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BIVARIATE CARDINAL INTERPOLATION BY
SPLINES ON A THREE-DIRECTION MESH

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Carl de Boor[1], Klaus Höllig[1,2] and Sherman Riemenschneider[3]

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## ABSTRACT

For a sequence $Z := (z_1, \ldots, z_{|Z|})$ in $\mathbf{z}^2$, the box spline $M_Z$ is the distribution given by the rule

$$M_Z\phi := \int \phi\left(\Sigma\, \lambda_j z_j\right) d\lambda .$$

We show that $P_Z(x) := \underset{k\in\mathbf{z}^2}{\Sigma} M_Z(k)\, e^{ikx} > 0$ iff $AZ \subseteq \{(1,0),\ (0,1),\ (1,1)\}$ for some linear map $A$ which is 1-1 on $\mathbf{z}^2$. This implies that, for such a $Z$, there exists, for any bounded continuous function $f$, a unique bounded spline $I_Z f \in \mathrm{span}\{M_Z(\cdot-k) : k\in\mathbf{z}^2\}$ which agrees with $f$ on $\mathbf{z}^2$. We give sufficient conditions in terms of the Fourier transform of $f$ for the convergence of $I_Z f$ as the degree $|Z|-2$ tends to infinity. E.g., we show, for the case that $Z$ contains each of the vectors $(1,0)$, $(0,1)$, $(1,1)$ exactly $n$ times, that $\|f - I_n f\|_{\infty,\mathbf{R}^2} \xrightarrow[n\to\infty]{} 0$ if $\mathrm{supp}\,\hat{f} \subset\subset \Omega := \mathrm{conv}\{(z,z),\ -(z,z),\ (-z,2z),\ -(-z,2z),\ (-2z,z),\ -(-2z,z)\}$, with $z := \dfrac{2\pi}{3}$. The converse of this result holds with "$\subset\subset$" replaced by "$\subseteq$".

AMS (MOS) Subject Classifications: 41A05, 41A15, 41A63

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## SIGNIFICANCE AND EXPLANATION

In a series of monographs (cf. MRC Technical Summary Reports #852, #1104, #1113, #1150, #1153, #1180, #1184, #1219, #1222, #1326, #1330, #1360, #1423, #1546) I. J. Schoenberg developed a comprehensive theory of univariate cardinal splines. His results strongly influenced the analysis of totally positive matrices.

In this report we extend two of his basic results on cardinal interpolation to bivariate box-splines, which have been introduced in MRC Technical Summary Reports #2320, #2415. We show that, for functions of exponential type, cardinal interpolation is a rapidly convergent approximation process as the degree tends to infinity. Being not restricted to a tensor product mesh gives a greater flexibility, and because of the exponential decay of the Lagrange functions, spline interpolation is suitable, e.g., for data smoothing. We also expect that bivariate cardinal splines have a similar significance for theoretical questions as in the univariate case.

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

BIVARIATE CARDINAL INTERPOLATION BY SPLINES ON A THREE-DIRECTION MESH

by

Carl de Boor[1], Klaus Höllig[1,2] and Sherman Riemenschneider[3]


Dedicated to I.J. Schoenberg

to whose insight and sense of beauty we are all indebted


**1. Introduction.** In this paper, we carry Schoenberg's beautiful cardinal spline theory $[S_{2,3}]$ over to a two-dimensional context which is not just the tensor product of the univariate situation. We find that we must work harder, yet must be satisfied with less precise results.

We are after a bounded cardinal interpolant to bounded data. This means that we are looking for a function of the form

$$If = \sum_{j \in \mathbb{Z}^2} a_j M(\cdot - j)$$

with $a \in \ell_\infty(\mathbb{Z}^2)$ which agrees with a given bounded function $f$ on $\mathbb{Z}^2$ . Here, $M$ is a fixed function of compact support. In Section 2, we follow Schoenberg $[S_1]$ in describing necessary and sufficient conditions on the Fourier transform of $M$ to insure the correctness of the interpolation problem, i.e., the existence and uniqueness of solutions.

We are particularly interested in using for $M$ a box spline , i.e., the two-dimensional "shadow" of an m-dimensional cube, as given explicitly in (1.1) below. We find it convenient to change the definition $[BH_1]$

$$M_\phi := \int_{[0,1]^{\mathbb{Z}}} \phi\left( \sum_{\zeta \in \mathbb{Z}} \lambda(\zeta)\zeta \right) d\lambda$$

of the box spline $M = M_Z$ to include an appropriate shift which makes the origin the center of the support of $M$ . This means that we use the definition

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$$M\phi \quad := \quad \int_{[-1/2,1/2]} \phi\Big( \sum_{\zeta \in Z} \lambda(\zeta)\zeta \Big) \, d\lambda \, . \tag{1.1}$$

This gives the Fourier transform $M^\wedge$ of $M$ the symmetric form

$$M^\wedge(x) \quad = \quad \prod_{\zeta \in Z} S(\zeta^* x) \tag{1.2}$$

with

$$S(t) \quad := \quad \frac{\sin t/2}{t/2} \, . \tag{1.3}$$

It is obvious from this formula that $M = M_Z$ is unchanged if one or more of the $\zeta \in Z$ are replaced by their negative, i.e.,

$$M_{AZ} = M_Z \tag{1.4}$$

if $A = \text{diag}(\pm 1, \dots, \pm 1)$ . Further, if $A$ is any matrix, then

$$M^\wedge_{AZ}(x) \quad = \quad M^\wedge_Z(A^* x) \quad \text{and} \quad M_{AZ}(Ax) = M_Z(x)/\det A \, . \tag{1.5}$$

This allows one to deduce symmetries in $M^\wedge$ in case $AZ$ equals $Z$ after, possibly, some elements of $AZ$ have been multiplied by $-1$ .

The set $Z$ of directions can, of course, be chosen arbitrarily. But since we are interested in having

$$S \quad := \quad \text{span} \, \Big( M(\cdot - j) \Big)_{j \in \mathbb{Z}^2}$$

be a simple piecewise polynomial space, we choose $Z$ from $\mathbb{Z}^2$ . It is shown in [BH$_1$] that the integer translates $M(\cdot - j)$ , $j \in \mathbb{Z}^2$ of the box spline are linearly dependent (when allowing for infinite linear combinations) in case the direction set $Z$ contains two vectors which span a proper sublattice of $\mathbb{Z}^2$ . Linear independence is an obvious necessary condition for the cardinal interpolation problem to be correct. Thus, up to obvious symmetries, this leaves the three vectors $(1,0)$, $(0,1)$ and $(1,1)$ as the only candidates for the directions $\zeta$ in $Z$ .

With this restriction, $S$ is a space of piecewise polynomial functions, of polynomial degree $|Z|-2$ or less, and with possible discontinuities only across the three types

$$x(1) = k \, , \quad x(2) = k \, , \quad x(1) - x(2) = k \, , \quad k \in \mathbb{Z}$$

of mesh lines. The overall smoothness of the elements of $S$ depends on the multiplicities of the directions in $Z$ . Such details, as well as the relationship of $S$ to the space of all piecewise polynomial functions on such a three-direction mesh, of degree $|Z|-2$ and of specified smoothness, are all discussed in [BH$_2$].

- 1.2 -

In Section 3, we supply certain detail concerning symmetries of such a three-direction box spline and its Fourier transform. We prove the correctness of cardinal interpolation with such a box spline in Section 4. We spend the major effort of this paper in Section 5 where we prove that, under reasonable conditions, the cardinal interpolant $If$ of any suitably smooth function $f$ converges to $f$ as $|Z| \longrightarrow \infty$. Specifically, we prove such convergence under the condition that $f$ is the Fourier transform of some compactly supported measure, following entirely the path established by Schoenberg [S] in the univariate case who showed that such convergence could be had whenever $\text{supp } f^{\wedge} \subseteq (-\pi, \pi)$. We find, though, that, in our bivariate setup, there are many different sets playing the role of this interval, and which of these sets is relevant depends on the manner in which $|Z|$ goes to infinity.

The final section is devoted to the many detailed estimates on which the arguments in Section 5 are based.

- 1.3 -

**2. Cardinal interpolation.** Let $M:\mathbb{R}^2 \longrightarrow \mathbb{R}$ be a continuous function with compact support, and denote by

$$S := S_M := \text{span } \{M(\cdot - j) : j \in \mathbb{Z}^2\}$$

the space generated by its integer translates. **Cardinal interpolation with** $M$ concerns inversion of the linear map

$$S \quad L_\infty \longrightarrow \ell_\infty : f \longmapsto f_{|\mathbb{Z}^2} \quad . \tag{2.1}$$

We say that cardinal interpolation with $M$ is **correct** if this map is 1-1 and onto, hence boundedly invertible, and denote its inverse by $I_M$ or $I$ . In other words, cardinal interpolation with $M$ is correct iff there exists, for every bounded sequence $f \in \ell_\infty(\mathbb{Z}^2)$ , a bounded function $If \in S$ which agrees with $f$ on $\mathbb{Z}^2$ . The interpolation problem, i.e., the determination of $If$ , is equivalent to the algebraic problem of determining the coefficient sequence $a$ for

$$If = \Sigma\, a_j M(\cdot - j)$$

so that

$$a \in \ell_\infty \qquad \text{and} \qquad \Sigma\, a_j M(\cdot - j) = f \quad \text{on} \quad \mathbb{Z}^2 \quad .$$

Hence the correctness of cardinal interpolation is equivalent to the invertibility of the matrix

$$A := \left(M(j-k)\right)_{j,k \in \mathbb{Z}^2} \tag{2.2}$$

as a map on $\ell_\infty$ . Since $A$ is a banded (bivariate) Toeplitz matrix, we have the following necessary and sufficient condition for the correctness of cardinal interpolation.

**Theorem 2.** <u>Cardinal interpolation with</u> $M$ <u>is correct iff</u>

$$P(x) := P_M(x) := \Sigma\, M(j)\, e^{ijx} \tag{2.3}$$

<u>does not vanish.</u>

**Proof.** If $P(x) = 0$ , then $\left(e^{-ijx}\right)_{j \in \mathbb{Z}^2} \in \ker A \cap \ell_\infty$ , and this contradicts the assumption that $A$ is 1-1. On the other hand, if $P$ does not vanish, then the inverse of $A$ can be expressed as a Toeplitz matrix,

$$(A^{-1})_{jk} := \int_{(-\pi,\pi)^2} \frac{e^{-i(j-k)x}}{P(x)}\, dx/2\pi \quad . \tag{2.4}$$

In view of the geometric decay of the Fourier series for $1/P$, we have

$$|(A^{-1})_{jk}| \leq \text{const } \lambda^{|j-k|} \qquad (2.5)$$

for some $\lambda = \lambda(P) \in (0,1)$. Therefore, $A^{-1}$ is bounded on $\ell_p(\mathbb{Z}^2)$ for any $p \in [1,\infty]$. |||

It is convenient to write the cardinal interpolant in Lagrange form:

$$If = \Sigma f_j L(\cdot - j)$$

with

$$L := L_M := I\delta = \Sigma (A^{-1})_{0j} M(\cdot - j) \qquad (2.6)$$

the **fundamental function** of the interpolation process. The Fourier transform $L^\wedge$ of $L$ is particularly simple. Combining (4) with (6), we obtain

$$L^\wedge = M^\wedge / P . \qquad (2.7)$$

We will also make use of the identity

$$P(x) = \Sigma M^\wedge(2\pi j - x) \qquad (2.8)$$

which follows from applying the Poisson summation formula $\Sigma f(j) = \Sigma f^\wedge(2\pi j)$ to (3).

