de Boor and Höllig also show that the approximation order of \(\pi_{3,\Delta}^1\) is \(3\) rather than \(4\) (see [BH 2]). Thus the approximation order of \(S\) may differ from \(m(k)\) by \(1\). Based on those investigations, de Boor and Höllig raised the following

**Conjecture** ([BH 3]). The approximation order of \(S = \pi_{k,\Delta}^0\) never differs from its upper bound \(m(k)\) by more than \(1\).

In this paper, we shall show that the approximation order of \(S = \pi_{k,\Delta}^0\) never differs from \(m(k)\) by more than \(2\). The proof of this result will be based on a quasi-interpolant scheme. For the record we state the following

**Theorem 1.** Suppose that \(B \in S\) with \(\text{supp } B\) finite. If the map

\[T : p \mapsto \sum_{z \in \mathbb{Z}^2} p(j)B(-j)\]

is one-to-one and onto \(\pi_n\), then

\[\text{dist}(f, S_h) = O(h^{n+1})\]

for all sufficiently smooth functions \(f\).

The argument in [BH 1; Section 6] essentially gives the proof for Theorem 1. We do not need to repeat the proof here.

To construct an element \(B \in S\) with the property required by Theorem 1, we shall employ box splines, which were introduced by [BD] and [BH 1]. In section 2, we develop some preliminary results from univariate B-spline
theory. In section 3, we elaborate some properties of box splines on the three-direction mesh $\Delta$. In section 4, we construct an element $B \in S$ with the property required by Theorem 1, and therefore prove our main results. In section 5, we show that the approximation order of $w_{4,\Delta}$ is 4. This illustrates that the approximation order of $w_{k,\Delta}$ might be exactly $k$ when $k = 2p+2$.

2. **Some preliminary results from univariate B-spline theory.**

Let $t = (t_i)_{i \in \mathbb{Z}}$ be a knot sequence. Recall that

$$M_{i,k}(x) := k[t_i, \ldots, t_{i+k}](x) +$$

is a normalized B-spline of order $k$ for each $i \in \mathbb{Z}$. Also we write

$$N_{i,k}(x) := (t_{i+k} - t_i)M_{i,k}(x)/k .$$

If $p$ is a polynomial of degree $< k$, then

$$p = \sum_{i \in \mathbb{Z}} (\lambda_i p)N_{i,k} ,$$

where $\lambda_i$ is the linear functional defined by

$$\lambda_i f := \sum_{j < k} (-)^{k-1-j}p_{i,k}(j)(t_i)$$

with

$$p_{i,k}(x) := (t_{i+1} - x) \cdots (t_{i+k-1} - x)/(k-1)!$$

and $t_i \in (t_i, t_{i+k})$ (see [BF]; also [B]). Now suppose $t_i = i$, all $i \in \mathbb{Z}$.

Then $N_{i,k} = M_{i,k}$ and

$$p_{i,k}(x) = (i+1-x) \cdots (i+k-1-x)/(k-1)! .$$

It is easily seen that there exist unique constants $a_{i,k-1}(\ell=0,1,\ldots,k-2)$ such that

$$p_{i,k}(x) = \sum_{\ell=0}^{k-2} a_{i,k-1}(\ell+1-x) \cdots (\ell+k-2-x)/(k-2)! . \tag{1}$$

Comparing the coefficient of $x^{k-2}$ on both sides of (1), we obtain

$$\sum_{\ell=0}^{k-2} a_{i,k-1} = 1 . \tag{2}$$
If $f$ is a polynomial of degree $< k-2$, then

$$p := \sum_{i \in \mathbb{Z}} f(i) N_{i,k-1}$$

is also a polynomial of degree $< k-2$. On the one hand,

$$f(i) = \lambda_{i,k-1} p .$$

On the other hand,

$$p = \sum_{i \in \mathbb{Z}} (\lambda_{i,k} p) N_{i,k} .$$

Pick $\tau_i \in (i+k-2,i+k-1)$ and calculate $\lambda_{i,k} p$ as follows:

$$\lambda_{i,k} p = \sum_{j<k} (-)^{k-1-j} \psi_{i,k}^{(k-1-j)}(\tau_i) p(j)(\tau_i)$$

$$= \sum_{j<k-1} (-)^{k-1-j} \psi_{i,k}^{(k-2-j)}(\tau_i) p(j)(\tau_i)$$

$$= \sum_{j<k} (-)^{k-1-j} (\psi_{i,k}^{(k-2-j)}(\tau_i) p(j)(\tau_i))$$

$$= \sum_{k=0}^{k-2} a_{i+k-1,k-1} \lambda_{i+k-1,k-1} p .$$

Therefore

$$\sum_{i \in \mathbb{Z}} f(i) N_{i,k-1} = p = \sum_{i \in \mathbb{Z}} (\lambda_{i,k} p) N_{i,k} = \sum_{i=0}^{k-2} a_{i,k-1} \sum_{i \in \mathbb{Z}} (\lambda_{i,k-1} p) N_{i,k}$$

$$= \sum_{i=0}^{k-2} a_{i,k-1} \sum_{i \in \mathbb{Z}} f(i) N_{i,k} .$$

We have proved

**Lemma 1.** For any polynomial of degree $< k-2$

$$\sum_{i \in \mathbb{Z}} f(i) N_{i,k-1} = \sum_{i \in \mathbb{Z}} f(i) (\sum_{i=0}^{k-2} a_{i,k-1} N_{i,k}) .$$
3. Box splines on a three-direction mesh.

