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THERMOELASTIC RESPONSE TO A SHORT LASER PULSE

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ABSTRACT

We consider a one-dimensional model of the thermoelastic response to laser heating by a pulse of very short duration. We treat the linear equations of thermoelasticity by perturbation methods using the pulse duration as the perturbation parameter. The perturbation analysis separates the problem into two time regimes, the short time scale, which is of the same order as the pulse duration, and the long time scale, which is much longer than the pulse duration. Convenient analytical expressions are obtained for the temperature change and displacement in both time regimes.

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SIGNIFICANCE AND EXPLANATION

In this paper we study the effects caused by heating a material with a single short laser pulse. We consider only a one-dimensional model but consider both temperature changes and elastic changes, which are coupled together. By use of perturbation analysis we can separate the problem into two time regimes, the short time scale, which is on the order of the pulse duration, and the long time scale, which is much longer than the pulse duration. Convenient expressions are given for evaluating the temperature changes and displacement.
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1. INTRODUCTION.

In this paper we study the temperature change and displacements induced in a solid material by a short laser pulse. The determination of the thermoelastic response of solids to laser radiation is an important problem for many applications, especially as laser pulses become increasingly powerful and of shorter duration.

We apply perturbation techniques to the equations of linear thermoelasticity using the pulse duration as the perturbation parameter. This approach has the advantage of yielding relatively simple expressions for the temperature change and displacement induced by a laser pulse of short duration.

In other work on this problem Bechtel (1975) and Ready (1965) considered the heating caused by laser pulses but did not consider displacement effects. Dunbar (1981) considered the effects of temperature on displacements but neglected the coupling effect of displacements on temperature. He also included the modified Fourier law of heat conduction. (Dunbar's results are in error due to an inappropriate application of his initial conditions.) All of the above used a Green's function approach to obtain explicit, though complicated, formulas for the temperature field. The perturbation method used here has the advantage of giving simpler expressions for the temperature and displacements. Moreover, we treat the full equations of linear thermoelasticity in one-dimension with the modified Fourier law. The method is limited by the requirement that the pulse duration be short. The method can be extended to two and three dimensional problems and to include nonlinear effects, however, the resulting equations will then have to be solved by numerical methods.

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Nayfeh (1977) has treated a problem similar to the one considered in this paper by the method of perturbation expansions. He considered an instantaneous heat source distributed on the surface of the material. We compare our results with Nayfeh's in Section 5.

The equations describing the thermoelastic response in one dimension to laser heating are

\begin{align}
(1.1) & \quad \left(1 + \frac{\tau_0}{\varepsilon}\right)\left(p c_v \frac{\partial \bar{T}}{\partial t} + (3\lambda + 2\mu) \frac{\gamma T_0}{\partial x} \frac{\partial^2 \bar{u}}{\partial x^2} - \kappa \frac{\partial^2 \bar{u}}{\partial x^2} \right) = \left(1 + \frac{\tau_0}{\varepsilon}\right)A(\varepsilon/b)e^{-\alpha x} \\
(1.2) & \quad \rho \frac{\partial^2 \bar{u}}{\partial t^2} = (\lambda + 2\mu) \frac{\partial^2 \bar{u}}{\partial x^2} + (3\lambda + 2\mu) \gamma \frac{\partial^2 \bar{T}}{\partial x^2} = 0
\end{align}

where \( \bar{T} \) is the temperature change measured from the mean state \( \bar{T}_0 \) and \( \bar{u} \) is the displacement. The solid material occupies the region with \( \bar{x} > 0 \). The normal stress is given by

\begin{equation}
\bar{\sigma} = (\lambda + 2\mu) \frac{\partial \bar{u}}{\partial x} - (3\lambda + 2\mu) \gamma \bar{T}.
\end{equation}

These equations are those derived by Lord and Shulman (1967) with the addition of the term involving \( A \) which represents the laser heating. (We have ignored the speed of propagation of the laser pulse which is sufficiently fast to be regarded as infinite.) The parameter \( b \) is a measure of the pulse width. The exponential damping of the laser heating in space is the result of the attenuation of electromagnetic radiation in a conductor. The parameter \( \tau_0 \) is the relaxation time for the modified Fourier law for which the heat flux \( \bar{Q} \) satisfies

\begin{equation}
\bar{Q} + \frac{\tau_0}{\varepsilon} \frac{\partial \bar{T}}{\partial t} = -\kappa \text{grad} \bar{T}
\end{equation}