## 3. Cardinal interpolation with a box spline.

In this section, we develop in some detail facts about cardinal interpolation with the box spline $M_Z$. Recall from Section 1 that $\left(M(\cdot - j)\right)_{j \in \mathbb{Z}^2}$ is linearly dependent if $Z$ contains two vectors which span a proper sublattice of $\mathbb{Z}^2$. Linear independence of $\left(M(\cdot - j)\right)$ is an obvious necessary condition for cardinal interpolation with $M_Z$ to be correct. Thus, up to obvious symmetries, the only relevant case to consider is the case when the only directions in $Z$ are

$$d_1 := (1,0), \quad d_2 := (0,1), \quad \text{and} \quad d_3 := (1,1) .$$

We show in Section 4 that, with this restriction, cardinal interpolation with $M_Z$ is always correct.

Assume from now on that

$$Z = (d_1 : r, \ d_2 : s, \ d_3 : t) .$$

In this case, $Z$ is characterized by the vector

$$n := (n_1, n_2, n_3) := (r, s, t)$$

of direction multiplicities, and we will freely write $n$ instead of $Z$ whenever it is necessary to indicate by subscript the dependence on $Z$ of some quantity. Further, the general formulae given in Section 1 simplify. For example,

$$\hat{M}_n(u,v) = S(u)^r \, S(v)^s \, S(u+v)^t , \tag{3.1}$$

with

$$S(t) := \frac{\sin(t/2)}{t/2} .$$

Further, the characteristic polynomial $P = P_n$ and the Fourier transform $\hat{L}$ of the fundamental spline $L = L_n$ have the representations

$$P(2\pi u, 2\pi v) = \pi^{|n|} \left(\sin(\pi u)\right)^r \left(\sin(\pi v)\right)^s \left(\sin(\pi(u+v))\right)^t \sum_{k,\ell} \frac{(-)^{rk+s\ell+t(k+\ell)}}{(u+k)^r (v+\ell)^s (u+v+k+\ell)^t} \tag{3.2}$$

and

$$1/\hat{L}(2\pi u, 2\pi v) = \sum_{k,\ell \in \mathbb{Z}} (-)^{rk+s\ell+t(k+\ell)} \left(\frac{u}{u+k}\right)^r \left(\frac{v}{v+\ell}\right)^s \left(\frac{u+v}{u+v+k+\ell}\right)^t . \tag{3.3}$$

Let $A^*m$ denote the transpose of $A$. The relation

$$\hat{M}_{AZ}(x) = \hat{M}_Z(A^* x) \tag{1.5}$$

valid for any matrix $A$ together with the fact that

$$M_{AZ} = M_Z \tag{1.4}$$

in case $A = \text{diag}(\epsilon_1, \ldots, \epsilon_{|n|})$ with $\epsilon_i \in \{-1, 1\}$, all $i$, implies certain symmetries of

M  and  M^  if the matrix  A  leaves the set

$$Z_{\pm} := \{d_1, d_2, d_3, -d_1, -d_2, -d_3\}$$

invariant. Denote by **A** the group of all such <u>invertible</u> matrices  A . Each  $A \in \mathbf{A}$  is

associated with a permutation  $\sigma_A \in \mathbf{\$}_3$  (:= symmetric group on 3 elements) by the condition

$$Ad_i \in \{d_{\sigma_A(i)}, -d_{\sigma_A(i)}\} , \quad i=1,2,3 .$$

From the two matrices corresponding to a given  $\sigma \in \mathbf{\$}_3$ , we choose one,  $A_\sigma$ , in such a way

that the six matrices form a group and we call this group  $A_+$ . Thus,

$$A_\sigma d_i \in \{d_{\sigma(i)}, -d_{\sigma(i)}\} , \quad \text{all} \quad \sigma \in \mathbf{\$}_3 , \tag{3.4}$$

and one choice for the group generators are the three matrices

$$A_{(12)} := \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} , \quad A_{(13)} := \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} , \quad A_{(23)} := \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

corresponding to the transpositions  (12), (13), and (23) . With the definition

$$\sigma(n) := (n_{\sigma(1)}, n_{\sigma(2)}, n_{\sigma(3)}) ,$$

it follows from (1.5) and from (4) that

$$M_n(x) = M_{\sigma(n)}(\pm A_\sigma x) , \tag{3.5}$$

$$M^\wedge_{\sigma(n)}(y) = M^\wedge_n(\pm A^*_\sigma y) .$$

This implies

$$P_n(\pm A^*_\sigma x) = P_{\sigma(n)}(x) \tag{3.6}$$

and

$$L_n(x) = L_{\sigma(n)}(\pm A_\sigma x) , \tag{3.7}$$

$$L^\wedge_{\sigma(n)}(y) = L^\wedge_n(\pm A^*_\sigma y) .$$

Of particular interest is the case

$$r = s = t ,$$

i.e., when the direction multiplicities are all equal. In this case,  $\sigma(n) = n$ , all  $\sigma$ ,

i.e., (5)-(7) hold with  $\sigma(n)$  replaced by  n . For example, writing out in detail the

relations (6) for  $P = P_{(s,s,s)}$ , we get

$$P(u,v) = P(-u,-v) = P(v,u) = P(u+v,-v) = P(-u,u+v) . \tag{3.6'}$$

The relations for  M^ ,  P^  and  L^  will be used frequently in the sequel. Since they

are given in terms of the transposes of the matrices in  **A** , we consider now

$$A^* := \{A^* : A \in A\}$$

in more detail. Set

$$d' := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} d ,$$

$$z_\pm' := \{d' : d \in z_\pm\} = \{(0,1), (-1,0), (-1,1), (0,-1), (1,0), (1,-1)\} .$$

Since $d'^* d = 0$ , we see from (4) that $A^*$ leaves $z_\pm'$ invariant. To further illustrate the action of the group $A^*$ , we divide $\mathbb{R}^2$ into the six cones $R_\sigma$ , $\sigma \in \$_3$ , as indicated in Figure 3.1.
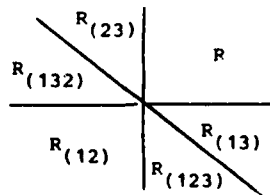


Figure 3.1.

It is easily checked that

$$A_\sigma^* R = R_\sigma , \text{ all } \sigma \in \$_3 . \tag{3.8}$$

**4. The correctness of cardinal interpolation with** $M_n$ . In this section, we show that cardinal interpolation with $M_n$ is correct for all choices of $n \in \mathbb{Z}_+^3$ .

**Theorem 4.** <u>For all</u> $n \in \mathbb{Z}_+^3$ , $P_n$ <u>is strictly positive.</u>

Since $P$ is $2\pi$-periodic, this amounts to the claim that

$$P_n(x) > 0 \quad \text{for all} \quad x \in [-\pi,\pi]^2 .$$

This is the bivariate analogue of Schoenberg's well known result for univariate cardinal spline interpolation. To recall this result, denote by

$$N_r$$

the univariate cardinal B-spline of degree $r$ , and by $Q_r$ the corresponding characteristic polynomial given by

$$Q_r(x) := \sum N_r(j)\, e^{ijx} .$$

Schoenberg showed in $[S_1]$ that

$$\min_x Q_r(x) = Q_r(\pi) = 2\left(\tfrac{2}{\pi}\right)^{r+1} \sum_{\nu=0}^{\infty} \frac{(-)^{\nu(r+1)}}{(2\nu+1)^{r+1}} . \qquad (4.1)$$

The fact that, for any $r$ , the minimum is attained at $x = \pi$ is a consequence of the total positivity of the matrix $\left(N_r(j-k)\right)_{j,k\in\mathbb{Z}}$ . In view of this result, one might think that, in the above theorem, $\min_{u,v} P(u,v) = P(\pi,\pi)$ . This is trivially true in the tensor product case, i.e., when $n = (r,r,0)$ . However, in general, the point at which $P_n$ attains its minimum depends on $n$ . It would be interesting to determine its location for special choices of $n$ . The nicest conjecture in this context (cf. Section 5) is that

$$\min P(u,v) = P(2\pi/3,\ 2\pi/3) \quad \text{in case} \quad n = (r,r,r) . \qquad (4.2)$$

In the proof of the theorem, we make use of (3.6). This allows us to assume without loss of generality that

$$r \geqslant s \geqslant t .$$

We first consider two cases which reduce to Schoenberg's result.

The <u>tensor product case</u> $n = (r,s,0)$ . Here, we have $M_n(u,v) = N_r(u)N_s(v)$ , and this implies that

$$P_n(u,v) = Q_r(u)Q_s(v) .$$

The case $\underline{n = (r,1,1)}$ . Since the open support of $M_n$ intersects exactly one mesh line of the form $(\cdot,\ell)$ , viz. the meshline $(\cdot,0)$ , it follows that, in this case,

$$M(k,\ell) = \begin{cases} N_r(k) , & \ell = 0 \\ 0 , & \ell \in \mathbb{Z} \setminus 0 \end{cases} .$$

This means that cardinal interpolation with $M_n$ reduces to univariate interpolation with $N_r$ on each of the lines $(\cdot,\ell)$ , $\ell \in \mathbb{Z}$ . In particular, $P(u,v) = Q_r(u)$ .

For the **proof of Theorem** 3, it remains to consider the cases where the multiplicities are all at least 1 , with equality for at most one. We make this assumption for the remainder of this section.

