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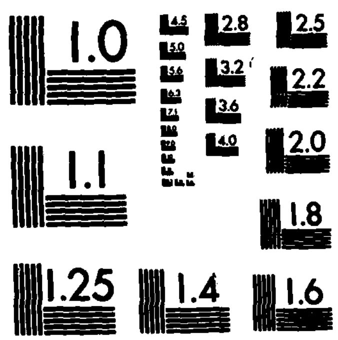

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A TUTORIAL ON THE DETERMINATION OF SINGLE-WEAPON-SYSTEM-TYPE KILL RATES FOR USE IN LANCHESTER-TYPE COMBAT MODELS

James G. Taylor

August 1982

Final Report for Period OCT 80-AUG 82

Approved for public release; distribution unlimited.

Prepared for:

U.S. Army Training & Doctrine Command
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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NPS55-82-021	2. GOVT ACCESSION NO. AD-A124 937	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A TUTORIAL ON THE DETERMINATION OF SINGLE-WEAPON-SYSTEM-TYPE KILL RATES FOR USE IN LANCHESTER-TYPE COMBAT MODELS		5. TYPE OF REPORT & PERIOD COVERED Final Report 1 OCT 80-30 AUG 82
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) James G. Taylor		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS MIPR CD 2-82
11. CONTROLLING OFFICE NAME AND ADDRESS U.S. Army Training & Doctrine Command Fort Monroe, Virginia 23651		12. REPORT DATE August 1982
		13. NUMBER OF PAGES 169
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Lanchester Attrition-Rate Coefficients; Force-on-Force Attrition Modelling; Single-Weapon-System-Type Kill Rates; Deterministic Combat Attrition; Lanchester Theory of Combat; Development of Operational Models; Combat Modelling; Analytical Models of Kill Rates

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report is a tutorial on basic analytical-modelling methodology for the determination of single-weapon-system-type kill rates (i.e. so-called Lanchester attrition-rate coefficients) for use in operational Lanchester-type combat models. It is oriented more towards the user of ~~such combat models~~ than towards the research specialist. Thus, the purpose of the tutorial is to facilitate the intelligent use and adaptation of such Lanchester-type combat models to defense-planning problems. It emphasizes those aspects

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) of the Lanchester theory of combat that have been most useful for developing operational combat models. The tutorial begins by reviewing the basic paradigm that stands at the heart of attrition calculations in contemporary Lanchester-type combat models and discussing its enrichment for the development of such operational models. It focuses on how the combat-attrition process is conceptualized and on the delineation of the assumptions involved with using each particular attrition-rate-coefficient expression (i.e. model of a single-weapon-system-type kill rate). Enrichments in both the target-acquisition process and also the line-of-sight process are discussed in detail. Those aspects and methodologies that appear to be important for future enrichments (e.g. detailed modelling of command and control) are emphasized. Both homogeneous-force combat and also heterogeneous-force combat are considered, as well as attrition-rate coefficients for different weapon-system types.

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A TUTORIAL ON THE DETERMINATION OF SINGLE-WEAPON-SYSTEM-TYPE KILL RATES FOR USE IN LANCHESTER-TYPE COMBAT MODELS*

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* This work was supported by the U.S. Army Training & Doctrine Command, Fort Monroe, Virginia.

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1. Introduction.

Lanchester-type combat models have been used in a number of important U.S. Army studies and are being considered for a number of pending ones. Such complex operational models[1*] are currently either maintained by U.S. Army agencies (e.g. AMSWAG by AMSAA or FOURCE by TRASANA) or available through contractors (e.g. VECTOR-2 from Vector Research, Inc. of Ann Arbor, Michigan). Because of the past use and potential prominent future use, this report will review the conceptual/operational basis for the assessment of casualties by such operational Lanchester-type combat models. Thus, this report has been written on the premise that there is a set of analytical models which are being used (and will continue to be used) in support of various U.S. Army/DoD decision makers, and that their underlying conceptual bases and assumptions are not as well understood as they should be. It will attempt to make these conceptual bases and assumptions more accessible and comprehensible to the users of these models.

Central to much of the practice of military operations research (OR) for defense-planning purposes is the use of combat models, of which a principal variety (especially as concerns land combat) are deterministic-differential-equation models that are commonly called Lanchester-type combat models, which are so-called after F.W. Lanchester's [36] pioneering work which was first published in 1914. The author has found it convenient to refer to

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*To avoid distracting the reader, all footnotes have been placed together just before the references.

any "force-on-force" differential-equation model of the combat-attrition process as a Lanchester-type combat model or as a system of Lanchester-type differential equations (or sometimes simply as Lanchester-type equations). The state variables are typically the numbers of the various different weapon-system types.

The goal of this tutorial is to review (and make more accessible to U.S. Army OR analysts) basic methodology for the determination of single-weapon-system-type kill rates for use in operational Lanchester-type combat models. The tutorial will highlight how the combat-attrition process is conceptualized and what are the assumptions involved with using each particular attrition-rate-coefficient expression (i.e. model of a single-weapon-system-type kill rate). In particular, those aspects and methodologies that appear to be important for command and control applications (e.g. methodology used in the VECTOR-2 model, the FOURCE model, or the TFECS model) will be emphasized. Thus, one might consider this tutorial to be (in some sense) a primer for studying VECTOR-2 or the TFECS model.

Finally, this tutorial is oriented towards the user of operational Lanchester-type combat models, not towards the research specialist. It is assumed, however, that the reader has a general familiarity with the material contained in the author's Force-on-Force Attrition Modelling [50]. Since the emphasis will be placed on communicating how the force-on-force attrition process has been conceptualized, derivations or proofs will by-and-large be omitted, except when some insight into the model-building process will be gained by their inclusion. In all cases when a proof has been omitted, the reader will be told where he can find such information if he wants it.

2. The Concept of "Models versus Modelling" and Its Implications for Model Appraisal.

William T. Morris [40] has emphasized for teaching purposes the intrinsic difference between models and modelling, the former being inanimate objects while the latter is an active process. He has conceptualized that the process of model building consists of the three basic ingredients shown in Table I. The basic idea is that a complex operational model is built in an evolutionary fashion by the process of model enrichment (see Table II) from a basic logical structure or paradigm.

Documentation and evaluation of complex computer-based models has only relatively recently been explicitly recognized as a very difficult and important problem[2]. Szymczak [48] has hypothesized that three different levels of documentation are required for such models:

(L1) decision-maker level,

(L2) analyst level,

and (L3) computer-programmer level.

In particular, he has pointed out the need for documenting the conceptual bases of a complex model to the analyst. This means explaining in plain language how the model operates overall and how each part of it functions individually (both in concept as well as in detail). It is the purpose of this tutorial to provide such analyst-level documentation on the determination of single-weapon-system-type kill rates (i.e. Lanchester attrition-rate coefficients) for use in Lanchester-type combat models.

TABLE I. Three Basic Ingredients of the Model-Building Process.

(1) The process of enriching or elaborating upon a basic logical structure

(2) The use of analogy or association with previously developed logical structures to determine the starting point for this enrichment process

(3) The interactive (i.e. "looping") nature of the model-building process

TABLE II. Elements of the Model-Enrichment Process.

(1) Making Constants into Variables

(2) Adding More Variables

(3) Using More Complicated (i.e. Nonlinear) Functional Relations Between Variables

(4) Using Weaker Assumptions and Restrictions

(5) Not Suppressing Randomness

For explaining to anyone how a complex operational model works, it is the author's hypothesis that a simple overview should be given and then each major part explained. In this tutorial we will focus on the modelling of one of the most crucial parts of any Lanchester-type model: namely, the Lanchester attrition-rate coefficients. For communicating to a reader how the force-on-force attrition process is conceptualized in a complex operational model, it is the author's hypothesis that the reader should be shown the simplest paradigm from which the complex model has been developed by the process of model enrichment. Thus, the reader should be shown the simplest paradigm to foster his conceptual comprehension, with the expectation that although the details may very well look different and be much more complicated in the operational model, the basic simple paradigm will have captured the basic idea of how the attrition process has been conceptualized. Thus, if an operational model has been built from a basic paradigm (or paradigms) by the process of model enrichment, then the inverse process of model simplification should be used to recreate the basic paradigm (or paradigms) for understanding the complex operational model's conceptual basis.

3. The Basic Lanchester-Type Paradigm.

Let us consider combat between two homogeneous forces: a homogeneous X force (for example, tanks) opposed by a homogeneous Y force (for example, anti-tank weapons). We will focus on the force-on-force attrition process in the combat between these two homogeneous forces (see Fig. 1). The basic Lanchester-type paradigm for modern warfare assumes that the casualty rate of such a homogeneous force is directly proportional to the number of enemy firers, e.g. the X-force casualty rate is given by

$$\frac{dx}{dt} = -ay, \tag{3.1}$$

where a denotes the rate at which a single typical Y firer kills X targets and is called a Lanchester attrition-rate coefficient. Here (as usual) $x(t)$ and $y(t)$ denote the numbers of X and Y combatants (respectively) at time t. According to the usual Bonder/Barfoot[3] attrition-rate-coefficient methodology, the Lanchester attrition-rate coefficient a (also referred to as the single-weapon-system-type kill rate) is given by

$$a = \frac{1}{E[T_{XY}]}, \tag{3.2}$$

where $E[\cdot]$ denotes mathematical expectation and

T_{XY} = the time for a Y firer type to kill an X target type (a r.v.).

Here the notation "a r.v." stands for "a random variable." Justification (according to Bonder and Farrell [11]) for taking the

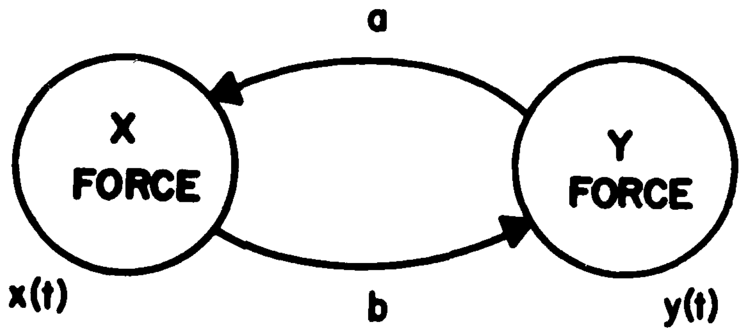

Figure 1. Combat between two homogeneous forces, as conceptualized by the basic Lanchester-type paradigm. The quantities a and b (here assumed to be constant) are called Lanchester attrition-rate coefficients. The coefficient a denotes the rate at which one Y firer kills X targets. Consequently, it represents the fire effectiveness of the weapon-system type used by the Y force in the operational circumstances of the battle under consideration.

Lanchester attrition-rate coefficient as the reciprocal of the expected time to kill a target, e.g. (3.2) above, is given in Appendix A below. At present, we have not been very specific about the variables upon which the attrition-rate coefficient a depends, but let us assume here that a is a constant for the engagement in question. For the sake of completeness, we will restate here the fundamental assumption behind the basic homogeneous-force Lanchester-type paradigm (3.1):

(A1) the casualty rate of a force is directly proportional to the number of enemy firers.

The above assumption (A1) could be stated in an equivalent form in more operational terms (or could be interpreted in these more operational terms) as follows: the Y force engages the X force with "aimed" fire, and the time for a single Y firer to acquire an X target is constant, independent of the number of enemy targets (see Taylor [50] for further details). For present purposes, however, it will be more fruitful to use assumption (A1). Moreover, within the present context of constant attrition-rate coefficients, an entirely equivalent (and even more useful for future purposes) form for assumption (A1) is as follows:

(A1') the casualty rate of a force is equal to the product of the single-weapon-system-type kill rate and the number of enemy firers.

Assumption (A1') may be considered to be the conceptual point of departure for the development of the VECTOR-2 model by the process of model enrichment (see Section 2 above) via its heterogeneous-force form.

The above basic paradigm (3.1) says that the attrition rate of a target type is proportional to only the number of enemy firers. Furthermore, it may be interpreted as saying that the total-force attrition rate ($- \frac{dx}{dt}$) is obtained by "scaling up" the single-weapon-system-type kill rate of a "typical" enemy firer through multiplication by the total number of firers. If one can only determine what is a "typical" firer and what are the environmental and operational circumstances of his employment, then use of this paradigm (3.1) presupposes that the corresponding total-force kill rate is simply obtained by "scaling up" this single-weapon-system-type kill rate.

In the above formulation it has been assumed that the single-weapon-system-type kill rate a is constant over time. Under many circumstances [e.g. fire effectiveness being range dependent and the range (distance) between firer and target changing over time due to changes in their positions], however, it is desirable to consider time-dependent attrition-rate coefficients, i.e.

$$\frac{dx}{dt} = - a(t)y. \qquad (3.3)$$

Although it is now considered to change over time, the Lanchester attrition-rate coefficient $a = a(t)$ is still given by (3.2) at any point in time. Also unchanged is the fact that we may still consider the basic total-force-casualty-rate paradigm to be based on assumption (A1) [equivalently, (A1')], the "scaling up" of the total-force casualty rate from the single-weapon-system-type kill rate.

A further enrichment of the basic Lanchester-type paradigm is involved in the complex operational models built by Vector Research, Inc. (VRI). If we assume that the single-weapon-system-type kill rate a depends not only on time t but also on the number of targets x (e.g. target detection depends on the number of targets), then one is led to the following further-enriched basic Lanchester-type paradigm for homogeneous-force combat:

$$\frac{dx}{dt} = - a(t,x)y. \tag{3.4}$$

Again, the Lanchester attrition-rate coefficient a is still given by (3.2), but weapon-performance characteristics have been allowed to depend on not only time t but also the number of targets x, i.e. $a = a(t,x)$. This version of the basic paradigm (3.4) may be considered to be the point of departure for the development of the maneuver-unit-attrition algorithms in the VECTOR-2 model. Moreover, it should be noted that the basic total-force-casualty-rate paradigm (3.4) may be considered to be based on only assumption (A1') and that assumption (A1) no longer holds for the X-force casualty rate. Thus, the total-force casualty rate is still "scaled up" as before, but because of the functional dependence of the single-weapon-system-type kill rate, i.e. $a = a(t,x)$, this "scaling up" must be now explicitly stated in order for one to fully grasp the dynamics of the force-on-force attrition process.

We can go even further in enriching the basic homogeneous-force Lanchester-type paradigm, however. If we assume that the single-weapon-system-type kill rate a depends not only on time t and the number of targets x but also on the number of firers y (e.g. too high a density of firers degrades their average effectiveness), then one is led to the following "fully-enriched" basic Lanchester-type paradigm for homogeneous-force combat:

$$\frac{dx}{dt} = -a(t,x,y)y. \tag{3.5}$$

Again, the Lanchester attrition-rate coefficient a is (as always) still given (3.2), but weapon-performance characteristics have now been allowed to depend on not only time t and the number of targets x but also the number of firers y, i.e. $a = a(t,x,y)$. As in the immediately preceeding case, the "scaling up" of the total-force casualty rate from the single-weapon-system-type kill rate can only be expressed in terms of assumption (A1'), i.e. assumptions (A1) and (A1') are no longer equivalent. Although apparently not corresponding to the basic attrition-rate paradigm of any current large-scale operational model, this attrition-rate-coefficient functional form has nevertheless been included here for the sake of completeness. It is the most general form of the basic Lanchester-type paradigm for combat between two homogeneous forces.

Thus, we have progressed in a step-by-step fashion via the process of model enrichment from the simplest basic homogeneous-force Lanchester-type paradigm to the most complicated one. This evolution towards more operational complexity is depicted in Table III. The increasing complexity of the functional dependence of the attrition-rate coefficient for a typical Y firer illustrates element (3) of the model-enrichment process as given in Table II.

We now turn to heterogeneous forces and will discuss how the above concepts may be extended still further. Modern combat is characterized by combined-arms operations involving (for example) tanks, anti-tank weapon systems, artillery, infantry (armed with several different types of weapons), etc. Unfortunately, the simple homogeneous-force paradigms considered above are inadequate to capture interactions armong different weapon-system types in modern combined-arms combat. Let us therefore consider combat between heterogeneous forces and briefly indicate how the above simple paradigms may be extended to such more complicated interactions.

For illustrative purposes, we will consider an engagement with m different types of weapon systems on the X side and n for Y (see Fig. 2). For notational convenience we will always let the subscript i refer to the X force and the subscript j refer to the Y force. Thus, the index i will always take on the integer values 1 through m, and the index j will always take on the integer values 1 through n. The generalization of (3.1) to heterogeneous-force combat is then given by

$$\frac{dx_i}{dt} = - \sum_{j=1}^{n} A_{ij} \, y_j, \tag{3.6}$$

TABLE III. Summary of the Step-by-Step Enrichment (Showing the Evolution Towards More Operational Complexity) of Attrition-Rate Coefficients in the Basic Homogeneous-Force Lanchester-Type Paradigm.

Total-Force Attrition-Rate Paradigm (Eq. No.)	Functional Form of Attrition-Rate Coefficient for a Typical Y Firer	Functional Dependence of Attrition-Rate Coefficient
(3.1)	a	constant over time
(3.3)	a(t)	depends only on time
(3.4)	a(t,x)	depends on time and also the number of targets
(3.5)	a(t,x,y)	depends on times, the number of targets, and the number of firers

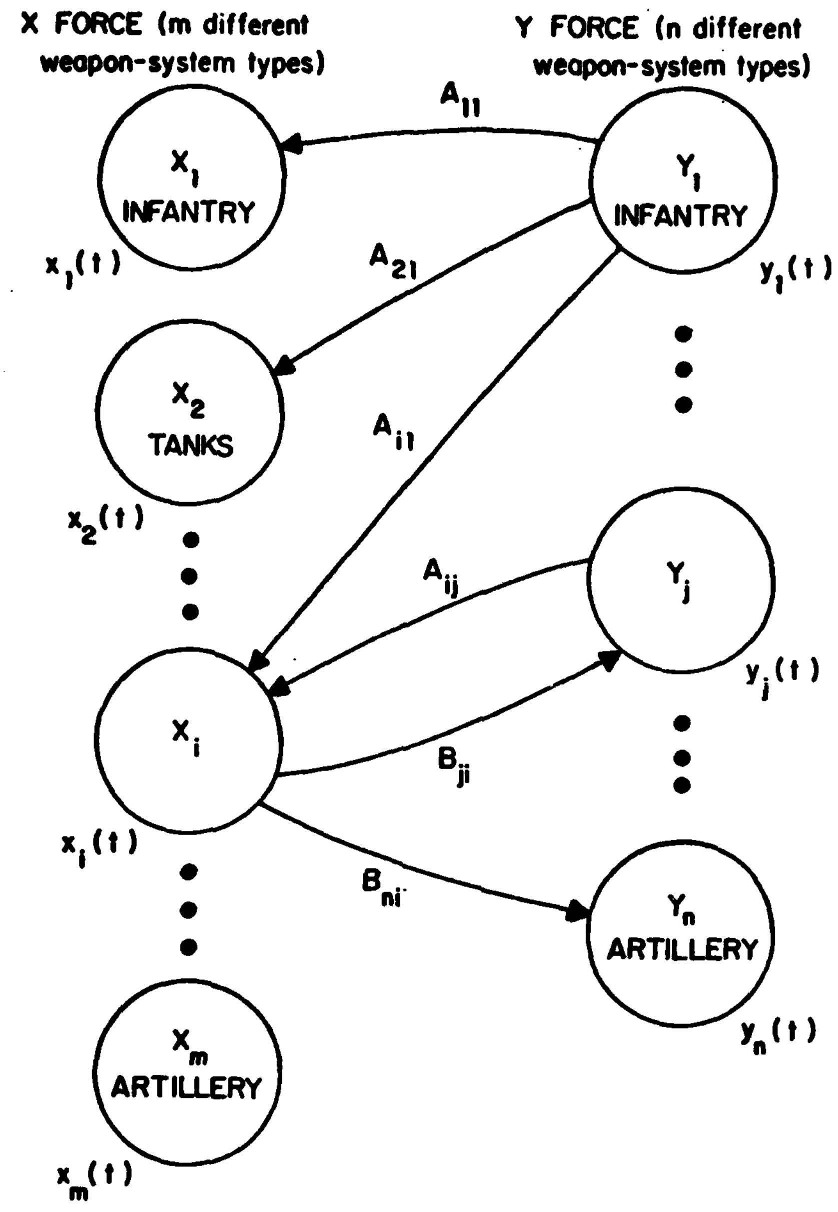

Figure 2. Schmatic showing notation convention for subscripts on attrition-rate coefficients in heterogeneous-force combat. The convention adopted here is that the first subscript will denote the target type and the second subscript will denote the firer type, e.g. A_{ij} denotes the rate at which a typical Y_j firer kills X_i targets in the opposing enemy force.

where A_{ij} denotes the rate at which a single typical Y_j firer kills X_i targets. In this heterogeneous-force case (according to the Bonder/Barfoot methodology), the Lanchester attrition-rate coefficient A_{ij} (also referred to as the single-weapon-system-type kill rate) is given by

$$A_{ij} = \frac{1}{E[T_{X_i Y_j}]}, \tag{3.7}$$

where

$T_{X_i Y_j}$ = the time for a Y_j firer type to kill an X_i target type (a r.v.).

