SOME REMARK ON THE CONTINUATION METHOD OF
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SOME REMARKS ON THE CONTINUATION METHOD OF LERAY-SCHAUDER-RABINOWITZ AND THE METHOD OF MONOTONE ITERATIONS

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ABSTRACT

In this paper, we consider the following abstract problem. Let $(E,P)$ be an ordered Banach space with cone $P$ having a nonempty interior $\overline{P}$. Let $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 < \lambda_2$, $a, b \in P$, such that $b - a \in \overline{P}$. Let the operator $K : [\lambda_1, \lambda_2] \times [a,b] \rightarrow [a,b]$ be compact, strongly increasing with respect to the second variable for fixed $x \in (\lambda_1, \lambda_2)$, strictly increasing with respect to the first variable for fixed $u \in [a,b]$. Moreover, assume that $a$ is the only fixed point of $K(\lambda_1, \cdot)$ and that $b$ is the only fixed point of $K(\lambda_2, \cdot)$. Consider the equation

(*) $u = K(\lambda, u)$.

Under the above assumptions, we prove that any closed connected subset of solutions of (*) in $[\lambda_1, \lambda_2] \times [a,b]$ which meets $(\lambda_1, a)$ and $(\lambda_2, b)$, contains the maximal and the minimal solutions of (*), which are obtained by monotone iterations. Such a subset of solutions is shown to exist. Applications to a semilinear elliptic eigenvalue problem are studied.

AMS (MOS) Subject Classifications: Primary 47H07, Secondary 47H10, 35J65

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SIGNIFICANCE AND EXPLANATION

Certain physical phenomena can be modelled by the nonlinear eigenvalue problem (P),

\[ \begin{align*}
-\Delta u &= \lambda g(u,u) \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \Gamma = \partial \Omega .
\end{align*} \]

Under minimal assumptions on the nonlinear term \( g \), which may be of interest in some applications, the existence of solutions can be obtained by several known methods. The purpose of this paper is to compare two such methods, namely the continuation method due to Leray and Schauder as extended by Rabinowitz, and the method of monotone iterations. Our results are then applied to problem (P).
1. INTRODUCTION.

Let \((E,P)\) be an ordered Banach space, see [1, p. 627]. For \(a, b \in E\), \(a \leq b\), \([a, b]\) denotes the order-interval \(\{u \in E| a \leq u \leq b\}\).

Let \(K : [a, b] + [a, b]\) be a compact mapping, i.e. \(K\) is continuous and the range of \(K\) is relatively compact in \([a, b]\).

Since \([a, b]\) is closed and convex in \(E\), it is a consequence of Schauder's theorem, that \(K\) possesses at least one fixed point in \([a, b]\). If \(K\) is also increasing, i.e. \(u \leq v\) implies \(K(u) \leq K(v)\), then the existence of a minimal (resp. maximal) fixed point of \(K\), which we denote by \(\bar{u}\) (resp. \(\bar{v}\)), is easily established by an iteration procedure [1, p. 639].

\[
\bar{u} = \lim_{n \to \infty} K^n(a); \quad \bar{v} = \lim_{n \to \infty} K^n(b).
\]

If we also assume that \((E,P)\) is normal [1, p. 627] and that \(P\) has a nonempty interior, then \([a, b]\) is a bounded set of \(E\), with nonempty interior \([a; b]\), provided that \(a \ll b\), i.e. \(b - a \in \bar{P}\).

The Leray-Schauder degree of \(I - K\) relative to \([a; b]\), \(d(I - K, [a; b])\), see for example [6], is then well-defined, whenever \(K\) has no fixed point on the boundary of \([a, b]\), e.g. when \(K\) maps \([a, b]\) into its interior. Note that this implies

\[(1.0) \quad a \ll K(a) \text{ and } K(b) \ll b.\]

Conversely if \(K\) satisfies \((1.0)\), then a sufficient condition for \(K\) to map \([a, b]\) into its interior is that, \(K\) is strongly increasing, i.e. \(u \leq v\) implies \(K(u) \ll K(v)\). We shall assume \(K\) strongly increasing and satisfying \((1.1)\). \(d(I - K, [a; b])\) is easily computed by considering the compact homotopy:

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\( (1.1) \quad H(t,u) := u - (1 - t)c - tK(u), \quad t \in [0,1], \quad u \in [a,b], \text{ with } c \in [a,b] \)

Then, \( d(I - K, [a,b]) = d(H(1, \cdot), [a,b]) = d(H(t, \cdot), [a,b]) = d(H(0, \cdot), [a,b]) = 1, \quad t \in [0,1], \) by noting that the solutions \((t, u)\) of

\( (1.2) \quad H(t,u) = 0, \quad t \in [0,1], \quad u \in [a,b] \)

satisfy \( u \in [a,b] \).

Since \( d(H(t, \cdot), [a,b]) \) is constant and \( \neq 0 \), for \( t \in [0,1] \), it follows from (6, Corollaire 10, p. 15 - 16), that there exists a subset \( C \) of solutions of the equation

\( (1.2) \) which is connected in \([0,1] \times [a,b]\) equipped with the product topology, and which meets \((0, c)\) and at least one point \((1, \bar{u})\) where \( \bar{u} \) is a fixed point of \( K \). A natural question arises, namely which fixed points \( \bar{u} \) can be "reached by the homotopy" or more precisely which fixed points \( \bar{u} \) of \( K \) belong to the component of \((0, c)\) in \([0,1] \times [a,b] \). In particular are \((1, \bar{u}), (1, \hat{u}) \in C? \)

In section 2, we shall prove that if \( a < c < K(c) < \bar{u} \), then the component \( C \) of \((0, c)\) in \([0,1] \times [a,b]\) meets \((1, \bar{u})\) and that \((t, u) \in C, \quad 0 \leq t \leq 1 \) implies \( u \ll \bar{u} \).