To prove the positivity of $P$ , we use the representation (2.10) in the form

$$P(2\pi x) = \Sigma \, \tilde{M}(x+j) , \tag{4.3}$$

with

$$\tilde{M}(x) := M^{\wedge}(2\pi x) .$$

Recall from (3.1) that, for $x = (u,v)$ and $j = (k,\ell)$ ,

$$\tilde{M}(x+j) = \pi^{|n|}(\sin \pi u)^r(\sin \pi v)^s(\sin \pi(u+v))^t \frac{(-)^{rk+s\ell+t(k+\ell)}}{(u+k)^r(v+\ell)^s(u+v+k+\ell)^t} . \tag{4.4}$$

It is sufficient to show the positivity of $P(2\pi\cdot)$ on $[0,1/2]^2$ for arbitrary $n$ . This follows from (3.6) since, by (3.8),

$$[-1/2,1/2]^2 \subseteq \bigcup_{A\in\mathbf{A}} A^*[0,1/2]^2 .$$

For $x \in [0,1/2]^2$ , we now show that the three positive terms

$$\tilde{M}(x) , \quad \tilde{M}(x - d_1) , \quad \text{and} \quad \tilde{M}(x - d_2) \tag{4.5}$$

dominate the sum in (3). To this end, we associate each of the other terms with one of these (even to the point of splitting one of the other terms between two of these) and show that the resulting three sums, when divided by their respective dominant term, each is less than 1 . For ease of argument, we actually split the sum into altogether ten parts, as indicated in part by the figure below.
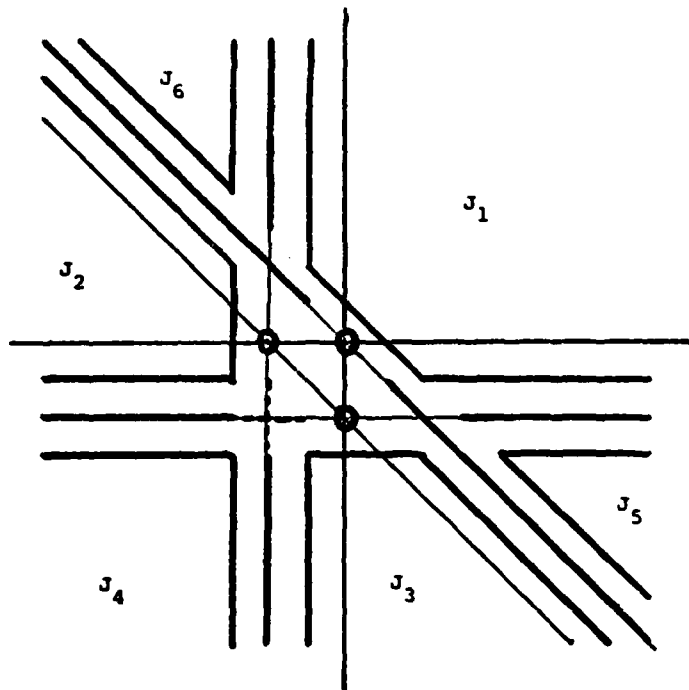
Figure 4.1

To simplify notation, we set

$$b_\nu(j) \; := \; b_{\nu,n}(j,x) \; := \; \frac{|\tilde{M}(x+j)|}{\tilde{M}(x+j_\nu)} \;\;, \; \nu=1,2,3 \;, \tag{4.6}$$

with

$$j_1 := 0 \;, \;\; j_2 := -d_1 \;, \; j_3 := -d_2 \;.$$

We now prove that

$$\Sigma_{j\in J_1\cup J_4} \; b_1(j) \;\; + \;\; \Sigma_{\ell\neq 0} \; b_1(\ell,-\ell) \;\; < \;\; 1 \tag{4.7}$$

$$\Sigma_{j\in J_2\cup J_5} \; b_2(j) \;\; + \;\; \Sigma_{|\ell|>0} \; b_2(-1,\ell) \;\; + \;\; b_2(-1,-1)/2 \;\; < \;\; 1 \tag{4.8}$$

$$\Sigma_{j\in J_3\cup J_6} \; b_3(j) \;\; + \;\; \Sigma_{|\ell|>1} \; b_3(\ell,-1) \;\; + \;\; b_3(-1,-1)/2 \;\; < \;\; 1 \tag{4.9}$$

Since each of the summands (divided by its appropriate dominant term) other than the three

dominant terms (5) occurs in (7-9) exactly once, we conclude from (7-9) the positivity of

P .

- 4.3 -

The estimation of the various sums in (7-9) is straightforward. In each case, we find a majorant which is independent of $x \in [0,1/2]^2$ and $n$. For this, recall that we are assuming that $r,s,t \geq 1$ with at most one equality.

We begin with the sum $\Sigma_{J_1} b_1(j)$. By (3.1,4.6), we have for $x = (u,v)$ and $j = (k,\ell)$,

$$b_1(j) = |\frac{u}{u+k}|^r |\frac{v}{v+\ell}|^s |\frac{u+v}{u+v+k+\ell}|^t .$$

Since $k, \ell \geq 0$ for $j = (k,\ell) \in J_1$ and we are assuming that $u, v \in [0,1/2]$, this quotient is largest when $u = v = 1/2$, i.e.,

$$b_1(j) < (\frac{1/2}{1/2+k})^r (\frac{1/2}{1/2+\ell})^s (\frac{1}{1+k+\ell})^t .$$

This bound is largest when the exponents $r$, $s$, $t$ are as small as possible, i.e., when $n = (1,2,2), (2,1,2)$ or $(2,2,1)$. Since

$$\sum_{\substack{k,\ell \geq 0 \\ (k,\ell) \neq 0}} (\frac{1/2}{1/2+k})^r (\frac{1/2}{1/2+\ell})^s (\frac{1}{1+k+\ell})^t = .1723...$$

for these values of $n$, we conclude that

$$\Sigma_{J_1} b_1(j) < .18 . \tag{4.10}$$

Similarly, one verifies that, for $j = (-k,-\ell) \in J_4$ and $x \in [0,1/2]^2$,

$$b_1(j) < (\frac{1/2}{k-1/2})^r (\frac{1/2}{\ell-1/2})^s (\frac{1}{k+\ell-1})^t$$

and so obtains that

$$\Sigma_{J_4} b_1(j) < .02 , \tag{4.11}$$

since

$$\sum_{k,\ell > 1} (2k-1)^{-r} (2\ell-1)^{-s} (k+\ell-1)^{-t} = .0101...$$

for $(r,s,t) = (1,2,2), (2,1,2)$, or $(2,2,1)$.

Finally, for $j = (-\ell,\ell)$ and $\ell \neq 0$, we have

$$b_1(-\ell,\ell) < \begin{cases} (\frac{1/2}{\ell+1/2})^r (\frac{1/2}{\ell-1/2})^s & , \quad \ell > 0 \\ (\frac{1/2}{-\ell-1/2})^r (\frac{1/2}{-\ell+1/2})^s & , \quad \ell < 0 \end{cases}$$

and so obtain

$$\Sigma_{\ell \neq 0} b_1(\ell,-\ell) < \Sigma_1^\infty [(\frac{1/2}{\ell+1/2})(\frac{1/2}{\ell-1/2})^2 + (\frac{1/2}{\ell+1/2})^2 (\frac{1/2}{\ell-1/2})] = .5 . \tag{4.12}$$

- 4.4 -

Combining (10-12) establishes (7).

The other inequalities are proved in a similar fashion and we only list the estimates involved.

**Proof of (8):**

$$b_2(j) = |\frac{u-1}{u+k}|^r |\frac{v}{v+\ell}|^s |\frac{u+v-1}{u+v+k+\ell}|^t \; .$$

$j = (-k,\ell) \in J_2$ :

$$b_2(j) < (\frac{1}{k})^r (\frac{1/2}{\ell+1/2})^s (\frac{1}{k-\ell})^t$$

and

$$\sum_{k=2}^{\infty}\sum_{\ell=0}^{k-1} (\frac{1}{k})^r (\frac{1/2}{\ell+1/2})^s (\frac{1}{k-\ell})^t = \begin{array}{ll} .25... & (2,2,1) \\ .23... \quad \text{for} \quad (r,s,t) = & (2,1,2) \\ .30... & (1,2,2) \end{array} \; .$$

Therefore

$$\sum_{J_2} b_2(j) < .35 \; .$$

$j = (k,-\ell) \in J_5$ :

$$b_2(j) < (\frac{1}{k})^r (\frac{1/2}{\ell-1/2})^s (\frac{1}{k-\ell})^t$$

and

$$\sum_{k=3}^{\infty}\sum_{\ell=2}^{k-1} (\frac{1}{k})^r (\frac{1/2}{\ell-1/2})^s (\frac{1}{k-\ell})^t = \begin{array}{ll} .027... & (2,2,1) \\ .081... \quad \text{for} \quad (r,s,t) = & (2,1,2) \\ .010... & (1,2,2) \end{array} \; .$$

Therefore

$$\sum_{J_5} b_2(j) < .1 \; .$$

$j = (-1,\ell) \, , \; |\ell| > 1$ :

$$b_2(j) < \begin{cases} (\frac{1/2}{\ell+1/2})^s (\frac{1}{\ell-1})^t & , \; \ell > 1 \\ (\frac{1/2}{-\ell-1/2})^s (\frac{1}{-\ell+1})^t & , \; \ell < -1 \end{cases}$$

$$\sum_{|\ell|>1} b_2(j) < \sum_{\ell>1}[(\frac{1/2}{\ell+1/2})(\frac{1}{\ell-1})^2 + (\frac{1/2}{\ell-1/2})(\frac{1}{\ell+1})^2] = .329...$$

$j = (-1,-1)$ :

$$b_2(-1,-1) < \frac{v}{1-v}\frac{1-u-v}{2-u-v} < 1/3 \quad (\text{at} \; (u,v) = (0,1/2)).$$

**Proof of (9):** Since

$$\hat{M}_{(r,s,t)}(u,v) = \hat{M}_{(s,r,t)}(v,u)$$

and therefore

$$b_{2,(r,s,t)}((u,v),(k,\ell)) = b_{3,(s,r,t)}((v,u),(\ell,k)) \; ,$$

- 4.5 -

the inequality (9) follows from (8) by interchanging the roles of  u  and  v  as well as those of  k  and  $\ell$ .

This completes the proof of Theorem 3.  |||

**5. Convergence of Cardinal Interpolation.** This section is devoted to the main goal of our paper, a study of the convergence of the cardinal interpolant to smooth functions as the degree tends to infinity. We prove the analogue of I.J. Schoenberg's basic result:

**Theorem [$S_2$].** _If_ $f$ _is the Fourier transform of a measure with support in_ $(-\pi, \pi)$ , _then its cardinal spline interpolant_ $I_r f$ _of degree_ $r$ _converges to_ $f$ _as the degree tends to infinity, i.e.,_

$$\| f - I_r f \|_\infty \xrightarrow[r \to \infty]{} 0 \ .$$

This theorem is a consequence of the fact that the Fourier transform $\hat{L}_r$ of the fundamental spline converges to the characteristic function of the interval $(-\pi, \pi)$ .