The fundamental assumptions behind the above basic heterogeneous-force Lanchester-type paradigm (3.6) are as follows (cf. the homogeneous-force case):

(A_{het}1) the attrition-rate effects of various different enemy weapon-system types against a particular friendly target type are <u>additive</u>,

and (A_{het}2) the loss rate of a particular friendly target type to each enemy weapon-system type is proportional to the number of enemy firers of that particular enemy-firer type.

Although assumption (A_{het}1) is fairly restrictive (it means that there is <u>no mutual support</u> among different weapon-system types, i.e. no synergistic effects), the author does not know of <u>any</u>

heterogeneous-force model that does not use it. It should be noted that (3.6) and (3.7) are straightforward generalizations of the basic homogeneous-force paradigm given by (3.1) and (3.2).

It is instructive to note that assumption ($A_{het}2$) may be also stated in the following equivalent form [cf. the restatement of the homogeneous-force assumption (A2) in the (not always) equivalent form (A2')]:

($A_{het}2'$) the loss rate of a particular friendly target type to each enemy weapon system type is equal to the product of the single-weapon-system-type kill rate and the number of enemy firers of that particular enemy-firer type.

<u>Assuming ($A_{het}1$) and ($A_{het}2'$)</u>, one can formulate the following enriched basic Lanchester-type *paradigm for* heterogeneous-force combat:

$$\frac{dx_i}{dt} = - \sum_{j=1}^{n} A_{ij}(t,\underset{\sim}{x})\, y_j, \tag{3.8}$$

where $\underset{\sim}{x}$ denotes the appropriately-sized vector of the number of each weapon-system type comprising the X force (i.e. an m-vector) and the Lanchester attrition-rate coefficient $A_{ij} = A_{ij}(t,\underset{\sim}{x})$ is still given by (3.7). Maneuver-unit attrition in VECTOR-2 is based on the above heterogeneous-force paradigm [19, pp. 51-52].

4. Additional Operational Factors to be Considered.

As discussed in the previous section, the homogeneous-force Lanchester-type paradigm (3.1) and (3.2) may be considered to be basic for building force-on-force combat models. Let us therefore consider this basic paradigm further and investigate what combat factors it may be thought of as representing and what factors it omits. This brief examination will set the stage for some important topics subsequently to be investigated in this report.

Let us accordingly consider the basic homogeneous-force Lanchester-type paradigm

$$\frac{dx}{dt} = -ay, \tag{4.1}$$

where the Lanchester attrition-rate coefficient a is given by

$$a = \frac{1}{E[T_{XY}]}, \tag{4.2}$$

and

T_{XY} = the time for a Y firer type to kill an X target type (a r.v.).

This paradigm has been hypothesized to apply when the Y force uses "aimed" fire against X targets and the time to acquire an X target is constant, independent of the X force level. Other sets of operational circumstances may be hypothesized, but they are not germane for our investigation here (see Taylor [50, pp. 23-28] for further details). In the simplest case in which the time for a Y firer to acquire an X target is negligible, the Lanchester attrition-rate coefficient a may be taken to be given by

$$a = \nu_Y P_{SSK_{XY}}, \tag{4.3}$$

where ν_Y denotes the firing rate of a "typical" Y firer and $P_{SSK_{XY}}$ denotes the single-shot kill probability for a Y firer engaging an X target.

For addressing any real operational problem of military OR the above simple model is woefully inadequate, since many significant operational factors have been omitted in abstracting the basic paradigm from the complex real-world details of modern combat. We can enrich this basic paradigm by considering additional operational factors such as (1) range-dependent weapon-system capabilities, (2) other temporal variations in fire effectiveness, (3) unit breakpoints, (4) the diversity of weapon-system types, (5) command, control, and communications, (6) suppressive effects of weapon systems, (7) the target-acquisition process, (8) the line-of-sight process, etc. In the tutorial at hand, however, we will focus on the last two operational factors: namely,

(F1) target-acquisition process,

and (F2) line-of-sight process.

5. Determination of Attrition-Rate Coefficients for Homogeneous-Force Combat.

Let us return to the consideration of the basic paradigm of Lanchester-type combat between two homogeneous forces (see Fig. 1 again)

$$\begin{cases} \dfrac{dx}{dt} = -ay & \text{with} \quad x(0) = x_0, \\ \dfrac{dy}{dt} = -bx & \text{with} \quad y(0) = y_0. \end{cases} \tag{5.1}$$

For present purposes it is not essential that we be explicit about the functional dependence of, for example, a. Thus, a may stand for a, $a(t)$, $a(t,x)$, or even $a(t,x,y)$. In any case, the fundamental relation for determining a numerical value for a Lanchester attrition-rate coefficient is given by, for example,

$$a = \frac{1}{E[T_{XY}]}, \tag{5.2}$$

where

$E[\cdot]$ denotes mathematical expectation and

T_{XY} = the time for a Y firer type to kill an X target type (a r.v.).

Thus, a Lanchester attrition-rate coefficient may be taken as the reciprocal of the expected time to kill a target, and thus determination of the expected time to kill a target $E[T]$ is a fundamental calculation required for the building of any operational Lanchester-type combat model.

Bonder and Farrell [11] have developed general methodology for determining the expected time to kill a target for a wide spectrum of weapon-system types. To facilitate analysis of the time to kill a target they have developed the taxonomy shown in Table IV for classifying the engagement of a particular target type by a specific weapon-system type. According to this taxonomy, weapon-system types are first classified according to the mechanism by which they kill particular target types (i.e. their lethality characteristics) as being either impact-to-kill systems or area-lethality systems. Within each of these two categories, Bonder and Farrell have further classified weapon-system types according to how they use firing information to control the system's aim point and their delivery characteristics, i.e. the firing doctrine employed. Expressions have been developed by Bonder and Farrell [11] for Lanchester attrition-rate coefficients corresponding to all the weapon-system-type classifications tagged with an * in Table IV.

Moreover, research since the mid-1960's (dating from the appearance of Bonder's Ph.D. thesis [6]) has led to the development of several methods for computing the expected time to kill a target $E[T]$ (see the author's treatise [51] for further details). For present purposes, it is convenient to focus on the following two methods for computing $E[T]$:

(M1) method based on sum of component event times,

and

(M2) method based on first-passage time in semi-Markov process.

TABLE IV. Classification of Weapon-System Types for the Development of Lanchester Attrition-Rate Coefficients for the Model (5.1) (From Bonder and Farrell [11]).

Lethality Mechanism

(1) Impact

(2) Area

Firing Doctrine

(1) Repeated Single Shot

(a)* Without Feedback Control of Aim Point

(b)* With Feedback on Immediately Preceding Round (Markov-Dependent Fire)

(c) With Complex Feedback

(2) Burst Fire

(a)* Without Aim Change or Drift in or Between Bursts

(b)* With Aim Drift in Bursts, Aim Refixed to Original Aim Point for Each Burst

(c) With Aim Drift, Re-aim Between Bursts

(3) Multiple Tube Firing: Feedback Situations (1a), (1b), (1c)

(a)* Salvo or Volley

(4) Mixed-Mode Firing

(a) Adjustment Followed by Multiple Tube Fire

(b)* Adjustment Followed by Burst Fire

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*Indicates that analysis of this category has been performed by Bonder and Farrell [11].

In simple cases the first method (M1) provides by far the most transparent model of the attrition process of a particular target type, while the second method (M2) is the basis for the manuever-unit attrition processes in VECTOR-2 and command-and-control processes in TFECS. Additionally, the first method (M1) may be used to determine rates of attrition for acquired targets (such rates are required in the calculation of attrition-rate coefficients in VECTOR-2). Finally, the first method (M1) provides a basis for better understanding the realm of applicability of attrition-rate coefficients calculated by the second method (M2). We will now compare these two basic methods for a special case of tactical interest: namely, the case of Markov-dependent fire.

For the case of Markov-dependent fire and an impact-to-kill lethality mechanism, Bonder [6-8] has shown that

$$E[T] = t_a + t_1 - t_h + \frac{(t_h + t_f)}{P(K|H)} + \frac{(t_m + t_f)}{P(h|m)} \left\{ \frac{[1 - P(h|h)]}{P(K|H)} + P(h|h) - p_1 \right\}, \qquad (5.3)$$

where all symbols are defined in Table V. This expression for $E[T]$ holds for the following assumptions:

(A1) Markov-dependent fire with parameters p_1, $P(h|h)$, and $P(h|m)$,

(A2) geometric distribution for the number of hits required for a kill with parameter $P(K|H)$.

TABLE V. Variables Contained in Expression for Lanchester Attrition-Rate Coefficient for Single-Shot Markov-Dependent-Fire Weapon Systems with a Geometric Distribution for the Number of Hits Required for a Kill.

Time to acquire a target, t_a

Time to fire first round after target acquired, t_1

Time to fire a round following a hit, t_h

Time to fire a round following a miss, t_m

Time of flight of the projectile, t_f

Probability of a hit on first round, p_1

Probability of a hit on a round following a hit, $P(h|h)$

Probability of a hit on a round following a miss, $P(h|m)$

Probability of destroying a target given it is hit, $P(K|H)$

For simplicity we have assumed that all the event times t_a, t_1, t_h, t_m, and t_f are deterministic quantities, although under the appropriate rather mild assumptions (5.3) still holds when they are random variables, with expected values replacing the deterministic quantities, e.g. $E[T_a]$ replacing t_a (see [51, Chapter 5] for further details). We will now investigate how (5.3) may be developed by each of the two methods (M1) and (M2) mentioned above. These developments should help further elucidate the general remarks made above about them.

We will first consider the development of (5.3) by method (M1). Accordingly, we consider the process by which a single firer engages and kills a single passive enemey target and conceptualize this process as consisting of the sequence of events from target acquisition to destruction shown in Table VI. It follows that the time to obtain z hits T_z is given by

$$T_z = \underbrace{t_a}_{\substack{\text{time to}\\ \text{acquire}\\ \text{target}}} + \underbrace{(t_1+t_f)}_{\substack{\text{time to}\\ \text{impact of}\\ \text{first round}\\ \text{after}\\ \text{acquisition}}} + \underbrace{(t_h+t_f)(z-1)}_{\substack{\text{total time}\\ \text{to impact}\\ \text{of total of}\\ (z-1)\text{ hits}}} + \underbrace{(t_m+t_f)(N_z-z)}_{\substack{\text{total time}\\ \text{to impact of}\\ \text{total of}\\ (N_z-z)\text{ misses}}}, \qquad (5.4)$$

where N_z (a random variable) denotes the number of rounds to obtain z hits and z is a parameter (realization of the random variable Z, the number of hits required to kill the target). Let us rearrange this expression to read

$$T_z = t_a + t_1 - t_h + (t_h-t_m)z + (t_m+t_f)N_z, \qquad (5.5)$$

TABLE VI. Sequence of Events from Target Acquisition to Destruction Which is Conceptual Basis of Model for Expected Time to Kill a Target with Markov-Dependent Fire.

(E1) The sequence begins with target acquisition which takes t_a minutes to occur.

(E2) The first round is then fired and arrives in the target area $(t_1 + t_f)$ minutes later.

(E3) If the first round misses, the next round will arrive $(t_m + t_f)$ minutes after the first.

(E4) If the first round hits the target and more than one hit is required (i.e. $z > 1$), the next round will arrive $(t_h + t_f)$ minutes later.

(E5) The above sequence of firing after hits and misses is continued until the final hit, which destroys the target, is obtained.

which is the basic model for the time to obtain z hits. Taking the expected value of (5.5), we obtain

$$E[T_z] = t_a + t_l - t_h + (t_h - t_m)z + (t_m + t_f)E[N_z], \tag{5.6}$$

which is more convenient to write in terms of conditional expectations as

$$E[T|Z=z] = t_a + t_l - t_h + (t_h - t_m)z + (t_m + t_f)E[N|Z=z]. \tag{5.7}$$

Unconditioning (i.e. multiplying both sides by $p_Z(z) = P[Z=z]$ and summing from $z = 1$ to $z = \infty$, where $p_Z(z)$ denotes the probability mass function for the discrete-valued random variable Z, the number of hits necessary to kill the target), we obtain

$$E[T] = t_a + t_l - t_h + (t_h - t_m)E[Z] + (t_m + t_f)E[N], \tag{5.8}$$

where it has been assumed that the hitting process is independent of the killing-with-hits process (i.e. the random variables N and Z are independent). Here Z (a r.v.) denotes the number of hits to kill the target, and N (a r.v.) denotes the total number of rounds expended to kill it (see [51, Chapter 5] for further details). Under the assumption (A1) of Markov-dependent fire, we have [51, Chapter 5]

$$E[N|Z=z] = z + \frac{(1-p_1)}{P(h|m)} + \frac{\{1 - P(h|h)\}}{P(h|m)}(z-1); \tag{5.9}$$

while under the assumption (A2) that the number of hits required to kill obeys a geometric probability law with parameter $P(K|H)$, we have

$$E[Z] = \frac{1}{P(K|H)}. \tag{5.10}$$

Unconditioning (5.9) and using (5.10), we find that

$$E[N] = \frac{1}{P(K|H)} + \frac{1}{P(h|m)}\left\{\frac{[1 - P(h|h)]}{P(K|H)} + P(h|h) - p_1\right\}. \tag{5.11}$$

Substitution of (5.10) and (5.11) into (5.8) then yields our desired result (5.3). Thus, we have shown how the method (M1) based on the appropriately weighted sum of component event times leads to the expression for the expected time to kill a target with Markov-dependent fire (5.3) via the basic model for the time to obtain z hits (5.5). This development is by far the more transparent of the two considered here and shows that (5.3) holds exactly and not in any limiting sense (see below).

We now turn to the development of (5.3) by method (M2) which is based on the first-passage time in a semi-Markov process. We will see that this second method is not nearly as transparent as the first, although it has been used to develop more general results that are used for engagement-outcome assessment in the VECTOR-2 and TFECS models. Loosely speaking, a semi-Markov process[4] (SMP) is a continuous-time Markov chain (MC) with general distributions for the times between transitions (i.e. not necessarily exponentially distributed). The SMP is completely described by a matrix of transition probabilities for an imbedded

MC and a matrix of distribution functions for the "wait" in a state before going to another state. No specific assumptions are made about these distribution functions for the "wait" in a state (except that they are indeed distribution functions). The basic idea behind this second method (M2) is to model the attrition process with a SMP in such a way that the expected time to kill a target is equivalent to the mean recurrence time for a given state (i.e. the mean time between successive visits to that state). This method (M2) uses the following important result by Barlow [4] that shows that the mean recurrence time for a state may be simply computed from the unconditional mean wait in each state and the stationary distribution for the imbedded Markov chain.

THEOREM 5.1 (Barlow [4], 1962). Consider a semi-Markov process (with J states $S_1, S_2, \ldots, S_J$) in which all states communicate. The mean recurrence time for state S_i, denoted as ℓ_{ii}, is then given by

$$\ell_{ii} = \frac{1}{\pi_i} \sum_{j=1}^{J} \pi_j \mu_j, \tag{5.12}$$

where μ_j denotes the unconditional mean wait in state S_j and π_j is an element (corresponding to state S_j) of the stationary distribution for the imbedded Markov chain. It follows that

$$\pi_j = \sum_{i=1}^{J} \pi_i p_{ij}, \tag{5.13}$$

and

$$\mu_j = \sum_{k=1}^{J} p_{jk}\, \mu_{jk} \tag{5.14}$$

where p_{ij} is the transition probability that the system goes from state S_i to state S_j when such a change does occur, and μ_{jk} denotes the mean time that the system remains in state S_j before it transitions to state S_k.

It should be noted that no assumption at all is made here about the distribution of waiting time in state S_j before the system transitions to state S_k.

We will now show how Barlow's theorem may be used to develop (5.3). Considering a single firer trying to engage and kill a single passive type of target, we see that a particular target can be categorized as

(1) undetected,

(2) hit,

(3) missed,

or (4) killed.

When one target has been killed[5], search immediately begins for a new target. We now seek to define system states for this attrition process in such a way that the conditions requisite for invoking Barlow's theorem are met (in particular, given any starting state, after sufficient lapse of time, the system can be in any state). Consequently, the "killed" state cannot be absorbing.

To accomplish such a defining of system states, we observe that the following two situations are mathematically equivalent: (I) a new target immediately appearing upon the destruction of the currently engaged target, and (II) the same target being repeatedly killed. Thus, we will define the following three system states:

S_1 = killed state (which lasts from the destruction of the previous target until the first round has been fired at a new target),

S_2 = hit state (in which the target has been hit but not killed by the last round fired),

and S_3 = missed state (in which the target has been missed and not killed by the last round fired).

These states and the corresponding transition probabilities for changes in system states are shown in Fig. 3. The transition probabilities for the imbedded Markov chain are given by

$$\begin{aligned}
p_{11} &= p_1 P(K|H), & p_{21} &= P(h|h)\ P(K|H), & p_{31} &= P(h|m)\ P(K|H), \\
p_{12} &= p_1\{1-P(K|H)\}, & p_{22} &= P(h|h)\{1-P(K|H)\}, & p_{32} &= P(h|m)\{1-P(K|H) \\
p_{13} &= 1-p_1, & p_{23} &= 1-P(h|h), & p_{33} &= 1-P(h|m),
\end{aligned} \tag{5.15}$$

Furthermore, the expected wait in each state is independent of

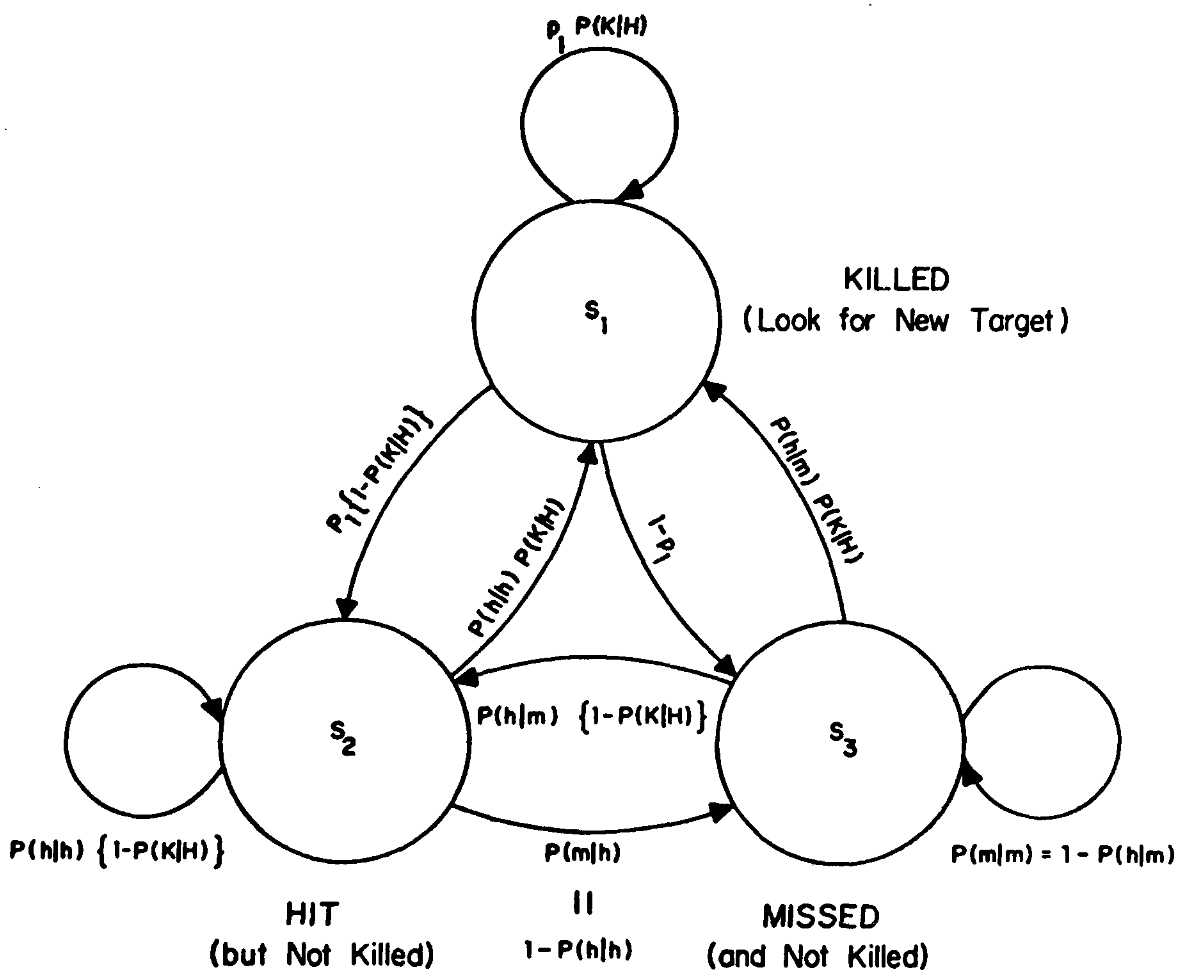

Figure 3. System states and transition probabilities used in the second method (M2) for the derivation of the expected time to kill a target by invoking Barlow's [4] result for the mean recurrence time of a semi-Markov process with an imbedded ergodic Markov chain.

the next state visited and given by

$$\mu_1 = t_a + t_1 + t_f,$$

$$\mu_2 = t_h + t_f, \tag{5.16}$$

and $\mu_3 = t_m + t_f.$

With the above definitions, all states communicate, and the expected time to kill a target is just the expected time between visits to state S_1, i.e. the mean recurrence time ℓ_{11} for state S_1. Hence, the expected time to kill a target $E[T]$ is given by

$$E[T] = \ell_{11} = \frac{1}{\pi_{11}} \sum_{j=1}^{3} \pi_j \mu_j, \tag{5.17}$$

where the stationary probabilities are given by the system of equations

$$\pi_j = \sum_{i=1}^{3} \pi_i p_{ij} \quad \text{for} \quad j = 1,2,3. \tag{5.18}$$

From (5.17) we see that what we need for computing the mean recurrence time for a target being killed ℓ_{11} is not the stationary probabilities π_j for $j = 1,2,3$ themselves but the ratios π_j/π_1 for $j = 2,3$. Accordingly, let us define

$$r_j = \frac{\pi_j}{\pi_1} \quad \text{for} \quad j = 2,3. \tag{5.19}$$

We may then write

$$E[T] = \ell_{11} = \mu_1 + r_2\mu_2 + r_3\mu_3, \tag{5.20}$$

where r_2 and r_3 are determined by the linear system of equations[6]

$$\begin{cases} (p_{22} - 1)r_2 + p_{32}r_3 = -p_{12}, \\ p_{23}r_2 + (p_{33} - 1)r_3 = -p_{13}. \end{cases} \tag{5.21}$$

The reader should note here that only two of the three equations (5.18) are linearly independent[7], since $\sum_{j=1}^{3} p_{ij} = 1$. The equations (5.21) are simply obtained from the last two of equations (5.18) by dividing both sides of each of them by $\pi_1 > 0$ and using (5.19). Solving (5.21), we find that

$$r_2 = \frac{p_{12}(1-p_{33}) + p_{13}p_{32}}{(1-p_{22})(1-p_{33}) - p_{23}p_{32}},$$

and (5.22)

$$r_3 = \frac{p_{13}(1-p_{22}) + p_{12}p_{23}}{(1-p_{22})(1-p_{33}) - p_{23}p_{32}}.$$

Substituting (5.15) into (5.22), we find that

$$r_2 = \frac{\{1 - P(K|H)\}}{P(K|H)},$$

and (5.23)

$$r_3 = \frac{1}{P(h|m)} \left\{ \frac{[1 - P(h|h)]}{P(K|H)} + P(h|h) - p_1 \right\}, \tag{5.23}$$

whence follows (5.3) from substitution of (5.16) and (5.23) into (5.20). Thus, we have developed an expression for the expected time for an individual firer to kill a target with Markov-dependent fire (5.3) by considering the first-passage time in the firer's target-destruction process modelled appropriately as a semi-Markov process. However, this approach may be used to develop an expression for $E[T]$ in much more complicated situations (see Appendix B for further details).