Similarly, one could consider the homotopy \( \tilde{H}(t,u) = u - (1 - t)K(u) - tc, \quad t \in [0,1], \quad u \in [a,b] \). Then provided that \( \hat{u} < K(c) < c < b \), then \( \tilde{C} \), the component of \((1, c)\) meets \((0, \tilde{u})\) and \((t, u) \in \tilde{C}, \quad 0 \leq t \leq 1 \) implies \( \hat{u} \ll u \).

If \( \bar{u} \ll \hat{\bar{u}} \) and if there exist \( u_1, u_2 \in [a,b] \) satisfying

\( (1.3) \begin{cases} \bar{u} < u_1 < u_2 < \hat{\bar{u}} \\ K(u_1) < u_1, u_2 < K(u_2) \end{cases} \)

then, Amann [2] proved that there exists a third fixed point \( \tilde{u} \) such that \( \bar{u} \ll \bar{u} \ll \hat{u} \) satisfying \( \tilde{u} \notin u_1 \) and \( u_2 \notin \tilde{u} \). See [1, Theorem 14.2, p. 666]. Consider the homotopy:

\( (1.4) \quad H(t,u) = \begin{cases} u - (1 - 2t)a - 2tK(u) \quad t \in [0, \frac{1}{2}] \\ u - 2(1 - t)K(u) - (2t - 1)b \quad t \in \left[\frac{1}{2}, 1\right] \end{cases}, \quad u \in [a,b], \) and define

\( s := \{(t,u) \in [0,1] \times [a,b] | H(t,u) = 0\} \).
then, if \( C_1 \) is the component of \((0, a)\) in \( S \), we know by what precedes that \( C_1 \) contains \( (\frac{1}{2}, \tilde{u}) \); similarly, \( C_2 \) the component of \((1, b)\) in \( S \) contains \( (\frac{1}{2}, \tilde{u}) \); we shall prove in section 2, that there exists a connected set \( C_2 \) in \( S \), which meets \( (\frac{1}{2}, \tilde{u}), (\frac{1}{2}, \tilde{u}) \) and at least a third point \( (\frac{1}{2}, \tilde{u}) \) where \( \tilde{u} \) is a fixed point of \( K \) satisfying \( \tilde{u} \notin u_1, u_2 \notin \tilde{u} \).

These results are special cases of Theorem 2.1 where a general homotopy

\[
(1.5) \quad u = K(\lambda, u), \quad \lambda \in [\lambda_1, \lambda_2], \quad u \in [a, b]
\]

is considered. There \( K \) is a compact mapping, which is strongly increasing "in \( u \)" for \( \lambda \in (\lambda_1, \lambda_2) \) and strictly increasing "in \( \lambda \)" for \( u \in [a, b] \). Then if \( a \) (resp. \( b \)) is the only fixed point of \( K(\lambda_1, \cdot) \) (resp. \( K(\lambda_2, \cdot) \)), we prove that \( C \) the component of \((\lambda, a)\) in \( S := \{(\lambda, u) \in [\lambda_1, \lambda_2] \times [a, b] | u = K(\lambda, u)\} \) meets \( (\lambda_2, b) \), and contains all maximal and minimal solutions of \((1.5)\) for \( \lambda \in (\lambda_1, \lambda_2) \). Thus Theorem (2.1) relates the solutions of \((1.5)\) obtained by applying the continuation method of Leray-Schauder-Rabinowitz [6] and the solutions of \((1.5)\) obtained by monotone iterations [1].

In section 3, we give an application of the results of section 2 to a semilinear elliptic problem:

\[
\begin{align*}
-\Delta u &= \lambda q(\ast, u) \quad \text{in} \quad \Omega \subset \mathbb{R}^N \\
u &= 0 \quad \text{in} \quad \partial \Omega
\end{align*}
\]

where we refine some results of [3], [4].

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2. THE MAIN RESULT.

Throughout this section, \((E,\mathcal{P})\) denotes an ordered Banach space with cone \(\mathcal{P}\) having a nonempty interior \(\mathcal{P}\), (We do not assume \((E,\mathcal{P})\) to be normal), \(a,b \in E\) such that \(a << b\), and \(\lambda_1, \lambda_2 \in \mathbb{R}\) such that \(\lambda_1 < \lambda_2\).

\(K : [\lambda_1, \lambda_2] \times [a,b] \to [a,b]\) is continuous and has a relatively compact range in \([a,b]\) (where \([\lambda_1, \lambda_2] \times [a,b]\) is equipped with the product topology). \(S\) denotes the set of solutions of

\[ u = K(\lambda, u) \quad (\lambda, u) \in [\lambda_1, \lambda_2] \times [a,b]. \]

For \(\lambda \in [\lambda_1, \lambda_2]\), \(\tilde{u}(\lambda)\) (resp. \(\hat{u}(\lambda)\)) denotes the maximal (resp. minimal) fixed point of \(K(\lambda, \cdot)\) in \([a,b]\), which are known to exist. We have

Theorem 2.1. Let \(K\) defined as above satisfy the following assumptions:

(i) For each \(\lambda \in (\lambda_1, \lambda_2), K(\lambda, \cdot)\) is strongly increasing.