The bivariate situation is more complicated. Here, the limit of $\hat{L}_n$ depends on just how $n$ goes to infinity. Recall from (3.3) that

$$1/\hat{L}(2\pi x) = 1 + \sum_{j \in \mathbb{Z}^2 \backslash 0} \varepsilon_j(x)\, a_{n,j}(x) \ , \tag{5.1}$$

with $\varepsilon_j(x) \in \{-1, 1\}$ and

$$a_{n,(k,\ell)}(u,v) := \left| \frac{u}{u+k} \right|^r \left| \frac{v}{v+\ell} \right|^s \left| \frac{u+v}{u+v+k+\ell} \right|^t \ . \tag{5.2}$$

Define the "middle component" $\mu(n)$ of $n$ by the requirement that it equal the middle or second number in any ordering of the three numbers $r, s, t$ , and set

$$n' := (r', s', t') := n/\mu(n) \ .$$

Then, the typical summand in the right hand side of (1) is, up to sign,

$$a_{n,j}(x) = \left( a_{n',j}(x) \right)^{\mu(n)} \ .$$

This shows that $\hat{L}(2\pi x)$ is close to 1 for large $\mu(n)$ provided $a_{n',j}(x) < 1$ for all $j \neq 0$ .

The set

$$\{ x : a_{n',j}(x) < 1 \text{ for all } j \in \mathbb{Z}^2 \backslash 0 \} \ .$$

depends on $n'$ . In particular, we cannot expect it to converge as $|n| \longrightarrow \infty$ unless $n' = n/\mu(n)$ converges, to some 3-vector $m$ , say. Here, we are willing to allow $m$ to have infinite components. For example, if $n = (1, s, s^2)$ , then $\mu(n) = s$ and $n' = (1/s, 1, s) \longrightarrow (0, 1, \infty)$ as $s \longrightarrow \infty$ . But not every $m \in [0, \infty]^3$ is a possible limit. By

- 5.1 -

construction of $\mu(n)$ , $n' = n/\mu(n)$ has exactly one component equal to 1 and, among the other two, one must be $\leqslant 1$ and the other must be $> 1$ . Thus the set

$$N \;:=\; \{n \in [0,\infty]^3 : \; n_{\sigma(1)} \leqslant 1 = n_{\sigma(2)} \leqslant n_{\sigma(3)} \quad \text{for some} \quad \sigma \in \$_3\}$$

makes up the collection of all possible limits. On this set, we set up a topology of sorts by defining the open ball of radius $r$ around $m \in N$ by

$$B_r(m) \;:=\; \{n \in N : \; \max_i \left\{ \begin{array}{ll} |n_i - m_i| & , \; m_i < \infty \\ 1/n_i & , \; m_i = \infty \end{array} \right\} < r \} \;.$$

We extend the definition (2) of $a_{n,j}$ to all $n \in N$ , by pointwise limit if necessary.
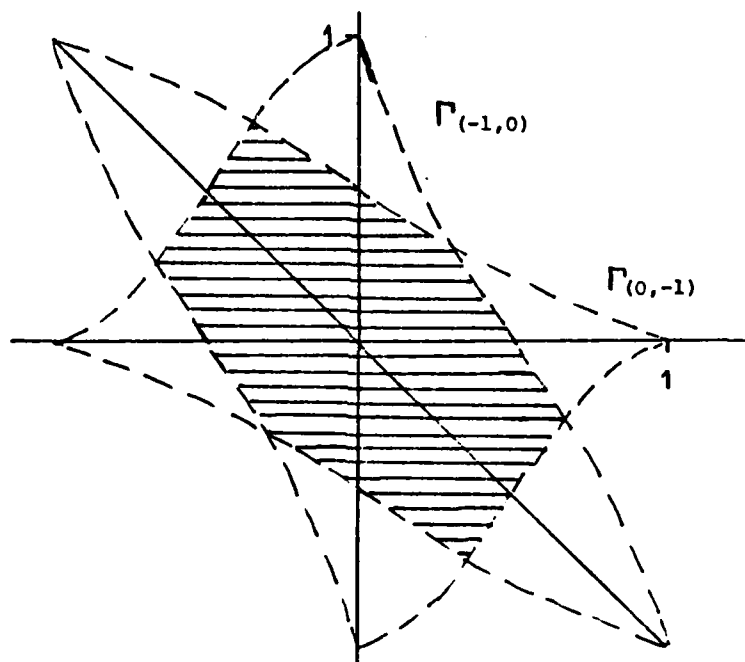


Figure 5.1

In what is to follow, the sets

$$\Omega_m \;:=\; \{x : a_{m,j}(x) < 1 \quad \underline{\text{for}} \; j \in J\} \;,$$

with

- 5.2 -

$$J := Z'_{\pm} = \{(1,0),(0,1),(-1,1),(-1,0),(0,-1),(1,-1)\}$$

play a major role. Note that

$$\Omega_n = \Omega_{n'} .$$

A qualitatively correct picture of $\Omega_n$ is given in Figure 5.1 which shows the roughly

hexagonal shape of $\Omega_n$ and also shows the six curves

$$\Gamma_{n,j} := \{x \in C_{-j} : a_{n,j}(x) = 1\} , \quad j \in J$$

which contribute to the boundary, $\partial\Omega_n$ . Here, $C_{-j}$ is the union of the two cones $R_\sigma$

which contain $-j$ .

**Theorem 5.1.** For $m \in N$ , let $\chi_m$ be the characteristic function of $\Omega_m$ . Then, for

any $d > 0$ , there exists $\epsilon > 0$ so that

$$|L_n^\wedge(2\pi x) - \chi_m(x)| < C_1\left(1 + C \, \text{dist}(x,\partial\Omega_m)\right)^{-\mu(n)} \tag{5.3}$$

for all $x$ with $\text{dist}(x, \partial\Omega_m) \geq d$ and all $n \in (0,\infty)^3$ with $n' := n/\mu(n) \in B_\epsilon(m)$ , and

with the positive constants $C_1$ and $C$ independent of $m$ , $n$, $d$, or $x$ .

**Proof.** The proof is based on a series of propositions which we merely state as needed

and prove at leisure later. We begin with the following

**Proposition 5.1.** $\Omega_n$ depends continuously on $n$ in the Hausdorff topology.

which is part of the Corollary to Lemma 6.4 below. This implies , given $d > 0$ , the

existence of $\epsilon > 0$ so that

$$\text{dist}(\Omega_{n'},\Omega_m) < d/2$$

for all $n' \in B_\epsilon(m)$ . Consequently,

$$\text{dist}(x,\partial\Omega_m) < 2\text{dist}(x,\partial\Omega_{n'})$$

for all $x$ with $\text{dist}(x,\partial\Omega_m) \geq d$ and for all $n' \in B_\epsilon(m)$ . It is therefore sufficient to

prove (3) with $m$ replaced by $n'$ .

For its proof, we use (1) and we consider two cases.

(i) $x \in \Omega_{n'}$ . We need

$$- 5.3 -$$

**Proposition 5.2.** Let

$$J' := A^+(1,1) = \{\pm(1,1), \pm(2,-1), \pm(-1,2)\} .$$

For $n \in \mathbb{N}$ and $x \in \Omega_{n'} = \Omega_n$,

$$a_{n,j}(x) < \begin{cases} \left(1 + C \operatorname{dist}(x, \partial\Omega_n)\right)^{-1} & , \ j \in J \cup J' \\ \left(1 + C |j|\right)^{-1} & , \ j \in \mathbb{Z}^2 \setminus (0 \cup J \cup J') \end{cases} , \qquad (5.4)$$

with the positive constant $C$ independent of $n$, $j$, or $x$.


This, together with (1), implies that

$$|1/L_n^\wedge(2\pi x) - 1| < \Sigma_{j \neq 0}\left(a_{n',j}(x)\right)^{\mu(n)}$$

$$< 12\left(1 + C \operatorname{dist}(x, \partial\Omega_n)\right)^{-\mu(n)} + \sum_{j \in \setminus 0 \cup J \cup J'} \left(1 + C|j|\right)^{-\mu(n)} \quad (5.5)$$

$$< c_1\left(1 + C \operatorname{dist}(x, \partial\Omega_{n'})\right)^{-\mu(n)}$$

and so proves (3) for this case.

(ii) $x \notin \Omega_{n'}$. For this case, we need


**Proposition 5.3.** The integer translates of $\Omega_n$ form, up to a set of measure zero, a partition of $\mathbb{R}^2$, i.e.,

$$\mathbb{R}^2 = \bigcup_{j \in \mathbb{Z}^2} j + \bar{\Omega}_n \ , \quad \Omega_n \cap (j + \Omega_n) = \emptyset \ \text{for} \ j \neq 0 .$$


We conclude that there is $j \neq 0$ so that $x = x' + j$ with $x' \in \bar{\Omega}_{n'}$. With this, we use the periodicity of the characteristic polynomial $P$ to write for such $x$

$$L^\wedge(2\pi x) = L^\wedge(2\pi(x'+j)) = \frac{\tilde{M}(x'+j)}{P(2\pi(x'+j))}\frac{\tilde{M}(x')}{\tilde{M}(x')} = L^\wedge(2\pi x')\frac{\tilde{M}(x'+j)}{\tilde{M}(x')} .$$

Therefore

$$|L_n^\wedge(2\pi x)| = |L_n^\wedge(2\pi x')| \, a_{n,j}(x') = |L_n^\wedge(2\pi x')|\left(a_{n',j}(x')\right)^{\mu(n)} .$$

By (5),

$$|L_n^\wedge(2\pi x')| < C$$

since $x' \in \bar{\Omega}_{n'}$. Thus (3) is proved for this case once we show

- 5.4 -

**Proposition 5.4.** <u>Let</u> $x = x' + j$ <u>with</u> $j \in \mathbb{Z}^2 \setminus 0$ <u>and</u> $x' \in \Omega_{n'}^-$ , <u>and with</u> $n \in \mathbb{R}^3$ .

<u>Then</u>

$$a_{n',j}(x') \leq \left(1 + C \, \text{dist}(x, \partial\Omega_{n'})\right)^{-1} \tag{5.6}$$

<u>for some positive constant</u> $C$ <u>independent of</u> $n$ <u>and</u> $x$ .