As we have already mentioned above, although the first method (M1) is more transparent, the second method (M2) is the one that has been used to determine attrition-rate coefficients for maneuver-unit combat in VECTOR-2 and rates of observations by information-collection resources in TFECS (i.e. C^3I capabilities). Thus, the reader who desires to understand the modelling of attrition in VECTOR-2 and command and control in TFECS must thoroughly understand the above simple derivation based on the first-passage time for a semi-Markov process.

It is also useful to have two such different perspectives on the determination of values for Lanchester attrition-rate coefficients. In particular, it is quite helpful to see two different derivations of the same expression for a single-weapon-system-type kill rate in order to better understand the modelling assumptions involved in its derivation. For example, the derivation of (5.3) by the first method (M1) clearly shows that this expression for the attrition-rate coefficient holds for all time (in particular, for the early stages of an engagement). Furthermore, no assumption has been made about the attrition process being in a steady state. Thus, although the second method (M2) does use the Markov-chain steady-state frequencies π_j, no assumption has been made (either implicitly or explicitly) by the use of this method concerning the modelled attrition process being in a steady state (cf. the statement made in the VECTOR-2 documentation [19, p. 56] about the "limiting value" of the attrition-rate coefficient).

In the next couple of sections we will examine how the additional operational factors (F1) and (F2) of Section 4 may be incorporated into Lanchester attrition-rate coefficients in homogeneous-force combat.

6. Target-Acquisition Process.

In this section we will investigate how the target-acquisition process may be represented in the attrition-rate coefficients in homogeneous-force Lanchester-type combat models. Although the target-acquisition and line-of-sight processes [i.e. the two factors (F1) and (F2) selected in Section 4 for further consideration] are certainly not independent of each other, for simplicity in this section we will assume that line of sight always exists between every firer-target pair (i.e. combat on so-called "billard-table" terrain) in order to focus on the target-acquisition process. Thus, we will emphasize here the modelling of the target-acquisition process in the special case in which line of sight always exists in order to most easily introduce to the reader the germane modelling concepts. In the next section we will extend these ideas to include the effects of the line-of-sight process on the target-acquisition process.

Although given within the context of homogeneous-force combat, the basic ideas presented here for modelling the target-acquisition process do extend to heterogeneous-force combat (in which they become quite complicated and tedious to follow). It is the author's intention to present the general principles for representing target acquisition in Lanchester-type combat models in as simple a setting as possible in order to make them accessible to the widest possible audience. Hence, we have suppressed here the added complexities of the line-of-sight process and heterogeneous forces (i.e. target priorities). Finally, the material presented in this section is basic for understanding the modelling of the target-acquisition process for maneuver units in VECTOR-2 and also for building more complicated target-acquisition models through the process of model enrichment (see Section 2 above).

An important distinction made in VRI's Lanchester-type combat models is whether the target-acquisition process of a single "typical" firer type is a serial process or a parallel process. In other words, a basic assumption about the target-acquisition process for developing expressions for Lanchester attrition-rate coefficients concerns the model according to which an observer acquires targets: whether the target-acquisition process is considered to be done in series or parallel with the target-engagement (i.e. destruction) process. The two modes for the target-acquisition process considered by VRI's models (including VECTOR-2) are then as follows:

(M1) serial acquisition,

and (M2) parallel acquisition.

The following conceptualizations are made about these two modes of target acquisition in VECTOR-2. Weapon-system types that employ parallel acquisition search continuously for targets, even while engaging other targets. When such a weapon-system type kills an enemy target, it can immediately shift its fire to a new target, provided that such a target was acquired during or before the engagement of the previous target just killed. On the other hand, a weapon-system type employing serial acquisition does not acquire targets while engaging another target. When such a serial acquirer ceases to engage a target (due to either killing the target or losing line of sight), he must acquire a new target. It is assumed that a serial acquirer does not remember any acquisitions made prior to engaging the target whose engagement has just terminated, and consequently he must begin the acquisition process all over again from scratch. Once a target has been acquired, though, it is actively engaged until killed, with only a kill or loss of line of sight terminating the engagement. For both modes of target acquisition, the VRI models assume that a firer can always correctly distinguish between effective and killed enemy weapon systems and never engages a killed system. We will now examine how each of these conceptual models of target acquisition may be analytically represented in homogeneous-force Lanchester-type models.

Let us therefore again consider the simplest Lanchester-type paradigm of combat between two homogeneous forces (see Fig. 1). An observer in the serial mode of target acquisition selects a new target whenever the previous target has been killed (or line of sight to the previous target has been lost). The analytical model of this acquisition-attrition process (shown for the Y force engaging the X-force target types with Markov-dependent fire) is given in Table VII. At the expense of being a little redundant, we will now explicitly spell out in the main text these results for serial acquisition in order for them to be available for ready reference and comparison with those for parallel acquisition. Thus, it should be noted that the total-force kill rate has been assumed to be just the single-weapon-system-type kill rate times the number of firers, e.g.

TABLE VII. Summary of Results Comprising Analytical Model of Acquisition-Attrition Process (shown for the Y Force Engaging the X-Force Target Types).

SERIAL ACQUISITION

$$\frac{dx}{dt} = -ay$$

$$a = \frac{1}{E[T_{XY}]}$$

T_{XY} = time for a Y firer type to kill an X target type

Markov-Dependent Fire:

$$E[T] = t_a + t_1 - t_h + \frac{(t_h + t_f)}{P(K|H)}$$

$$+ \frac{(t_m + t_f)}{P(h|m)} \left\{ \frac{[1 - P(h|h)]}{P(K|H)} + P(h|h) - p_1 \right\}$$

$$\left(-\frac{dx}{dt}\right) = ay, \tag{6.1}$$

since each serial acquirer on a side operates independently and line of sight always exists between every firer-target pair. Furthermore, the single-weapon-system-type kill rate for a system using serial acquisition and Markov-dependent fire is given by (for example)

$$a = \frac{1}{E[T_{XY}]}, \tag{6.2}$$

where

T_{XY} = the time (a r.v.) for a Y firer type to kill an X target type,

and

$$E[T] = t_a + t_1 - t_h + \frac{(t_h + t_f)}{P(K|H)}$$
$$+ \frac{(t_m + t_f)}{P(h|m)} \left\{ \frac{[1 - P(h|h)]}{P(K|H)} + P(h|h) - p_1 \right\}, \tag{6.3}$$

since in order to cause attrition to enemy targets a firer must acquire a new target from scratch after the previous one has been killed (i.e. the expected time between kills includes the time to acquire the target). Thus, the results for serial acquisition are just the ones given previously in Section 3 (where the distinc-) tion between serial and parallel acquisition was not made).

On the other hand, an observer in the <u>parallel mode of target acquisition</u> continues to acquire new targets, even while he is engaging a given target. Once an enemy target has been killed, such a parallel-acquisition system can immediately shift fire to a new target provided that one was acquired while some previous target was being engaged and line of sight still exists. The analytical model of this acquisition-attrition process (again shown for a homogeneous Y force engaging homogeneous X-force target types with Markov-dependent fire) is given in Table VIII. In this case, the total-force kill rate is given by the product of the kill rate of a single weapon system against acquired targets and the expected number of firers who have already acquired one or more targets, e.g.

$$\left(-\frac{dx}{dt}\right) = f_{XY}\, \alpha\, y, \tag{6.4}$$

where

$$f_{XY} = 1 - \exp\left\{-x \int_0^t \lambda_{XY}(s)\,ds\right\}, \tag{6.5}$$

$$f_{XY} = \mathrm{Prob}\begin{bmatrix}\text{at random point in time } Y \text{ weapon-} \\ \text{system type employing parallel acqui-} \\ \text{sition is firing at an } X \text{ target type}\end{bmatrix},$$

$\lambda_{XY}(t)$ denotes the rate at which a Y firer (i.e. observer) acquires X targets at time t when there is a single target present and it is continuously visible, and α denotes his kill rate against acquired targets. In the case of homogeneous forces

TABLE VIII. Summary of Results Comprising Analytical Model of Acquisition-Attrition Process (Shown for the Y Force Engaging the X-Force Target Types).

PARALLEL ACQUISITION (First Cut)

$$\frac{dx}{dt} = - f_{XY} \alpha y$$

$$f_{XY} = \text{Prob}\left[\begin{array}{l}\text{at random point in time } Y \text{ weapon-}\\ \text{system type empolying parallel acqui-}\\ \text{sition is firing at } X \text{ target type}\end{array}\right]$$

$$f_{XY} = 1 - \exp\left\{- x \int_0^t \lambda_{XY}(s)\,ds\right\}$$

$$\alpha = \frac{1}{E[T'_{XY}]}$$

T'_{XY} = time for a Y firer type to kill an acquired X target type (conditional kill time)

Markov-Dependent Fire:

$$E[T'] = t_1 - t_h + \frac{(t_h + t_f)}{P(K|H)}$$

$$+ \frac{(t_m + t_f)}{P(h|m)} \left\{\frac{[1 - P(h|h)]}{P(K|H)} + P(h|h) - p_1\right\}$$

considered here, we could have equally well denoted the probability f_{XY} as the probability that a Y firer (who is a parallel acquirer) has available one or more acquired targets at which to fire $p_{A_{XY}}$, i.e.

$$p_{A_{XY}}(t) = f_{XY}(t), \tag{6.6}$$

where

$$p_{A_{XY}}(t) = \text{Prob}\left[\begin{array}{l}\text{a typical } Y \text{ firer (parallel acquirer)} \\ \text{has available one or more acquired } X \\ \text{targets at which to fire at time } t\end{array}\right].$$

However, we have chosen to use the notation f_{XY} here, since it provides a bridge to the heterogeneous-force developments of Section 10. Moreover, it is frequently useful to consider the probability that a firer using the parallel mode of target acquisition has available one or more acquired targets of a particular type at which to fire, and thus we have also introduced $p_{A_{XY}}$ here. It should also be noted that $f_{XY}y$ then represents the expected number of Y firers who have already acquired one or more X targets. Furthermore, the single-weapon-system-type kill rate against acquired targets with Markov-dependent fire is given by (for example)

$$\alpha = \frac{1}{E[T'_{XY}]}, \tag{6.7}$$

where

T'_{XY} = the time (a r.v.) for a Y firer type to kill an acquired X target type.

Here T' does not include the time to acquire a target, and hence

$$E[T'] = t_1 - t_h + \frac{(t_h + t_f)}{P(K|H)} + \frac{(t_m + t_f)}{P(h|m)} \left\{ \frac{[1 - P(h|h)]}{P(K|H)} + P(h|h) - p_1 \right\}, \quad (6.8)$$

Understanding the above simple model is essential for understanding maneuver-unit attrition processes in VECTOR-2, which uses α_{ij}'s (i.e. heterogeneous-force single-weapon-system-type kill rates against acquired targets). A derivation of (6.4) is provided for the interested reader in Appendix C.

It is worthwhile to note that originally the VRI models (e.g. BONDER/IUA, AIR CAV, AMSWAG, etc.) considered both

(I) nonfiring acquisition (due to stimuli of nonfiring targets),

and (II) firing acquisition (due to pinpointing the flashes of the enemy's firing targets),

in the parallel-acquisition mode. When both processes are present, the X-force attrition rate is given by

$$(-\frac{dx}{dt}) = \{1 - e^{-x\Lambda_{XY}(t)}\}\alpha y, \quad (6.9)$$

where α and $E[T']$ are still given by (6.7) and (6.8), and

$$\Lambda_{XY}(t) = \int_0^t \left(\lambda_{XY}(s) - \{\ln(1 - p_{PP_{XY}}(s)\}\nu_X(s) \right) ds. \qquad (6.10)$$

Here

$\lambda_{XY}(t)$ = the rate at which a Y firer (i.e. observer) acquires X targets by nonfiring acquisition at time t when there is a single target present and it is continuously visible,

$p_{PP_{XY}}(t)$ = the probability that a Y observer pinpoints an X weapon system when it fires one round,

and $\nu_X(t)$ = the rate of fire of a single X weapon system.

This latter firing rate may be computed as (see [19, pp. 71-72] for further details)

$$\nu_X(t) = \left\{ 1 - e^{-y\Lambda_{YX}(t)} \right\} \bar{\nu}_X, \qquad (6.11)$$

where $\{1 - e^{-y\Lambda_{YX}(t)}\}$ denotes the probability that at a random point in time an X firer will be firing, and

$\bar{\nu}_X(t)$ = the firing rate of an X weapon system when it is firing at enemy targets.

From (6.10) we see that (6.11) does not yield an explicit expression for $\nu_X(t)$. One approximation $\hat{\nu}_X(t)$ that one could use is

$$\hat{\nu}_X(t) = \left\{ 1-\exp\left[-y \int_0^t \left(\lambda_{YX}(s) - \bar{\nu}_X\{\ln(1-p_{PP_{YX}}(s))\} \right) ds \right] \right\} \bar{\nu}_X. \tag{6.12}$$

The distinction made in VRI's Lanchester-type combat models is important because quite different total-force-kill-rate expressions arise, depending on whether targets are acquired in series or in parallel with the firing-at-acquired-targets process. The reader can see this difference in total-force-kill-rate expressions by contrasting the results shown in Table VII with those in Table VIII. Thus, an important decision in developing (i.e. applying) any operational Lanchester-type combat model is whether to model a particular weapon-system type as a serial or parallel acquirer. Unfortunately, no information concerning how to decide the appropriate type of target-acquisition process for a particular weapon-system type (i.e. whether the weapon-system type is a series or parallel acquirer) has been given in the literature. We have shown the reader the importance of this distinction in the simplest context here, and in the next section we will extend these ideas to include the effects of the line-of-sight (LOS) process on the target-acquisition process.

7. Line-of-Sight Process.

In this section we will investigate how the line-of-sight (i.e. intervisibility) process is represented in the attrition-rate coefficients in homogeneous-force Lanchester-type combat models. Representing the line-of-sight (LOS) process allows one to model terrain effects that limit the firing activity due to loss of acquisition capability. The target-acquisition results of the previous section should be thought of as holding when continuous LOS exists between each and every firer-target pair. In the current section we will add a model of the LOS process to that of the acquisition-attrition process in order to investigate the interaction and combined influence of the LOS and target-acquisition processes on the attrition process. Our developments here will build rather heavily on those of the previous section.

The two general methods that have been used in VRI models for representing the effects of terrain on the line-of-sight process may be described as follows:

(TM1) mathematical simulation of actual terrain (actual terrain simulated as if it were a topographic map with three-dimensional relief and LOS determined between two points on this map as needed),

and (TM2) stochastic modelling of LOS process (actual terrain not simulated but its effects on LOS in a statistical sense represented as a stochastic process).

Furthermore, the exact form of the corresponding attrition-rate coefficients for homogeneous-force Lanchester-type combat will also depend on whether target acquisition is modelled as a serial or a parallel process (i.e. whether weapon systems employ serial acquisition or parallel acquisition as described in Section 6 above). Thus, there are actually four cases to be considered for investigating the modelling of the single-system kill rate of a particular weapon-system type against enemy targets:

(C1) mathematical simulation of actual terrain and serial acquisition of targets by weapon-system type,

(C2) mathematical simulation of actual terrain and parallel acquisition of targets by weapon-system type,

(C3) stochastic model of LOS process and serial acquisition of targets by weapon-system type,

(C4) stochastic model of LOS process and parallel acquisition of targets by weapon-system type.

For our purpose here, it will again suffice to consider combat between two homogeneous forces (see Fig. 1 again) and focus on the attrition process of the X force being engaged with Markov-dependent fire because (as already discussed above) the conceptualization of heterogeneous-force combat is developed from this homogeneous-force construct. Before presenting results for each of the four cases (C1) through (C4) above, it seems appropriate to discuss in general terms the two general methods (TM1) and (TM2) for modelling the line-of-sight process.

According to the first terrain-modelling method (TM1), the mathematical simulation of actual terrain, the terrain is represented in the computer by a topographic map (with three-dimensional relief) of the region in which the engagement takes place and LOS determined between two points on this map as required. For such a computer-based model, there are several approaches for simulating the topographic map on the digital computer (e.g. see [28, 49]). For not only LOS determination but also determination of all other parameter values for such a (homogeneous-force) Lanchester-type model, the location of each force is represented by a single point on the topographic map. Consequently, such a Lanchester-type model is sometimes called a lumped-parameter (as opposed to distributed-parameter) model, since all parameter values are determined by the engagement-attribute values at the two reference points, i.e. spatial variations in engagement attributes are ignored and lumped into a single (vector) value for the engagement at time t. Let us denote these two points as P_X and P_Y, where P_X denotes the location of the X force on the topographic map and P_Y that of the Y force. The model then determines whether LOS exists between these two points (e.g. see [28]). It is convenient for us to introduce the following notation concerning existence of LOS:

$$I_{LOS}(P_X,P_Y) = \begin{cases} 1 & \text{if LOS exists between } P_X \text{ and } P_Y, \\ 0 & \text{if LOS does not exist between } P_X \\ & \text{and } P_Y, \end{cases} \tag{7.1}$$

where the reader should bear in mind that P_X and P_Y are functions of time to reflect the movement of the two opposing forces over time, i.e. $P_X = P_X(t)$ and $P_Y = P_Y(t)$. Consequently, for convenience let us denote $I_{LOS}(P_X(t),P_Y(t))$ simply as the intervisibility function $I(t)$, i.e.

$$I(t) = I_{LOS}(P_X(t),P_Y(t)). \tag{7.2}$$

It should be noted that on physical grounds the LOS indicator function $I_{LOS}(P_X,P_Y)$ is symmetric in its arguments, i.e. $I_{LOS}(P_X,P_Y) = I_{LOS}(P_Y,P_X)$. In other words, existence of LOS between two points does not depend on whether an observer is at P_X and looking towards P_Y or at P_Y and looking towards P_X. We will see below how the LOS function $I(t)$ is used to turn on and turn off force-on-force attrition in our Lanchester-type model.