where \( \kappa \) is the thermal conductivity of the material. The other parameters in equations (1.1), (1.2) and (1.3) are the Lamé elastic constants \( \lambda \) and \( \mu \), the density \( \rho \), the specific heat at constant volume \( c_v \), and the coefficient of thermal expansion \( \gamma \). The bars on the several variables indicate that these are dimensional quantities.
We now non-dimensionalize the equations. The natural length scale for this problem is the attenuation length $a^{-1}$, so define $x = \frac{x}{x_{ref}}$ where $x_{ref}$ is $a^{-1}$. There are several possible time scales and we choose the time scale of heat conduction as the reference time,

$$t_{ref} = \frac{\rho c}{x_{ref}^2}.$$

The perturbation parameter is to be $\beta$ which is $b/t_{ref}$, the non-dimensional pulse width. The relaxation time is non-dimensionalized as

$$\tau = \frac{\tau_0}{(t_{ref} \beta)} = \frac{\tau_0}{b}.$$

The pulse magnitude is normalized to have a non-dimensional integral equal to unity.

Therefore we define $\lambda_{ref}$ by

$$\int_{-\infty}^{\infty} \lambda(x/b)dx = \lambda_{ref}.$$

then

$$\lambda(x/b) = \lambda_{ref} \frac{1}{\beta} A(t/\beta)$$

and thus

$$\int_{-\infty}^{\infty} \lambda(s)ds = 1.$$

The reference temperature and displacement are chosen as

$$T_{ref} = \lambda_{ref}(x_{ref})^2/K$$

and

$$u_{ref} = \beta^2 \gamma(3\lambda + 2\mu)t_{ref}^2/\lambda_{ref}^2.$$

The natural reference velocity is

$$V_{ref} = \lambda_{ref}(x_{ref})^2/\lambda_{ref}.$$

We thus obtain for the non-dimensionalized equations

$$(1.5) \quad [1 + \beta t \frac{3}{2t} + \frac{3}{2} \frac{\partial u}{\partial x}] - \frac{3 \lambda}{2} = \frac{1}{\beta} \frac{A(t/\beta)}{e^{-X}}$$

$$(1.6) \quad \beta^2 \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial \eta}{\partial x} = 0.$$
where
\[ c = \left( \frac{\lambda + 2\mu}{\rho} \right)^{1/2} \text{vel}_{ref} \]
and
\[ d = \left( \frac{3\lambda + 2\mu}{\rho} \right) \left( \frac{c_v}{c_v^*} \right)^{1/2} \text{vel}_{ref} \]

The non-dimensional stress is
\[ \Sigma = \frac{\delta u}{\delta x} - \frac{1}{c^2} T. \]  

For many common materials the non-dimensional parameters \( c \) and \( d \) are of order zero in \( \beta \), that is,
\[ \beta \ll c, \quad d \ll \beta^{-1}. \]

The relaxation parameter \( \tau \) plays an unimportant role in the analysis and satisfies
\[ 0 < \beta \tau \ll 1. \]

The justification of our treatment of these equations and the non-dimensionalization we have used rests on the above observations and on the reasonableness of the results.

The boundary conditions for equations (1.6) and (1.7) are the conditions of no radiative heat loss
\[ \frac{\partial T}{\partial x} = 0 \quad \text{at} \quad x = 0, \]
and no stress at the wall
\[ \frac{\delta u}{\delta x} - \frac{1}{c^2} T = 0 \quad \text{at} \quad x = 0. \]

The initial conditions are that \( T(t,x) \) and \( u(t,x) \) and all their derivatives vanish as \( t \) approaches negative infinity.

We will solve equations (1.6)-(1.11) using perturbation expansions with \( \beta \) as the perturbation parameter. Since \( \beta \) is on the order of \( 10^{-10} \) for many problems, excellent results can be obtained with only one term.

The paper is organized in the following manner. In section 2 we derive the perturbation expansion for the short time scale, that is times on the order of the pulse width, and in section 3 we consider the long time scale behavior. In section 4 we solve for the boundary layer correction at the material surface which is needed to complete the
solution considered in section 2. In section 5 we consider some explicit examples and compare our results with those of Bechtel (1975), Ready (1965), and Nayfeh (1977).
2. THE SOLUTION ON THE SHORT TIME SCALE

In this section we consider the solution of equations (1.6) and (1.7) for times on the order of $\beta$, the pulse width. We rescale the time variable using

$$t_1 = t/\beta$$

obtaining the equations

$$\left( \frac{\partial}{\partial t_1} + 1 \right) \left( \frac{\partial^2}{\partial t_1^2} - \lambda(t_1)e^{-x} + d^2 \frac{\partial^2 u_1}{\partial x^2} \right) = \beta \frac{\partial^2 \tau_1}{\partial x^2}$$