This finishes the proof of Theorem 5.1. |||


**Theorem 5.2.** <u>Let</u> $f$ <u>be the Fourier transform of a measure with support strictly</u> <u>inside</u> $2\pi\Omega_m$ <u>for some</u> $m \in N$ , <u>i.e.,</u>

$$d := \text{dist}(\text{supp } f^\wedge, \partial(2\pi\Omega_m)) > 0 .$$

<u>Then there exists</u> $\varepsilon > 0$ <u>so that, for all</u> $n \in B_\varepsilon(m)$ ,

$$\|f - I_n f\|_\infty \leq C_1\left(1 + Cd\right)^{-\mu(n)} \|f^\wedge\|_1 ,$$

<u>with</u> $\|f^\wedge\|_1$ <u>the total variation of</u> $f^\wedge$ . <u>The positive constants</u> $C$ , $C_1$ <u>do not depend</u> <u>on</u> $m$ , $d$ , <u>or</u> $n$ .


**Proof.** Fix $d > 0$ and choose $\varepsilon > 0$ so that supp $f^\wedge \subseteq (2\pi\Omega_{n'})$ and

dist(supp $f^\wedge$, $\partial(2\pi\Omega_{n'})) \geq d/2$ for all $n' \in B_\varepsilon(m)$ . We have to estimate

$$f(x) - \Sigma f(j) L_n(x-j) = f(x) - \Sigma(2\pi)^{-2} \int_{\mathbb{R}^2} f(j)e^{-ijy}L_n^\wedge(y) e^{ixy}dy .$$

Since $2\pi\Omega_{n'}$ is a fundamental domain, i.e., its integer translates form a partition of

unity (by Proposition 5.3), and supp $f^\wedge \subseteq \Omega_{n'}$ , $\left(f(j)\right)_{j\in\mathbb{Z}^2}$ are the Fourier coefficients

of the periodic extension $f^\wedge_p$ of the measure $f^\wedge$ . Using the weak convergence of the

Fourier series of a measure, we obtain

$$f(x) - (I_n f)(x) = (2\pi)^{-2}\int_{\mathbb{R}^2} \left[f^\wedge(y) - f^\wedge_p(y) L_n^\wedge(y)\right] e^{ixy} \, dy .$$

Applying Theorem 5.1 yields, for $n' \in B_\varepsilon(m)$ ,

$$\|f - I_n f\|_\infty \leq (2\pi)^{-2}C_1\left(1+Cd\right)^{-\mu(n)}\|f^\wedge\|_1 + \underset{j\neq 0}{\Sigma} \|L_n^\wedge(\cdot+2\pi j)\|_{\infty,\text{supp } f^\wedge}\|f^\wedge\|_1$$

$$\leq \left[C_1(1+Cd)^{-\mu(n)} + \underset{j\in\backslash\,\Omega\cup J\cup J'}{\Sigma} (1+C|j|)^{-\mu(n)}\right] \|f^\wedge\|_1$$

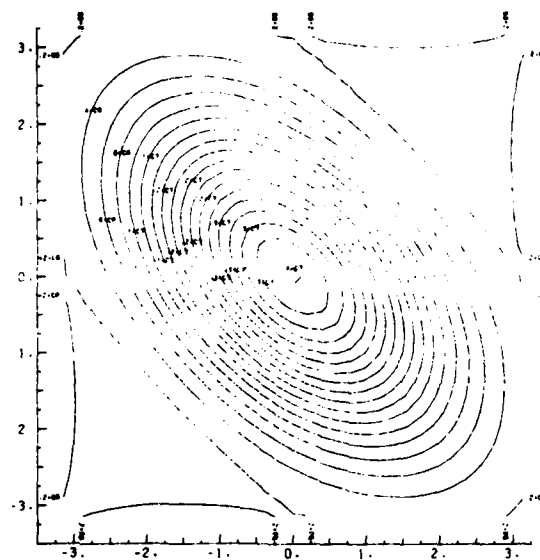$$\leq C_1(1+Cd)^{-\mu(n)}\|f^\wedge\|_1 . \quad |||$$

Figure 5.2



Figure 5.3
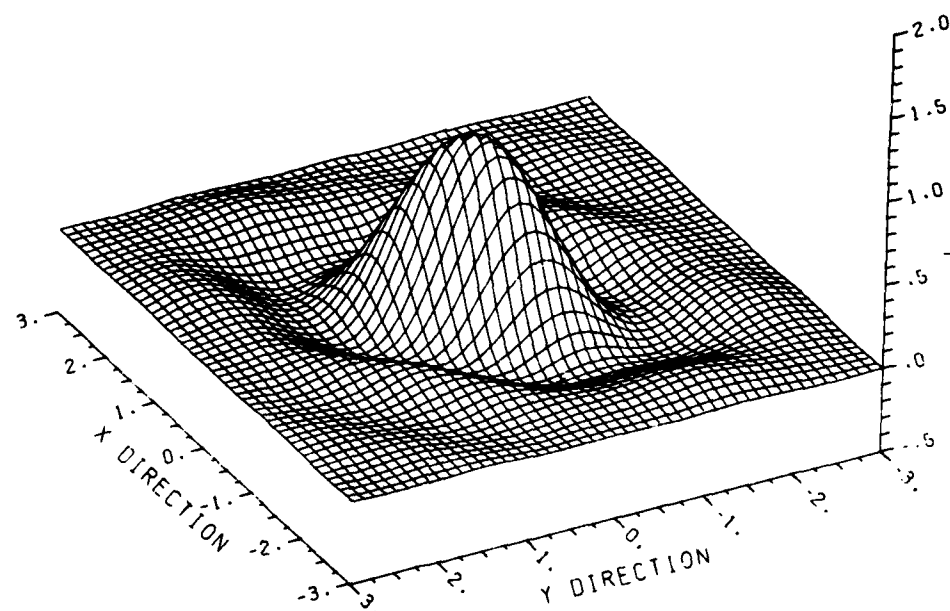
- 5.6 -

We now discuss briefly the particularly symmmetric and special case

$$n = (r,r,r) .$$

Figure 5.2 shows the level lines for $P_r := P_{(r,r,r)}$ for $r = 3$ . Note that its minimum seems to occurs at $(2\pi/3, 2\pi/3)$ , and this can be verified analytically for $r < 4$ . We conjecture that this is no accident, but is the case for all $r$ .

Figure 5.3 shows $L_r := L_{(r,r,r)}$ for $r = 3$ (?) . The fast decay is quite striking, making plain that cardinal interpolation with this $L$ would be strongly essentially local.

If we assume that $f^\wedge \in L_2$ with $\text{supp } f^\wedge \subseteq \Omega_m$ , then the convergence of $I_r f$ can be stated in a particularly nice way. We define a bivariate "Whittaker" operator

$$W: \ell_2 \longrightarrow L_2 : f \longmapsto \sum_{j \in \mathbb{Z}^2} f(j) \chi^\wedge(\cdot - j) ,$$

with

$$\chi^\wedge(u,v) := \frac{9}{2\pi^2} \left[ \frac{\cos \frac{2\pi}{3}(u+v)}{(u-2v)(v-2u)} + \frac{\cos \frac{2\pi}{3}(2u-v)}{(u+v)(u-2v)} + \frac{\cos \frac{2\pi}{3}(v-2u)}{(u+v)(v-2u)} \right]$$

the Fourier transform of $\chi := \chi_{(1,1,1)}$ . Note that the translates of $\chi^\wedge$ are orthogonal in $L_2$ . As in the univariate case [$S_2$] , the "Whittaker" series provides the limiting operator for cardinal interpolation $I_{(r,r,r)}$ as $r \to \infty$ . More precisely, we have

$$\| W - I_r : \ell_2 \longrightarrow L_2 \| \xrightarrow[r \to \infty]{} 0 . \tag{5.7}$$

If $f \in L_2$ and $\text{supp } f^\wedge \subseteq \Omega_{(1,1,1)}$ , we have $W\left((f(j))_{\mathbb{Z}^2}\right) = f$ and hence (7) is an $L_2$-version of Theorem 5.2.

To prove (7), we first show that the cardinal interpolation maps $I_r := I_{(r,r,r)}$ are bounded as maps from $\ell_2$ to $L_2$ uniformly in $r$ : For $f \in \ell_2$ , we have

$$\| I_r f \|_2 = \| \Sigma f(j) e^{-ij\cdot} L_r \|_2 \leq \| f \|_2 \left( \Sigma \| L_r^\wedge(\cdot + 2\pi j) \|_{\infty, \Omega}^2 \right) \leq C \| f \|_2 .$$

By the uniform boundedness principle, it is therefore sufficient to check the convergence (7) for the unit vector $e_j \in \ell_2$ . But this is an obvious consequence of (3). |||


In the univariate setting [R] proved the convergence in $L_p$ for all $p \in (1,\infty)$ , but we leave the corresponding bivariate problem to a later paper.

[MRR] extend Schoenberg's univariate result to include the possibility that $f$ has support at $\pm\pi$. This requires the realization that $L_r(\pm\pi) \longrightarrow 1/2$. Theorem 5.1 says

- 5.7 -

nothing about the limit of $L_r^{\wedge}(x)$ in case $x \in \partial\Omega_m$ . For the special choice $n = (r,r,r)$ , such a statement is relatively easy to make.

**Corollary** . <u>For the special choice</u> $n = (r,r,r)$ ,

$$\lim_{r \to \infty} L_n^{\wedge}(x) = \begin{cases} 1/3 \ , \ x \in Az \\ 1/2 \ , \ x \in \partial\Omega \setminus Az \end{cases} \tag{5.8}$$

<u>with</u> $z := (2\pi/3, 2\pi/3)$ .

Clearly, our result concerning the convergence of $I_n$ still holds if $f$ is a measure, absolutely continuous in a neighborhood of $\partial\Omega_m$ and supported in $\Omega_m$ . Our result is best possible in the following sense. If $\text{supp } f^{\wedge} \cap \partial\Omega \neq \emptyset$ , then, in general, $I_n f$ does not converge to $f$ . For example, if $f(x) = \cos(z^* x)$ , then

$$I_r f(x) \xrightarrow[r \to \infty]{} \Sigma_{A \in A} e^{i(Az)^* x} = \left[ \cos \frac{2\pi}{3}(u+v) + \cos \frac{2\pi}{3}(2u-v) + \cos \frac{2\pi}{3}(2v-u) \right] .$$

This follows from the Corollary. However, it requires slightly more precise information about the convergence asserted in (8). The heuristic argument is that

$$(I_n f)^{\wedge} = f_p^{\wedge} L_n^{\wedge} = \frac{1}{2} (2\pi)^2 \Sigma \ \delta_{z+2\pi j} L_n^{\wedge}(z+2\pi j)$$

and therefore

$$I_n f \xrightarrow[n \to \infty]{} \Sigma_{A \in A^*} \left( \delta_{Az} \cdot 1/3 \right) .$$

Here, $\delta_\xi$ denotes the Dirac measure at $\xi$ .