According to the second terrain-modelling method (TM2), the <u>stochastic modelling of LOS process</u>, the location of each of the two forces is again represented by a single point (again denoted as P_X for the X force and P_Y for the Y force) on a conceptual topographic map, but the topographic features of terrain are not directly used to determine whether intervisibility (i.e. LOS) actually exists between the two points P_X and P_Y on this conceptual topographic map. Rather the LOS process is represented by periods of time during which the target at, for example, P_X is visible from P_Y being sandwiched between periods of time during which the target is not visible. The length of such a time period of target visibility or invisibility is taken to be a random variable, influenced not only by the physical LOS process between P_X and P_Y but also by the motion and changes in posture of both the observer and the target. It has been empirically determined [19, p. 53] that it is not an unreasonable assumption to take that the lengths of these time intervals are exponentially distributed random variables. Thus, we may conceptualize this stochastic LOS process in the following manner: periods of target invisibility alternate with periods of target visibility; the length of time that the target is invisible during a period of target invisibility is an exponentially-distributed random variable (with parameter η), and the length of time that the target is visible during a period of target visibility is also an exponentially-distributed random variable (with parameter μ). Furthermore, we will assume that these random variables are all mutually independent.

Thus, the intervisibility process may be represented by two sequences of mutually independent random variables $\{T_1^I, T_2^I, \ldots\}$ and $\{T_1^V, T_2^V, \ldots\}$, exponentially distributed with parameters η and μ. The target can be in either of two states (either invisible or visible), and T_i^I denotes the length of time that the target spends in the invisible state the i^{th} time that it enters this state (with T_i^V being similarly defined for the visible state). The random variables T_1^I, T_2^I, ... are independent and identically distributed (i.i.d.) random variables (with common distribution exponential T^I), and the random variables T_1^V, T_2^V, ... are similarly i.i.d. random variables (with common exponential distribution T^V). Thus, $1/\eta$ is the expected time that the target spends in the invisible state each time that it enters this state, i.e.

$$\frac{1}{\eta} = E[T^I], \tag{7.3}$$

and $1/\mu$ is the expected time that the target spends in the visible state each time that it enters this state, i.e.

$$\frac{1}{\mu} = E[T^V]. \tag{7.4}$$

If, for example, the target starts out by being invisible (i.e. in the invisible state), there will be a transition to the visible state at time T_1^I, a transition back to the invisible state after a further time T_1^V, and so on (see Fig. 4). As a consequence of the assumptions made above (i.e. each of the sequences $\{T_1^I, T_2^I, \ldots\}$ and $\{T_1^V, T_2^V, \ldots\}$ is composed of i.i.d. exponential random variables and the two sequences themselves are independent), this two-state LOS model is a continuous-parameter (i.e. continuous-time) Markov chain (see Fig. 5). It is a straightforward matter to write down the forward-Kolmogorov equations which govern this Markov chain's probabilistic evolution:

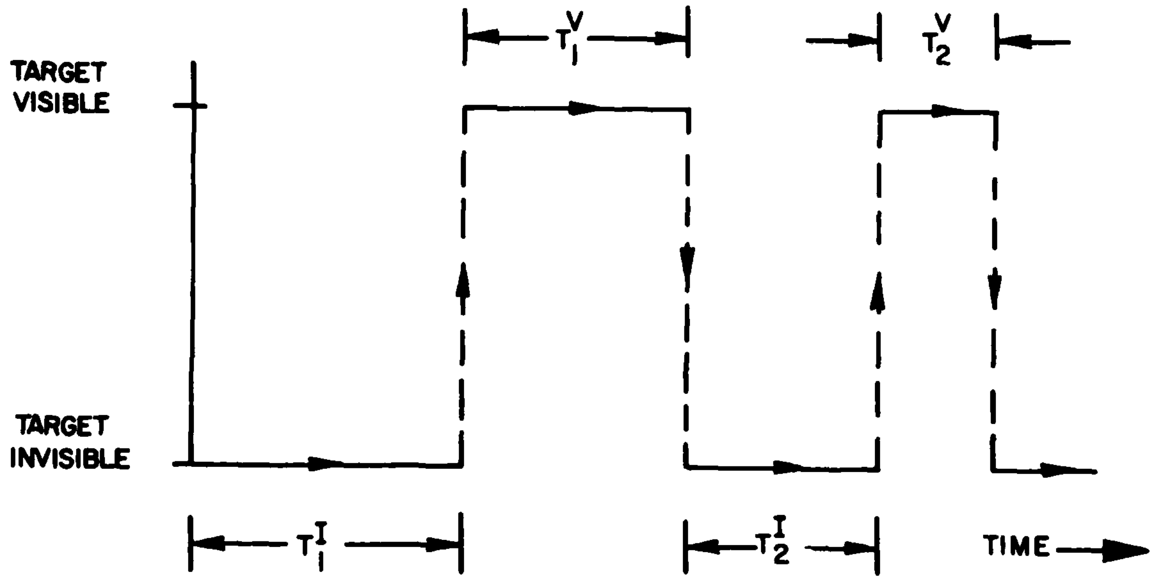

Figure 4. Two-state stochastic model of intervisibility process (technically called alternating-renewal process or alternating-Poisson process). As explained in the text, the model parameters η and μ are defined by $\eta = 1/E[T^I]$ and $\mu = 1/E[T^V]$.

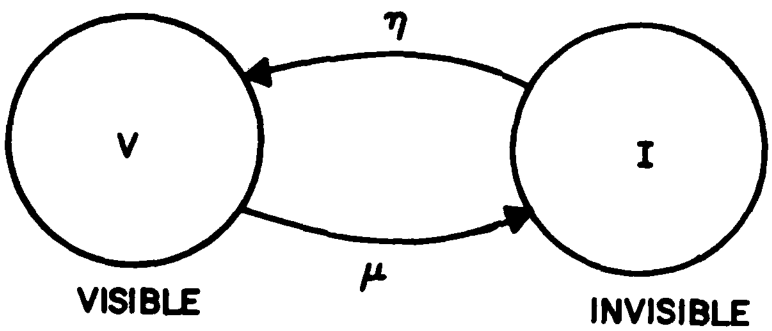

Figure 5. State space and transition structure for two-state continuous-time Markov-chain model of LOS process. Here η denotes the rate at which an invisible target becomes visible, i.e. Prob[target transitions to visible state in Δt] = $\eta\Delta t$, and μ denotes the rate at which a visible target becomes invisible.

let $p_I(t)$ denote the probability that the target is invisible at time t, and let $p_V(t)$ denote the probability that the target is visible at time t; it follows that

$$\begin{cases} \dfrac{dp_I}{dt} = -\eta\, p_I + \mu\, p_V, \\ \\ \dfrac{dp_V}{dt} = \eta\, p_I - \mu\, p, \end{cases} \tag{7.5}$$

which readily yields

$$p_V(t) = \frac{\eta}{\eta + \mu} + \left\{ p_V(0) - \frac{\eta}{\eta + \mu} \right\} e^{-(\eta+\mu)t}. \tag{7.6}$$

The equilibrium (or steady-state) probability of the target being visible $p_V(\infty)$ is easily seen to be given by

$$p_V(\infty) = \frac{\eta}{\eta + \mu}. \tag{7.7}$$

The VECTOR-2 model uses what is equivalent to this steady-state probability, but the exact details differ depending on whether target acquisition is done in the series or parallel mode.

We will now present attrition-rate results for each of the four cases (C1) through (C4) above. As discussed above, we will consider combat between two homogeneous forces (cf. Fig. 1) and will focus on the attrition process of the X force taking casualties inflicted by the Y force. We first consider the mathematical simulation of actual terrain (TM1). For both serial and parallel acquisition of targets, the basic idea is simply to "turn off" the attrition process when LOS is broken (i.e. the targets are all not visible).

<u>Case (C1): Actual terrain and serial acquisition</u>. In this case, (6.1) basically applies, but no attrition can occur when LOS does not exist between the two opposing forces. Additionally, there will be a delay in the starting of the attrition process after an interval of continuous LOS begins, since a target must be acquired, the first round fired, and the round must impact in the target area before any attrition can occur. For illustrative purposes (and also simplicity) let us assume that uninterrupted LOS exists between the two opposing forces in the time interval [0,T]. The corresponding attrition for this situation may be modelled by adding the intervisibility function $I(t)$, defined by (7.1) and (7.2), to the model (6.1) and also introducing the "unit step function" $H(x)$, defined by

$$H(\xi) = \begin{cases} 0 & \text{for} \quad \xi < 0, \\ 1 & \text{for} \quad \xi \geq 0, \end{cases} \tag{7.8}$$

i.e.

$$\left(-\frac{dx}{dt}\right) = I(t)H(t - t_d)\, a\, y, \tag{7.9}$$

where $0 \leq t \leq T$, $t_d = t_a + t_1 + t_f$, and a is again given by (6.2) and (6.3). Here the product $I(t)H(t-t_d)$ "turns on" the X-force attrition when LOS between the two opposing forces exists after a time delay of magnitude t_d (during which time LOS is assumed to continuously exist), and it "turns off" the attrition whenever LOS is lost. Furthermore, the attrition of enemy targets (which occurs in series with the acquisition process) may be thought of as being an "interval" process in the sense that the total-force attrition rate is governed (at least in the simple example considered here) by the length of the time interval during which uninterrupted LOS has existed through the "switch" $I(t)H(t-t_d)$. It is therefore necessary not only to determine whether LOS currently exists but also to keep track of time intervals during which uninterrupted LOS exists.

<u>Case (C2): Actual terrain and parallel acquisition</u>. In this case, (6.4) basically applies, but not only can no attrition occur when LOS does not exist between the two opposing forces but also acquisition of new targets cannot occur. Furthermore, the killing of acquired targets may be thought of as being a "point" process in the sense that whether or not it is "turned on" and operating depends on only whether or not LOS exists at the given instant of time, but the acquisition of enemy targets (which occurs in parallel with the attrition process) may be thought of as being an "interval" process in the sense that the fraction of the firing force that has acquired targets available to engage depends on the acquisition probability accumulated over an interval of time (during which it is assumed that uninterrupted LOS has existed). It is therefore necessary not only to determine whether or not LOS currently exists but also to keep track of those intervals of time during which uninterrupted LOS has existed

and make assumptions about the ability of an observer to remember a target's last-known location when LOS is temporarily broken[8]. For illustrative purposes let us assume that uninterrupted LOS exists between the two opposing forces in the time interval [0,T]. It follows from the results of Section 6 that the attrition rate of the X force is given by

$$\left(-\frac{dx}{dt}\right) = I(t)\alpha\{1 - \exp[-x \int_0^t \lambda_{XY}(s)ds]\}y, \tag{7.10}$$

where $0 \leq t \leq T$ and α is given by (5.4) and (6.6). When LOS exists only intermittently, additional assumptions concerning the ability of an observer to remember last-known locations of enemy targets are required in model building.

We now turn our attention to the method of <u>stochastic modelling of the LOS process</u> (TM2). We recall that this method represents the effects of terrain on the LOS process in terms of the durations of alternating periods of target invisibility and visibility to a single observer. The lengths of these time intervals are assumed to be exponentially-distributed random variables with parameters η and μ [see (7.3) and (7.4) above]. In the parlance of stochastic processes, such a process is technically called an <u>alternating Poisson process</u> (also called an <u>alternating Markov process</u> [19, p. 53]). Let us further assume now that the LOS process is stochastically independent and identical for all observer-target pairs, and that acquisition also occurs independently. As far as the attrition process is concerned, a firer can kill an enemy target only during one of its periods of visibility to him. In other words, the firer must kill an acquired enemy target before LOS is lost. Thus, the stochastic LOS process influences the attrition process both by limiting the availability of targets to be acquired and also by sometimes terminating an engagement before the target has been killed. However, we must now (as usual) treat serial and parallel acquisition separately.

<u>Case (C3): Stochastic LOS and serial acquisition</u>. In serial acquisition a firer must kill an acquired target before he can acquire a new one, and such a kill must occur before LOS is lost. Thus, the stochastic LOS process both limits the availability of targets to be acquired and also sometimes causes an engagement (always assumed to be one-on-one) to be terminated before the target has been killed. For modelling the total-force attrition rate of the X force, we will focus on a single (typical) Y firer and will ask ourselves what is the expected time required for this Y firer to kill an X target $E[T_{XY}]$. The single-weapon-system-type kill rate is simply the reciprocal of this time (see Section 5), and consequently the kill rate of the entire Y force against the X force is given by this single-weapon-type kill rate times the number of Y firers (cf. Section 6 above), i.e.

$$\left(-\frac{dx}{dt}\right) = a(t,x)\, y, \tag{7.11}$$

where (as always)

$$a(t,x) = \frac{1}{E[T_{XY}]}, \tag{7.12}$$

and

T_{XY} = the time (a r.v.) required for a Y firer to kill an X target.

Here we have denoted the single-weapon-system-type kill rate for a Y firer as $a(t,x)$, since the expected time for a Y firer to kill an X target will turn out to depend on the number of targets present as well as possibly changing over time.

Let us now observe that an engagement may be terminated either by the target being killed or by LOS being lost. Considering a single firer trying to engage and kill a single passive target, we see that a particular such engagement can be categorized as

(1) an engagement that ends with the target being killed,

or (2) an engagement that ends with LOS to the target being lost.

When one target has been killed, a new target is immediately engaged, with such a new engagement beginning with search for the new target. One can now define system states for the thusly described attrition process in such a way that the conditions requisite for invoking Barlow's theorem are met. Thus, we will define the following two system states:

S_1 = target-engaged-until-killed state (which lasts from the end of the engagement of the previous target until the present target is killed before LOS to it is lost),

and S_2 = target-engaged-until-LOS-lost state (which lasts from the end of the engagement of the previous target until LOS to the present target is lost without it being killed).

Let us observe that the system will transition to state S_1 (irrespective of where it is now) if the next engagement ends with the target being killed before LOS is lost. If we let p denote the probability that a target is killed before LOS is lost, i.e.

$$p = \text{Prob[target killed before LOS lost]}, \qquad (7.13)$$

then the transition probabilities for the imbedded Markov chain will be given by

$$p_{11} = p_{21} = p \quad \text{and} \quad p_{12} = p_{22} = 1 - p. \qquad (7.14)$$

The above states and the corresponding transition probabilities for changes in system states are shown in Fig. 6. Let us further assume that the time to kill an acquired target (for which uninterrupted LOS exists) is an exponentially distributed random variable with parameter α. In this case, we know that

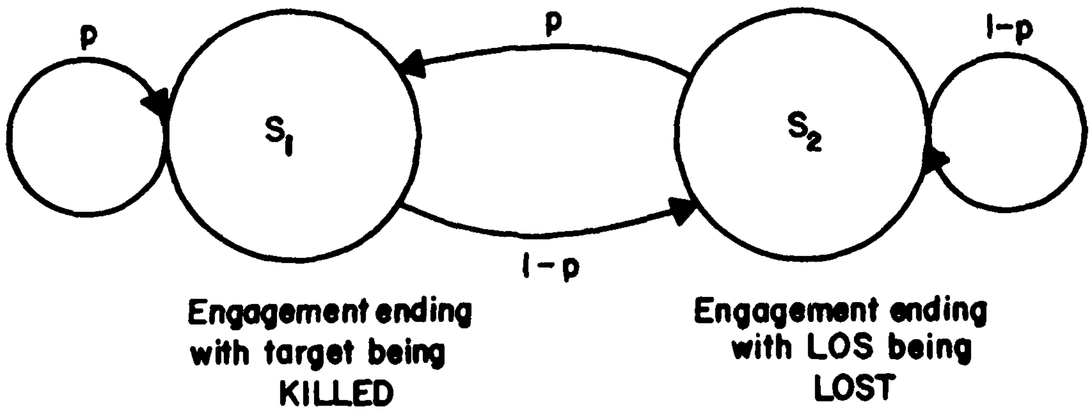

Figure 6. System states and transition probabilities used to derive expression for the expected time to kill a target by invoking Barlow's theorem for a serial acquirer and stochastic LOS. Here p denotes the probability that a target is killed before LOS to it is lost, i.e. p = Prob[target killed before LOS lost].

$$\frac{1}{\alpha} = E[T_{ka_{XY}}] \tag{7.15}$$

where

$T_{ka_{XY}}$ = the time (a r.v.) for a Y firer to kill an acquired X target (given that the target is continuously visible).

One can now invoke Barlow's theorem (see Appendix D for details) and show that

$$E[T] = \frac{1}{p}\left\{E[T_a] + E[T_{ea}]\right\}, \tag{7.16}$$

where

T_a = the time (a r.v.) required to acquire a target,

and T_{ea} = the time (a r.v.) to engage an acquired target until either the target is killed or LOS lost.

For a Y firer engaging an X target (again, see Appendix D for details), it may be shown that $p = \alpha/(\alpha+\mu)$ and $E[T_{ea}] = 1/(\alpha+\mu)$, and hence

$$a = \frac{\left(\frac{\alpha}{\alpha + \mu_{XY}}\right)}{\left\{E[T_{a_{XY}}] + \frac{1}{\alpha + \mu_{XY}}\right\}}, \tag{7.17}$$

where $T_{a_{XY}}$ denotes the time required for a Y firer to acquire an X target and μ_{XY} denotes the reciprocal of the expected time that an X target is visible to a Y firer. If we assume that the target-acquisition process is Markovian with rate parameter λ and that there are N enemy targets present within the acquisition range of the firer and the targets all behave independently, then (see Appendix E for derivation)

$$E[T_a] = \frac{\eta + \mu}{\eta \lambda N}, \tag{7.18}$$

where λ denotes the rate of acquiring a particular type of target when there is a single type of target present and it is continuously visible. When all the X targets are within the acquisition range of the Y firers, (7.17) and (7.18) yield

$$a = \frac{\left(\frac{\alpha}{\alpha + \mu_{XY}}\right)}{\left\{\frac{\eta_{XY} + \mu_{XY}}{\eta_{XY} \lambda_{XY} x} + \frac{1}{\alpha + \mu_{XY}}\right\}}, \tag{7.19}$$

where η_{XY} denotes the reciprocal of the expected time that an X target is invisible to a Y firer and λ_{XY} denotes the rate at which a Y firer acquires X targets when there is a single target present and it is continuously visible. Two limiting cases of the above Lanchester attrition-rate coefficient are particularly noteworthy: (I) when $E[T_{a_{XY}}] = 0$, then $a = \alpha$; and (II) when the X targets are continuously visible to the Y firers (i.e. $\mu_{XY} = 0$), then $a = 1/\{(1/(\lambda_{XY}x)) + (1/\alpha)\}$. To summarize: for <u>stochastic LOS and serial acquisition</u>, the X-force attrition rate is given by (7.11) with $a(t,x)$ given by (7.19).

Case (C4): Stochastic LOS and parallel acquisition. In parallel acquisition an observer continues to acquire new targets while engaging a given target. Once an engagement has been terminated (either by the target being killed or by LOS being lost), such a parallel-acquisition system can immediately shift fire to a new target provided that one was acquired while some previous target was being engaged and LOS still exists. In this case, by the usual "scaling-up" assumption [i.e. (A1') of Section 3], the total-force kill rate is given by the product of the kill rate of a single weapon system against acquired targets and the expected number of firers who have acquired one or more targets, e.g.

$$\left(-\frac{dx}{dt}\right) = p_{A_{XY}}(t)\ \alpha\ y, \tag{7.20}$$

where

$$p_{A_{XY}}(t) = \text{Prob}\begin{bmatrix}\text{a typical } Y \text{ firer (parallel acquirer)} \\ \text{has available one or more acquired } X \\ \text{targets at which to fire at time } t\end{bmatrix},$$

which is exactly the same as (6.4) above. The availability of acquired targets, however, is different for the two different models of the LOS process. For the case in which LOS is modelled by the stochastic process described above, $p_{A_{XY}}(t)$ is given by

$$p_{A_{XY}}(t) = 1 - \{1 - p_{VA_{XY}}(t)\}^{x}, \tag{7.21}$$

since we have assumed that the acquisition process is stochastically independent for all observer-target pairs. Here

$$p_{VA_{XY}}(t) = \text{Prob}\left[\begin{array}{l}X \text{ target visible and } Y \text{ firer} \\ \text{has acquired this given } X \\ \text{target at time } t\end{array}\right].$$

Let us now limit our discussion to only nonfiring acquisition (see Section 6 above).