(2.2)

$$\frac{\partial^2 u_1}{\partial t_1^2} - c^2 \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial \tau_1}{\partial x} = 0$$

(2.3)

with

$$\frac{\partial \tau_1}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u_1}{\partial x} - c^2 \tau_1 = 0 \quad \text{at} \quad x = 0$$

(2.4)

and

$$\frac{\partial^k \tau_1}{\partial t_1^k}, \frac{\partial^k u_1}{\partial t_1^k} = 0 \quad \text{as} \quad t \to \infty \quad \text{for} \quad k > 0.$$ $$(2.5)$$

We seek solutions to these equations in the form of power series in the perturbation parameter $\beta$ i.e.

$$T = T^0(t_1,x) + \beta T^1(t_1,x) + \beta^2 T^2(t_1,x) + \ldots$$

$$u = u^0(t_1,x) + \beta u^1(t_1,x) + \beta^2 u^2(t_1,x) + \ldots$$

Substituting these expansions into equations (2.2) and (2.3) and equating like powers of $\beta$ gives the following system of equations

$$\left( \frac{\partial}{\partial t_1} + 1 \right) \left( \frac{\partial^2}{\partial t_1^2} - \lambda(t_1)e^{-x} + d^2 \frac{\partial^2 u_1}{\partial x^2} \right) = 0$$

(2.6)

$$\frac{\partial^2 u_1}{\partial t_1^2} - c^2 \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial \tau_1}{\partial x} = 0, \quad \text{for} \quad k > 0$$

(2.7)

and
(2.6) \[
- \frac{y_x^2 y}{y_t^4} + 1 \left( \frac{y_x^2 y}{y_t^4} + \frac{y^2 u_x}{y_t^4} \right) = \frac{y^2}{y_t^4} \cdot k > 0
\]

Equation (2.6) can be integrated to

\[
\frac{y_t^2}{y_x} - \Lambda(t_1) e^{-x} + \frac{y^2}{y_t^4} \frac{y_t^2}{y_t^2} = C(x) e^{-t_1/\tau}
\]

for some function \( C(x) \). The initial conditions (2.6) imply that \( C \) is identically zero.

Another integration in \( t \) and using (2.6) again gives

(2.9) \[
\frac{y^0}{y_t^4}(t_1, x) = e^{-x} \int_0^{t_1} \Lambda(s) ds - c^2 \int_{-\infty}^{t_1} (t_1, x).
\]

Differentiating (2.9) with respect to \( x \) and substituting in (2.7) for \( k = 0 \) gives

(2.10) \[
\frac{y^0}{y_t^4} - (c^2 + d^2) \frac{y^0}{y_t^4} = e^{-x} \int_0^{t_1} \Lambda(s) ds
\]

with

(2.11) \[
\frac{y^0}{y_t^4} = \frac{1}{(c^2 + d^2)} \int_0^{t_1} \Lambda(s) ds \text{ at } x = 0.
\]

The solution to (2.10) and (2.11) satisfying the initial condition (2.5) is

(2.12) \[
\frac{y^0}{y_t^4}(t_1, x) = e^{-x} \int_0^{t_1} \frac{1}{a} \left( \cosh a(t_1 - s) - 1 \right) \Lambda(s) ds
\]

\[
\int_0^{t_1} \frac{1}{a} \sinh a(t_1 - s) \Lambda(s - x/a) ds
\]

where \( a = (c^2 + d^2)^{1/2} \). From (2.9) we obtain

(2.13) \[
\frac{y^0}{y_t^4}(t_1, x) = c^2 / a^2 e^{-x} \int_0^{t_1} \Lambda(s) ds + \frac{2}{a^2} \int_0^{t_1} \cos \left( \frac{t_1 - s}{a} \right) \Lambda(s - x/a) ds.
\]
Formulas (2.12) and (2.13) can be rewritten as

\[ u_1^0(t_1, x) = -a^2 e^{-x} \int_{-\infty}^{t_1} A(s) ds + a^2 e^{-x} \int_{-\infty}^{-st_1} e^{as} A(s) ds \]

\[ + a^2 e^{-x} \int_{t_1-x/a}^{t_1} \cosh \alpha(t_1-s) \Lambda(s) ds \]

and

\[ T_0^0(t_1, x) = \frac{c^2}{a^2} e^{-x} \int_{-\infty}^{t_1} A(s) ds - \frac{d^2}{a^2} e^{-x} \int_{-\infty}^{-at_1} e^{as} A(s) ds \]

\[ + \frac{d^2}{a^2} e^{-x} \int_{t_1-x/a}^{t_1} \cosh \alpha(t_1-s) \Lambda(s) ds \]