**6. Detailed estimates.** In this section, we prove Propositions 5.1 - 5.4 and various lemmas needed in the proofs. This amounts to a detailed study of the functions $a_{n,j}$ and the set $\Omega_n$ and how they depend on $n \in N$. In particular, we need to study the boundary of $\Omega_n$. This boundary is made up of pieces of curves given implicitly by the equation

$$a_{n,j}(x) = 1$$

for some $j \in J$.

We use the symmetries of the given situation. Recall the notation

$$\sigma(n) := (n_{\sigma(1)}, n_{\sigma(2)}, n_{\sigma(3)}) .$$

We conclude from (3.5) or directly from (2) that

$$a_{n,j}(x) = a_{\sigma(n),A_\sigma^* j}(A_\sigma^* x) . \tag{6.1}$$

This implies that

$$\pm A_\sigma^* \Omega_n = \Omega_{\sigma(n)} \tag{6.2}$$

and therefore

$$A_\sigma^* (\Omega_n \cap R) = \Omega_{\sigma(n)} \cap R_\sigma , \tag{6.3}$$

where, to recall from Section 4,

$$R = R_{(1)} = R_+^2 .$$

Next, we consider the boundary of $\Omega_n$ (cf. Figure 5.1). Each $-j \in J$ lies in two cones $R_\sigma$. With $C_{-j}$ their union, we define the curve

$$\Gamma_{n,j} := \{x \in C_{-j} : a_{n,j}(x) = 1\} . \tag{6.4}$$

The boundary of $\Omega_n$ is made up of segments of these curves. It follows from (1) that

$$A_\sigma^* \Gamma_{n,j} = \Gamma_{\sigma(n),A_\sigma^* j} . \tag{6.5}$$

**Lemma 6.1.** _For_ $j \in J$ , _denote by_ $j_0, j_1$ _the vectors spanning the union_ $C_j$ _of the two cones_ $R_\sigma$ _containing_ $j$ . _For_ $n \in (0,\infty)^3$ , _the curve_ $\Gamma_{n,j}$ _passes through the points_ $-j_0, -j/2, -j_1$ _and is symmetric with respect to the point_ $-j/2$ . _Moreover, it is monotone (in an appropriate coordinate system) and is Lipschitz continuous, uniformly in_ $n$ .

**Proof.** The symmetry with respect to the point $-j/2$ follows from the relations

$$a_{n,j}(x-j) = 1/a_{n,-j}(x) = 1/a_{n,j}(-x)$$

which are immediate consequences of the definition (5.2) of $a_{n,j}$. For the rest, it is, in view of (5), sufficient to consider $j = (-1,0)$. In this case, the curve $\Gamma_{n,j}$ is given by the equation

$$\left|\frac{u}{u-1}\right|^r \left|\frac{u+v}{u+v-1}\right|^t = 1 . \tag{6.6}$$

Using the fact that $u > 0$, $u+v > 0$, and solving for $v$, we obtain

$$v = -u + 1/\left[1 + \left(\frac{u}{1-u}\right)^\alpha\right] , \quad 0 < u < 1 , \tag{6.7}$$

where $\alpha := r/t$. This shows that, for any $\alpha \in (0,\infty)$, the points $-j_0 = (1,-1)$, $-j/2 = (1/2,0)$, and $-j_1 = (0,1)$ lie on the curve. Moreover we have

$$dv/du = -1 - \left[1 + \left(\frac{u}{1-u}\right)^\alpha\right]^{-2} \alpha \left(\frac{u}{1-u}\right)^{\alpha-1} (1-u)^{-2} \tag{6.8}$$

which shows that

$$dv/du < -1 , \tag{6.9}$$

with equality only if $u = 0$. This proves the remaining assertions of the Lemma. |||

**Lemma 6.2.** Let $j_0$, $j_1$ be the two vectors which span the cone $R_\sigma$. For $n \in (0,\infty)^3$, the curves $\Gamma_{n,-j_0}$ and $\Gamma_{n,-j_1}$ intersect at a unique point $z_{n,\sigma} \in R_\sigma$. The boundary of $\Omega_n$ consists of the segments of the curves $\Gamma_{n,j}$ connecting the points in $J/2$ and $z_{n,\sigma}$, $\sigma \in \$_3$.

**Proof.** In view of (3) and (5), it is sufficient to consider the case $\sigma = (1)$, $R_\sigma = R$ and $j_0 = (1,0)$, $j_1 = (0,1)$. By (9), $\Gamma_{n,(-1,0)}$ has slope $< -1$, with equality only at the point $(0,1)$. Similarly, a direct computation shows that $\Gamma_{n,(0,-1)}$ has slope between $-1$ and $0$. Since $\Gamma_{n,(-1,0)}$ and $\Gamma_{n,(0,-1)}$ pass through the points $(0,1)$, $(1/2,0)$ and $(0,1/2)$, $(1,0)$ respectively, they intersect at a unique point $z_{n,(1)} \in (0,1/2)^2$. To show that the boundary of $\Omega_n$ intersected with $R$ consists of the segments connecting $(0,1/2)$ with $z_{n,(1)}$, and $z_{n,(1)}$ with $(1/2,0)$, we prove that, for $x = (u,v) \in R$,

$$a_{n,(-1,0)}(x), a_{n,(0,-1)}(x) < 1 \text{ implies that } a_{n,j}(x) < 1 \text{ for all } j \in J .$$

- 6.2 -

Indeed,

$$a_{n,(-1,0)}(x) = |\frac{u}{u-1}|^r |\frac{u+v}{u+v-1}|^t < 1$$

implies $u < 1/2$ , and, from $a_{n,(0,-1)}(x) < 1$ , it follows that $v < 1/2$ . This implies

$$a_{n,(-1,1)}(x) = |\frac{u}{u-1}|^r |\frac{v}{v+1}|^t < 1 ,$$

and the other cases can be checked just as easily. |||

**Lemma 6.3.** <u>For all</u> $m \in N$ , $\Gamma_{m,j} = \lim_{n \to m} \{\Gamma_{n,j}: n \in (0,\infty)^3 \cap N\}$ . <u>Hence Lemma 1 is</u> <u>valid for all</u> $n \in N$ .

**Proof.** Without loss of generality, we consider only the case $j = (-1,0)$ . We claim that,

$$\Gamma_{m,j} = \begin{cases} BL\big((0,1),(0,1/2),(1/2,0),(1,-1/2),(1,-1)\big) & \text{if } m_1 = 0 \text{ and/or } m_3 = \infty \\ BL\big((0,1),(1/2,1/2),(1/2,0),(1/2,-1/2),(1,-1)\big) & \text{if } m_1 = \infty \text{ and/or } m_3 = 0 \end{cases},$$

where $BL(x_1,\ldots,x_m)$ denotes the broken line with vertices $x_1, \ldots, x_m$ . Consider, e.g., the first case. By (9) and the symmetry of the curves $\Gamma_{n,j}$ and $\Gamma_{m,j}$ , we have for $n \in (0,\infty)^3$ that

$$dist(\Gamma_{m,j}, \Gamma_{n,j}) < u_n , \text{ with } u_n \text{ s.t. } a_{n,j}(u_n,1/2) = 1 .$$

From (6), we obtain

$$\gamma := t/r = \frac{\ln(1-u_n) - \ln u_n}{\ln(u_n+1/2) - \ln(1/2-u_n)} .$$

If $n \longrightarrow m$ with $m_1 = 0$ and/or $m_3 = \infty$ , we must have $\gamma \longrightarrow \infty$ and, by the above equation, this implies that $u_n \longrightarrow 0$ . The second case can be handled similarly. |||

Lemma 3 is a particular case of the next lemma which states that $\Gamma_{n,j}$ depends continuously on $n$ .

**Lemma 6.4.** <u>For</u> $n, m \in N$ , $dist(\Gamma_{n,j},\Gamma_{m,j}) \longrightarrow 0$ <u>as</u> $n \longrightarrow m$ .

**Proof.** In view of Lemma 3, we may assume that $m \in (0,\infty)^3$ . Moreover, it is sufficient to consider $j = (-1,0)$ . In this case, it follows from (7) that $\Gamma_{n,j} \longrightarrow \Gamma_{m,j}$ pointwise,

- 6.3 -

both curves being viewed as functions of  u . By the uniform Lipschitz continuity of the
curves, this implies the assertion of the lemma. |||

Lemmas 1-4 give a qualitative description of the boundary of  $\Omega_n$ . We summarize the
main features in the following

**Corollary.** (i)  $\partial\Omega_n$  _consists of segments of the curves_  $\Gamma_{n,j}$  _connecting the points
in_  $J/2$  _with the intersection points_  $z_{n,\sigma}$  ,  $\sigma \in \$_3$  .

(ii)  $\partial\Omega_n$  _is piecewise monotone and is Lipschitz continuous, uniformly in_  n .

(iii)  $\Omega_n$  _depends continuously on_  n  _in the Hausdorff topology._

Note that this provides the **proof of Proposition 5.1.**

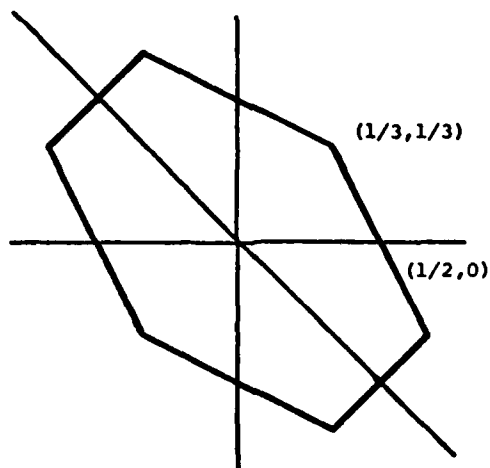To give a few examples, we list below all cases for which  $\Omega_n$  has a piecewise linear
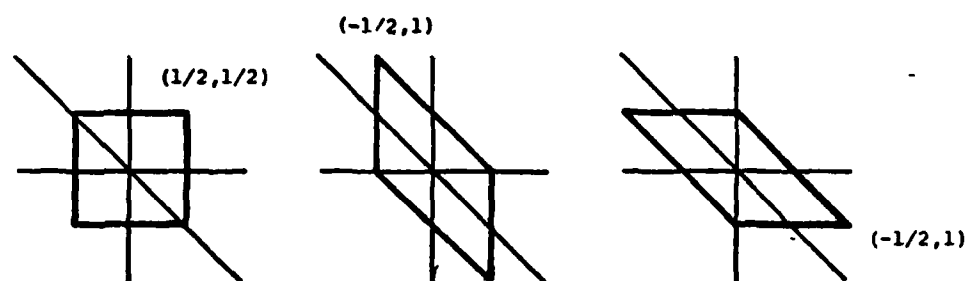boundary.