As defined in [BH 1], the box spline $M_\Xi$ is the distribution on $\mathbb{R}^n$
given by the rule:

$$M_\Xi : \phi \mapsto \int_{[0,1]^n} \phi(\sum_{i=1}^{n} \lambda(i) \xi_i) d\lambda$$

for some sequence $\Xi := (\xi_i)_1^n$ in $\mathbb{R}^n$. In our case, $m = 2$. Let $e_i$ be the
unit vector along $x_i$-axis ($i=1,2$), and

$$d_1 := e_1, \quad d_2 := e_1 + e_2, \quad d_3 := e_2.$$ 

For positive integers $r$, $s$ and $t$, let $\Xi = (\xi_i)_{r+s+t}^n$ be the sequence in
$\mathbb{R}^2$ given by

$$\xi_1 = \ldots = \xi_r = d_1, \quad \xi_{r+1} = \ldots = \xi_{r+s} = d_2 \quad \text{and} \quad \xi_{r+s+1} = \ldots = \xi_{r+s+t} = d_3.$$ 

From now we will write $M_{r,s,t}$ instead of $M_\Xi$. Caution! Our notation is
slightly different from [BH 3]. In [BH 3], $d_2 = e_2$ and $d_3 = e_1 + e_2$. Thus
our $M_{r,s,t}$ is just $M_{r,s,t}$ in the sense given by [BH 3].

The smoothness of $M_{r,s,t}$ depends on the direction multiplicities.

From [BH 3] we have

$$M_{r,s,t} \in L_{(d)} \subseteq C^{d-1}$$ 

with $d = \min(r, s, t) + r - 2$.

Now we define

$$B_{r,s,t}(x_1, x_2) := \sum_{\lambda_1 = 0}^{r-2} \sum_{\mu_1 = 0}^{s-2} \sum_{\nu_1 = 0}^{t-2} \sum_{\mu_{s-1} = 0}^{r-1} \sum_{\nu_{t-1} = 0}^{s-1} \sum_{\nu_{t-1} = 0}^{t-1}$$

$$\left( a_{\lambda_1, 1} \ldots a_{\lambda_{r-1}, 1} \ldots a_{\lambda_{r-1}, 1} a_{\mu_1, 1} \ldots a_{\mu_{s-1}, 1} \ldots a_{\mu_{s-1}, 1} \ldots a_{\nu_1, 1} \ldots a_{\nu_{t-1}, 1} \ldots a_{\nu_{t-1}, 1} \right)$$

\[(3)\]

$$M_{r,s,t}(x_1 + \lambda_{r-1} + \ldots + \lambda_{r-1} + \mu_1 + \ldots + \mu_{s-1} + x_2 + \mu_1 + \ldots + \mu_{s-1} + \nu_1 + \ldots + \nu_{t-1})$$

for $(r,s,t)$ with $\min(r,s,t) > 1$,

where $a$ has the meaning determined by (1).
The reason for introducing $B_{r,s,t}$ will be clear after we prove the following

**Lemma 2.** For any bivariate polynomial of degree $r+s+t-2$, we have

1° $D_1(D_1+D_2)^s \sum_{j \in \mathbb{Z}^2} p(j)(B_{r,s,t} - B_{r,s,t-1})(-j) = 0$ ;

2° $D_1D_2 \sum_{j \in \mathbb{Z}^2} p(j)(B_{r,s,t} - B_{r,s-1,t})(-j) = 0$ ;

3° $(D_1+D_2)^s D_2 \sum_{j \in \mathbb{Z}^2} p(j)(B_{r,s,t} - B_{r-1,s,t})(-j) = 0$ .

Here, as usual, $j = (j_1, j_2) \in \mathbb{Z}^2$, $x = (x_1, x_2) \in \mathbb{R}^2$, $D_1 = \frac{\partial}{\partial x_1}$,

$\forall_{i} f = f \cdot (x_{-i})$ ($i = 1,2$).

Proof. By symmetry, 2° and 3° follow from 1°. Thus we only need to prove 1°. Suppose

$B_{r,s,t-1} = \sum_{i \in \mathbb{Z}^2} b_i M_{r,s,t-1}(x_{+i})$ .

Then, by the definition of $B_{r,s,t}$, we have

$B_{r,s,t} = \sum_{i \in \mathbb{Z}^2} \sum_{l=0}^{t-2} a_i \sum_{i \in \mathbb{Z}^2} M_{r,s,t}(x_{+i+\xi_2})$ .

Hence

$$D_1(D_1+D_2)^s \sum_{j \in \mathbb{Z}^2} p(j)(B_{r,s,t} - B_{r,s,t-1})(-j)$$

$$= \sum_{i} b_i \sum_{j \in \mathbb{Z}^2} (p_1(p_1+\xi_2)^s) p(j)[\sum_{l=0}^{t-2} a_i \sum_{i \in \mathbb{Z}^2} M_{r,s,t}(x_{+i+\xi_2})] - M_{0,0,t-1}(x_{+j+i}) .$$

For any test function $\phi$, one can easily check that

$$\langle M_{0,0,t}, \phi \rangle = \int M_t(x_2) \phi(0,x_2) dx_2 .$$

Thus
< ∑ \( (V_1^{t} + V_2^{t})^{p} \) \( \phi \) \( f \) \( a_{t-1} M_{0,0,t-1} (-j+1+\delta_{2}) - M_{0,0,t-1} (-j+1) \rangle, \phi > \\
\sum_{j=0}^{t-2} \int_{\mathbb{R}^{2}} \}< \sum_{j=0}^{t-2} \int_{\mathbb{R}^{2}} (V_1^{t} + V_2^{t})^{p} \phi \int_{a_{t-1} M_{0,0,t-1} (x_2 - j_2 + 1 + \delta_{2}) - M_{0,0,t-1} (x_2 - j_2 + 1)} \phi (0, x_2) dx_2 = 0 \\
by Lemma 1, since \( V_1^{t} + V_2^{t} \) has degree < t-2. This completes the proof of Lemma 2.