Since \( A(t_1) \to 0 \) as \( t \to +\infty \) we obtain from (2.12') and (2.13')

\[ \lim_{t_1 \to +\infty} u_1^0(t_1, x) = -a^2 e^{-x} \int_{-\infty}^{t_1} A(s) ds = -a^2 e^{-x} \]

by (1.5) and

\[ \lim_{t_1 \to +\infty} T_0^0(t_1, x) = \frac{c^2}{a^2} e^{-x} - \frac{c^2}{c^2 + d^2} e^{-x} \]

We also obtain from the last two terms in (2.12') and (2.13') the limiting form of the compression wave induced by the laser heating. We have for large \( t_1 \) and \( x \)

\[ u_1^0(t_1, x) \sim \frac{1}{2a^2} f(t_1 - x/a) \]

and

\[ T_0^0(t_1, x) \sim \frac{d^2}{2a^2} f'(t_1 - x/a) \]
where
\[ \phi(r) = e^{-ar} \int_{-\infty}^{r} e^{as} \Lambda(s) ds + e^{ar} \int_{r}^{\infty} e^{-as} \Lambda(s) ds , \]
\[ = \int_{-\infty}^{r} e^{-a|r-s|} \Lambda(s) ds . \]

Note that the temperature change associated with the compression wave is first positive then negative as the wave passes any point.

The functions $T_k^j$ do not satisfy the boundary condition (1.10). A boundary-layer expansion is needed to satisfy this boundary condition and this expansion is the topic of section 4.
3. THE SOLUTION ON THE LONG-TIME SCALE

In this section we consider the thermoelastic response on the long time scale, that is for times much larger than the laser pulse width. The equations to be solved are (1.6) and (1.7) which we rewrite as

\[ \frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial T}{\partial t} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^3 u}{\partial x^3} \right) = \frac{1}{\beta} \left( A(t/\beta) + \tau A'(t/\beta) \right) e^{-\alpha t}, \]

\[ -c^2 \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + c^2 \frac{\partial^2 u}{\partial t^2} = 0, \]

with the initial conditions that \( T \) and \( u \) vanish as \( t \) approaches negative infinity, and the boundary conditions (1.10) and (1.11). We will obtain the solution as a perturbation expansion in the variable \( \beta \). However, since the right-hand side of (3.1) has a singularity at \( \beta = 0 \), the terms of the perturbation expansion will be distributions and we must consider the weak form of (3.1) with regard to derivatives in \( t \).

First, let us consider equation (3.2). Writing

\[ T(t,x) \sim \sum_{j=0}^{\infty} \rho^j T^j(t,x) \]

and

\[ u(t,x) \sim \sum_{j=0}^{\infty} \rho^j u^j(t,x) \]

we obtain

\[ -c^2 \frac{\partial^2 u^j}{\partial x^2} + \frac{\partial u^j}{\partial x} = -c^2 \frac{\partial^{j-2} u^j}{\partial t^2} \quad j = 0, 1, \ldots, \]

where \( u^{-2} = u^{-1} = 0 \).

When \( j \) is 0 and 1, we have, by integrating,

\[ -c^2 \frac{\partial \psi^0}{\partial x} + \psi^0 = 0 \]

(3.4)

and

\[ -c^2 \frac{\partial \psi^1}{\partial x} + \psi^1 = 0 \]
which satisfy both the no stress boundary condition (1.11) at \( x = 0 \), and the decay of \( u \) and \( T \) as \( x \to \pm \infty \). Thus to within \( O(\delta^2) \) there is no stress on the long time scale.

The weak form of (3.1) which we consider is

\[
\int_{-\infty}^{\infty} (-\varphi'(t)T - \varphi(t) \frac{dx}{\delta x} - d^2 \varphi'(t) \frac{\partial u}{\partial x} + \beta T \varphi'(t)(T + d^2 \varphi'(t))dt
\]

\[
= \int_{-\infty}^{\infty} (\varphi(t) - \beta \varphi(t))A(t/\beta)dt e^{-\delta x}
\]

for all \( \varphi \in C_0^\infty(\mathbb{R}) \). To continue we must expand the right hand side of (3.5) as a series in \( \delta \). Making the change of variable \( t = \beta \tau \) we have

\[
\int_{-\infty}^{\infty} (\varphi(t) - \beta \varphi(t))A(t/\beta)dt
\]

\[
= \int_{-\infty}^{\infty} (\varphi(\beta \tau) - \beta \varphi'(\beta \tau))A(s)ds .
\]

Now for any positive integer \( N \) we will obtain an expansion of the above integral with error \( O(\delta^N) \). First choose \( R \) so that

\[
| \int_{|s|>R} \beta^k A(s)ds | < \delta^{N-k} \text{ for } k = 0, ..., N .
\]