(1/3,1/3)

(1/2,0)

Figure 6.1: n = (1,1,1)

- 6.4 -

**Figure 6.2:** n = (1,1,0), (1,0,1), (0,1,1)



**Figure 6.3:** n = (1,1,∞), (1,∞,1), (∞,1,1)

- 6.5 -

Moreover, we have

$$\Omega_{(1,1,0)} = \Omega_{(\infty,1,0)} = \Omega_{(1,\infty,0)}$$

$$\Omega_{(1,0,1)} = \Omega_{(\infty,0,1)} = \Omega_{(1,0,\infty)} \qquad (6.10)$$

$$\Omega_{(0,1,1)} = \Omega_{(0,\infty,1)} = \Omega_{(0,1,\infty)}$$

We take the occasion to prove the following observation which stresses the underlying hexagonal structure.

**Proposition 6.** $\Omega_0 := \bigcap_n \Omega_n =$ int conv $J/2$ , $\Omega_1 := \bigcup_n \Omega_n = \bigcup_{A \in \mathbf{A}_+} A^*(0,1/2)^2$ .
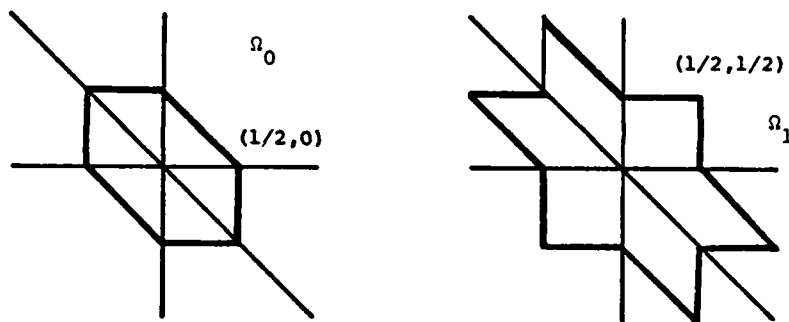


Figure 6.4

**Proof.** We claim that, for any $n$ ,

$$\{(u,v) : u, v \geqslant 0, \ u+v < 1/2\} \subseteq R \cap \Omega_n \subseteq [0,1/2)^2 . \qquad (6.11)$$

This follows from Lemma 1, in particular from the fact that the curves $\Gamma_{n,(-1,0)}$ and $\Gamma_{n,(0,-1)}$ pass through the points $(1/2,0)$ and $(0,1/2)$ and, as functions of $u$ , have slopes $< -1$ and $> -1$ , respectively. To complete the proof, note that $n = (0,1,1)$

- 6.6 -

gives equality in the first inclusion of (11) while $n = (1,1,0)$ gives equality in the second. |||

We are now also ready for the

**Proof of Proposition 5.3.** Because of the continuity of $\Omega_n$ as a function of $n$, it is sufficient to consider $n \in (0,\infty)^3$ . In this situation, Figure 5.1 gives a qualitatively correct description of $\Omega_n$ . Because of the geometry of $\Omega_n$ and the symmetry relations (2), it is sufficient to establish the following claims:

(i) $j + \Gamma_{n,j} = \Gamma_{n,-j}$ for all $j \in J$ .

(ii) The curve $(1,0) + \Gamma_{n,(1,-1)}$ passes through the point $z_{n,(1)}$ .

The first assertion follows from the relation

$$a_{n,j}(x-j) = 1/a_{n,-j}(x)$$

alluded to earlier and directly derivable from the definition (5.2) of $a_{n,j}$ . As to (ii), note that

$$a_{n,(-1,0)}(z_{n,(1)}) = 1 = a_{n,(0,-1)}(z_{n,(1)})$$

implies that

$$1 = a_{n,(0,-1)}/a_{n,(-1,0)}(z_{n,(1)})$$
$$= a_{n,(1,-1)}(z_{n,(1)} - (1,0)) ,$$

i.e., $z_{n,(1)} \in (1,0) + \Gamma_{n,(1,-1)}$ . |||


The next three lemmas state various estimates for the functions $a_{n,j}$ needed for the proof of Proposition 5.2.


**Lemma 6.5.** <u>For</u> $n \in [0,\infty)^3$ <u>with at most one component less than</u> 1 , <u>we have</u>

$$a_{n,j}(x) \leq \left[ 1 + c \, \text{dist}(x, \Gamma_{n,j} \cap \Omega_n) \right]^{-1} , \quad x \in \Omega_n^- , \; j \in J , \qquad (6.12)$$

<u>with</u> $c$ <u>a positive constant which does not depend on</u> $x$ , $n$ , <u>or</u> $j$ .


**Proof.** we may assume that $x := (u,v) \in R$ , in particular that $u, v \in [0,1/2]$ . We consider each $j \in J$ separately and suppress all references to $n$ .

(i) $j = (1,0)$ . We have $a_j(x) = |\frac{u}{1+u}|^r |\frac{u+v}{1+u+v}|^t \leq 1/2$ . This proves (12) since

dist$(x, \Gamma_{n,j} \cap \Omega_n) \leqslant 1$ . For the estimate, we have used the fact that  min $\{r,t\} \geqslant 1$ .

The case  $j = (0,1)$  is similar.

(ii)  $j = (-1,0)$ . The following figure may be of help in following the argument.



Figure 6.5
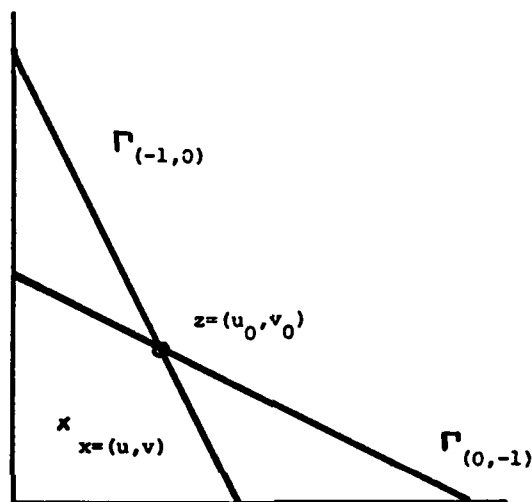
Let  $i := (0,-1)$  and let  $z =: (u_0,v_0) := z_{(1)}$  be the intersection of the two curves  $\Gamma_j$ and  $\Gamma_i$ . We consider two cases.

For  $v \leqslant v_0$ , there exists  $\varepsilon \geqslant$ dist$_\infty(x, \Gamma_j \cap \Omega)$  such that  $(u+\varepsilon,v) \in \Gamma_j$ , i.e., $a_j(u+\varepsilon,v) = 1$ . It follows that

$$
\begin{aligned}
a_j(u,v) &= a_j(u,v)/a_j(u+\varepsilon,v) \\
&= \left(\frac{u}{u+\varepsilon}\right)^r \left(\frac{1-u-\varepsilon}{1-u}\right)^r \left(\frac{u+v}{u+v+\varepsilon}\right)^t \left(\frac{1-u-v-\varepsilon}{1-u-v}\right)^t \\
&\leqslant \left(\frac{u+v}{u+v+\varepsilon}\right)^{\max\{r,t\}} \leqslant (1 + \varepsilon)^{-1} .
\end{aligned}
$$

- 6.8 -

where we have used the fact that $s, t > 1$ and the last inequality is easily checked.

Since

$$|u - u_0| \leq |u| + |u_0| \leq 2|u| ,$$

this proves (15) for this case.

(b) $x \in \Gamma_{(0,-1)} \cap \Omega$, i.e., $0 < u < u_1 = |u_0|$. We assume first that

$$v_0 - 1/2 > |u_0|/3 . \qquad (6.18)$$

Since $z_0 \in \Gamma_j$, we have

$$a_j(x) = a_j(z_0)/a_j(x) = \left|\frac{(u+1)/u}{(u_0+1)/u_0}\right|^r \left|\frac{(1-v)/v}{(1-v_0)/v_0}\right|^s$$

$$> \left| 1 + \frac{(v_0-v)/(vv_0)}{(1-v_0)/v_0} \right| > 1 + (v_0-v) .$$

In view of

$$|u - u_0| \leq 2|u_0| \leq 6(v_0 - 1/2) \leq 6(v_0 - v) ,$$

this proves (15) under the assumption (18).

Next, suppose that

$$v_0 - 1/2 \leq |u_0|/3 . \qquad (6.19)$$

We claim that

$$|(u_0+1)/u_0|^r > 1 + c|u_0| , \qquad (6.20)$$

and this finishes the proof of (15) for this case, in view of (16) and the inequalities

$$1/a_j(x) > \left|\frac{u+1}{u}\right|^r > \left|\frac{u_0+1}{u_0}\right|^r > 1 + c|u_0| > 1 + (c/2)|u - u_0| .$$

To prove (20), we use the fact that $z_0 \in \Gamma_* = \Gamma_{(-1,0)} - j$, therefore

$$|(u_0+1)/u_0|^r |(u_0+v_0)/(1-u_0-v_0)|^t = 1 .$$

Solving this for $r$, we obtain

$$r = t \frac{\ln |(u_0+v_0)/(1-u_0-v_0)|}{\ln |(u_0+1)/u_0|} > -\ln \left|\frac{1/2 - (2/3)|u_0|}{1/2 + (2/3)|u_0|}\right| / \ln |(u_0+1)/u_0| .$$

Here we used the assumption (19) and that $t > 1$. Therefore, we have

$$r \ln |(u_0+1)/u_0| > \ln \left|\frac{1 + (4/3)|u_0|}{1 - (4/3)|u_0|}\right| > \ln |1 + (4/3)|u_0||$$

- 6.11 -

which establishes (20). This completes the case $j = (1,-1)$ .

The case $j = (-1,1)$ is treated similarly. |||


From the statement of Proposition 5.2, recall the definition
$$J' \quad := \quad A^*(1,1) = \{\pm(1,1), \pm(2,-1), \pm(-1,2)\} .$$


**Lemma 6.6.** <u>For</u> $n \in (0,\infty)^3 \cap N$, $x \in \Omega_n^-$ , <u>and</u> $j \in J'$
$$a_{n,j}(x) \quad \leqslant \quad \left[1 + C \text{ dist}(x, -j/2)\right]^{-1} ,$$
<u>with</u> C <u>a positive constant which does not depend on</u> $n$ , $x$ , <u>or</u> $j$ .