(ii) For each \(u \in [a,b], K(\cdot, u)\) is strictly increasing

\((\lambda < \mu \Rightarrow K(\lambda, u) < K(\mu, u)).\)

(iii) \(a, b\) a (resp. \(b\)) is the only fixed point of \(K(\lambda_1, \cdot)\) (resp. \(K(\lambda_2, \cdot))\). Then

(1) \(C\) the component of \((\lambda_1, a)\) in \(S\) meets \((\lambda_2, b)\).

(2) Any closed connected set \(D\) in \(S\) which meets \((\lambda_1, a)\) and \((\lambda_2, b)\) contains all maximal \(\tilde{u}(\lambda)\) and minimal \(\hat{u}(\lambda)\) fixed points of \(K(\lambda, \cdot)\), \(\lambda \in (\lambda_1, \lambda_2)\). Moreover, for each \(\lambda \in (\lambda_1, \lambda_2),\)

\[ \tilde{u}(\lambda) = \sup_{\lambda_1 < \mu < \lambda} \tilde{u}(\mu) = \lim_{\mu \uparrow \lambda} \tilde{u}(\mu). \]

\[ \hat{u}(\lambda) = \inf_{\lambda_2 < \mu < \lambda} \hat{u}(\mu) = \lim_{\mu \downarrow \lambda} \hat{u}(\mu). \]

(3) If for some \(\lambda \in (\lambda_1, \lambda_2), \tilde{u}(\lambda) < \tilde{d}(\lambda), \) and if there exist \(u_1, u_2\) satisfying:

\[ \tilde{u}(\lambda) < u_1 < u_2 < \tilde{d}(\lambda). \]

Then, any closed connected set \(D\) in \(S\) which meets \([\lambda_1, a]\) and \([\lambda_2, b]\) contains a point \([\lambda, \tilde{u}]\) where \(\tilde{u} \notin [u_1, u_2] \supset u.\)

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Remark 1. If $K$ satisfies (i), (ii), and (iii), $a$ is the only fixed point of $K(\lambda_1)$, but $b$ is not the only fixed point of $K(\lambda_2)$, then one can apply the theorem on $[\lambda_1, \lambda_2] \times [a, b]$ provided that $a \lessdot b$. Indeed, $u \lessdot b$ implies $K(\lambda, u) < K(\lambda, b) < K(\lambda, a) = b$, for $\lambda \in [\lambda_1, \lambda_2]$. Then $b$ plays the role of $b$. Similarly, when $a$ is not the only fixed point of $K(\lambda_1)$ but $b$ is for $K(\lambda_2)$.

Remark 2. When $K$ satisfies (i), (ii), and (iii), and $a < b$, then from what proceeds we know that there is a connected set $C_1$ in $S \cap [\lambda_1, \lambda_2] \times [a, b]$ which meets $(\lambda_1, a)$ and $(\lambda_2, b)$. A priori the component $C$ of $(\lambda_1, a]$ in $S$ may not be contained in $[\lambda_1, \lambda_2] \times [a, b]$. The following lemma, which will be useful in the proof of Theorem 2.1, shows that $(\lambda_1, u) \in C$ implies $u \in [a, b]$. Let $C_1$ denote the component of $(\lambda_1, c)$ in $S \cap [\lambda_1, \lambda_2] \times [a, b]$. Then for each $(\lambda, u) \in C$,

$$u \lessdot d \text{ holds.}$$

Similarly, let $C_d$ denote the component of $(\lambda_2, d)$ in $S \cap [\lambda_1, \lambda_2] \times [a, b]$. Then for each $(\lambda, u) \in C$,

$$c \lessdot u \text{ holds.}$$

Proof of Lemma 2.2.

Let $A := \{(\lambda, u) \in C_1 | u \lessdot d\}$. Then $A \neq \emptyset$, since $(\lambda_1, c) \in A$, moreover $A$ is closed in $C_1$. For $(\lambda, u) \in A$, we have $u \lessdot d$. Otherwise $u = K(\lambda, u) \lessdot K(\lambda, d) \lessdot K(\lambda_2, d) = d$, a contradiction. Thus $u = K(\lambda, u) \lessdot K(\lambda, d) < K(\lambda_2, d) = d$ and $u \lessdot d$. This and the fact that $\lambda < \lambda_2$ for $(\lambda, u) \in A$ imply that $A$ is open in $C_1$. Thus $A = C_1$ and (2.4) holds. The second part of the lemma is proved by exchanging the role of $(\lambda_1, c)$ and $(\lambda_2, d)$ and reversing the inequalities.

Remark. The proof of Lemma 2.2 is similar to the proof of part 2 of Theorem 1 of [4].
Proof of Theorem 2.1.

We first prove assertion (2). Let $\bar{\lambda} \in (\lambda_1, \lambda_2)$. We denote by $D_{\lambda}$ the component of $(\lambda_1, \lambda_2)$ in $D \cap ((\lambda_1, \lambda) \times [a, b])$. We claim that

a) $\sup_{(\lambda, u) \in D_{\lambda}} \lambda = \bar{\lambda}$

b) $(\lambda, u) \in D_{\lambda}$ implies $u < \tilde{u}(\bar{\lambda})$.