Then, by the Taylor series expansion of \( \varphi(\beta \tau) \),

\[
\int_{-\infty}^{\infty} \varphi(\beta \tau)A(s)ds = \int_{-R}^{R} \varphi(\beta \tau)A(s)ds + \int_{|s|>R} \varphi(\beta \tau)A(s)ds
\]

\[
= \sum_{j=0}^{N-1} \int_{-R}^{R} \frac{1}{j!} \varphi^{(j)}(0)(\beta \tau)^j A(s)ds
\]

\[
+ \delta^N \int_{-R}^{R} \frac{A(s)}{(N-1)!} \int_{0}^{s} \varphi^{(N)}(\beta \tau')(s - \tau')^{N-1}d\tau' ds + \int_{|s|>R} \varphi(\beta \tau)A(s)ds
\]

\[
= \sum_{j=0}^{N-1} \frac{\beta^j}{j!} \varphi^{(j)}(0) \int_{-R}^{R} s^j A(s)ds + O(\delta^N) \psi^{(N)} \delta \psi
\]

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where \( \| \phi \|_{w, M} = \sup_{s \in J} \| \phi^{(3)}(s) \|. \) For convenience define

\[
(3.6) \quad a_j = \frac{1}{j!} \int_{-\infty}^{\infty} s^j \phi(s) ds \quad j = 0, 1, \ldots
\]

with \( a_{-1} = 0. \) By the normalization of \( A(t) \) we have that \( a_0 \) is 1.

Therefore,

\[
\frac{1}{B} \int_{-\infty}^{\infty} \phi(t) a\left(\frac{s}{B}\right) dt = \sum_{j=0}^{N-1} B^j a_j \phi^{(j)}(0) + O(B^N) \| \phi \|_{w, M}
\]

and similarly

\[
\int_{-\infty}^{\infty} \phi'(t) A(t/B) dt = \sum_{j=1}^{N-1} B^j a_{j-1} \phi^{(j)}(0) + O(B^N) \| \phi \|_{w, M}
\]

Thus the right-hand side of (3.5) is

\[
\sum_{j=0}^{N-1} B^j (a_j - \tau a_{j-1}) \phi^{(j)}(0) + O(B^N) \| \phi \|_{w, M}.
\]

Now expanding the left-hand side of (3.5) in powers of \( \beta \) we have for \( j = 0 \)

\[
\int_{-\infty}^{\infty} \phi'(t)^2 - \frac{2\phi'^2}{2x^2} - \frac{d^2 \phi'}{dx^2} \int_{-\infty}^{\infty} \phi(t) e^{-x} dt = \phi(0)e^{-x}
\]

and using (3.4) we obtain

\[
\int_{-\infty}^{\infty} \phi'(1 + \frac{d^2}{c^2}) T^0 - \frac{2\phi'(0)}{2x} dt = \phi(0)e^{-x}
\]

Since the right-hand side of this equation depends on \( \phi(t) \) only at \( t = 0 \) we have that \( T^0 \) is differentiable for all \( t \) other than \( t = 0. \) Integrating by parts then gives us

\[
\int_{-\infty}^{\infty} \phi(t) \left[ \left(1 + \frac{d^2}{c^2} \right) \frac{\phi'(0)}{2x} - \frac{2\phi'(0)}{2x^2} \right] dt + \phi(0) \left[ T^0(0+, x) - T^0(0-, x) \right] = \phi(0)e^{-x}.
\]

Thus \( T^0 \) is a solution to the heat equation for \( t > 0 \) and \( t < 0, \) with the homogeneous boundary condition (1.10). Since \( T^0 \) vanishes as \( t \to -\infty \) the uniqueness of the initial boundary value problem implies that \( T^0 \) is identically zero for \( t < 0. \) Therefore for \( t > 0, \) \( T^0 \) satisfies

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\[ \frac{\partial T^0}{\partial t} - \frac{k}{2\pi} \frac{\partial^2 T^0}{\partial x^2} = 0 \]

with \( T^0(0,x) = ke^{-x} \) and \( \frac{\partial T^0}{\partial x}(t,0) = 0 \) where \( k = c^2/(c^2 + d^2) \).