**Proof.** Assume without loss that $x = (u,v) \in R$ .

We consider first $j = (-1,-1)$ . Assume, e.g., that $1/2 - u = \text{dist}_\infty(x, (1,1)/2)$
$=: \varepsilon$ . Then we have
$$a_{(-1,-1)}(x) \ = \ |\tfrac{u}{1-u}|^r |\tfrac{v}{1-v}|^s |\tfrac{u+v}{2-u-v}|^t \ \leqslant \ |\tfrac{1/2-\varepsilon}{1/2+\varepsilon}|^r |\tfrac{1-\varepsilon}{1+\varepsilon}|^t \ \leqslant \ 1/(1+\varepsilon) .$$

In the remaining cases $j \in J' \setminus (-1,-1)$ , we have $\text{dist}_\infty(-j/2,R) \leqslant 2$ and therefore it
is sufficient to bound $a_j(x)$ by an absolute constant less than 1 . This is
straightforward, using the fact that $a_{(-1,0)}(x)$ , $a_{(0,-1)}(x) \leqslant 1$ . We list only the
estimates:

<u>$j = (1,1)$ :</u>
$$|\tfrac{u}{u+1}|^r |\tfrac{v}{v+1}|^s |\tfrac{u+v}{u+v+2}|^t \ \leqslant \ 3^{-r} 3^{-s} 2^{-t} .$$

<u>$j = (2,-1)$ :</u>
$$a_{(2,-1)}(x) \ \leqslant \ a_{(2,-1)}(x)/a_{(0,-1)}(x) = |\tfrac{u}{2+u}|^r |\tfrac{1-u-v}{2-u-v}|^t \ \leqslant \ 5^{-r} 2^{-t} .$$

<u>$j = (-2,1)$ :</u>
$$a_{(-1,2)}(x) \ \leqslant \ a_{(-1,2)}(x)/a_{(-1,0)}(x) = |\tfrac{v}{2+v}|^s |\tfrac{1-u-v}{2-u-v}|^t \ \leqslant \ 5^{-s} 2^{-t} .$$

The remaining two cases , $j = (-2,1)$ , $(1,-2)$ are similar. |||


**Lemma 6.7.** <u>For</u> $n \in N$ , $x \in \Omega_n^-$ , <u>and</u> $j \in \mathbb{Z}^2 \setminus (0 \cup J \cup J')$ ,
$$a_{n,j}(x) \quad \leqslant \quad \left[1 + C|j|\right]^{-1} ,$$
<u>with</u> C <u>a positive constant which does not depend on</u> $n$ , $x$ , <u>or</u> $j$ .

**Proof.** Let $j = (k,\ell)$ and $x = (u,v)$ and, without loss of generality, $x \in R$, hence $u, v \in [0,1/2]$. We consider several cases.

(i) $k, \ell \neq 0, -1$, and $k+\ell \neq 0, -1, -2$: Then

$$\left|\frac{u}{u+k}\right| < (1 + |k|)^{-1}, \quad \left|\frac{v}{v+\ell}\right| < (1 + |\ell|)^{-1}, \quad \left|\frac{u+v}{u+v+k+\ell}\right| < (1 + |k+\ell|/3)^{-1}.$$

Using the fact that $(1+p)^{-1}(1+q)^{-1} < (1+p+q)^{-1}$ for $p,q > 0$ and that $|p| + |p+q| \geq \max\{|p|,|q|\}$, we see that the product of any two of the above lefthand terms is bounded by $(1 + |j|_\infty/3)^{-1}$. Since at most one of $r$, $s$, $t$ is less than $1$, this yields

$$a_{n,j}(x) < (1+|k|)^{-r}(1+|\ell|)^{-s}(1+|k+\ell|/3)^{-t} < (1 + |j|_\infty/3)^{-1}.$$

(ii) $j = (0,-2)$: Since $x \in \Omega_n$, we have

$$a_{n,(0,-1)}(x) = \left|\frac{v}{1-v}\right|^s \left|\frac{u+v}{1-u-v}\right|^t < 1$$

and therefore

$$a_{n,j}(x) < a_{n,j}(x)/a_{n,(0,-1)}(x) = \left|\frac{1-v}{2-v}\right|^s \left|\frac{1-u-v}{2-u-v}\right|^t < 2^{-s}2^{-t} < 1/2 = (1 + |j|_\infty/2)^{-1}.$$

(iii) $k = 0$, $\ell \neq -2,-1,0$: Here

$$a_{n,j}(x) = \left|\frac{v}{v+\ell}\right|^s \left|\frac{u+v}{u+v+\ell}\right|^t < (1+|\ell|)^{-s}(1+|\ell|/3)^{-t} < (1 + |j|_\infty/3)^{-1}.$$

(iv) $k = -1$, $\ell \neq -1, 0, 1, 2$. Here

$$\left|\frac{u}{1-u}\right|^r \left|\frac{v}{v+\ell}\right|^s \left|\frac{u+v}{u+v+\ell-1}\right|^t < 1 \cdot (1+|\ell|)^{-s}(1+|\ell-1|/3)^{-t} < (1 + |j|_\infty/5)^{-1}.$$

(v) $k = 0, -1$ and $\ell = 0, -1$: Treated in analogy to cases (ii) – (iv).

(vi) $\ell = -k$, $k \neq -1, 0, 1$: Here

$$a_{n,j}(x) = \left|\frac{u}{u+k}\right|^r \left|\frac{v}{v-k}\right|^s < (1+|k|)^{-r}(1+|k|)^{-s} < (1 + |j|_\infty)^{-1}.$$

(vii) $\ell = -k-1$, $k \neq -2, -1, 0, 1$: Here

$$a_{n,j}(x) < a_{n,j}(x)/a_{n,(-1,0)}(x) < \left|\frac{1-u}{u+k}\right|^r \left|\frac{v}{v-k-1}\right|^s < |k|^{-r}(1+|k-1|)^{-s} < (1+|k|/2)^{-1}$$

$$< (1 + |j|_\infty/3)^{-1}.$$

(viii) $\ell = -k-2$, $k \neq -2,-1,0$: Here

$$a_{n,j}(x) < a_{n,j}(x)/a_{n,(-1,0)}(x) = \left|\frac{u-1}{u+k}\right|^r \left|\frac{v}{v-k-2}\right|^s \left|\frac{u+v-1}{u+v-2}\right|^t < (1+|k|/3)^{-1}$$

$$< (1 + |j|_\infty/9)^{-1}.$$

This covers all $j \in \mathbb{Z}^2 \setminus (0 \cup \mathbb{J} \cup \mathbb{J}')$. |||

**Proof of Proposition 5.2.** Lemmas 5 – 7 prove (5.4) for $n \in N \cap [0,\infty)^3$, with the constants independent of $n$. Since $\Omega_n$ depends continuously on $n \in N$ (by Proposition

- 6.13 -

5.1), this proves (5.4) for all $n \in N$. |||


**Proof of Proposition 5.4.** To prove (5.6), we consider three cases.

(i) $j \in J$ : From the geometry of the set $\Omega_n = \Omega_{n'}$ (cf. Figure 5.1 which gives a qualitatively accurate description of the general situation) and, in particular, from the estimates of the slopes of the curves $\Gamma_{n,j}$ , we can see that in this case

$$\text{dist}(x, \partial\Omega_n) = \text{dist}(x', \Gamma_{n,j} \cap \Omega_n) .$$

Therefore, (5.6) is a consequence of Lemma 6.5 in this case.

(ii) $j \in J'$ : Assume, e.g., that $x' \in R \cap \Omega_n$ . Then, for $j \in J' \setminus (-1,-1)$ , we have $\text{dist}_\infty(x', -j/2) > 1/2$ and (5.6) follows from Lemma 6.6 since $\text{dist}_\infty(x, \partial\Omega_n) \leq C$ .

It remains to consider the case $j = (-1,-1)$ . From the bounds on the slopes of the curves $\Gamma_{n,(1,0)}, \Gamma_{n,(0,1)}$ , we see that $\Omega_n$ lies in the half space

$$\{y : y^*(1,1) > -(u_0,v_0)^*(1,1)\}$$

where $(u_0,v_0) := z_{(1)}$ is the point of intersection of the curves $\Gamma_{n,(-1,0)}$ and $\Gamma_{n,(0,-1)}$ . Since $\Omega_n \cap R \subseteq [0,1/2]^2$ , it follows that, for $x' \in \Omega_n \cap R$ ,

$$\text{dist}_1(x, \partial\Omega_n) = \text{dist}_1\left(x', -z_{(1)}+(1,1)\right) . \tag{6.21}$$

Here, $\text{dist}_1$ denotes the $\ell_1$-distance. Moreover, we have, in view of $[-z_{(1)} + (1,1)] - (1/2,1/2) = (1/2,1/2) - z_{(1)}$ , that

$$\text{dist}_1(x', -z_{(1)}+(1,1)) = \text{dist}_1(x', z_{(1)}) + 2\text{dist}_1(z_{(1)}, (1/2,1/2))$$
$$\leq 2\text{dist}_1(x', -j/2) . \tag{6.22}$$

This, together with (21) and Lemma 6, proves (5.6) for this case.

(iii) $j \in \mathbb{Z}^2 \setminus (0 \cup J \cup J')$ : In this case, (5.6) follows from Lemma 7 since, for any $j \neq 0$ ,

$$\text{dist}(x, \partial\Omega_n) \leq C|j| .$$

This completes the proof of Proposition 5.4. |||

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For a sequence $Z := (z_1, \ldots, z_{|Z|})$ in $\mathbf{Z}^2$, the box spline $M_Z$ is the distribution given by the rule

$$M_Z \phi := \int \phi(\Sigma \lambda_j z_j)\, d\lambda .$$

We show that $P_Z(x) := \Sigma_{k \in \mathbf{Z}^2} M_Z(k)\, e^{ikx} > 0$ iff $AZ \subseteq \{(1,0), (0,1), (1,1)\}$

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20.  ABSTRACT - cont'd.


for some linear map  A  which is 1-1 on  $\mathbf{Z}^2$.  This implies that, for such a

Z,  there exists, for any bounded continuous function  f,  a unique bounded spline

$I_Z f \in \text{span}\{M_Z(\cdot-k) : k \in \mathbf{Z}^2\}$  which agrees with  f  on  $\mathbf{Z}^2$.  We give sufficient

conditions in terms of the Fourier transform of  f  for the convergence of

$I_Z f$  as the degree  $|Z|-2$  tends to infinity.  E.g., we show, for the case

that  Z  contains each of the vectors  (1,0), (0,1), (1,1)  exactly  n  times,

that  $\|f - I_n f\|_{\infty, \mathbf{R}^2} \xrightarrow[n \to \infty]{} 0$  if

supp $\hat{f}$ CC $\Omega := \text{conv}\{(z,z), -(z,z), (-z,2z), -(-z,2z), (-2z,z), -(-2z,z)\}$,

with  $z := \dfrac{2\pi}{3}$ .  The converse of this result holds with  "CC"  replaced by  "$\underline{C}$".

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