4. Quasi-interpolant scheme.

In the following, \( r, s \) and \( t \) are always integers. Let

\[ I := \{ (r,s,t) \mid r+s+t = 2p+4 \text{ and } 2 < r, s, t < p+1 \} \]

\[ J_1 := \{ (r,s,t) \mid r+s+t = 2p+3 \text{ and } 2 < r, s, t < p+1 \} \]

\[ J_2 := \{ (r,s,t) \mid r+s+t = 2p+3 \text{ and } 2 < r, s, t < p \} \]

\[ K := \{ (r,s,t) \mid r+s+t = 2p+2 \text{ and } 2 < r, s, t < p \}. \]

We have

\[ I = \{ (r,2,t) \mid r+2+t = 2p+4, 2 < r, t < p+1 \} \cup \{ (r,s,t) \mid r+s+t = 2p+4, 3 < s < p+1, 2 < r, t < p+1 \} \]

\[ = \{ (r+1,2,p+1) \} \cup \{ (r,s,t) \mid r+s+t = 2p+4, 3 < s < p+1, 2 < r, t < p \} \cup \]

\[ \{ (r,s,t) \mid r+s+t = 2p+4, 3 < s < p+1, 2 < r, t \text{ and } \max(r,t) = p+1 \} \]

\[ = \{ (p+1,2,p+1) \} \cup \{ (r,s,t) \mid r+s+t = 2p+4, 4 < s < p+1, 2 < r, t < p \} \cup \]

\[ \{ (r,s,t) \mid r+s+t = 2p+4, 3 < s < p+1, 2 < r, t \text{ and } \max(r,t) = p+1 \}. \]

Similarly,

\[ J_1 = \{ (r,s,t) \mid r+s+t = 2p+3, 3 < s < p+1 \text{ and } 2 < r, t < p \} \cup \]

\[ \{ (r,s,t) \mid r+s+t = 2p+3, 2 < s < p, 2 < r, t \text{ and } \max(r,t) = p+1 \}, \]

and

\[ J_2 = \{ (r,s,t) \mid r+s+t = 2p+3, 3 < s < p \text{ and } 2 < r, t < p \}. \]

Therefore

\[ |I| = |J_1| + |K| = 1. \]

Here, by |E| we mean the cardinality of the set E.
In the following we use the convention that the empty sum has value 0.

Now we construct $B$ as follows:

$$B := \sum_{(r,s,t) \in I} B_{r,s,t} - \sum_{(r,s,t) \in J_1} B_{r,s,t} - \sum_{(r,s,t) \in J_2} B_{r,s,t} + \sum_{(r,s,t) \in K} B_{r,s,t}. \quad (8)$$

**Lemma 3.** $\sum_{j \in \mathbb{Z}^2} B(-j) = 1$.

**Proof.** From [BH 1] we have

$$\sum_{j \in \mathbb{Z}^2} M_{r,s,t}(-j) = 1.$$ 

Then (2) and (3) yield

$$\sum_{j \in \mathbb{Z}^2} B_{r,s,t}(-j) = 1.$$ 

Therefore

$$\sum_{j \in \mathbb{Z}^2} B(-j) = |I| - |J_1| - |J_2| + |K| = 1.$$ 

The following lemma plays an essential role in this paper.

**Lemma 4.** For $k = 2p+2$, $B \in \mathbb{P}_k, \Delta$ and $p - \sum_{j \in \mathbb{Z}^2} p(j)B(-j)$ is a polynomial of degree $< \deg p$ for any polynomial of degree $< k-1$.

**Proof.** We first show that

$$D_1 D_2 \left[ \sum_{j \in \mathbb{Z}^2} p(j)B(-j) \right]$$

is a constant for any $(q_1, q_2) \in \mathbb{Z}^2$ with $q_1 > 0$, $q_2 > 0$ and $q_1 + q_2 = \deg p < k-1$. \quad (9)
Fix \( q_1 \) and \( q_2 \). Consider the following index sets:

\[
E_1 := \{ (r,s,t) \mid r > q_1 \text{ and } t > q_2 \} \\
E_2 := \{ (r,s,t) \mid r < q_1 \text{ and } t < q_2 \} \\
E_3 := \{ (r,s,t) \mid r < q_1 \text{ and } t > q_2 \} \\
E_4 := \{ (r,s,t) \mid r > q_1 \text{ and } t < q_2 \} 
\]

Then \( \{ E_i \mid i = 1,2,3,4 \} \) forms a partition of \( \mathbb{Z}^3 \). To prove (10) it is sufficient to show that

\[
D_1^{-1} D_2^{-1} \sum_j p(j) \left( \sum_{(r,s,t) \in E_1} B_{r,s,t} - \sum_{(r,s,t) \in E_1 \cap (r,s,t) \in E_1'} B_{r,s,t} \right) = \sum_{(r,s,t) \in E_2} B_{r,s,t} + \sum_{(r,s,t) \in E_4} B_{r,s,t} \tag{**-j}
\]

is a constant for each \( i = 1,2,3 \) or 4. Thus we have to split our consideration into the four cases: \( i = 1,2,3 \) or 4.

Case \( i = 1 \). Then \( r > q_1 \), and \( t > q_2 \). We have

\[
D_1^{-1} D_2^{-1} \sum_j p(j) M_{r,s,t} \tag{**-j} = \sum_j (\tilde{q}_1 \tilde{q}_2 p)(j) M_{r-q_1,s,t-q_2} \tag{**-j},
\]

which is a constant, since \( \tilde{q}_1 \tilde{q}_2 p \) is a constant. It follows that

\[
D_1^{-1} D_2^{-1} \sum_j p(j) B_{r,s,t} \tag{**-j} \text{ is a constant for any } (r,s,t)
\]

with \( r > q_1 \) and \( t > q_2 \).