First we prove a). Assume that $\sup_{(\lambda, u) \in D_{\lambda}} \lambda < \bar{\lambda}$. Then, there exists $\tilde{u} \in (\lambda_1, \lambda)$ such that $(\lambda, u) \in D_{\lambda}$ for some $u$. Set $A := D \cap ((\tilde{u}) \times [a, b])$. $A \neq \emptyset$ since $\tilde{u} \in (\lambda_1, \lambda)$.

If we define $C := D \cap ((\tilde{u}) \times [a, b])$ then $C$ is a compact metric space, and $A$, $D_{\lambda}$ are closed disjoint subsets of $C$. There is not connected set $\tilde{D}$ meeting both $A$ and $D_{\lambda}$, otherwise $D_{\lambda} \supset C$ by using the maximality of $D_{\lambda}$ and $D_{\lambda} \cap A \neq \emptyset$, a contradiction. Thus by a lemma of point set topology, see for instance [5, Lemma 1.9], there are closed disjoint subsets of $C$, $C_1$ and $C_2$ such that $D_{\lambda} \subset C_1$, $A \subset C_2$ and $C = C_1 \cup C_2$. Then define $C_3 := \{(u, v) \in D | u > \lambda\}$. $C_2 \cup C_3$ is closed.

Thus, $\sup_{(\lambda, u) \in D_{\lambda}} \lambda = \bar{\lambda}$. Next we prove b). Observe that b) is a consequence of Lemma 2.2, $(\lambda, u) \in D_{\lambda}$ with $\lambda_2$ replaced by $\bar{\lambda}$, $c$ replaced by $a$ and $d$ replaced by $\tilde{u}(\bar{\lambda})$. Note that $a < \tilde{u}(\bar{\lambda})$, since $a = K(\lambda_1, a) < K(\lambda, a) < K(\lambda_2, \tilde{u}(\bar{\lambda})) = \tilde{u}(\bar{\lambda})$. Then $K(\lambda, a) < K(\lambda_2, \tilde{u}(\bar{\lambda}))$ and $a < K(\lambda, a) < K(\lambda_2, \tilde{u}(\bar{\lambda})) = \tilde{u}(\bar{\lambda})$. This proves b). By using a), b) and the fact that the range of $K$ is relatively compact in $[a, b]$, there exist $\bar{u} \in [a, b]$ and a sequence $((\lambda_n, u_n)) \in D_{\lambda}$ such that $\lim_{n \to \infty} \lambda_n = \bar{\lambda}$ and $\lim_{n \to \infty} u_n = \bar{u}$. By using the continuity of $K$, $\bar{u} = K(\lambda, \bar{u})$ and thus $\tilde{u}(\bar{\lambda}) < \bar{u}$. But from b) it follows that $\bar{u} \leq \tilde{u}(\bar{\lambda})$, thus $\bar{u} = \tilde{u}(\bar{\lambda})$.

Since $(\lambda_n, u_n) \in D$, $(\lambda_n, \tilde{u}(\bar{\lambda})) \in D$. For the $u_n$ we could have chosen $\tilde{u}(\lambda_n)$. Since for each sequence $\tilde{u}(\mu_n)$ such that $\mu_n \to \bar{\lambda}$ as $n \to \infty$, there exists a subsequence which converges to $(\lambda, \tilde{u}(\bar{\lambda}))$ we have also proven that $\lim_{n \to \infty} \tilde{u}(\lambda) = \sup_{\lambda \in \lambda} \tilde{u}(\lambda) = \tilde{u}(\bar{\lambda})$. The second part of the assertion of (2) is proven in a "dual" fashion.

Next we prove assertion (1). From (iii) we and the compactness of $K$ it follows that $\lim_{\lambda \to \lambda_1} \tilde{u}(\lambda) = a$ and $\lim_{\lambda \to \lambda_2} \tilde{u}(\lambda) = b$. Thus, since $b - a \in \mathcal{F}$, there are $\alpha \in (0, \frac{1}{2})$ and $\lambda \in \lambda_1 \lambda_2$. 

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\[ \varepsilon \in (0, \lambda_2 - \lambda_1) \text{ such that } d(\lambda) < a + d(b - a) \text{ for } \lambda \in (\lambda_1, \lambda_1 + \varepsilon) \text{ and } \]

\[ \tilde{u}(\lambda) > b - e(b - a) \text{ for } \lambda \in (\lambda_2 - \varepsilon, \lambda_2). \text{ Hence } d(\lambda) < \tilde{u}(\mu) \text{ for } \lambda \in (\lambda_1, \lambda_1 + \varepsilon) \text{ and } \mu \in (\lambda_2 - \varepsilon, \lambda_2). \text{ We claim that for each } \lambda \in (\lambda_1, \lambda_1 + \varepsilon) \text{ and } \mu \in (\lambda_2 - \varepsilon, \lambda_2), \text{ there is } \]