We will further assume that the length of time required to acquire a visible target is stochastically independent of the LOS process with parameter λ, i.e. $1/\lambda$ is the expected time to acquire a visible target. Combining these assumptions with those for the alternating-Poisson-process for LOS, we may determine the probability that a given target is visible and acquired $p_{VA}(t)$ from a three-state continuous-time Markov-chain model (see Fig. 7) with the following forward-Kolmogorov equations

$$\left\{\begin{array}{lcl}\dfrac{dp_I}{dt} & = & -\eta p_I + \mu p_{VNA} + \mu p_{VA}, \\[2ex] \dfrac{dp_{VNA}}{dt} & = & \eta p_I - (\lambda+\mu) p_{VNA}, \\[2ex] \dfrac{dp_{VA}}{dt} & = & \lambda p_{VNA} - \mu p_{VA},\end{array}\right. \tag{7.22}$$

where $p_I(t)$ denotes the probability that the target is invisible at time t, $p_{VNA}(t)$ denotes the probability that the target is visible but not acquired at time t, and $p_{VA}(t)$ denotes the probability that the target is visible and has been acquired at time t. It follows that

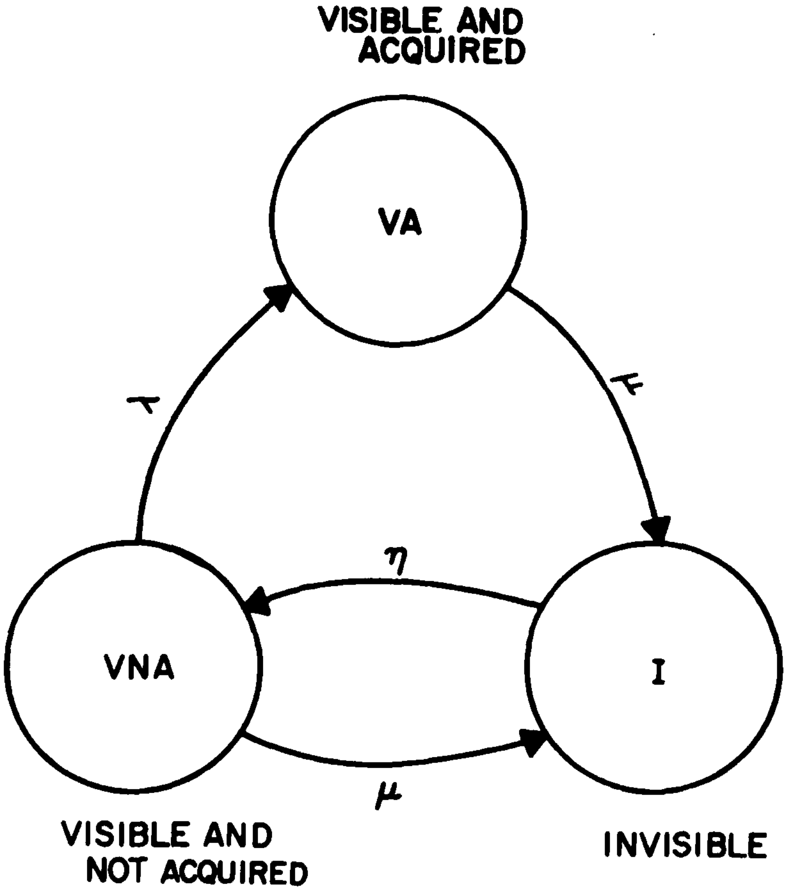

Figure 7. State space and transition structure for three-state continuous-time Markov-chain model of target-acquisition process imbedded in line-of-sight (LOS) process. Here η denotes the rate at which an invisible target becomes visible, i.e. Prob[target transitions to visible state in Δt] = $\eta \Delta t$, λ denotes the rate at which a visible target is acquired, and μ denotes the rate at which a visible target enters the invisible state.

$$p_{VA}(t) = \frac{\eta\lambda}{(\eta+\mu)(\lambda+\mu)} + \left\{ p_{VA}(0) - \frac{\eta\lambda}{(\eta+\mu)(\lambda+\mu)} \right\} e^{-\mu t}$$

$$+ \left\{ p_{VNA}(0) - \frac{\eta\mu}{(\eta+\mu)(\lambda+\mu)} \right\} (1-e^{-\lambda t}) e^{-\mu t}$$

$$+ \left\{ p_I(0) - \frac{\mu}{\eta+\mu} \right\} \left(\frac{\eta\lambda}{\lambda-\eta} \right) \left\{ \left(\frac{1-e^{-\eta t}}{\eta} \right) - \left(\frac{1-e^{-\lambda t}}{\lambda} \right) \right\} e^{-\mu t}, \qquad (7.23)$$

whence the equilibrium (or steady-state) probability of the target being visible and acquired $p_{VA}(\infty)$ is given by

$$p_{VA}(\infty) = \frac{\eta\lambda}{(\eta+\mu)(\lambda+\mu)}. \qquad (7.24)$$

VECTOR-2 currently uses this steady-state probability in ground-force-maneuver-unit-attrition calculations, but use of (7.23) with the appropriate initial conditions would seem to be more appropriate. Returning now to (7.23), we will assume that no targets are initially acquired, i.e. $p_{VA}(0) = 0$. However, targets are distributed between the invisible state and the visible state (with all visible targets being unacquired). We will further assume that when the engagement begins at $t = 0$, the equilibrium distribution between the invisible and visible states has already been reached, i.e. $p_I(0) = \mu/(\eta+\mu)$ and $p_V(0) = \eta/(\eta+\mu) = p_{VNA}(0)$. It follows that

$$p_{VA}(t) = \frac{\eta\lambda}{(\eta+\mu)(\lambda+\mu)} \left\{ 1 - e^{-(\lambda+\mu)t} \right\}. \qquad (7.25)$$

Using this result for the calculation of the target-availability probability (7.21), we will find it convenient to write

$$p_{VA_{XY}}(t) = \frac{\eta_{XY}\lambda_{XY}}{(\eta_{XY}+\mu_{XY})(\lambda_{XY}+\mu_{XY})} \quad 1 - e^{-(\lambda_{XY}+\mu_{XY})t} \quad . \tag{7.26}$$

To summarize: for <u>stochastic LOS and parallel acquisition</u>, the X-force attrition rate is given by (7.20) with α given by (6.5) and (6.6), $p_{A_{XY}}(t)$ given by (7.21), and $p_{VA_{XY}}(t)$ given by (7.26).

We will now close this section by briefly discussing certain aspects concerning the implementation of these ideas in various VRI models. It will also be convenient to touch upon "validation" of such model results against those from a high-resolution Monte-Carlo combat simulation in this context. Consequently, Table IX contrasts the conceptual implementation of these ideas in the BONDER/IUA model with that in VECTOR-2. The reader should bear in mind that the BONDER/IUA model is (in some sense) the conceptual ancestor of VECTOR-2 and is approximately ten years older than it. Thus, this table in some sense depicts the evolution of modelling ideas at VRI (and it has been substantial and virtually unknown beyond a very small circle) and shows their current status in VECTOR-2. Although both models base a ground-force attrition algorithm on parallel acquisition, in many significant ways the details are quite different in the two models, with (for example) firing acquisition apparently not considered in VECTOR-2. Finally, it should be noted that results from

TABLE IX. Some Differences in the Implementation of Selected Modelling Ideas in VRI Models.

	BONDER/IUA	VECTOR-2
LOS Process		
Deterministic Simulation of Actual Terrain	X	
Stochastic Model of LOS		X
Target-Acquisition Process		
Serial		X
Parallel	X	X
Comparison Against High-Resolution-Simulation Results	X	

BONDER/IUA have been compared with those obtained from a high-resolution Monte-Carlo combat simulation [11-12] but that such a comparison apparently has not been carried out for VECTOR-2. Thus, the stochastic LOS model and allied aspects of the attrition algorithms for maneuver units in VECTOR-2 apparently have not been compared with such detailed-simulation results.

8. Attrition-Rate Coefficients for Different Weapon-System Types.

As we have seen above, the time for a single firer to kill an acquired target (equivalently, the rate at which a single firer kills acquired targets of a particular type) may be considered to be a fundamental quantity in any Lanchester-type model for assessing combat attrition. It depends on the following factors:

(F1) firer type,

(F2) target type,

(F3) range between firer and target,

(F4) engagement conditions.

Although we are discussing here attrition-rate coefficients within the context of homogeneous-force combat, it does seem appropriate to show some important connections with current operational models which are all heterogeneous-force models. In particular, VECTOR-2 requires as part of its input data base single-weapon-system-type kill rates against acquired targets (also known as conditional single-weapon-system-type kill rates), denoted here as α_{ij} for the j^{th} weapon-system type of Y firing at the i^{th} weapon-system type of X. These conditional single-weapon-system-type kill rates (i.e. the α_{ij}'s) are externally computed (in a manner consistent with the internal dynamics of VECTOR-2) according to formulae that consider only measurable (i.e. observable) weapon-system characteristics. They are taken to be range dependent (but apparently constant within a given range band). However, the important point to note is that the only essential difference between our discussion here concerning homogeneous-force conditional single-weapon-system-type kill rates and the handling of heterogeneous-force conditional single-weapon-system-type kill rates is merely a notational one (i.e. the adding of double subscripts to identify firer-type/target-type pairs, e.g. α_{ij} instead of merely α).

It seems appropriate for us to say here a few words about the distinction between the "inherent" single-weapon-system-type kill rate a (the rate at which one Y firer kills X targets when he is engaging only them) and the conditional single-weapon-system-type kill rate α (the rate at which one Y firer kills acquired X targets). We saw above that

$$a = \frac{1}{E[T_{XY}]}, \tag{8.1}$$

where

$E[T]$ = the expected time for a single firer to kill a target,

and thus

$$\alpha = \frac{1}{E[T'_{XY}]}, \tag{8.2}$$

where

$E[T']$ = the expected time for a single firer to kill an acquired target.

From the definitions of $E[T]$ and $E[T']$, it follows that

$$E[T] = \bar{t}_a + E[T'], \tag{8.3}$$

where $\bar{t}_a$ denotes the expected time to acquire a target. We may also write (8.3) as

$$E[T] = \bar{t}_a + \frac{1}{\alpha}. \tag{8.4}$$

which shows that α may be considered to be the basic descriptor of "raw" weapon-system kill capability. In the rest of this section we will focus on giving expressions for $E[T']$ (equivalently, α) for different weapon-system types.

Experience has shown that the conditional single-weapon-system-type kill rate is given by quite different expressions for different types of weapon systems. Bonder and Farrell [11] developed their taxonomy for different weapon-system types (see Table IV above) to help structure the general modelling requirements for attrition-rate coefficients. We will now summarize various basic attrition-rate-coefficient results that have been developed for different weapon-system types. These results are the basic ones that are apparently used by the preprocessor to VECTOR-2 for computing values for the conditional single-weapon-system-type kill rates (i.e. α_{ij}'s). Such results have been developed for weapon-system types operating under the following conditions:

(C1) Markov-dependent fire and impact-lethality mechanism,

(C2) Markov-dependent fire and lethality mechanism by which a target can be killed not only by a hit but also by a miss, (C3) burst fire and impact-lethality mechanism,

(C4) multivolley fire and area-lethality mechanism.

We will merely summarize results for $E[T']$ or α here, with the reader being directed to [51, Chapter 5] for a more thorough discussion of the assumptions upon which each expression is based, a derivation of each, and a citation of the original-source literature.

For the case of Markov-dependent fire and an impact-lethality mechanism, the expected time for a single firer to kill an acquired target $E[T']$ is given by

$$E[T'] = \bar{t}_1 - \bar{t}_h + \frac{(\bar{t}_h+\bar{t}_f)}{P(K|H)} + \frac{(\bar{t}_m+\bar{t}_f)}{P(h|m)} \left\{ \frac{[1-P(h|h)]}{P(K|H)} + P(h|h) - p_1 \right\}, \tag{8.5}$$

where all symbols are as explained in Table V, with the exception that $\bar{t}$ denotes average time (e.g. $\bar{t}_1$ denotes the average time to fire the first round after the target has been acquired).

For the case of Markov-dependent fire and a lethality mechanism by which a target can be killed not only by a hit but also by a miss, the expected time for a single firer to kill an acquired target $E[T']$ is given by

$$E[T'] = \bar{t}_1 + \bar{t}_f + \frac{(\bar{t}_h+\bar{t}_f)\{1-P(K|H)\}\{[1-P(K|M)][P(h|m)-p_1]+p_1\}}{P(h|m)P(K|H)\{1-P(K|M)\}+P(K|M)\{1-P(h|h)[1-P(K|H)]\}}$$

$$+ \frac{(\bar{t}_m+\bar{t}_f)\{1-P(K|M)\}\{1-P(h|h)+[P(h|h)-p_1]P(K|H)\}}{P(h|m)P(K|H)\{1-P(K|M)\}+P(K|M)\{1-P(h|h)[1-P(K|H)]\}}, \tag{8.6}$$

where $P(K|M)$ denotes the probability that a miss kills the target and all other symbols are as defined before.

For the case of <u>burst fire and an impact-lethality mechanism</u>, there are two modes of fire to be considered:

(M1) repeated-burst fire [multiple (short) bursts independently fired],

and (M2) mixed-mode fire [repeated-single-shot-Markov-dependent fire until first hit after which there is an immediate switch to burst fire (one long burst)].

For <u>repeated-burst fire</u>, i.e. multiple (short) bursts independently fired, the expected time for a single firer to kill an acquired target $E[T']$ is given by

$$E[T'] = \bar{t}_{B1} + \bar{t}_{Bs} \left\{ \frac{1-P_{SBK_1}}{P_{SBK_s}} \right\}, \tag{8.7}$$

where

$\bar{t}_{B1}$ denotes the average time to fire the first burst after the decision to engage the target has been made,

$\bar{t}_{Bs}$ denotes the average time between the firings of any two successive bursts,

P_{SBK_1} denotes the probability of killing the target with the first burst,

and P_{SBK_s} denotes the probability of killing the target with any subsequent burst.

The simplest model for P_{SBK} is to assume that all rounds within the burst have stochastically independent effects, and then $P_{SBK} = 1 - (1 - P_{SSKB})^n$, where n denotes the number of rounds in the burst and P_{SSKB} denotes the single-shot hit probability for any round in the burst.

For <u>mixed-mode fire</u>, i.e. repeated-single-shot-Markov-dependent fire until first hit after which there is an immediate switch to burst fire (one long burst), the expected time *for a single firer* to kill an acquired target $E[T']$ is given by

$$E[T'] = \bar{t}_1 + \bar{t}_f + (\bar{t}_m + \bar{t}_f)\left\{\frac{1 - p_1}{P(h_1|m)}\right\}$$

$$+ 1 - P(K|H)\left[\bar{t}_h + \bar{t}_f + \bar{t}_b\left\{\frac{1-P_{SSKB}}{P_{SSKB}}\right\}\right], \tag{8.8}$$

where

$\bar{t}_1, \bar{t}_f, \bar{t}_h, \bar{t}_m, p_1$, and $P(K|H)$ are all as previously defined above,

$P(h_1|m)$ denotes the conditional probability of a hit following a miss before the first hit has been obtained,

$\bar{t}_b$ denotes the average time between the firings of any two successive rounds in the burst-fire model,

and $P_{SSKB} = P_{SSHB}P(K|H)$ denotes the probability of killing the target with any one round in the burst-firing mode and P_{SSHB} denotes the corresponding hit probability.

Finally, <u>for the case of multivolley fire and an area-lethality mechanism</u>, the conditional single-weapon-system-type kill rate α

is (approximately) given by

$$\alpha = \nu_Y \{\ln(1 - \lambda S_1)\}x, \tag{8.9}$$

where ν_Y denotes the constant firing rate of the Y weapon system, λ denotes the conditional kill probability for the circular-cookie-cutter damage function with damage radius of R_p, R_t denotes the radius of the circular area target,

$$S_1 = \frac{1}{R_t^2} \int_0^{R_t} P(R_p, r) r dr,$$

$$P(R_p, r) = e^{-r^2/2} \int_0^{R_p} \xi e^{-\xi^2/2} I_0(\xi r) d\xi,$$

and $I_0(\xi)$ denotes the modified Bessel function of the first kind of zero order. Here the function $P(R_p, r)$ is called the circular coverage function.

For the reader's convenience, the various conditions under which different expressions have been developed for the conditional single-weapon-system-type kill rate (equivalently, the expected time for a single firer to kill an acquired target) are summarized in Table X. The equation number(s) of the corresponding formula(e) to each set of conditions is (are) also cited in this table. These formulae allow one to compute all the required conditional-kill-rate inputs[9] to VECTOR-2 and are used for this purpose by the model's preprocessor.

TABLE X. Summary of Various Conditions Under Which Different Expressions Have Been Developed for the Conditional Single-Weapon-System-Type Kill Rate, With Equation Number of Each Expression Given.

(C1) Markov-dependent fire and impact-lethality mechanism: Eq. (8.5)

(C2) Markov-dependent fire and lethality mechanism by which a target can be killed not only by a hit but also by a miss: Eq. (8.6)

(C3) Burst fire and impact-lethality mechanism:

repeated-burst-fire mode--Eq. (8.7)

mixed-fire mode--Eq. (8.8)

(C4) Multivolley fire and area-lethality mechanism: Eq. (8.9)

9. Model-Validation Considerations.

It seems appropriate to briefly discuss the extent to which the Lanchester-type combat models and submodels discussed in this report have been verified (or validated[10]) against empirical combat data. Such historical validation of Lanchester-type models is reviewed in some detail in the author's treatise [51, Section 7.22]. Basically, since essentially only aggregated data for large-scale operations is available, only very simple aggregated Lanchester-type models (i.e. homogeneous-force models that do not consider variation of weapon-system-fire-effectiveness capabilities with range) have been investigated for their scientific validity. Results have been somewhat mixed, with the general consensus being that such simple aggregated models do not have a particularly bad correlation with the available historical data but that there is too much stochastic variability in the parameters estimated from the historical data for the resultant models to have any predictive value.

The detailed models for Lanchester attrition-rate coefficients discussed above compute their numerical values for these coefficients from input values for measurable weapon-system characteristics and may have a high degree of prima facie (or face) validity, but they have never been validated against historical combat data. The primary reason for this lack of empirical verification is that the detailed combat data that is required for such verification [e.g. positions of all combatants (and hence firer-target ranges) as a function of time, firing rate as a function of time, etc.] simply does not exist and the prospects for obtaining it in the future are not at all bright (see [30] for further details). Additionally, even though all weapon-system-type-subsystem inputs (e.g. target-acquisition time, lethality, etc.) can be measured under simulated combat conditions, it does not follow that the overall system in actual combat will (in some conceptual sense) behave as the sum of these parts modelled with simulated combat data. Thus, although the detailed models for Lanchester attrition-rate coefficients have been very logically developed from very plausible assumptions, there is still some uncertainty in the scientific validity of their functional form, and their predictive capability in any absolute sense should not be uncritically accepted. On the other hand, they are at least consistent with the state of the art for combat models, and no other type of combat model (e.g. high-resolution Monte-Carlo combat simulation, firepower-score model, etc.) has been any more scientifically validated against real combat data.

Additionally, the VRI methodology is quite explicit, provides transparent so-called audit trails, and does not rely on any unspecified external inputs or "tuning parameters." There is much to be said for Bonder's [6-7] (see also [19]) methodological approach of determining numerical values for Lanchester attrition-rate coefficients only based on measurable weapon-system-performance characteristics, and this same philosophy has apparently been used in developing the TFECS model, which quantifies the contributions of command and control, intelligence, communications and electronic warfare to the ability of a theater force to attain its objectives [17-18].

10. Determination of Attrition-Rate Coefficients for Heterogeneous-Force Combat.

The modern battlefield contains many different weapon-system types that operate together with complementary capabilities as "combined-arms teams." For example, there might be both mounted and dismounted infantry, infantry with rifles, infantry with machine guns, tanks, different types of anti-tank weapon systems, artillery, mortars, other types of fire-support systems, etc. Since each of these various different weapon-system types would generally inflict and sustain casualties at different rates, when one wants to model the attrition process for combat between such combined-arms teams, one is obliged to keep track of the number of each type of casualty and consider combat between heterogeneous forces.

For such heterogeneous-force combat, the natural generalization of the simple homogeneous-force Lanchester-type paradigm (3.1) [in which the casualty rate of a homogeneous force is equal to the product of the single-weapon-system-type kill rate and the number of opposing homogeneous enemy firers] is given by (3.5). Let us therefore consider the following Lanchester-type paradigm for combat between two heterogeneous forces (see Fig. 2 again)

$$\begin{cases} \dfrac{dx_i}{dt} = -\displaystyle\sum_{j=1}^{n} A_{ij} y_j \quad \text{with} \quad x_i(0) = x_i^0, \\ \\ \dfrac{dy_j}{dt} = -\displaystyle\sum_{i=1}^{m} B_{ji} x_i \quad \text{with} \quad y_j(0) = y_j^0, \end{cases} \tag{10.1}$$

where $x_i(t)$ (for $i = 1,2,\ldots,m$) denotes the number of the i^{th} weapon-system type of the X force at time t, B_{ji} denotes the rate[11] at which one X_i firer kills Y_j targets, and the quantities $y_j(t)$ (for $j = 1,2,\ldots,n$) and A_{ij} are similarly defined for the Y force. Here (as above in Section 3) we will always let the subscript i refer to the X force (and take on the integer values 1 through m) and the subscript refer to the Y force (and take on the integer values 1 through n). The interested reader can find a discussion of the basic assumptions behind the above fundamental heterogeneous-force Lanchester-type paradigm in Section 3 above (see also [50, Section 6.6; 51, Section 7.7]). For present purposes it is not essential that we be explicit about the functional dependence of, for example, A_{ij}. Thus, A_{ij} may stand for a constant A_{ij}, a function of time $A_{ij}(t)$, a function of time and the numbers of targets $A_{ij}(t,\underset{\sim}{x})$, or even $A_{ij}(t,\underset{\sim}{x},\underset{\sim}{y})$.