Hence

\[
D_1^{-1} D_2^{-1} \sum_j p(j) \left( \sum_{(r,s,t) \in E_1} B_{r,s,t} - \sum_{(r,s,t) \in E_1 \cap E_1'} B_{r,s,t} \right) = \sum_{(r,s,t) \in E_2} B_{r,s,t} + \sum_{(r,s,t) \in E_4} B_{r,s,t} \tag{**-j}
\]

is a constant.
Case i = 2. In this case, \( r < q_1 \) and \( t < q_2 \). Note that 
\[
(p+1, 2, p+1) \notin E_2 
\]
This is true, because \( r < q_1 \) and \( t < q_2 \) imply that 
\[
 r + t < q_1 + q_2 < 2p + 1 
\]
Now (4), (5) and (6) tell us that 
\[
\sum_{(r,s,t) \in E_1} B_{r,s,t} - \sum_{(r,s,t) \in E_2} B_{r,s,t} = \sum_{(r,s,t) \in E_1} B_{r,s,t} - \sum_{(r,s,t) \in E_2} B_{r,s,t} 
\]
\[
\sum_{(r,s,t) \in E_2} B_{r,s,t} = \sum_{s=4}^{p+1} \sum_{2r,t \leq p} \sum_{r+t=2p+4-s} \sum_{r \leq q_1, t \leq q_2} (B_{r,s,t} - B_{r,s-1,t}) + 
\]
\[
\sum_{s=3}^{p+1} \sum_{2r,t \leq p} \sum_{r+t=2p+3-s} \sum_{r \leq q_1, t \leq q_2} (B_{r,s,t} - B_{r,s-1,t}) \qquad (10) 
\]
while 
\[
\mathbf{D}_1 \mathbf{D}_2 \left[ \sum_{j \in \mathbb{Z}^2} p(j) (B_{r,s,t} - B_{r,s-1,t}) (-j) \right] = 
\]
\[
\mathbf{D}_1 \mathbf{D}_2 \left[ \sum_{j \in \mathbb{Z}^2} p(j) (B_{r,s,t} - B_{r,s-1,t}) (-j) \right] = 0 
\]
by Lemma 2. Therefore 
\[
\mathbf{D}_1 \mathbf{D}_2 \left[ \sum_{j \in \mathbb{Z}^2} p(j) \left( \sum_{(r,s,t) \in E_1} B_{r,s,t} - \sum_{(r,s,t) \in E_2} B_{r,s,t} \right) (-j) \right] = 0 
\]
\[
\mathbf{D}_1 \mathbf{D}_2 \left[ \sum_{j \in \mathbb{Z}^2} p(j) \left( \sum_{(r,s,t) \in E_2} B_{r,s,t} + \sum_{(r,s,t) \in E_2} B_{r,s,t} (-j) \right) \right] = 0 
\]
Case i = 3. Then \( r < q_1 \) and \( t > q_2 \). We have
\[ q_1 q_2 = D_1^{-r} q_1^{-r} q_2 \]
\[ D_1 = D_1^{(D_1 + D_2)} D_2 \]
\[ q_1^{-r} q_2 \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1^{-r} q_2 \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1+q_2-r-t \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1^{-r} q_2 \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1+q_2-r-t \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1^{-r} q_2 \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1+q_2-r-t \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1^{-r} q_2 \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1+q_2-r-t \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1^{-r} q_2 \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1+q_2-r-t \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1^{-r} q_2 \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1+q_2-r-t \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1^{-r} q_2 \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1+q_2-r-t \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1^{-r} q_2 \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1+q_2-r-t \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1^{-r} q_2 \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1+q_2-r-t \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1^{-r} q_2 \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1+q_2-r-t \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1^{-r} q_2 \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1+q_2-r-t \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1^{-r} q_2 \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1+q_2-r-t \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1^{-r} q_2 \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]
\[ q_1+q_2-r-t \]
\[ = D_1^{-r} \left[ \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \right] \]

where the differential operators \( H_{r,s} \) and \( G_{r,t} \) are defined as follows:

\[ H_{r,s} := \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \]
\[ q_1+q_2-r-t \]
\[ G_{r,t} := \sum_{l=0}^{r} (-1)^{l} (D_1 + D_2)^l D_2 \]
\[ q_1+q_2-r-t \]

For the third term in (11) we observe that

\[ D_1^{-r} (D_1 + D_2) D_2 \left( \sum_{l=0}^{s-1} \frac{p(j)m_{r,s,t}(-j)}{l_2!} \right) \]
\[ = \sum_{l=0}^{s-1} \left( \psi_1^{\psi_1+\psi_2} q_1+q_2-r-t-l \right) \]
\[ \frac{p(j)m_{s,t}(-j)}{l_2!} \]

is a constant for \( l \in \{q_1+q_2-r-t+1, s-1\} \), because

\[ s-l > 0 \] and \( t-(q_1+q_2-r-t-l) > 0 \).
Thus we can omit the third term of (11) in the following discussion. Now we want to prove that

$$\sum_{j \in \mathbb{Z}} p(j) \left( \sum_{E_1} - \sum_{E_2} - \sum_{E_3} + \sum_{E_4} \right) \left[ \frac{D_{r_{1}+D_{2}}^{r}(D_{2})}{r_{1}+D_{2}} + G_{r_{1}+D_{2}}^{r_{2}} \right] B_{r_{1},2} \frac{B_{r_{2},1}}{r_{2},1} (\cdot \cdot \cdot j)$$

is a constant. Let

$$U := \sum_{j \in \mathbb{Z}} p(j) \left( \sum_{E_1} - \sum_{E_2} - \sum_{E_3} + \sum_{E_4} \right) \left[ H_{r_{1},s} D_{r_{1}+D_{2}}^{r_{1}} (D_{2})^{3} B_{r_{1},2} \frac{B_{r_{2},1}}{r_{2},1} (\cdot \cdot \cdot j) \right]$$

and

$$V := \sum_{j \in \mathbb{Z}} p(j) \left( \sum_{E_1} - \sum_{E_2} - \sum_{E_3} + \sum_{E_4} \right) \left[ G_{r_{1}+D_{2}}^{r_{2}} \right] B_{r_{1},2} \frac{B_{r_{2},1}}{r_{2},1} (\cdot \cdot \cdot j) \right].$$