Note that the initial condition for \( T^0 \) is the same as the limit of \( T_1 \) as \( t \) becomes infinite for the short scale (2.15). Similarly for \( u^0 \), from (3.4) and the initial value for \( T^0 \) we have

\[ u^0(0,x) = -\frac{1}{c^2 + d^2} e^{-x}, \]

which agrees with (2.14). The explicit solution to (3.7) is

\[ T^0(t,x) = \sqrt{\frac{k}{4\pi t}} \int_0^\infty \left( e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt} \right) e^{-y} dy \]

\[ = \frac{k}{2} e^{kt} (1 - \text{erf}(\sqrt{k}t - x/) + e^{x}(1 - \text{erf}(\sqrt{k}t + x/\sqrt{4kt})) , \]

and so

\[ u^0(t,x) = \frac{1}{c^2 + d^2} \left( -1 + \frac{1}{2} \int_0^\infty \text{erf}((x - y)/\sqrt{4kt}) + \text{erf}((x + y)/\sqrt{4kt}) e^{-y} dy \right) . \]

From (3.4) it is seen that \( u^0 \) is determined only up to a function of \( t \). In deriving (3.10) we have used the condition that \( u^0 \) vanishes as \( x \) becomes larger. From (3.10) we see that

\[ u^0(t,x) = \frac{1}{c^2 + d^2} \text{ as } t = \infty. \]

This non-zero limit for the displacement is the result of neglecting any restraining forces on the material which would restore it to its original position. The equation for \( T^1 \), using (3.4), is

\[ \int_0^\infty \left( -\phi' \frac{\partial T^1}{\partial t} - \frac{k}{2\pi} \frac{\partial^2 T^1}{\partial x^2} + \phi'' \xi^2 \right) dt = \phi'(0)k(a + b) e^{-x}. \]

As before \( T^1 \) is zero for \( t < 0 \). \( T^1(t,x) \) can be written as the sum of a distribution and a smooth function \( T^1(t,x) \). We have

\[ T^1(t,x) = -ka_1 \delta(t)e^{-x} + T^1(t,x) \]

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where $\delta(t)$ is the usual Dirac delta function and $\dot{T}(t,x)$ satisfies

$$\frac{\partial^2 \dot{T}}{\partial t^2} - k \frac{\partial^2 \dot{T}}{\partial x^2} + \tau \frac{\partial \dot{T}}{\partial t} = 0$$

with $\dot{T}(0,x) = -k(a, k + \tau)e^{-\lambda x}$ and $\frac{\partial \dot{T}}{\partial t}(t,0) = 0$. The equation for $u^1(t,x)$ from (3.4) is

$$u^1(t,x) = \frac{1}{c^2 + d^2} k a \delta(t)e^{-\lambda x} - \frac{1}{c^2} \int_0^t T^1(t,y)dy.$$ 

In a similar way one may derive expressions for $T^2$ and $u^2$, however, for most applications the first two terms should suffice.
4. THE BOUNDARY CORRECTION ON THE SHORT TIME SCALE.

In this section we calculate the correction terms which are needed to satisfy the radiation boundary condition (1.10) for the short time scale. As in section 2 the time variable is \( t_1 \), i.e. \( t/\theta \), and we write the temperature function as

\[
T(t_1, x, \theta) = T_1(t_1, x, \theta) + \theta^{1/2} T_1(t_1, x \theta^{-1/2}, \theta).
\]

Setting \( x_1 = x \theta^{-1/2} \), we have by (1.10)

\[
\frac{\partial T_1}{\partial x_1}(t_1, 0, \theta) + \frac{\partial T_1}{\partial x_1}(t_1, 0, \theta) = 0.
\]

\( T_1 \) satisfies the same initial conditions as \( T \), that is \( T_1 \) and all its derivatives vanish as \( t_1 \) tends to negative infinity.

To obtain meaningful equations for \( T_1 \) and \( u_1 \) we set

\[
u(t_1, x, \theta) = u_1(t_1, x, \theta) + \theta u_1(t_1, x, \theta)
\]

and, from (2.2) and (2.3), the equations for \( T_1 \) and \( u_1 \) are

\[
\frac{\partial^2 T_1}{\partial t_1^2} + \frac{\partial T_1}{\partial t_1} + \frac{\partial^2 u_1}{\partial t_1 \partial x_1} - \frac{\partial^2 T_1}{\partial x_1^2} = 0
\]

\[
\frac{\partial^2 u_1}{\partial t_1^2} - \theta^2 \frac{\partial^2 u_1}{\partial t_1 \partial x_1} + \frac{\partial u_1}{\partial x_1} = 0.
\]