As we have seen above, a nonnegative quantity such as, for example, A_{ij} is called a <u>heterogeneous-force Lanchester attrition-rate coefficient</u>. It represents the fire effectiveness of one Y_j firer against X_i targets and denotes the rate at which a typical Y_j firer kills X_i targets in the opposing heterogeneous enemy force (see Fig. 2 again). Bonder and Farrell [11] (see also Section 3 above) have argued that one should take such a heterogeneous-force Lanchester attrition-rate coefficient to be given, for example, by

$$A_{ij} = \frac{1}{E[T_{X_iY_j}]}, \tag{10.2}$$

where $E[T_{X_i Y_j}]$ denotes the expected time for a single Y_j firer to kill an X_i target. All the VRI models (including VECTOR-2) have been based on this fundamental premise and the concept of building detailed submodels (based on only measurable inputs) of all required Lanchester attrition-rate coefficients. The development of credible methodology for computing numerical values for such Lanchester attrition-rate coefficients has made possible the use of Lanchester-type combat models as defense-planning tools.

Heterogeneous-force attrition-rate coefficients such as A_{ij} and B_{ji} in the model (10.1) reflect a much greater complexity in the attrition process than do homogeneous-force attrition-rate coefficients such as a and b in the model (5.1): besides being complex functions of weapon-system-type capabilities and target-type characteristics, the attrition-rate coefficients A_{ij} and B_{ji} also depend on additional operational factors such as the distribution of target types, relative rates of target-type acquisition for the various different types of firer-target pairs, procedures and priorities for assigning weapon-system types to target types, etc. In other words, not only must one consider how a given weapon-system type causes attrition to a particular engaged-enemy-weapon-system type (as one does in modelling homogeneous-force-on-force combat attrition), but also one must account for different such pairings occurring at different times and places on the battlefield and also possible changes in these pairings over time. Thus, attrition-rate coefficients for heterogeneous-force combat must reflect much greater complexities of the attrition process than those for homogeneous-force combat. It is of fundamental importance, though, that all approaches known to this author for modelling heterogeneous-force attrition-rate coefficients take homogeneous-force results [e.g. (8.5) through (8.9)] as key "building blocks" for constructing their heterogeneous-force results. In particular, the VRI models use the same conceptual approach (i.e. an individual firer engaging a single passive enemy target) that was used in Section 5 above to develop homogeneous-force-attrition-rate-coefficient results (but now set in the combined-arms-team environment). Furthermore, they take as their basic input the appropriate conditional single-weapon-system-type kill rates that have been computed for firer-type--target-type pairs in essentially a homogeneous-force environment. Moreover, it should be noted here that the use of such values for single-weapon-system-type kill rates, each of which has been computed under conditions independent from the other weapon-system types in the combined-arms operation, implicitly assumes that there are no synergistic effects between different weapon-system types. Thus, although there will occasionally be some minor modifications, we will use (in the appropriate way) all the above homogeneous-force-attrition-rate-coefficient results for developing heterogeneous-force attrition-rate coefficients.

In our discussion here about determining numerical values for heterogeneous-force Lanchester attrition-rate coefficients, we will focus on methodology developed by VRI, since two of the purposes of this tutorial are (1) to foster a greater understanding of the conceptual bases of the assessment of maneuver-unit force-on-force attrition in VECTOR-2 and (2) to be a primer for studying VECTOR-2 and the TFECS model. The interested reader can find corresponding details for other operational Lanchester-type combat models (e.g. IDAGAM, COMANEW, etc.) in the author's treatise on Lanchester-type models [51, Section 5.16]. The principals at VRI (e.g. see [11, pp. 15-16] or [16, pp. 6-7] have found it convenient for modelling attrition-rate coefficients to reflect such complexities of heterogeneous-force combat as discussed above by partitioning the attrition process into four distinct subprocesses:

(SP1) the fire effectiveness of weapon-system types firing at live targets,

(SP2) the allocation process of assigning weapon-system types to target types,

(SP3) the inefficiency of fire when weapon-system types engage other than live targets,

and (SP4) the effects of terrain on limiting firing activities of weapon-system types and on the mobility of the systems.

Exactly now these effects are included in Lanchester attrition-rate coefficients depends in an essential way on how the target-acquisition process is conceptualized: whether one considers so-called <u>serial acquisition</u> of targets or <u>parallel acquisition</u>. Here serial acquisition means that a weapon system is assumed not to acquire targets while engaging other targets. On the other hand, parallel acquisition means that the weapon system is assumed to search continuously for targets, even while engaging other targets (see Section 6 for further details about the distinction between serial and parallel acquisition).

Although they are not strictly mutually exclusive, it will be convenient for future purposes to consider two general ways in which the effects of the above four subprocesses (SP1) through (SP4) have been included in Lanchester attrition-rate coefficients in each of two periods of model development. Moreover, throughout the rest of this section we will always focus on A_{ij}, with B_{ji} being symmetrically determined. Thus, focusing on A_{ij}, we present these four ways as follows:

(PI) Before Development of VECTOR-0

(W1) $$A_{ij} = \psi_{ij} f_{ij}^{Y} a_{ij}, \tag{10.3}$$

or (W2) $$A_{ij} = \psi_{ij} g_{ij}^{Y} \alpha_{ij}, \tag{10.4}$$

and

(PII) During Evolution of VECTOR Series (Currently VECTOR-2)

(W3) $$A_{ij} = F_{ij}^{Y}\left(\alpha_{ij}, \begin{array}{l}\text{all other variables describing}\\ \text{the acquisition and engagement}\\ \text{of targets}\end{array}\right), \tag{10.5}$$

(W4) $$A_{ij} = \psi_{ij} \alpha_{ij}, \tag{10.6}$$

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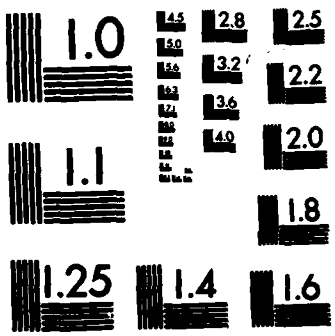

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where

ψ_{ij} denotes the allocation factor (the fraction of Y_j assigned to engage X_i),

a_{ij} denotes the "inherent" single-firer weapon-system serial kill rate (the rate at which one Y_j firer type kills X_i target types when it is engaging only them in an engagement process in which periods of acquiring and firing at a target alternate, with acquisition not going on during firing),

f_{ij}^Y denotes a factor aggregating the effects of all other variables that are not included in either the allocation factor ψ_{ij} or the "inherent" single-firer weapon-system serial kill rate a_{ij} and modifying the effectiveness of an individual Y_j firer type against X_i target types,

α_{ij} denotes the conditional single-firer weapon-system kill rate (the rate at which one Y_j firer type kills acquired X_i target types when it is engaging only them),

g_{ij}^Y denotes a factor aggregating the effects of all other variables that are not included in either the allocation factor ψ_{ij} or the conditional single-firer weapon-system kill rate α_{ij} and modifying the effectiveness of an individual Y_j firer against X_i target types,

and F_{ij}^Y denotes a function that yields the attrition-rate coefficient for a Y_j firer type engaging X_i target types (with arguments as indicated).

Ways (W1) and (W3) are for serial acquisition, while ways (W2) and (W4) are for parallel acquisition. The reader should note (see also Sections 6 and 8 above) the distinction between the "inherent" single-firer serial kill rate a_{ij} (the rate at which one Y_j firer type kills X_i target type when it is engaging only them in an engagement process in which periods of acquiring and firing at a target alternate, with acquisition not going on during firing) other hand, parallel acquisition means that the weapon system is assumed to search continuously for targets, even while engaging other targets (see Section 6 for further details about the distinction between serial and parallel acquisition).

Although they are not strictly mutually exclusive, it will be convenient for future purposes to consider two general ways in which the effects of the above four subprocesses (SP1) through (SP4) have been included in Lanchester attrition-rate coefficients in each of two periods of model development. Moreover, throughout the rest of this section we will always focus on A_{ij}, with B_{ji} being symmetrically determined. Thus, focusing on A_{ij}, we present these four ways as follows:

(PI) <u>Before Development of VECTOR-0</u>

(W1) $$A_{ij} = \psi_{ij} f^{Y}_{ij} a_{ij}, \tag{10.3}$$

or (W2) $$A_{ij} = \psi_{ij} g^{Y}_{ij} \alpha_{ij}, \tag{10.4}$$

and

(PII) <u>During Evolution of VECTOR Series (Currently VECTOR-2)</u>

(W3) $$A_{ij} = F^{Y}_{ij}\left(\alpha_{ij}, \begin{matrix}\text{all other variables describing} \\ \text{the acquisition and engagement} \\ \text{of targets}\end{matrix}\right), \tag{10.5}$$

(W4) $$A_{ij} = \psi_{ij} \alpha_{ij}, \tag{10.6}$$

and the <u>single-firer kill rate against acquired targets</u> α_{ij} (the rate at which one Y_j firer type kills acquired X_i target types when it is engaging only them). In other words, $a_{ij} = \alpha_{ij}$ when the time to acquire a target is equal to zero (see Section 8 for further details).

As we have discussed in Section 8, conditional single-weapon-system-type kill rates (e.g. the α_{ij}'s) are required inputs into the VRI models. Thus, we may consider an α_{ij} to be the fundamental descriptor of inherent weapon-system fire effectiveness which is then modified by the circumstances (i.e. acquisition process, terrain effects, target priorities, etc.) of the engagement. Values for these conditional kill rates are computed external to the model according to the formulas given in Section 8 (see Table X). Although these formulas were given in Section 8 for attrition-rate coefficients in homogeneous-force combat, they are immediately extendable to heterogeneous-force combat simply by adding double subscripts to denote the firer-type--target-type pair to which the conditional single-firer kill rate corresponds (e.g. α_{ij} denotes the conditional kill rate of a single Y_j firer against acquired X_i targets). For example, the conditional single-firer kill rate for a weapon-system type using Markov-dependent fire and an impact-lethality mechanism is given by

$$\frac{1}{\alpha_{ij}} = \bar{t}_1 - \bar{t}_h + \frac{\bar{t}_h+\bar{t}_f}{P(K|H)} + \left\{\frac{\bar{t}_m+\bar{t}_f}{P(h|m)}\right\}\left\{\frac{[1-P(h|h)]}{P(K|H)} + P(h|h) - p_1\right\}, \quad (10.7)$$

where we have suppressed on the right-hand side of (10.7) the double subscripts denoting dependence on the firer-target pair and all symbols are as defined in Sections 5 and 8. Thus, a value for the conditional single-weapon-system-type kill rate α_{ij} for each particular firer-type--target-type pair that is played in the model can be calculated under the engagement conditions of interest by using data for this pair together with the appropriate attrition-rate-coefficient formula given above. In the VRI models, these coefficient values are pre-calculated for all engagement conditions (e.g. firer moving and target stationary, etc.) likely to be encountered when the model is run. Since firer-target range is a continuous-valued variable, a conditional attrition-rate coefficient is computed at a discrete number of ranges and linear interpolation used to generate values for other ranges as needed.

Before providing some detailed results on the modelling of heterogeneous-force-attrition-rate coefficients A_{ij} in two general forms in each of two periods of model development, we will present a brief overview (see Table XI). As we saw in Section 6, depending on whether target acquisition is modelled as a process that is in series or in parallel with the firing process, a fundamentally different mathematical expression is obtained for a Lanchester attrition-rate coefficient. This same basic dichotomy between results for serial and parallel acquisition is reflected in Table XI: ways (W1) and (W3) are for serial acquisition, while ways (W2) and (W4) are for parallel acquisition. We will now provide some detailed heterogeneous-force-attrition-rate-coefficient results for VRI models in the two periods of model development:

TABLE XI. General Ways in which Attrition-Rate Coefficients have been Represented in VRI Lanchester-Type Combat Models in Two Stages of Model Evolution: (I) before VECTOR-0 and (II) in VECTOR-2. Here Ways (W1) and (W3) are for <u>Serial Acquisition</u>, and Ways (W2) and (W4) are for <u>Parallel Acquisition</u>.

(I). <u>Pre-VECTOR-0</u>

(W1) $$A_{ij} = \psi_{ij} f^{Y}_{ij} a_{ij}$$

(W2) $$A_{ij} = \psi_{ij} g^{Y}_{ij} \alpha_{ij}$$

(II). <u>VECTOR-2</u>

(W3) $$A_{ij} = F^{Y}_{ij}\left(\alpha_{ij}, \begin{array}{l}\text{all other variables des-}\\ \text{cribing the acquisition and}\\ \text{engagement of targets}\end{array}\right)$$

(W4) $$A_{ij} = \psi_{ij} \alpha_{ij}$$

(PI) before development of VECTOR-0,

and (PII) during evolution of VECTOR series (currently VECTOR-2).

During the first period (PI), allocation factors (e.g. ψ_{ij}'s) were more explicitly used, while during the second period (PII), target priorities have been more explicitly modelled in the dynamics of the engagement process.

<u>Period (PI): Before Development of VECTOR-0</u>.

The basic concept upon which casualty assessment for direct-fire weapon-system types is based in BONDER/IUA [12] and its many derivatives such as AIRCAV [57], BLDM [2], AMSWAG [29], and FAST [13] is to represent the effects of the above first three subprocesses (SP1) through (SP3) in an attrition-rate coefficient such as A_{ij} with the following functional form (see also [51, Section 7.7] where the basic heterogeneous-force Lanchester-type paradigm with $A_{ij} = \psi_{ij}a_{ij}$ is developed):

$$A_{ij} = \psi_{ij} I_{ij}^{Y} a_{ij}, \tag{10.8}$$

where ψ_{ij} and a_{ij} are as defined after equations (10.3) through (10.6), and I_{ij}^{Y} denotes the <u>intelligence factor</u> (the fraction of those Y_j allocated against X_i who are actually engaging live X_i target types). This intelligence factor, however, has apparently not been considered in any applications (at least through 1975 [16, p. 7]). In other words, I_{ij}^{Y} has been taken to be equal to 1.0 for all i and j, and in this case (10.8) reduces to

$$A_{ij} = \psi_{ij} a_{ij}. \tag{10.9}$$

We could consider (10.9) to apply to both serial and parallel acquisition (with the time to acquire a target being set equal to zero for parallel acquisition), but for better understanding subsequent developments in the VECTOR series of models it is more convenient (as we have done above) to have a_{ij} refer to only the single-weapon-system-type kill rate for serial acquisition and to introduce α_{ij} to refer to the single-weapon-system-type kill rate against acquired targets (see Section 8). Again, we bring to the reader's attention that $a_{ij} = \alpha_{ij}$ when the time to acquire a target is taken to be equal to zero. Thus, although apparently never actually played in BONDER/IUA and its derivatives, equation (10.9) would be used for <u>serial acquisition</u>. The corresponding heterogeneous-force attrition-rate coefficient would be given by

$$A_{ij} = \psi_{ij} \alpha_{ij} \tag{10.10}$$

for <u>parallel acquisition</u>. Again, equation (10.10) is the only way in which values for heterogeneous-force attrition-rate coefficients were actually determined in the BONDER/IUA and its derivatives.

In these models, the Lanchester-type equations that are used for maneuver-unit casualty assessment in direct-fire engagements are numerically integrated in a stepwise fashion (see [51, Appendix E]). In other words, one considers the force levels to evolve (i.e. casualties to occur) in an engagement according to a system of so-called difference equations at discrete points in time instead of the differential equations (10.1) which describe the battle dynamics continuously over time. Thus, battle time is divided into discrete increments called time steps and a complete calculation cycle (see Fig. 8) performed at each time step. In slightly more detail, this calculation cycle is composed of the following steps:

(S1) update clock (time),

(S2) update the position of each group of weapon systems played on the model's battlefield,

(S3) determine whether or not line of sight exists between each pair of such opposing groups,

(S4) determine the attribute values for each engagement between such opposing groups,

(S5) compute the values of the attrition-rate coefficients (e.g. A_{ij}'s),

(S6) assess casualties in time step for each such engagement between opposing groups.

In some sense Fig. 8 also holds in the VECTOR models, although VECTOR-2 has eight different clocks to control the sequence of events in a much more complicated fashion (see [19] for further

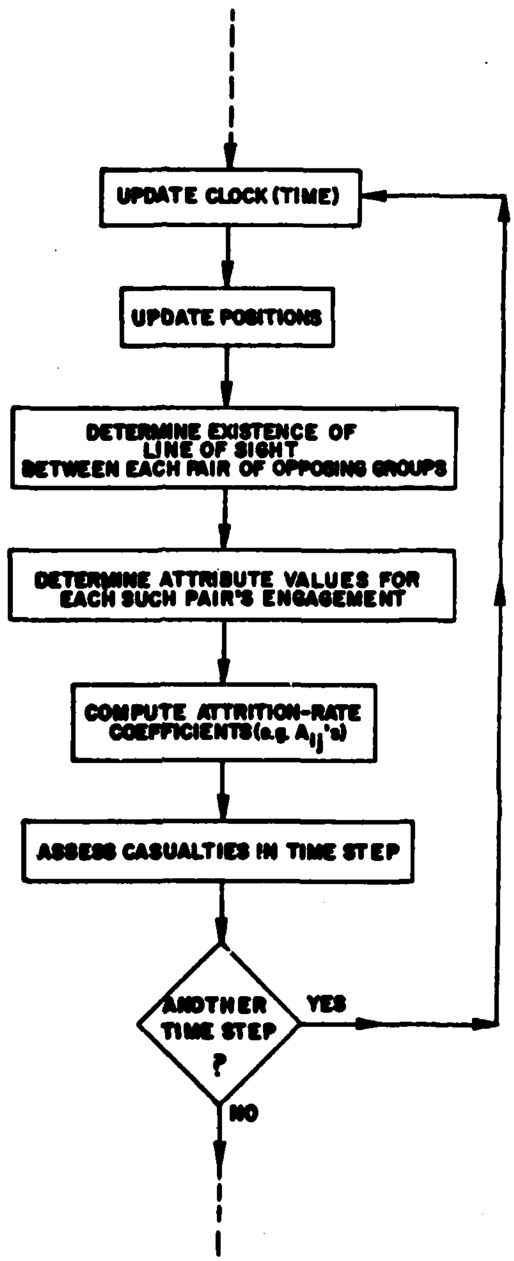

Figure 8. Schematic of basic calculation cycle in typical operational differential combat model. For such calculations, however, the combat dynamics are taken to be represented by a system of difference equations.

details). In BONDER/IUA and its derivatives, the line-of-sight (LOS) process is modelled through the mathematical simulation of actual terrain as discussed in Section 7 above. Thus, the position of each modelled group of combatant forces is represented by a single point on the simulated topographic map. More importantly, the positions of each pair of groups of opposing firers (or potential firers) are represented by a pair of points located on this simulated topographic map, and the existence or nonexistence of LOS between each such pair is determined at each time step. A submodel based on target-acquisition considerations is then used to determine numerical values for the allocation factors ψ_{ij} at each such time step through which the model sequentially moves. The computational procedure used in the original version of the BONDER/IUA was similar to that used in AMSWAG, which is discussed in more detail below[12]. In all these models these allocation factors were calculated based on the assumption of parallel acquisition of targets and a target-priority list, with the AIRCAV and BLDM models using a target-priority list in which more than one type of target was allowed to be tied at the same level of priority to a firing weapon-system type. In actual computation, an algorithm based on a simplifying approximation was used to compute numerical values for such allocation factors (see [57, pp. 29-32] or [2, pp. III-6 through III-8]). Attrition of weapon-system types in direct-fire engagements is then assessed using a finite-difference approximation to the basic Lanchester-type paradigm (10.1) with the attrition-rate coefficients (fo[illegible] am[illegible]e, A_{ij}'s) computed according to (10.10) in the BONDER/IUA. However, various enrichments have subsequently been added during the evolution of derivative models such as BLDM, AMSWAG, and FAST, which compute a value for an A_{ij} according to a formula like (10.4). We will now focus on the calculation of values for heterogeneous-force attrition-rate coefficients (e.g. A_{ij}'s) in AMSWAG.

In the AMSWAG [29] model, attrition-rate coefficients are modelled as

$$A_{ij} = \psi_{ij} U_j \alpha_{ij}, \tag{10.11}$$

where U_j denotes the fraction of the firer-type Y_j that are unsuppressed. Submodels are used for

(a) the suppression factor U_j [29, pp. 15-17],

and (b) the fire-allocation factor ψ_{ij} [29, pp. 18-21].

We will now discuss in detail the fire-allocation submodel used in AMSWAG.

The following factors influence which target types will be engaged by a particular firer type in AMSWAG and what allocation of fire they will receive[13]

(F1) target-type priority,

(F2) range to target,

(F3) intervisibility,

(F4) round choice,

and (F5) target-type acquisition.

In AMSWAG each firing weapon-system type has its own target priority scheme which allows different target types to have the highest priority at various ranges. An example of one such firer-type target-priority scheme is shown in Fig. 9. It is assumed that a firer type will attempt to allocate its fire-power against the enemy target type currently having the higher priority, with the closest target not necessarily having the highest priority (see Fig. 9). However, if two potential targets are of the same type, the one at the shortest range always has the higher priority. Besides being an important factor in target priority, the range (distance) between firer and target also determines firing feasibility, i.e. no firing event can take place beyond the specified maximum effective range of the firing weapon-system type. Moreover, no target (regardless of priority or proximity) can receive any fire allocation if line of sight from the firer to that particular target (i.e. intervisibility) does not exist. However, if line of sight does exist, the fact that a target is seen either partially exposed or fully exposed does not affect either the target's priority or its allocation.