Then we can argue separately for $U$ and $V$. Interchange $s$ and $t$ in (4), (5) and (6). We can write down (cf. (10))

$$U = \sum_{j \in \mathbb{Z}} p(j) H_{p+1,2}^{r+1} D_{p+1}^{r+1} (D_{2})^{p+1} B_{p+1,2} (\cdot \cdot \cdot j)$$

and

$$V := \sum_{j \in \mathbb{Z}} p(j) \sum_{t=4}^{p+1} \sum_{3< r, s < p} H_{r,s} D_{r,s}^{r} (D_{2})^{r} \left[ B_{r,s} \frac{B_{s,t}}{s,t} - B_{r,s,t} \right] (\cdot \cdot \cdot j)$$

and

$$\sum_{j \in \mathbb{Z}} p(j) \sum_{t=3}^{p+1} \sum_{2< r, s < p} H_{r,s} D_{r,s}^{r} (D_{2})^{r} \left[ B_{r,s} \frac{B_{s,t}}{s,t} - B_{r,s,t-1} \right] (\cdot \cdot \cdot j)$$

For the second term of the above expression, we have
\[
\sum_{p+j} \sum_{t=4}^{p+1} \sum_{r+s=2p+4-t}^{3r, s \leq p} \sum_{r<q_1, t>q_2} H_{r, s} D_1^{(D_1+D_2)}(B \sum_{r, s, t} - B_{r, s, t-1})(\cdots-j) = 0
\]

by Lemma 2. The third and fourth terms of (12) are also zero by the same argument. Thus \( U = 0 \). Similarly we can show \( V = 0 \). Therefore

\[
q_1 q_2 \left( \sum_{j \leq 2} p(j) \left( \sum_{E \subseteq \mathbb{E}_3} \sum_{J_1 \subseteq \mathbb{E}_3} \sum_{J_2 \subseteq \mathbb{E}_3} \sum_{K \subseteq \mathbb{E}_3} B_{r, s, t}(\cdots-j) \right) \right)
\]

is a constant.

Case 4. In this case \( r > q_1 \) and \( t < q_2 \), and the argument is as in Case 3.

So far we have proved statement (9). Now

\[
p - \sum_{j \leq 2} p(j) B(\cdots-j)
\]

is a polynomial of degree \( \leq \deg p \). For \((q_1, q_2)\) with \( q_1 > 0 \), \( q_2 > 0 \) and \( q_1 + q_2 = \deg p \) we have

\[
q_1 q_2 (p - \sum_{j \leq 2} p(j) B(\cdots-j)) = q_1 q_2 p - \sum_{j \leq 2} p(j) q_1 q_2 B(\cdots-j)
\]

\[
= q_1 q_2 p - \sum_{j \leq 2} (q_1 q_2 p(j) B(\cdots-j))
\]

However, \( q_1 q_2 p \) is a constant. Hence

\[
\sum_{j \leq 2} (q_1 q_2 p(j) B(\cdots-j)) = q_1 q_2 p
\]

by Lemma 3. Therefore
\[q_1 q_2 (p - \sum_{j \in \mathbb{Z}^2} p(j)B(-j)) = 0, \text{ for any } (q_1, q_2) \text{ with } q_1 > 0, q_2 > 0 \text{ and } q_1 + q_2 = \deg p.\]

This shows that \( p - \sum_{j \in \mathbb{Z}^2} p(j)B(-j) \) is a polynomial of degree < \( \deg p \). Thus Lemma 4 is proved.

Now we can prove

**Theorem 2.** The mapping \( T \) defined by

\[
T : p \mapsto \sum_{j \in \mathbb{Z}^2} p(j)B(-j), \quad p \in \pi_{k-1}
\]

is one-to-one and onto \( \pi_{k-1} \).

**Proof.** \( \pi_{k-1} \) is a linear space of finite dimension, and \( T \) is a linear mapping from \( \pi_{k-1} \) to \( \pi_{k-1} \) by Lemma 4. If \( p \neq 0 \), then \( \deg p > 0 \). Lemma 4 tells us that \( \sum_{j} p(j)B(-j) \) has the same degree as \( p \); that is

\[
\sum_{j} p(j)B(-j) \neq 0. \text{ This shows that } T \text{ is one-to-one. Since } \pi_{k-1} \text{ is finite-dimensional, } T \text{ is also onto. The proof of Theorem 2 is complete.}
\]

Now combining Theorem 1 and Theorem 2 gives

**Theorem 3.** If \( k = 2p+2 \) and \( S = \pi_{k, \Delta}^\rho \), then

\[
\text{dist}(f, S_h) = O(h^k)
\]

for any sufficiently smooth function \( f \).

Remark. From the above arguments we see that Theorem 3 remains true for \( k > 2p+2 \).

We show in Section 5 that the approximation order of \( \pi_{4, \Delta}^1 \) is 4. Thus, in general, Theorem 3 cannot be improved.

For the general case, we also have

**Theorem 4.** If \( S = \pi_{k, \Delta}^\rho \) and \( \rho < \rho(k) := \lfloor (2k-2)/3 \rfloor \), then

\[
\text{dist}(f, S_h) = O(h^m(k)-2)
\]

for any sufficiently smooth function \( f \).
Proof. From [BH 1] we already know

\[ \text{dist}(f, S_h) = O(h^{p+2}) \]

If \( 2k \leq 3p+4 \), then

\[ m(k) - 2 < 2(k-p) - 2 = 2k - 2p - 2 < p+2 \]

Hence Theorem 4 holds for \( 2k \leq 3p+4 \). If \( k > 2p+2 \), then

\[ m(k) - 2 < k-1 \]

Thus Theorem 4 follows from Theorem 3. Now assume \( 2k > 3p+5 \) and \( k < 2p+2 \).