The system (4.4) and (4.5) with the boundary condition (4.2) and the initial conditions determine \( T_1 \) and \( u_1 \). We again solve these equations using perturbation expansions in \( \theta \). We write

\[
T_1(t_1, x_1, \theta) \sim \sum_{k=0}^\infty \theta^k T_1^k(t_1, x_1)
\]

\[
u_1(t_1, x_1, \theta) \sim \sum_{k=0}^\infty \theta^k u_1^k(t_1, x_1).
\]
The equations for \( \tau_0 \) and \( u_0 \) are

\[
\tau \frac{\partial \tau_0}{\partial t_1} + \frac{\partial^2 \tau_0}{\partial x_1^2} + \frac{2}{c^2} \frac{\partial u_1}{\partial t_1} + \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_1^2} = 0
\]

and

\[
- c^2 \frac{\partial^2 u_0}{\partial x_1^2} + \frac{\partial \tau_0}{\partial x_1} = 0.
\]

Integrating this last equation, we have

\[
- c^2 \frac{\partial u_0}{\partial x_1} + \tau_0 = 0,
\]

which shows that there is no correction to the stress to the first order in \( \delta \). We then have

\[
\frac{\partial^2 \tau_0}{\partial t_1^2} + \frac{\partial \tau_0}{\partial t_1} - k \frac{\partial^2 \tau_0}{\partial x_1^2} = 0
\]

where, as in section 3, \( k = c^2 / (c^2 + d^2) \). The boundary condition for \( \tau_0 \) is, from (4.2) and (2.13),

\[
\tau_0 (t_1, 0) = k \int_{-\infty}^{t_1} A(t) + (1 - k) e^{-st_1} \int_{-\infty}^{t_1} e^{-st} A(s) ds = \eta_0 (t_1).
\]

The solution to (4.7) and (4.8) is

\[
\tau_0 (t_1, x_1) = -c_1 \int_{-\infty}^{t_1} e^{-s} I_0 \left( \frac{1}{2 \sqrt{t_1}} \left( (t_1 - s)^2 - \frac{x_1^2}{c^2} \right)^{1/2} \right) \eta_0 (s) ds.
\]

where \( c_1 \) is \( \sqrt{k/c} \) and \( I_0 \) is the modified Bessel function of order zero. The formula for the displacement correction is

\[
u_0 (t_1, x_1) = \frac{c_1}{c} \int_{-\infty}^{t_1} e^{-s} U(t_1 - s, x_1/c) \eta_0 (s) ds.
\]

where
For the case of $t = 0$, these formulas are

\[ u_0(x, t, \xi) = \int_0^\infty I_0 \left( \frac{1}{2t} (s^2 - \pi^2)^{1/2} \right) ds. \]

and

\[
T_0^0(t_1, x_1) = -\int_{-\infty}^{t_1} \frac{S}{\sqrt{k(t_1 - s)}} b^0(s) ds
\]

and

\[
u_1^0(t_1, x_1) = \frac{1}{2a^2} \int_{-\infty}^{t_1} \left( 1 - \text{erf}(x_1 / \sqrt{4k(t_1 - s)}) \right) b^0(s) ds.
\]

Note that the effect of the correction is to decrease the temperature at the surface by $O(\beta^{1/2})$ and to decrease the magnitude of the displacement by $O(\beta)$. This is as expected since the correction accounts for the effects of radiation.
5. EXAMPLES AND CONCLUSIONS

We now present some explicit results of the preceding analysis. We consider two pulse shapes, the first pulse has the shape of a Gaussian

\[ A(s) = \frac{1}{\sqrt{2\pi}} e^{-s^2} \]  \hspace{1cm} (5.1)

and the second has an exponential decay

\[ A(s) = \begin{cases} 
0 & s < 0, \\
se^{-s} & s > 0.
\end{cases} \]  \hspace{1cm} (5.2)

The analysis of sections 2 and 3 shows that one may take \( A(s) \) to be any function with unit integral satisfying

1) \[ \int_{-\infty}^{\infty} s^k A(s) ds < \infty \quad k = 0, 1, \ldots \]

2) \[ e^{at} \int_{-\infty}^{t} e^{-as} A(s) ds \to 0 \quad \text{as} \quad t \to \infty, \]

where \( a = (c^2 + d^2)^{1/2} \).

Graphs of \( T^0_1 \) and \( u^0_1 \), the short time scale temperature change and displacement, are shown in figures 1 to 5 for several values of \( x \) as functions of \( t_1 \). The values of \( T^0_1 \) and \( u^0_1 \) are computed using equations (2.12) and (2.13) for \( x = 0, 1, 2, \ldots \). The values of \( t_1 \) run up to 10.0 in each figure.