a maximal connected set \( C_{\lambda,\mu} \) in \( S \cap ([\lambda,\mu] \times [a,b]) \) which meets \( (\lambda, d(\lambda)) \) and \( (\mu, \tilde{u}(\mu)) \). For \( t \in [\lambda,\mu] \), define \( O_t := [K(t,a), K(t,b)] \). Note that \( K(t,a) \triangleleft \tilde{u}(t) \triangleleft K(t,b) \), for \( t \in [\lambda,\mu] \). Then \( O := \bigcup_{t \in [\lambda,\mu]} \{t\} \times O_t \) is an open subset of \( [\lambda,\mu] \times [a,b] \) containing no solution of (2.1) on its boundary (as a subset of \( [\lambda,\mu] \times [a,b] \)). We know that \( d(I - K(t,\ast), O_t) = 1 \), \( t \in [\lambda,\mu] \). By [7, Corollaire 10, V - 6], there exists a component \( C_{\lambda,\mu} \) of \( S \cap ([\lambda,\mu] \times [a,b]) \) which meets

\[ (\lambda) \times S_\lambda \text{ and } (\mu) \times S_\mu \text{ where } S_\lambda := \{u \in [a,b] \mid (t,u) \in S, \ t \in [\lambda_1,\lambda_2]\}. \text{ Next we prove that } C_{\lambda,\mu} \text{ contains } (\lambda, \tilde{u}(\lambda)) \text{ and } (\mu, \tilde{u}(\mu)). \text{ We denote by } c \text{ any element of } \]

\[ (\lambda) \times S_\lambda \cap C_{\lambda,\mu}. \text{ Note that } C_{\mu}, \text{ the component of } (\lambda, c) \text{ in } S \cap ([\lambda,\mu] \times [a,b]) \text{ is } C_{\lambda,\mu}. \text{ Next we define } C_c, \text{ the component of } (\lambda, c) \text{ in } S \cap ([\lambda,\mu] \times [a,b]) \text{ and as in the proof of part (2) one proves that } \sup_{t \in [\lambda,\mu]} t = \mu, \text{ and by applying the lemma 2.2, with } \]

\[ \lambda_1, \lambda_2, d \text{ and } D \text{ replaced by } \lambda_1, \mu, \tilde{u}(\mu) \text{ and } C_c, \text{ noting that } c < d(\lambda) < \tilde{u}(\mu), \text{ we obtain } u < \tilde{u}(\mu) \text{ for } (t,u) \in C_c. \text{ Then one chooses a sequence } (t_n, u_n) \to (\mu, \tilde{u}(\mu)) \text{ such that } t_n + u_n + u \text{ and one proves as in part (2) that } \tilde{u} = \tilde{u}(\mu). \text{ Thus } (\mu, \tilde{u}(\mu)) \text{ belongs to the closure of } C_c \text{ and hence to } C = C_{\lambda,\mu}. \text{ Similarly one proves that } \]

\[ (\lambda, \tilde{u}(\lambda)) \in C_{\lambda,\mu} \text{ is also connected and that its closure in } [\lambda_1, \lambda_2] \times [a,b] \text{ satisfies } \]

\[ \lambda \in (\lambda_1, \lambda_1 + \varepsilon) \text{ and } \mu \in (\lambda_2 - \varepsilon, \lambda_2) \text{ the requirements of assertion (1). } \]

Finally we prove assertion (3). We define \( O_1 := (\lambda_1, \lambda_1 + u_1] \times [a,u_1] \) and \( O_2 := (\lambda_2, \lambda_2 + u_2] \times [a,u_2]. \) \( O_1 \) and \( O_2 \) are open in \( [\lambda_1, \lambda_2] \times [a,b]. \) Let \( \lambda \in (\lambda_1, \lambda_1 + \varepsilon) \), then \( (\lambda, \tilde{u}(\lambda)) \in O_1. \) Indeed,

\[ (2.7) \quad a < \tilde{u}(\lambda) = K(\lambda, \tilde{u}(\lambda)) < K(\lambda_1, u_1) < K(\lambda_2, u_2) < b. \]

Similarly, let \( \mu \in (\lambda_2, \lambda_1), \text{ then } (\mu, \tilde{u}(\mu)) \in O_2. \) We know that \( (\lambda, \tilde{u}(\lambda)) \in D \text{ and } (\mu, \tilde{u}(\mu)) \in D. \) Since \( O_1 \) and \( O_2 \) are disjoint and \( D \) is closed and connected in
\[[\lambda_1,\lambda_2] \times [a,b],\] there exists \(\tilde{D}\) closed and connected in \([\lambda_1,\lambda_2] \times [a,b]\) such that

a) \(\tilde{D} \subset O_1^C \cap O_2^C\)

b) \(\tilde{D}\) meets \(\gamma_1\) and \(\gamma_2\), where \(O_i^C = ([\lambda_1,\lambda_2] \times [a,b]) \setminus O_i,\ i = 1,2\) and \(\gamma_i\) is the boundary of \(O_i\) as subset of \([\lambda_1,\lambda_2] \times [a,b]\), i = 1,2.

\(\gamma_1 = (\lambda_1,\lambda_2) \times \partial([a,u_1] \cup [\lambda] \times [a,u_1])\)

\(\gamma_2 = (\lambda_1,\lambda_2) \times \partial([u_2,b] \cup [\lambda] \times [u_2,b])\)

\(D \cap \gamma_1 = \{(\lambda_1,a)\} \cup B_1\)

\(D \cap \gamma_2 = \{(\lambda_2,b)\} \cup B_2\)

where

\(B_1 := \{(\lambda,u) \in D| \lambda = \lambda_1 u \in [a,u_1]\}\)

\(B_2 := \{(\lambda,u) \in D| \lambda = \lambda_2 u \in [u_2,b]\}\).

Note that the component of \(O_1^C \cap O_2^C\) which contains \((\lambda_1,a)\) is \(\{(\lambda_1,a)\}\). Similarly for \((\lambda_2,b)\).