Let

\[ \sigma := 2p+2-k, \quad k' := k-3\sigma, \quad \rho' := \rho-2\sigma \]

Then

\[ \rho' = \rho-2\sigma = \rho-2(2p+2-k) = 2k-3p-4 > 1 \]

and

\[ k' = k-3\sigma = 4k-6p-6 = 2(2k-3p-4) + 2 = 2p'+2 \]

Let

\[ I' := \{(r,s,t) | r+s+t = 2p'+4 \text{ and } 2 < r, s, t < p'+1 \} \]
\[ J_1' := \{(r,s,t) | r+s+t = 2p'+3 \text{ and } 2 < r, s, t < p'+1 \} \]
\[ J_2' := \{(r,s,t) | r+s+t = 2p'+3 \text{ and } 2 < r, s, t < p' \} \]
\[ K' := \{(r,s,t) | r+s+t = 2p'+2 \text{ and } 2 < r, s, t < p' \} \]

Define

\[ \tilde{B} = \sum_{(r,s,t) \in I'} \frac{1}{(r,s,t) \in J_1'} \sum_{(r,s,t) \in J_2'} \frac{1}{(r,s,t) \in K'} B_{r+s+t+\sigma} \]

Then \( \tilde{B} \in \mathfrak{P}_{k,\Delta}^\rho \). An argument similar to that used for Lemma 4 shows that

\[ p - \sum_{j \in \ell^2} p(j) \tilde{B}(\cdot-j) \]

is a polynomial of degree \( < \deg p \) for any polynomial \( p \) with \( \deg p < k'-1+2\sigma \). However,

\[ k'-1+2\sigma = k-3\sigma-1 + 2\sigma = k-\sigma-1 = 2k-2p-3 \]

Thus the mapping
\[ p \mapsto \sum_{j \in \mathbb{Z}^2} p(j)B(\cdot-j) \]

is one-to-one and onto \( \mathbb{N}_{2k-2p-3} \). Now Theorem 1 gives the required result:

For any sufficiently smooth function \( f \),

\[ \text{dist}(f,S_h) = O(h^{2k-2p-2}) \]

This ends the proof of Theorem 4.

5. **Approximation order from bivariate \( C^1 \)-quartics**

In this section we will show that for \( S = \mathbb{N}^1_{4,\Delta} \) and

\[ f : \mapsto x^2_1 x^3_2, \quad x = (x_1, x_2) \in \mathbb{R}^2 \]

there exists a positive constant such that

\[ \text{dist}(f,S_h) > \text{const} h^4 \]

To this end we shall follow \([BH 2]\) and discuss B-nets in the following.

Given a triangle \( \tau \) with vertices \( U, V \) and \( W \), we associate each point \( x \) with its barycentric coordinates, i.e. with \((u,v,w)\) for which

\[ x = uU + vV + wW, \quad \text{and} \quad u + v + w = 1 \]

Any polynomial \( p \) of degree \(< n \) can be represented by

\[ p = \sum_{i+j+k=n} b_{ijk} \phi_{ijk} \]

with

\[ \phi_{ijk}(x) = \frac{n!}{i!j!k!} \frac{i \cdot j \cdot k}{u \cdot v \cdot w} \]

where \( b_{ijk} \) are uniquely determined by \( p \). This representation gives rise to a function

\[ b : x_{ijk} \mapsto b_{ijk}, \quad x_{ijk} = (iu+jv+kw)/n \quad \text{and} \quad i+j+k = n \]

This function is called the B(ernstein or ezier)-net for \( p \) (with respect to \( \tau \)). (See \([BH 2]\).)

To a given function \( f \in \mathbb{N}^1_{4,\Delta} \) we associate a function \( b_f \) so that \( b_f \) is defined on

\[ J_4 := (\mathbb{Z}/4)^2 \]

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and \( b_f \) agrees with the B-net for \( f \) on each triangle of \( \Delta \). Obviously, \( b_f \) is well defined. We also call \( b_f \) the B-net for \( f \) with respect to \( \Delta \).

Let us now introduce some linear functionals on \( \pi_{4, \Delta}^0 \). Define

\[
\lambda_{i1}^{(m,n)} f := b_f(m + \frac{i-1}{4}, n) + b_f(m + \frac{i}{4}, n) - b_f(m + \frac{i-1}{4}, n - \frac{1}{4}) - b_f(m + \frac{i}{4}, n + \frac{1}{4}),
\]

\[
\lambda_{i2}^{(m,n)} f := b_f(m, n + \frac{i-1}{4}) + b_f(m + \frac{i}{4}, n) - b_f(m, n + \frac{i-1}{4}) - b_f(m + \frac{i}{4}, n + \frac{i}{4}),
\]

\[
\lambda_{i3}^{(m,n)} f := b_f(m + \frac{i-1}{4}, n + \frac{i-i}{4}) + b_f(m + \frac{i}{4}, n + \frac{i}{4}) - b_f(m + \frac{i-1}{4}, n + \frac{i}{4}) - b_f(m + \frac{i}{4}, n + \frac{i}{4}),
\]

\[
b_f(m + \frac{i}{4}, n + \frac{i}{4}), \quad i = 1, 2, 3, 4; \quad m, n \in \mathbb{Z}.
\]

Let

\[
\Lambda_{ij} := \{ \lambda_{ij}^{(m,n)} \mid m, n \in \mathbb{Z} \}, \quad i = 1, 2, 3, 4; \quad j = 1, 2, 3
\]

and

\[
\Lambda := \bigcup_{i=1}^{4} \bigcup_{j=1}^{3} \Lambda_{ij}.
\]

If \( f \in \pi_{4, \Delta}^1 \) then \( \lambda b_f = 0 \) for any \( \lambda \in \Lambda \) (see [F] and [BH 2]).