The availability of ammunition of the appropriate type also influences the allocation of fire in AMSWAG: a proper round choice must exist before a firer type can allocate its fire against a particular target type. Round choice is modelled for each firer-type--target-type combination by a table of first and second choices of rounds at both short and long ranges, plus a threshold range used to determine whether the current firer-target range will be classified as either short or long (see Table XII). If

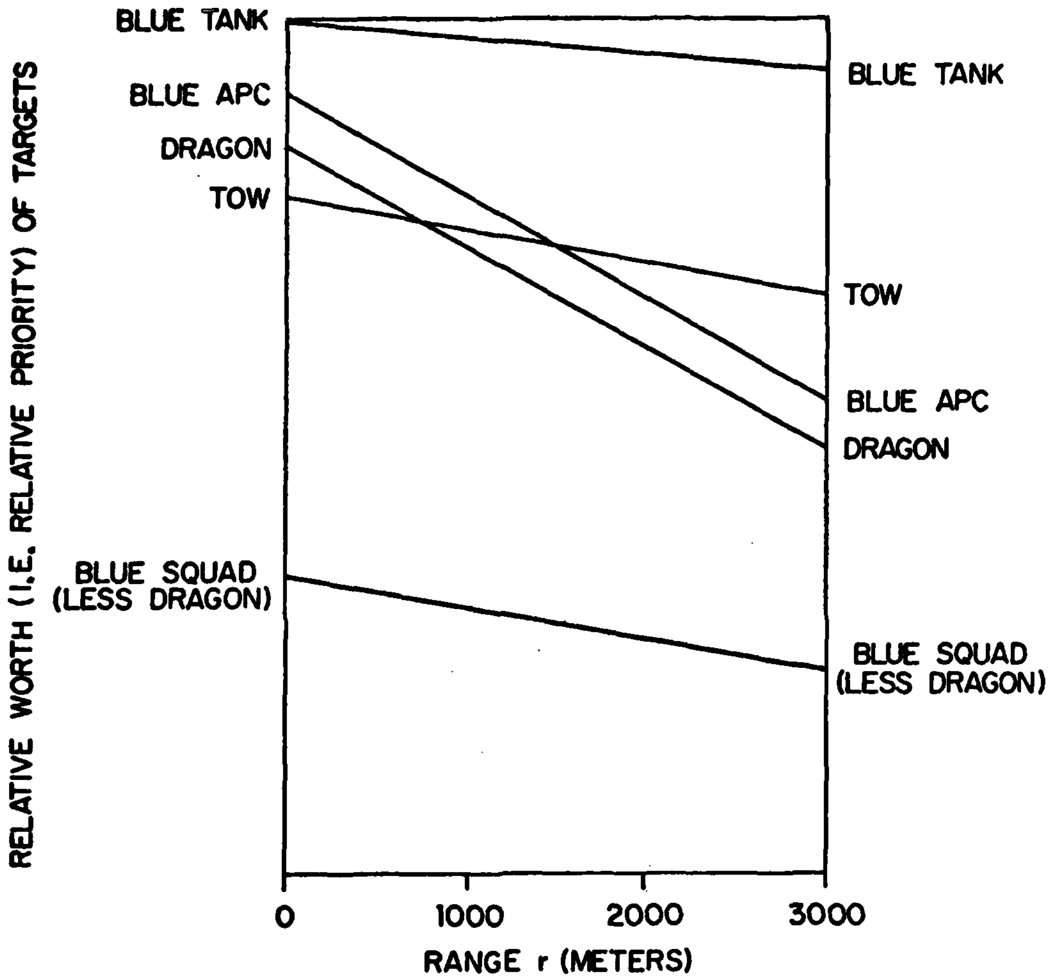

Figure 9. Typical target-type priorities used in AMSWAG for a BMP firer in Europe with Blue on the attack (from [29]).

TABLE XII. Sample Round Choices Used in AMSWAG (from [29]).

Firer Type	Target Type	First Choice at Shcrt Range	First Choice at Long Range	Second Choice at Short Range	Second Choice at Long Range	Threshold Range (in meters) Used to Distinguish between Short and Long Range
M60A3	BMP	HEAT	APDS	APDS	HEAT	1500
M60A3	SQUAD	COAX	HEP	HEP	HEAT	1000
MICV	ATGM	COAX	HE	HE	COAX	1000
BMP	MICV	73mm HEAT	SAGGER	73mm HEAT	SAGGER	800
M113/ TOW	T62	TOW	TOW	TOW	TOW	1250

for some reason the first choice of round type cannot be fired, the model tries to carry out the firing event with the second-choice round type. If neither round type can be fired, the target type receives no allocation of fire during this time interval. [Here the term time interval refers to the fact that the battle has been segmented into a large number of small time steps (i.e. intervals) for computational reasons as per the numerical integration of the Lanchester-type attrition equations (see [51, Appendix E], especially, Figure E.1).] Currently in ASWAG, there are two reasons why a particular round type might not be used: (1) the particular firer type does not have available that type of round, and (2) the firer is moving and that type of round cannot be fired from a moving platform. Thus, a target type will receive an allocation of fire only when all the following conditions have been met:

($\overline{C}1$) the firer type has not allocated more than ninety-eight percent of its firepower;

($\overline{C}2$) the target type is the highest priority target type that has not already received an allocation;

($\overline{C}3$) the target type is within the maximum effective range of the firer type;

($\overline{C}4$) line of sight exists;

and ($\overline{C}5$) a proper choice of round type exists.

Finally, target-acquisition probabilities determine in the following way exactly what the allocation by a firer type against a particular target type will be when all the above conditions have been met. The cumulative detection probability for each firer type (say the i^{th}) against each target type (say the j^{th}) is computed at each time step since the existence of intervisibility. If we let p_{ij} denote this cumulative detection probability, then in such an "expected-value" model as AMSWAG p_{ij} is interpreted as representing the fraction of the i^{th} firer type that has detected the j^{th} target type. Then the fraction of fire allocated by the i^{th} firer type against the j^{th} target type cannot exceed p_{ij} times the unallocated portion of the firer type's fire. A firer type continues to allocate its fire until it runs out of target types or has allocated more than ninety-eight percent of its firepower [see [29, p. 21] for further details).

Period (PII): During Evolution of VECTOR Series (Currently VECTOR-2).

VECTOR-2 [19, 38] also considers a conditional single-firer kill rate (e.g. an α_{ij}) to be the fundamental descriptor of a weapon-system type's inherent fire effectiveness and uses different formulas to compute numerical values for the attrition-rate coefficients A_{ij} according to whether the target-acquisition process is done in series with or in parallel with the killing of acquired targets[14]. The two major factors determining the numerical value of an attrition-rate coefficient in VECTOR-2 are

(F1) the acquisition and selection of targets, and (F2) the conditional single-firer kill rate against acquired target types, α_{ij}.

The acquisition and selection of targets in VECTOR-2 is conceptualized as consisting of the following three processes:

(P1) the line-of-sight process, which determines whether a given target type is visible or not to a particular firer type,

(P2) the target-acquisition process, which determines the time for a firer type to acquire a particular target type,

and (P3) the target-selection process, which represents how a particular target type is selected for engagement from among those acquired.

The interaction of these three processes depends on whether target acquisition is done in series or in parallel. In both cases each firer type orders all opposing enemy target types into a priority list, which the model uses to determine which target types are to be engaged first. Terrain effects are played stochastically through the LOS process in essentially the same way as discussed above in Section 7, only here within the setting of heterogeneous forces. Thus, LOS effects on force-on-force attrition were considered within the context of homogeneous forces in Section 7, but we are considering combat between heterogeneous forces in the section at hand. However, in such heterogeneous-force combat all firers and targets are assumed to operate independently and identically, and consequently the basic LOS-modelling ideas of Section 7 essentially carry over directly to the heterogeneous-force case considered here. Moreover, because of the diversity of weapon-system types, additional factors such as target-type priorities have been incorporated into the attrition-rate coefficient models of a single firer engaging an enemy target.

In serial acquisition in VECTOR-2 (cf. the discussions of the target-acquisition and LOS processes in Sections 6 and 7 above) the acquired target type of highest priority is engaged by a particular firer type until it has been destroyed or until line of sight has been lost. At this time the serial acquirer must acquire a new target. Moreover, past acquisitions are not remembered by the serial acquirer. Also, in searching for a new target, the timeliness of acquisition is given consideration through a series of search-cutoff times. When there are m target types, the selection of the next target type involves a sequence of $(m-1)$ search-cutoff times. Prior to the k^{th} cutoff time (where $k < m$), the observer looks for only target types of priorities 1 through k and ignores any lower priority targets. If the observer has not acquired a target by the $(m-1)^{st}$ cutoff time, he will then engage the first target acquired (regardless of its priority). Once a target is acquired in serial acquisition, it cannot be preempted by a higher priority target, and only its destruction or loss of line of sight can cause fire to be shifted away from it. In parallel acquisition search for new targets continues even during the engagement of acquired targets. When the target has been destroyed, a higher priority target type has been acquired, or line of sight has been lost; fire is instantaneously shifted to the highest priority acquired enemy target type. A parallel acquirer does remember all past target-type acquisitions. It should be noted here that these two different conceptual models of target acquisition lead to two completely different expressions for the Lanchester attrition-rate coefficient: the attrition-rate coefficient for serial acquisition may be developed using the mean-first-passage-time result given in Section 5 for a continuous-time-semi-Markov process (cf. the use of Barlow's theorem in Section 7 for homogeneous-force combat), while that for parallel acquisition may be developed by straightforward probability arguments.

The following is a summary of the assumptions made in VECTOR-2 concerning target-type acquisition and selection in maneuver-unit combat [19, pp. 53-54]:

(A1) the time to acquire a target, given that it is continuously visible, is an exponentially distributed random variable with parameter λ_{ij}, where i is an index denoting the weapon-system type of the target and j is an index denoting the weapon-system type of the firer;

(A2) the line-of-sight process between a pair of opposing weapon-system types in an alternating Markov process with two states--visible and invisible;

(A3) the line-of-sight process for an observer-target pair is independent of that for all other pairs;

(A4) there are two modes of acquiring targets; an observer using the parallel mode acquires targets continuously, even while engaging other targets; an observer using serial acquisition can acquire only between engagements of targets;

(A5) when an observer in the parallel mode acquires a target of higher priority than the one being engaged, he shifts his fire instantaneously to the target of higher priority;

and (A6) an observer in the serial mode selects a new target whenever he loses line of sight to the previous target or the previous target is killed (the model assumes that the firer can perfectly distinguish between active and killed weapon systems and never engages killed systems); there is a sequence of cutoff times to limit the time spent searching for certain target types, such that prior to the n^{th} cutoff time only weapon-system types of priorities 1 through n are eligible as targets.

Thus, the target-acquisition-and-selection process transforms a Y weapon-system type's (say the j^{th}) kill rates against acquired X target types (α_{ij} for $i = 1,2,\ldots,m$) into an achieved kill rate against a particular enemy target type (say the i^{th}) A_{ij} that accounts for target priorities and the various competing activities in which a single firer may be engaged over time. Moreover, the amount of attrition actually assessed against a force is limited by a tactically acceptable maximum attrition rate (see [19, pp. 54-55] for further details). We will now give attrition-rate-coefficient results for the two cases

(CA1) serial acquisition of targets,

and (CA2) parallel acquisition of targets.

For the former case (CA1), it is additionally assumed for the derivation of an expression for A_{ij} that the time to kill an acquired target is exponentially distributed [with parameter α_{ij}, where i is an index denoting the weapon-system type of the target (here X_i) and j is an index denoting the weapon-system type of the firer (here Y_j)]. Also, in VECTOR-2 the maximum number of weapon-system types in a maneuver element is currently 11, i.e. with a homogeneous portion of the battlefield $m = n = 11$ where m and n are X- and Y-force integer index limits appearing in (for example) summations below.

For serial acquisition of targets in VECTOR-2, the heterogeneous-force LANCHESTER attrition-rate coefficient A_{ij} is taken to be given by

$$A_{ij} = \frac{h_{ij}P_{ij}}{\sum_{k=1}^{m} E[T_{kj}^{as}] + \left\{\frac{1}{\alpha_{kj} + \mu_{kj} + \tilde{A}_{kj}}\right\}}, \tag{10.12}$$

where

$$h_{ij} = \text{Prob}\left[\begin{array}{l}\text{a group-i target (here } X_i\text{) being fired} \\ \text{upon a acquired by a group-j firer (here} \\ Y_j\text{) will be destroyed by that firer} \\ \text{before either line of sight is lost or} \\ \text{the target is destroyed by another} \\ \text{firer}\end{array}\right],$$

$$P_{ij} = \text{Prob}\left[\begin{array}{l}\text{a group-j weapon which employs serial} \\ \text{acquisition acquires and selects a} \\ \text{group-i target type when it selects} \\ \text{a target}\end{array}\right],$$

$E[T_{ij}^{as}]$ = expected time on a given acquisition that a group-j weapon spends acquiring and selecting a group-i target [here $T_{ij}^{as} = 0$ if the acquisition is of a non-group-i target; also if $T_{ij}^{as} > 0$ for some i, then $T_{ij}^{as} = 0$ for all other i],

$\frac{1}{\alpha_{ij}}$ = expected time that a group-j weapon firing at a group-i target requires to achieve a kill, i.e. the single-firer weapon-system kill rate against an acquire target [it should be recalled that the corresponding time to achieve a kill (a r.v.) has been assumed to be exponentially distributed with parameter α_{ij}],

$\frac{1}{\mu_{ij}}$ = expected time that a weapon system in group i spends in the visible state (for a weapon in a group j) each time that it enters that state [it is assumed that the corresponding time (a r.v.) is exponentially distributed with parameter μ_{ij}],

$\frac{1}{\eta_{ij}}$ = corresponding value for the invisible state,

and $\frac{1}{\tilde{A}_{ij}}$ = expected time for any firer other than the single group-j firer in question to kill a particular target in group i.

In somewhat simpler words, P_{ij} denotes the selection probability of an X_i-type target by a Y_j-type firer, and h_{ij} denotes the corresponding destruction probability. Similar to the homogeneous-force case considered in Section 7 (see Appendix D for details), the above expression (10.12) was developed by taking the Lanchester attrition-rate coefficient to be the reciprocal of the expected time to kill a target [i.e. (10.2)] and then by invoking Barlow's [4] mean-first-passage-time result for a continuous-time semi-Markov process (see Theorem 5.1) to determine this expected time. Consequently, in VECTOR-2 the target-destruction process has been conceptualized in such a way that this latter result could be invoked (see [19, pp. 55-67] for further details). Because of the complexity of (10.12), we will not derive this expression here. It should be emphasized, however, that except for some differences in modelling details due to "heterogeneous-force effects" (e.g. target-type priorities) the conceptual basis of (10.12) is essentially the same as that for the Lanchester-attrition-rate-coefficient expression for homogeneous-force combat (7.17). The reader will therefore find it instructive to compare (10.12) with (7.17).

We will now give expressions for all the remaining computed quantities in (10.12) (again, see [19] for further details). Accordingly, we have

$$h_{ij} = \frac{\alpha_{ij}}{\alpha_{ij} + \mu_{ij} + \tilde{A}_{ij}}, \qquad (10.13)$$

and

$$P_{IJ} = D_{IJ}(t^{CO}_{I-1,J})\{\prod_{i=1}^{I-1} \bar{D}_{iJ}(t^{CO}_{t-1,J})\}$$

$$\times \exp\{-\sum_{i=1}^{I-1} R_{iJ}N_{iJ}[t^{CO}_{I-1,J} - t^{CO}_{i-1,J}]\}$$

$$+ R_{IJ}N_{IJ} \sum_{\ell=I-1}^{m-1} \{\prod_{k=1}^{\ell+1} \bar{D}_{k,J}(t^{CO}_{k+1,J})\}$$

$$\times \exp\{\sum_{k=0}^{\ell} R_{k+1,J}N_{k+1,J}t^{CO}_{RJ}\}$$

$$\times \frac{1}{Z_{\ell J}}\{[\exp(-Z_{\ell J}t^{CO}_{\ell J})] - [\exp(-Z_{\ell J}t^{CO}_{\ell+1,J})]\}, \tag{10.14}$$

where

$$D_{IJ}(t) = \text{Prob}\begin{bmatrix}\text{observer in group } J \text{ (here } Y_j) \\ \text{has a target in group } I \text{ (here} \\ X_I) \text{ under surveillance at time} \\ t \text{ after initial of search}\end{bmatrix},$$

$\bar{D}_{IJ}(t) = 1 - D_{IJ}(t)$,

t^{CO}_{ij} = cut-off time for an observer in group J searching for targets to exclusively engage acquired targets of priority classes 1 through I (i.e. a target of priority class $I+1$ will not be engaged in acquired before $t^{CO}_{IJ} < t^{CO}_{I+1,J}$) (see Table XIII; also Karr [31, pp. 32-33]),

N_{IJ} = expected number of currently surviving group-I targets within range of a group-J firer,

$$R_{IJ} = \frac{\lambda_{IJ}\eta_{IJ}}{\eta_{IJ} + \mu_{IJ}}, \tag{10.15}$$

TABLE XIII. Rules for Target Selection by Serial Acquirer in VECTOR-2.

Time	Priorities of Targets to be Engaged Immediately Upon Acquisition	Priorities of Targets to be Engaged if Previously Acquired and Still Visible
$[0, t_{1J}^{CO})$	1	
t_{1J}^{CO}		2
$(t_{1J}^{CO}, t_{2J}^{CO})$	1, 2	
t_{2J}^{CO}		3
$(t_{2J}^{CO}, t_{3J}^{CO})$	1, 2, 3	
$\vdots$		
$(t_{m-2,J}^{CO}, t_{m-1,J}^{CO})$	1,2, ... , m-1	
$t_{m-1,J}^{CO}$		m
$(t_{m-1,J}^{CO}, +\infty)$	1, 2, ... , m-1, m	

$\frac{1}{\lambda_{IJ}}$ = expected time for a weapon in group J (here Y_J) to detect a visible target in group I (here X_I) [it should be recalled that the corresponding time to detect (a r.v.) has been taken by assumption (A1) to be exponentially distributed with parameter λ_{IJ}],

and

$$Z_{\ell J} = \sum_{k=1}^{\ell+1} R_{kJ} N_{kJ}. \tag{10.16}$$

Here the two conventions have been followed that (1) a summation over an empty index set is always taken to be equal to zero, and (2) a product taken over an empty index set is always taken to be equal to one, e.g. $\sum_{k=1}^{0} T_k = 0$ and $\Pi_{k=1}^{0} T_k = 1$. Also, the complement of a cumulative distribution function like (for example) $D_{IJ}(t)$ has been denoted as $\overline{D}_{IJ}(t)$, and we then (of course) have $\overline{D}_{IJ}(t) = 1 - D_{IJ}(t)$. Let us observe that $0 \le N_{IJ} \le x_I$. The target types have been indexed in such a way that X_1 denotes the highest priority target, X_2 denotes the next highest, etc. It remains for us to give an expression for $D_{IJ}(t)$ in order that P_{IJ} as given by (10.14) may be computed: the following expression has been developed for $D_{IJ}(t)$ (see [19, pp. 62-63] for further details)

$$D_{IJ}(t) = 1 - \left[1 - \frac{R_{IJ}}{\mu_{IJ} + \tilde{A}_{IJ} - R_{IJ}}\{\exp(-R_{IJ}(t) - \exp[-(\mu_{IJ} + \tilde{A}_{IJ})t]\}\right]^{N_{IJ}}. \tag{10.17}$$

Returning now to the computation of the Lanchester attrition-rate coefficient A_{ij} by (10.12), we see that it remains for us to give expressions for the expected time to acquire and select a target $E[T_{IJ}^{as}]$ and the single-firer kill rate of X_i-type targets by other than Y_j-type firers $\tilde{A}_{ij}$. The following expression has been developed for $E[T_{IJ}^{as}]$ (see [19, pp. 65-66] for further details)

$$
\begin{aligned}
E[T_{IJ}^{as}] \quad = \quad & t_{I-1,J}^{CO} D_{IJ}(t_{I-1,J}^{CO}) \{ \prod_{i=1}^{I-1} \bar{D}_{iJ}(t_{i-1,J}^{CO}) \} \\
& \times \exp\{ \sum_{i=1}^{I-1} R_{iJ} N_{iJ} [t_{I-1,J}^{CO} - t_{i-1,J}^{CO}] \} \\
& + R_{IJ} N_{IJ} \sum_{\ell=I-1}^{m-1} \{ \prod_{k=1}^{\ell} \bar{D}_{k+1,J}(t_{kJ}^{CO}) \} \\
& \times \exp\{ \sum_{k=0}^{\ell} R_{k+1,J} N_{k+1,J} t_{kJ}^{CO} \} \\
& \times \frac{1}{Z_{\ell J}^2} \{ (Z_{\ell J} t_{\ell J}^{CO} + 1) \exp(-Z_{\ell J} t_{\ell J}^{CO}) \\
& - (Z_{\ell J} t_{\ell+1,J}^{CO} + 1) \exp(-Z_{\ell J} t_{\ell+1,J}^{CO}) \} .
\end{aligned}
\tag{10.18}
$$

Finally, the following approximation has been developed for $\tilde{A}_{ij}$ and is used in VECTOR-2

$$
\tilde{A}_{ij}(t + \Delta t) \quad = \quad \sum_{\substack{\ell=1 \\ \ell \neq j}}^{n} A_{i\ell}(t) \; f_{\ell}^{j}(t) , \tag{10.19}
$$

where

$$f_{\ell}^{j}(t) = y_{\ell}(t)/\{\sum_{\substack{k=1\\k\neq j}}^{n} y_k(t)\}$$ = fraction of total Y weapons exclusive of group j that Y weapons of group ℓ comprise.