Thus \(\tilde{D} \cap B_1 \neq \phi\), and \(\tilde{D} \cap B_2 \neq \phi\). We want to prove that

\(\tilde{D} \cap ((\lambda) \times ([a,u_1]^C \cup [u_2,b]^C)) \neq \phi\), where \([a,u_1]^C = [a,b] \setminus [a,u_1],\ l = 1,2\). Define

\(A_1 := ([\lambda_1,\lambda_2] \times [a,u_1] \cup [\lambda_1,\lambda_2] \times [u_2,b]^C)\)

\(A_2 := ([\lambda_1,\lambda_2] \times [a,u_1]^C \cup [\lambda_1,\lambda_2] \times [u_2,b])\)

Then \(A_1, A_2\) are closed subsets of \([\lambda_1,\lambda_2] \times [a,b]\), and \((\tilde{D} \cap A_1) \cup (\tilde{D} \cap A_2) = \tilde{D}\).

\(\tilde{D} \cap A_1 \neq \phi\), since \(\phi \neq \tilde{D} \cap B_1 \subset \tilde{D} \cap A_1\). Similarly \(\tilde{D} \cap A_2 \neq \phi\).

Thus \(\tilde{D} \cap A_1 \cap A_2 \neq \phi\)

by using the connectedness of \(\tilde{D}\). From the observation that \((t,u) \in \tilde{D}\) and \(t < \lambda, u \in [a,u_1]\) (resp. \(t > \lambda, \ [u_2,b]\)) implies \(u \in [a^*,u_1]\) (resp. \([u_2^*,b]\)), it follows that \(\tilde{D} \cap A_1 \cap A_2 = \tilde{D} \cap ([\lambda] \times ([u_2,b]^C \cap [a^*,u_1]^C))\). Thus there is \((\lambda,\tilde{u}) \in \tilde{D} \subset D\) such that \(\tilde{u} \notin u_1\) and \(u_2 \notin \tilde{u}\). If \(\tilde{u} \leq u_1\), then \(\tilde{u} \leq \tilde{u}_1\). Indeed \(\tilde{u} < u_1\) implies \(\tilde{u} = K(\lambda,\tilde{u}) < K(\lambda,u_1) \leq u_1\). Thus \(\tilde{u}\) satisfies \(\tilde{u} \notin u_1\) and \(u_2 \notin \tilde{u}\). This completes the proof of the assertion (3) and of the Theorem 2.1.
3. AN EXAMPLE.

We consider the nonlinear eigenvalue problem:

\[
\begin{align*}
-\Delta u &= \lambda g(u,u) \quad \text{in } \Omega \\
0 &= u \quad \text{on } \Gamma = \partial \Omega
\end{align*}
\]

where \( \Omega \) is a bounded domain of \( \mathbb{R}^N \) with smooth boundary \( \Gamma \).

\( g : \bar{\Omega} \times \mathbb{R} \times \mathbb{R} \) is continuous and \( g_u \) exists and is continuous. A solution of \((P)\) is a part \((\lambda, u) \in \mathbb{R} \times W^{2,p}(\Omega)\) with \( p > N \) satisfying \((P)\). Let \( \bar{u} \) be a positive, superharmonic, bounded, lower-semicontinuous function on \( \Omega \) such that \( g(x, \bar{u}(x)) = 0 \) a.e. in \( \Omega \) and such that \( g(x, u) > 0 \) for \( 0 < u < \bar{u}(x) \), \( x \in \Omega \). It was shown in \([4]\), that if \( S \) denotes the set of solutions of \((P)\) in \( \mathbb{R}^+ \times W^{2,p} \) equipped with the \( \mathbb{R} \times C^1 \) topology, and if \( C \) is the component of \( S \) containing \((0,0)\), then \( C \) satisfies:

1. \((\lambda, u) \in C \setminus (0,0) \) implies \( u \) is positive, superharmonic and \( u(x) < \bar{u}(x) \), \( x \in \Omega \).

2. For every \( \lambda > 0 \), \( C \) has a minimal solution \( \bar{u}(\lambda) \).

3. \( \lim_{\lambda \to 0} \bar{u}(\lambda) - \bar{u} = 0 \) \( p < \infty \).

For \( \lambda > 0 \), we shall say that \( \bar{u}(\lambda) \) is the minimal (resp. maximal) solution of \((P)\) in \([0, \bar{u}]\) if \((\lambda, \bar{u}(\lambda))\) (resp. \((\lambda, \bar{u}(\lambda))\) is a solution of \((P)\) and for any solution \((\lambda, u)\) satisfying \( 0 < u(x) < \bar{u}(x) \), \( x \in \Omega \), \( u(x) > \bar{u}(\lambda)(x) \), \( x \in \Omega \) (resp. \( u(x) < \bar{u}(\lambda)(x) \), \( x \in \Omega \)).

The aim of this section is to prove the following:

**Theorem 3.3.** \( C \) above defined contains for each \( \lambda > 0 \) the minimal and the maximal solutions in \([0, \bar{u}]\).