We extend each \( \lambda \in \Lambda \) to the continuous linear functional \( \lambda I \) on \( C(\mathbb{R}^2) \) with the aid of the local linear map \( I \) which associates \( f \) with the unique element \( If \) of \( \pi_{4, \Delta}^0 \) which agrees with \( f \) on \( J_4 \). Let \( T \) be the mapping \( f \mapsto b_{If} \) for \( f \in C(\mathbb{R}^2) \), and let \( T_j \) be the shift operator \( f \mapsto f (+j) \).

We have the following

**Lemma 5.** \( T \) is a linear mapping and commutes with any shift \( T_j, \ j \in \mathbb{Z}^2 \).

**Proof.** It is obvious that \( T \) is a linear mapping. To prove the second statement we first show that \( I \) commutes with any \( T_j \). Indeed,

\[
T_j(If)(i) = If(i+j) = f(i+j) \quad \text{for any } i \in J_4,
\]

\[
I(T_if)(i) = I(f(+j))(i) = f(i+1) \quad \text{for any } i \in J_4.
\]

This shows that \( T_j I = IT_j \). Next, we have to show that the mapping

\[
g \mapsto b_g, \quad g \in \pi_{4, \Delta}^0
\]

commutes with any \( T_j \). Let \( \tau \) be a triangle of \( \Delta \). Then
It follows that
\[ g_{\tau} = \sum_{p+q+r=4} b_{pqr} \phi_{pqr}. \]

Hence the mapping \( g \mapsto b_g \) commutes with any shift. The Lemma is proved.

**Corollary.** If \( f \in \mathfrak{g}^1_5 \) and \( \lambda \in \Lambda \), then \( \lambda b_{f} \) is invariant under translates.

**Proof.** By Lemma 5
\[ \lambda (b_{f} (\ast + j) - b_{f}) = \lambda (T_{\mathfrak{g}} f - f) = \lambda (T_{\mathfrak{g}} f - f) \]

However, \( T_{\mathfrak{g}} f - f \in \mathfrak{g}^1_4 \); hence \( \lambda (T_{\mathfrak{g}} f - f) = 0 \). This shows that
\[ \lambda b_{f} (\ast + j) = \lambda b_{f} \]

for any \( j \in \mathbb{Z}^2 \).

The Corollary is proved.

Now let
\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4 \\
1 & 2 & 3 & 4
\end{pmatrix} = 
\begin{pmatrix}
-1 & 1 & -1 & 1 \\
0 & 1 & -1 & 0 \\
0 & -1 & 1 & 0
\end{pmatrix}
\]

and
\[
\mu_{h} := \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \sum_{i=1}^{4} \sum_{j=1}^{3} a_{ij} \lambda_{i}^{(m,n)} \sigma_{4_{i}1_{j}} \] with \( h > 0 \) and \( N = \lceil 1/h \rceil \). (13)

Since \( S_{h} \subseteq \ker \mu_{h} \), we have
\[
\text{dist}(f, S_{h}) > \text{dist}(f, \ker \mu_{h}) = |\mu_{h} f|/|\mu_{h}| .
\]

By the above Corollary
\[
\mu_{h} f = N^2 \sum_{i=1}^{4} \sum_{j=1}^{3} \lambda_{ij}^{(0,0)} \sigma_{4_{i}1_{j}} f \text{ for } f \in \mathfrak{g}^1_5 .
\]

For \( f : x \mapsto x_{1}^{2} x_{2}^{3} \) we have \( \sigma_{4_{i}1_{j}} f = h^{5} f \). Hence
\[
\mu_{h} f = h^{5} N^2 \sum_{i=1}^{4} \sum_{j=1}^{3} \lambda_{ij}^{(0,0)} \text{ for } (14)
\]

It is easy to verify that
Let \( \tau_1 \) be the triangle with vertices \( U = (0,0) \), \( V = (0,1) \) and \( W = (1,1) \). Then

\[
(x_1, x_2) = u(0,0) + v(0,1) + w(1,1) \quad \text{with} \quad u + v + w = 1.
\]

It follows that

\[
u = 1-x_2, \quad v = x_2-x_1 \quad \text{and} \quad w = x_1.
\]

Hence

\[
x_{ijk} = (iu + jv + kw)/4 = (k, i+jk)/4
\]

and

\[
\phi_{ijk} = \frac{4i}{ijijk} (1-x_2)^i x_1^j (x_2-x_1)^k.
\]

Thus

\[
\phi_{ijk} = \frac{4i}{ijijk} (1-x_2)^i x_1^j (x_2-x_1)^k
\]

By Lemma 5 we have

\[
I(f(-e_2))_{\tau_1} = \sum_{p=0}^{4} \frac{P}{q=0} b_{If(4',4')} \phi_{4-q , q-p , p}
\]

Therefore,

\[
\sum_{p=0}^{4} \frac{P}{q=0} \{ b_{If(4',4')} - b_{If(4',4') - 1} \} \phi_{4-q , q-p , p} = I(f - f(-e_2))_{\tau_1}
\]

On the other hand

\[
(f - f(-e_2))(x_1, x_2) = x_1^2 x_2^3 - x_1^2 (x_2 - 1)^3 = x_1^2 (3x_2^2 - 3x_2 + 1)
\]

\[
= x_1^4 + 2x_1^3 (x_2 - x_1) + x_1^2 (x_2 - x_1)^2 - x_1^3 (1-x_2) - x_1^2 (x_2 - x_1)(1-x_2) + x_1^2 (1-x_2)^2
\]

\[
= \phi_{0,0,4} + \frac{1}{2} \phi_{0,1,3} + \frac{1}{6} \phi_{0,2,2} - \frac{1}{4} \phi_{1,0,3} - \frac{1}{12} \phi_{1,1,2} + \frac{1}{6} \phi_{2,0,2}
\]