Figures 1a and 1b show the temperature change and displacement for the Gaussian pulse (5.1) for values of \( c = 1.0 \) and \( d = 1.0 \). Figures 2a and 2b show similar results for the exponential pulse (5.2) with \( c = 1.0 \) and \( d = 1.0 \). Comparing figures 1 and 2 one can see the effect on the temperature and displacement of the different rates of energy deposition due to the different pulse shapes. The exponential pulse (5.2) deposits energy more slowly than does the Gaussian pulse (5.1) and therefore the temperature and displacement approach their limiting values more slowly. Figures 3a and 3b show results for the Gaussian pulse (5.1) for \( c = 0.7 \) and \( d = 1.0 \). Figures 4a and 4b display the temperature change and
Figure 1. The temperature change (a) and displacement (b) on the short time scale for a Gaussian pulse for $x=0(1)5$ and $t_1$ less than 10, $c=1.0$ and $d=1.0$. 
Figure 2. The temperature change (a) and displacement (b) on the short time scale for an exponential pulse for $x=0(1.5)$ and $t_1$ less than 10, $c=1.0$ and $d=1.0$. 
Figure 3. The temperature change (a) and displacement (b) on the short time scale for a Gaussian pulse for $x=0(1)5$ and $t_1$ less than 10, $c=0.7$ and $d=1.0$. 
Figure 4. The temperature change (a) and displacement (b) on the short time scale for a Gaussian pulse for $x = 0(1)5$ and $t_1$ less than 10, $c = 0.7$ and $d = 0.7$. 
Figure 5. The temperature change on the short time scale for a Gaussian pulse for $x=0(1)5$ and $t_1$ less than 10, $c=1.0$ and $d=0.7$.

Figure 6. The temperature change on the long time scale for $x=0(1)5$ and $t$ less than 40, $c=1.0$ and $d=1.0$. 
displacement for the Gaussian pulse (5.1) with $c = 0.7$ and $d = 0.7$. Figure 5 shows the
temperature change for the Gaussian pulse (5.1) with $c = 1.0$ and $d = 0.7$. The
displacement corresponding to figure 5 is the same as figure 3b. In each case the limit of
the temperature for large $t$, is $c^2(c^2 + d^2)^{-1}e^{-x}$ and the limit of the displacement is
$-(c^2 + d^2)^{-1}e^{-x}$.

One clearly sees in these figures the beginning of the compression wave given by
(2.16) and (2.17). The negative temperatures associated with the compression wave are a
consequence of the strong thermoelastic coupling and comparative insignificance of
diffusion on the short time scale. Similar behavior is not seen in Nayfeh's results
(Nayfeh 1977) since in that work the thermoelastic coupling coefficient, i.e. $d$, is used
as the perturbation parameter.

The different figures also show the effect of the parameters $c$ and $d$ on the
relative magnitudes of temperature and displacement of the compression wave as compared to
the temperature and displacement on the surface. Further, figures 1, 2, and 4 illustrate
the decrease in speed of the compression wave as $c$ and $d$ are decreased. The
temperature on the surface is independent of $c$ and $d$ except through a multiplicative
factor, therefore because of our scaling, the temperature curves for $x = 0$ are identical
in figures 1, 2, 4 and 5.

Figure 6 shows the long term behavior of the temperature for values of $c$ and $d$ of
1.0, hence $k$ is 0.5. Equation (3.9) is plotted for values of $x$ of 0, 1.0, 2.0, etc.
and $t$ ranges up to 40.

The results displayed in these figures agree qualitatively with the results of Bechtel
(1975) and Ready (1965). Moreover, the final results are, we believe, easier to compute
and more informative than previous results since the perturbation analysis emphasizes the
dominant effects.

The original system of equations (1.1) and (1.2) has two propagation speeds associated
with it when the relaxation time is non-zero. In our analysis these show up on two
different scales. On short time scale, as discussed in section 2, the effects propagate
with the non-dimensional velocity $a = (c^2 + d^2)^{1/2}$, and in the boundary correction for
the short time discussed in section 4 the thermal disturbances propagate with speed
\[ c_s \beta^{-1/2} = c(a^2 \tau \beta)^{1/2}. \]
Moreover, the disturbances propagating with the faster speed
\[ c_s \beta^{-1/2} \]
die out rapidly and those propagating with the slower speed do not die out.
These results agree with those of Nayfeh (1977) and are compatible with the results of
Norwood and Warren (1969). On the long time scale of section 3, there is no propagation as
such since diffusion is the dominant phenomena.
REFERENCES


Thermoelastic Response to a Short Laser Pulse

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We consider a one-dimensional model of the thermoelastic response to laser heating by a pulse of very short duration. We treat the linear equations of thermoelasticity by perturbation methods using the pulse duration as the perturbation parameter. The perturbation analysis separates the problem into two time regimes, the short time scale, which is of the same order as the pulse duration, and the long time scale, which is much longer than the pulse duration. Convenient analytical expressions are obtained for the temperature change and displacement in both time regimes.