**Proof.**

a) \( C \) contains the minimal solution \( \bar{u}(\lambda) \) for each \( \lambda > 0 \). Let \( \lambda > 0 \) and \( w(\lambda) > 0 \) be such that \((3.1)\) \( w(\lambda) + \lambda g_u(x,u) > 0 \) for \( \lambda \in [0, \lambda] \), and for \( 0 < u < \bar{u}(x) \), \( x \in \Omega \). Then we rewrite \((P)\) as

\[
\begin{align*}
-\Delta u + w(\lambda) u &= w(\lambda) u + \lambda g(u,u) \quad \text{in } \Omega \\
0 &= u \quad \text{on } \Gamma
\end{align*}
\]

\((P')\) is then equivalent with
u = K(\lambda, u)

u \in E := \{v \in C^1(\Omega) | v = 0 \text{ on } \Gamma\}
equipped with the \(C^1\) norm, and

(3.2) \quad K(\lambda, u)(x) := \int_\Omega G_w(x, y)(w(\lambda)u(y) + \lambda g(y, u(y)))dy

where \(G_w(\cdot, \cdot)\) denotes the Green function relative to \(-Au + w(\lambda)u\) on \(\Omega\) with Dirichlet boundary conditions.

Note that by (3.2) \(K\) is defined on \(R \times L^\infty(\Omega)\) and takes its values in \(E\). In \(E\) we introduce the cone

\[ P := \{u \in E | u(x) > 0, x \in \Omega\} \]
of positive solutions; it is standard that \(P\) has a nonempty interior \(\hat{P}\). Next we define

\[ \tilde{u}(\lambda) := K(\lambda, u). \]

Then \(\tilde{u}(\lambda) \in \hat{P}\) and \(\tilde{u}(\lambda) < \tilde{u}\) in \(\Omega\). By our choice of \(w(\lambda)\), this implies that

\[ K(\lambda, \tilde{u}(\lambda)) < \tilde{u}(\lambda). \]

Since \(K(\lambda, +)\) is increasing in \(u\), \(K(\lambda, +) : [0, \tilde{u}(\lambda)] + [0, \tilde{u}(\lambda)]\) and thus \((P)\) has a minimal solution \(\tilde{u}(\lambda)\). Note that \(\tilde{u}(\lambda) \in \hat{P}\). By our choice of \(w(\lambda)\), \(u + K(\lambda, u)\) is strongly increasing for \(\lambda \in (0, \tilde{\lambda})\), increasing for \(\lambda = 0\), and \(K(\lambda, 0) > 0\) for \(\lambda \in (0, \tilde{\lambda})\) and \(K(\lambda, \tilde{u}(\lambda)) < \tilde{u}(\lambda)\) for \(\lambda \in (0, \tilde{\lambda})\). Moreover \(K(\lambda, u)\) is strictly increasing in \(\lambda\) for each \(u \in [0, \tilde{u}(\lambda)]\). Thus, \(K : [0, \tilde{\lambda}] \times [0, \tilde{u}(\lambda)] + [0, \tilde{u}(\lambda)]\)

satisfies the assumptions of Theorem 2.1. There exists a connected set \(D\) of solutions in \(R \times C^1\) which meets \((0, 0)\) and \((\tilde{\lambda}, \tilde{u}(\lambda))\). Since \(D \subset C\), \(C\) contains \((\tilde{\lambda}, \tilde{u}(\lambda))\) and obviously, \(\tilde{u}(\lambda) = \tilde{u}(\tilde{\lambda})\).

b) \(C\) contains the maximal solution \(u(\lambda)\) in \([0, \tilde{u}]\) for each \(\lambda > 0\). Let \(\tilde{\lambda} > 0\) and \(w(\lambda) > 0\) be chosen as in a). We denote by \(S(\lambda) := \{u \in [0, \tilde{u}] | (\lambda, u)\) is a solution of \((P)\). \(K\) being defined as in a)\}, we know that \(u \in S(\lambda)\) implies \(u < K(\lambda, u) = \tilde{u}(\lambda)\), and \(u \geq 0\). Thus \(S(\lambda) \subset [0, \tilde{u}(\lambda)] = [0, K(\lambda, \tilde{u}(\lambda))]. \) Define \(\lambda_n := \tilde{\lambda} + n, n \in N\). Then, since

\[ \tilde{u}(\lambda_n) < \tilde{u}(\lambda_{n+1}), \quad n \in N \text{ (easily verified), we have } K(\lambda_n, \tilde{u}(\lambda_n)) < K(\lambda_{n+1}, \tilde{u}(\lambda_{n+1})). \]

Moreover \(K(\lambda_n, \tilde{u}(\lambda_n))\) follows from statement 3) before Theorem 3.1. We claim that
Indeed let \( v \in [0, \bar{K}(\lambda, \bar{u})] \). By definition, there is \( \beta > 0 \) such that \( K(\lambda, \bar{u}) - v > \beta e \) where \( e \) is an element of \( P \). Moreover

\[
K(\lambda, \bar{u}) = \lim_{n \to \infty} K(\lambda, \bar{u}(\lambda_n)) \quad \text{in } C^1
\]
implies the existence of a sequence \( \{\beta_n\} \) with

\[
\lim_{n \to \infty} \beta_n = 0
\]

such that

\[
K(\lambda, \bar{u}) - K(\lambda, \bar{u}(\lambda_n)) < \beta_n e, \quad n \in \mathbb{N}.
\]

Thus \( K(\lambda, \bar{u}(\lambda_n)) - v > (\alpha - \beta_n) e, \quad n \in \mathbb{N} \) and there are \( N \in \mathbb{N} \) and \( c > 0 \), such that

\[
K(\lambda, \bar{u}(\lambda_N)) - v > ce.
\]