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This yields the following result:

\[ b_{If}(0, \frac{3}{4}) - b_{If}(0, -\frac{1}{4}) = 0 \]
\[ b_{If}(\frac{1}{4}, \frac{3}{4}) - b_{If}(\frac{1}{4}, -\frac{1}{4}) = 0 \]
\[ b_{If}(\frac{2}{4}, \frac{3}{4}) - b_{If}(\frac{2}{4}, -\frac{1}{4}) = -\frac{1}{12} \]
\[ b_{If}(\frac{3}{4}, \frac{3}{4}) - b_{If}(\frac{3}{4}, -\frac{1}{4}) = -\frac{1}{4} \]

Now we consider another triangle \( \tau_2 \) with vertices \( U = (0,0), V = (1,0) \) and \( W = (1,1) \). Then

\[ u = 1 - x_1, \ v = x_1 - x_2 \text{ and } w = x_2 \]

\[ x_{ijk} = \left( \frac{1}{4} \right)^{i+j+k} \]
\[ \phi_{ijk} = (1-x_1)^i (x_1 - x_2)^j x_k \]

Moreover, we have

\[ f(x_1, x_2) - f(x_1 - 1, x_2) = (2x_1 - 1)x_2 = -(1-x_1)x_2 + (x_1 - x_2)x_2 + x_2^4 \]
\[ = -\frac{1}{4} \phi_{1,0,3} + \frac{1}{4} \phi_{0,1,3} + \phi_{0,0,4} \]

It follows that

\[ b_{If}(1,0) - b_{If}(0,0) = 0 \]
\[ b_{If}(1, \frac{1}{4}) - b_{If}(0, \frac{1}{4}) = 0 \]
\[ b_{If}(\frac{3}{4}, \frac{1}{4}) - b_{If}(\frac{3}{4}, -\frac{1}{4}) = 0 \]
\[ b_{If}(\frac{3}{4}, \frac{2}{4}) - b_{If}(\frac{3}{4}, -\frac{2}{4}) = 0 \]

In conclusion we obtain

\[ \sum_{i=1}^{4} \sum_{j=1}^{3} a_{ij} \lambda_{ij}^{0,0} \mid_{If} = \frac{1}{12} - \frac{1}{4} = -\frac{1}{6} \]  \hspace{1cm} (15)
Thus

$$|u_h f| = \frac{1}{6} h^5 N^2 > \frac{1}{12} h^3 \text{ for } h < \frac{1}{4} .$$  \hspace{1cm} (16)

Furthermore, we have, for any $g \in C(R^2),$

$$u_h g = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} \sum_{i=1}^{4} \sum_{j=1}^{3} a_{ij} \lambda_i^{(m,n)} I_{1/h}^g$$

$$= \sum_{m=0}^{N-1} [b_{1/4} I_{1/h}^g (m, -1/4) - b_{1/4} I_{1/h}^g (m+1/4, -1/4) + b_{1/4} I_{1/h}^g (m+2/4, -1/4) - b_{1/4} I_{1/h}^g (m+3/4, -1/4)] \hspace{1cm} (17)$$

$$- \sum_{m=0}^{N-1} [b_{1/4} I_{1/h}^g (m, N-1/4) - b_{1/4} I_{1/h}^g (m+1/4, N-1/4) + b_{1/4} I_{1/h}^g (m+2/4, N-1/4) - b_{1/4} I_{1/h}^g (m+3/4, N-1/4)]$$

$$+ \sum_{n=0}^{N-1} [-b_{1/4} I_{1/h}^g (0, n) + b_{1/4} I_{1/h}^g (0, n+1/4) - b_{1/4} I_{1/h}^g (-1/4, n+1/4) + b_{1/4} I_{1/h}^g (-1/4, n+2/4)] \hspace{1cm} (17)$$

$$- \sum_{n=0}^{N-1} [-b_{1/4} I_{1/h}^g (N, n) + b_{1/4} I_{1/h}^g (N, n+1/4) - b_{1/4} I_{1/h}^g (-1/4, n+1/4) + b_{1/4} I_{1/h}^g (-1/4, n+2/4)].$$

It is easily seen that

$$|b_{1/4} I_{1/h}^g| < \text{const } g \in C$$

where the const is independent of $h$. Hence (17) implies that

$$|u_h g| < 4N \text{ const } g \in C < \text{const } \frac{1}{h} g \in C .$$

This shows that

$$u_h = 0(\frac{1}{h}).$$  \hspace{1cm} (18)

Now (14), (16) and (18) yield the desired result:

$$\text{dist}(f, S_h) > \text{const } h^4$$

for some positive constant and the function $f : x \mapsto x_1^2 x_2^3$.

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RQJ/jvs
**Approximation by Smooth Bivariate Splines on a Three-Direction Mesh**

**Title**: Approximation by Smooth Bivariate Splines on a Three-Direction Mesh

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**Keywords**: B-splines, bivariate, degree of approximation, pp, quasi-interpolants, linear functionals, smooth, spline functions.

**Abstract**: Let $S := \mathcal{P}_{k,\Delta}^p$ be the space of bivariate piecewise polynomial functions in $C^0$, of degree $\leq k$, on the mesh $\Delta$ obtained from a uniform square mesh by drawing in the same diagonal in each square.

De Boor and Höllig have given the following upper bound

$$m \leq m(k) := \min\{2(k-p), k+1\}$$

for the approximation order $m$ of $S$. 

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