Thus \( v \in [0, K(\lambda, \bar{u}(\lambda_N))] \subset \bigcup_{n=1}^m [0, K(\lambda, \bar{u}(\lambda_n))] \). Hence \( S^\lambda \subset \bigcup_{n=1}^m [0, K(\lambda, \bar{u}(\lambda_n))] \). Next we observe that \( S^\lambda \) is compact in \( C^1 \). Hence there is \( m \in \mathbb{N} \), such that

\[
S^\lambda \subset \bigcup_{n=1}^m [0, K(\lambda, \bar{u}(\lambda_n))] \subset [0, K(\lambda, \bar{u}(\lambda_m))].
\]

Note that \( \bar{v} := K(\lambda, \bar{u}(\lambda_m)) \) satisfies

\[
\begin{align*}
-\Delta \bar{v} + w(\lambda) \bar{v} &= w(\lambda) \bar{u}(\lambda_m) + \lambda g(\bar{u}, \bar{u}(\lambda_m)) \quad \text{in } \Omega \\
\bar{v} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

and \( \bar{u}(\lambda_m) \) satisfies

\[
\begin{align*}
-\Delta \bar{u}(\lambda_m) + w(\lambda) \bar{u}(\lambda_m) &= w(\lambda) \bar{u}(\lambda_m) + \lambda g(\bar{u}, \bar{u}(\lambda_m)) \quad \text{in } \Omega \\
\bar{u}(\lambda_m) &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

Since \( \lambda_m > \lambda \) and \( g(\bar{u}, \bar{u}(\lambda_m)) > 0 \), we have

\[
\begin{align*}
-\Delta (\bar{v} - \bar{u}(\lambda_m)) + w(\lambda)(\bar{v} - \bar{u}(\lambda_m)) &< 0 \quad \text{in } \Omega \\
\bar{v} - \bar{u}(\lambda_m) &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

Thus \( \bar{v} < \bar{u}(\lambda_m) \) and

\[
S^\lambda \subset [0, \bar{u}(\lambda_m)].
\]

Next, choosing \( w(\lambda_m) > 0 \) such that

\[
w(\lambda_m) + \lambda g(\bar{u}, \bar{u}(\lambda_m)) > 0 \quad \text{for } \lambda \in [0, \lambda_m],
\]

\( 0 < \bar{u} < \bar{u}(x), \quad x \in \Omega \), one defines \( K \) as in a) and verifies that with this choice of
w, $K$ satisfies the assumptions of Theorem 2.1 on $[0, \lambda_m] \times [0, \tilde{u}(\lambda_m)]$. Then, there is a connected set $D$ of solutions of (P) in $\mathbb{R} \times C^1$ which contains the maximal solution $\tilde{u}(\lambda)$ in $[0, \tilde{u}(\lambda_m)]$. But since $S_{\lambda} \subset [0, \tilde{u}(\lambda_m)]$, $\tilde{u}(\lambda)$ is the maximal solution in $[0, \tilde{u}]$. Since $D \subset C$, $C$ contains the maximal solution $\tilde{u}(\lambda)$ of (P) in $[0, \tilde{u}]$. This completes the proof of Theorem 3.1.

Remark 1. It is also a consequence of the proof that if we denote by $C_{\lambda}$ the component of solutions of (P) in $[0, \lambda) \times C^1$ which contains $(0,0)$, then $C = \bigcup_{\lambda \in \mathcal{A}} C_{\lambda}$. 

Remark 2. In the "bifurcation case", i.e., when $g$ satisfies $g(x,0) = 0$, $x \in \Omega$ but $g_u(x,0) > 0$, $x \in \Omega$, then a similar analysis shows that $C = \bigcup_{\lambda > \lambda_1} C_{\lambda}$ when $C$ is the component of positive solutions "emanating" from $(\lambda_1,0)$, bifurcation point. Then for $\lambda > \lambda_1$, $C$ contains all maximal solutions in $[0, \tilde{u}]$. Note that in this case the minimal solution in $[0, \tilde{u}]$ is $0$, but it is shown in [3], [4], that $C$ possesses a minimal solution for each $\lambda > \lambda_1$. 

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REFERENCES

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In this paper, we consider the following abstract problem. Let \((E, P)\) be an ordered Banach space with cone \(P\) having a nonempty interior \(\mathring{P}\). Let \(\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 < \lambda_2, a, b \in P,\) such that \(b - a \in \mathring{P}\). Let the operator \(K : [\lambda_1, \lambda_2] \times [a, b] \to [a, b]\) be compact, strongly increasing with respect to the second variable for fixed \(\lambda \in (\lambda_1, \lambda_2)\), strictly increasing with respect (continued)
to the first variable for fixed $u \in [a, b]$. Moreover, assume that $a$ is the only fixed point of $K(\lambda^1, \cdot)$ and that $b$ is the only fixed point of $K(\lambda^2, \cdot)$. Consider the equation

\begin{equation}
(*) \quad u = K(\lambda, u) .
\end{equation}

Under the above assumptions, we prove that any closed connected subset of solutions of (*) in $[\lambda^1, \lambda^2] \times [a, b]$ which meets $(\lambda^1, a)$ and $(\lambda^2, b)$, contains the maximal and the minimal solutions of (*), which are obtained by monotone iterations. Such a subset of solutions is shown to exist. Applications to a semilinear elliptic eigenvalue problem are studied.