Here, the fact that the differential-equation force-on-force attrition model is numerically integrated by discretizing time into time steps (see [51, Appendix E] has been used to develop this approximation, with the right-hand side of (10.19) being evaluated at the old time step and the left-hand side being taken at the new one. In way of summary, <u>the computation of A_{ij} for weapons that employ serial acquisition requires the following inputs</u>: α_{ij}, μ_{ij}, η_{ij}, λ_{ij}, N_{ij}, y_j, and t_{ij}^{CO}.

The interested reader can find the derivation of the above serial-acquisition attrition-rate-coefficient results sketched in [19, pp. 55-68] (see also Karr [31, pp. 38-44]). It will be instructive, however, for us to briefly consider the development of the expression (10.14) for P_{IJ}, the probability of selecting a target from target-type group I. This probability is given by

$$\begin{aligned} P_{IJ} = {} & D_{IJ}(t_{I-1,J}^{CO}) \prod_{i=1}^{I-1} \{\bar{F}_{T_{iJ}^{a}}(t_{I-1,J}^{CO} - t_{i-1,J}^{CO})\bar{D}_{iJ}(t_{i-1,J}^{CO})\} \\ & + \sum_{\ell=I-1}^{m-1} \int_{t_{\ell J}^{CO}}^{t_{\ell+1,J}^{CO}} \{\prod_{k=I}^{\ell} \bar{D}_{k+1,J}(t_{kJ}^{CO})\bar{F}_{T_{k+1,J}^{a}}(t - t_{kJ}^{CO})\} \\ & \times \{\prod_{k=0}^{I-2} \bar{F}_{T_{k+1,J}^{a}}(t - t_{kJ}^{CO})\bar{D}_{k+1,J}(t_{kJ}^{CO})\} \\ & \times \bar{D}_{IJ}(t_{I-1,J}^{CO})\, dF_{T_{IJ}^{a}}(t - t_{I-1,J}^{CO}), \end{aligned} \tag{10.20}$$

where

T^a_{ij} = the time (a r.v.) for an observer in group i to acquire a target in group j, with cumulative distribution function

$$F_{T^a_{ij}}(t) = \text{Prob}[T^a_{ij} \leq t].$$

The <u>first term</u> on the right-hand side of (10.20) represents the probability that a target in group I (here X_I) is under surveillance at time $t^{CO}_{I-1,J}$ and that no higher priority target was ever under surveillance at a time before $t^{CO}_{I-1,J}$ at which time it would have been engaged, while the <u>second term</u> represents the probability that a target in group I was acquired at some time t after $t^{CO}_{I-1,J}$ and that neither a higher priority target nor a lower priority one was ever under surveillance at a time before t at which time it would have been engaged. It follows from assumptions (A1) through (A3) above that

$$F_{T^a_{ij}}(t) = 1 - \exp(-R_{ij}N_{ij}t), \tag{10.21}$$

whence substitution of (10.21) into (10.20) yields (10.14). The expression (10.18) for $E[T^{as}_{IJ}]$ may be developed in a similar fashion. Finally, it is worthwhile to observe here that $\eta_{ij}/(\eta_{ij} + \mu_{ij})$ gives the probability that a target of type i is visible. Recalling that λ_{ij} denotes the rate of acquisition of a group-i target by a group-j observer, we then immediately see the justification of (10.21). Finally, the reader should note the great similarity between (10.21) and the corresponding homogeneous-force result (E.7) given in Appendix E. The fact that these two expressions are (except for the obvious difference in notation) identical stems from the assumption made that all observers and firers may be considered to behave independently on VECTOR-2's heterogeneous-force battlefield.

For parallel acquisition of targets in VECTOR-2, the heterogeneous-force Lanchester attrition-rate coefficient A_{ij} is taken to be given by

$$A_{ij} = f_{X_iY_j}\,\alpha_{ij}, \tag{10.22}$$

where

$$f_{X_iY_j} = \text{Prob}\begin{bmatrix}\text{at random point in time any } Y_j \\ \text{weapon-system type employing} \\ \text{parallel acquisition is firing at} \\ \text{an } X_i \text{ target type}\end{bmatrix}.$$

We further have that this probability that a Y_j weapon-system type is firing at an X_i target type $f_{X_iY_j}$ is given by

$$f_{X_iY_j}\begin{cases} p_{A_{X_1Y_j}} & \text{for } i = 1, \\ p_{A_{X_iY_j}} \cdot \prod\limits_{k=1}^{i-1} (1 - p_{A_{X_kY_j}}) & \text{for } i = 2,\ldots,m, \end{cases} \tag{10.23}$$

where

$$p_{A_{X_iY_j}} = \text{Prob}\begin{bmatrix}\text{a typical } Y_j \text{ firer (parallel} \\ \text{acquirer) has available one or} \\ \text{more acquired } X_i \text{ target types} \\ \text{at which to fire}\end{bmatrix},$$

and this target-type-availability probability $p_{A_{X_iY_j}}$ is given by

$$p_{A_{X_iY_j}} = 1 - \left[1 - \frac{\eta_{ij}\lambda_{ij}}{(\eta_{ij}+\mu_{ij})(\lambda_{ij}+\mu_{ij})}\right]^{N_{ij}}, \qquad (10.24)$$

Here (as above) N_{ij} denotes the expected number of currently surviving X_i targets that are within acquisition and firing range of a Y_j observer/firer. The reader should note the similarity between the previously given homogeneous-force results for stochastic LOS and parallel acquisition [i.e. the combination of (7.20) and (7.21) with $p_{VA}(t)$ given by (7.24)] and the above heterogeneous-force results. Their similarity again stems from the assumption made that all observers and firers may be considered to behave independently on VECTOR-2's heterogeneous-force battlefield.

We may consider the above probability that at a random point in time a Y_j weapon-system type is firing at an X_i target type $f_{X_iY_j}$ to be an allocation factor ψ_{ij}. Furthermore, the expression for $f_{X_iY_j}$ has been derived from a model of the LOS, target-acquisition, and target-selection (i.e. target priorities) processes, and this model combines all these factors into the probability that a Y_j weapon system is firing at an X_i target. It is worthwhile to note that the expression for the heterogeneous-force Lanchester attrition-rate coefficient for parallel acquisition in VECTOR-2, i.e. A_{ij} as given by (10.22), is of the same form as that used in the BONDER/IUA model, i.e. A_{ij} as given by either (10.6) or (equivalently) (10.10). Moreover, the expressions for these allocation factors are different in the two models, since (for example) target acquisition is considered to be time-homogeneous in VECTOR-2 and firing acquisition (see Section 6) is considered in BONDER/IUA.

The above attrition-rate-coefficient results for stochastic LOS and parallel acquisition of targets in heterogeneous-force combat may be developed in exactly the same manner as we developed the homogeneous-force ones, i.e. (7.20) and (7.21), since all firers and targets are assumed to behave stochastically independently in both cases. The expression for $f_{X_iY_j}$ (10.23) may be obtained by observing that the probability that a Y_j weapon-system type is firing at an X_i target type is simply given by the product of the probability that an X_i target type is available and the probability that no higher priority target type is available (i.e. any X_k target type for $1 \leq k \leq i-1$). The expression for $p_{A_{X_iY_j}}$ (10.24) may be developed in exactly the same way as (7.21) with $p_{VA_{XY}}(t) = p_{VA_{XY}}(\infty)$ and $p_{VA}(\infty)$ given by (7.25). Thus, the expression for $p_{A_{X_iY_j}}$ (10.24) has embedded in it the assumption that the target-acquisition/LOS process (see Fig. 7) has reached its steady state[15].

Finally, let us give a brief overview of the data-base requirements for computation of attrition-rate coefficients in VECTOR-2. Current values of the following parameters are required for the calculation of attrition-rate coefficients at each time step:

(P1) number of survivors in each weapon-system-type group;

(P2) conditional single-firer kill rate, α_{ij} or β_{ji};

(P3) acquisition rate for each weapon-system type in each observing and observed group[16], λ_{ij}^{XY} or λ_{ij}^{YX},

(P4) rates for the alternating-MARKOV-renewal line-of-sight process, μ_{ij} and η_{ij};

(P5) fraction of targets within range for every pair of firer type and target type;

and (P6) rate of fire for each weapon-system type.

The parameters (P1) are obtained from other parts of VECTOR-2, while (P6) is an external-user input. Parameters (P2) through (P5) are internally computed in the model. These computations involve more detailed input data from the following four classes (see [19, pp. 70-71] for further details):

(DC1) scenario data expressing differences in force employment (e.g. between armored, mechanized, and dismounted infantry units); such data reflect the initial geometry and maneuver patterns of forces and the making of such tactical decisions as, for example, when to mount and dismount infantry into APCs, (DC2) movement data consisting of the speed of each weapon-system type (indexed on terrain trafficability),

(DC3) line-of-sight data consisting of the rates of entering and leaving the visible state in each of the terrain visibility classes,

(DC4) weapon-system-performance data (including the firing rate for each weapon-system type) used to compute the conditional single-firer kill rate, acquisition rate, and the fraction of the target group within range for each firer-type/target-type pair.

From the above brief sketch, the reader undoubtedly senses that the data-base requirements for VECTOR-2 are rather demanding. In fact, upwards of 350,000 pieces of input data are required for its running (see Bonder [9, p. 36]), and many man-months of effort are involved in the use of this much data in such a complex operational model, e.g. the time required to acquire the input data, the time required to structure this data into the model's input format, the time required to run the model, and the time required to analyze and evaluate the model's results (see [5] for further details).

11. Final Remarks.

The goal of this tutorial has been to review the most salient points concerning the conceptual and operational bases for assessing casualties with Lanchester-type combat models [particularly complex operational models such as BONDER/IUA and VECTOR-2 developed by Vector Research, Inc. (VRI) or other models that have evolved from these]. This tutorial has focused on the currently existing methodology for the calculation of numerical values for single-weapon-system-type kill rates (or Lanchester attrition-rate coefficients), whose numerical determination stands at the heart of casualty assessment in such models. The reader is reminded, however, that casualty assessment is only one of many important, interrelated combat processes (e.g. movement; command, control, communications, and intelligence; etc.) that are represented in a force-on-force combat model. Furthermore, such attrition-rate-coefficient methodology is important for the following three reasons. Firstly, Lanchester-type models are currently more widely used in various U.S. Army and DoD planning activities than ever before, and such current acceptance points to even more increased use in the future. Secondly, a single-weapon-system-type kill rate is a basic element of any Lanchester-type conventional-ground-combat model, and besides the inherent firepower of a weapon-system type such a Lanchester-type attrition-rate coefficient reflects (at least in the methodology presented here) line of sight, acquisition of targets, and selection of targets, which are processes deemed to be of great significance by military tacticians for the effective application of firepower.

Thirdly, significant new developments have occurred (many produced by VRI) in methodology for developing more tactically realistic Lanchester attrition-rate coefficients, and these important results have not been accessible to a very wide audience.

Operational Lanchester-type models of ground combat (e.g. FOURCE or VECTOR-2) are very complicated to say the very least, but (as we have emphasized in Section 2) they may be viewed as having been developed from a simple basic Lanchester-type paradigm through the process of model enrichment. Thus, in the process of model building, one starts with a simple basic idea and enriches it in details in an evolutionary fashion. It is the author's basic hypothesis in writing this tutorial that in order to explain the conceptual bases of such a complex operational model (especially one whose development is based on significant new methodology that is essentially unknown to the military OR community at large) one should return to the basic paradigms from which the model has evolved and try to capture the fundamental modelling philosophy that has guided the model-enrichment process. The basic philosophy behind the determination of Lanchester attrition-rate coefficients in VRI's models is to consider how a single typical firer of a particular type engages and kills a single enemy target of a particular type. This process is analyzed in detail and a model constructed out of only measurable quantities. Thus, we have not sought to recreate here the VECTOR-2 model itself but have tried to capture in a simple setting the basic ideas of the methodology for determining single-weapon-system-type kill rates and the modelling philosophy that has guided the enrichment of these ideas in an evolutionary fashion. This philosophy of explaining the basic methodology (out of which a complicated model has been built) in simple terms that nevertheless capture the essence of the basic ideas may also be useful for model evaluation and especially model documentation.

The author has been particularly impressed by VRI's philosophy (going back to Bonder's Ph.D. thesis [6]) of building process models (not only for attrition but also for other combat processes such as command, control, communications, and intelligence) that contain only measurable quantities (i.e. inputs). This model-building approach is truly scientific (within the epistemological limitations inherent in such combat analysis) and in this respect is unique among the modelling philosophies formally articulated by the builders of such models. Although the resulting combat model takes only input data that can be generated by some type of military field test, there are two epistemological dangers present here (especially for the unwary): (1) such data can only be obtained from <u>simulated</u> combat and not real combat, and (2) although all the subsystem inputs have an empirical origin, the basic paradigm that combines them all together does not. With regard to this last point, the basic problem is that real (i.e. historical) combat data is at a much more aggregated level than is the basic paradigm on which the determination of a Lanchester attrition-rate coefficient is based. Such detailed combat data simply does not exist (see McQuie [37] for further details). Moreover, such criticism applies to <u>all</u> detailed combat models, irrespective of the modelling methodology upon which their development has been based, and creates a severe epistemological dilemma which is far beyond the scope of our current discussion.

Some people have criticized the VRI models because of their requirement for a huge data base of a very detailed nature. This situation is certainly true, but it is undoubtedly the price that one must pay for the explicit (as opposed to implicit) treatment of factors such as line of sight or target acquisition. Some models (e.g. IDAGAM or TACWAR) claim to have such factors implicit in their data base (with apparently no well-defined methodology for regulating the dosage of such implicit factors into the input data, or even documented and examinable), but cause-and-effect relations between changes in militarily-relevant weapon-system characteristics, doctrine, and weapon-system deployment and model are simply not present in them. Thus, such a model's internal dynamics are more static than dynamic (see Farrell [21] for an excellent discussion of implicit versus explicit treatment of such factors in combat models). Since any combat model used repeatedly in U.S. Army analyses will be required to examine a multitude of questions about interactions of weapon-system capabilities, the combat environment, tactics, and doctrine under a wide spectrum of circumstances, such a model must be general purpose and fairly rich in internal dynamics, not a highly specialized (or limited) model calibrated for a single set of circumstances. Thus, it appears that fairly general models that in some sense duplicate many of the real world's micro-combat interactions are required by U.S. Army analyses. If one wants to duplicate on the computer the complexity of modern large-scale conventional air-ground combat, then one is going to have a very complicated model with a very large required data base. A basic problem in the documentation of such a model is that the visibility (i.e. explicit representation) of model logic produces clutter [21, pp. 93-94], and a lot of clutter when everything is explicit. Nevertheless, the author believes that a complicated model can be documented in a hierarchical fashion to retain transparency and capability for so-called "audit-trail analysis" [e.g. see [45; 48]), provided that the reader of such documentation is familiar with the methodologies used to build its various pieces.

Thus, since recent developments in attrition-rate-coefficient methodology apparently are not very widely known, this tutorial has attempted to popularize the basic ideas that are involved in order that the users of these models may better understand them. The author feels that the acceptability of such models (both by the decision maker and also by the analyst) will dramatically increase as their basic paradigms become better known.

FOOTNOTES

[1]Here we mean a Lanchester-type (i.e. rate-of-change-based) model that represents enough of the complexities of actual combat operations to be used to address planning/operational problems. Because of the size and scope of such combat operations, all such operational Lanchester-type models require a modern large-scale digital computer for their implementation and approximate the differential equations (which conceptually represent the rates of change of things on the battlefield) with difference equations (which are numerically solved in a step-by-step fashion). For the reader's convenience and ready reference, we will collect here references to the documentation of all the models considered in this tutorial, and we will generally omit further reference to such documentation (unless it is to give a page citation to substantiate some point) in order to enhance the tutorial's readability by reducing clutter. We should warn the reader that (as emphasized by Shubik and Brewer [44]; see also [15]) even when it does exist, documentation of an operational combat model is generally weak, poor, uneven, incomplete, inadequate, and all too frequently unavailable. However, the following documentation and information is exceptionally good for this field. General information about contemporary combat models (and primarily focusing on conventional ground combat) in the United States is available in [5; 50-52; 55]. Information about the BONDER/IUA may be found in [11-12] (see also [13]), but for the many important

subsequent developments one should consult documentation on its various derivative models: BONDER AIRCAV (or IHA) [57], BLDM [2] (see also [5]), AMSWAG [29], and FAST [13]. Documentation of DIVOPS is given by [59], while that of FOURCE by [54] (see also [53]). VECTOR-2 is documented in [38] and [19] (see also [10]), but the reader may still want to consult documentation on VECTOR-0 [58] and VECTOR-1 [60] (see also [5; 20]) out of which it has evolved. TFECS is documented in [17-18]. IDAGAM is documented in [1] (see also [45]), while TACWAR is in [32; 34] (see also [5; 33]). For the reader's convenience, we summarize here (according to level-of-combat represented) all the above operational Lanchester-type combat models:

battalion-level combat:	BONDER/IUA and its many derivatives such as BONDER AIRCAV (or IHA), BLDM, AMSWAG, FAST,
division-level combat:	DIVOPS, FOURCE,
theater-level combat:	VECTOR-0, VECTOR-1, VECTOR-2, IDAGAM, TACWAR, TFECS (C^3I processes only).

[2]For general (fairly comprehensive) background on large-scale general-purpose-force combat models (especially concerning problems related to documenting and evaluating them), the reader should consult the recent GAO report [55]. Various aspects of the problems of documenting and evaluating complex models are discussed in the articles by Brewer and Hall [14], Strauch [47], Hall [27], Gass [25], and Gass and Thompson [26].

[3]The basic references here are Bonder [6; 8] and Barfoot [3] (see [51, Section 5.3 and Footnotes for Chapter 5] for a review of historical developments in the evolution of this important methodology).

[4]Our discussion here more or less follows that of Barlow [4] (see [51, Section 5.9] for further background and a brief guide to the literature about semi-Markov processes).

[5]To be precise, we should say that the firer believes that the target has been killed. However, for simplicity (and following developments in the field) we will assume that the firer possesses perfect perception of the target's state. Basic scientific research on the behavior and perceptions of firers is required before more realistic values of such kill rates can be estimated.

[6]From the theory of nonnegative matrices (e.g. see Gantmacher [24]) via (5.18), it follows that any r_2 and r_3 satisfying (5.21) must be nonnegative. More precisely, since the stationary probabilities π_j for $j = 1,2,3$ defined by (5.18) always exist [24, p. 98], we know that the ratios $r_j = \pi_j/\pi_1$, which may be considered to be defined by (5.21), are also guaranteed to be nonnegative.

[7]We additionally have the condition that

$$\sum_{i=1}^{3} \pi_i = 1,$$

which has not been used to obtain (5.21) from (5.18) via (5.19). Although the above condition must be used to determine the stationary probabilities π_j for $j = 1,2,3$, it is not needed to determine r_2 and r_3.

[8] Strictly speaking, when the Markov assumption (i.e. the future is independent of the past and only dependent on the present) is made concerning target acquisition (as it is here), an observer loses all acquired targets when LOS is lost. This assumption is currently employed in the BONDER/IUA and all its derivatives (e.g. the AMSWAG model [29]) which directly simulate actual terrain [i.e. use method (TM1)], but it has been questioned by combat-experienced military OR analysts as to whether an observer ever really loses all knowledge about the last-known enemy target locations when LOS is temporarily broken (see J. Smoler [46, pp. 30-31] for further details). More research is clearly needed on the modelling of target acquisition when LOS is temporarily broken.

[9] The mixed-mode version of burst fire (8.8) is currently not considered by VECTOR-2.

[10] We are using here the words "verification" and "validation" interchangeably. Many authors distinguish between the terms "the verification of a model" and "the validation of a model," but there is apparently no consistent use of these terms in the literature (see, for example, Morris [39], Bonder [6, pp. 30-31], Fishman and Kiviat [23], Van Horn [56], and Naylor and Finger

[41]). For our present purposes, however, such a distinction does not seem to be warranted, especially since there is no consistent use of these terms in the literature.

[11] It is not assumed here that B_{ji} is constant. In fact, for present purposes one need not make any assumption about the variables upon which B_{ji} depends, i.e. no particular functional dependence is assumed here.

[12] Smoler [46, pp. 10-11] has pointed out that both the detection and fire-allocation submodels in AMSWAG contain several features that are at variance with military experience and judgment. He has consequently proposed an alternative fire-allocation procedure [46, pp. 31-36]. See also Footnote 8 above.

[13] Our discussion here is drawn from Hawkins [29]. Also, see Footnotes 8 and 12 above.

[14] See Karr [31, pp. 31-47] for a critique of the determination of the expressions for the attrition-rate coefficients in VECTOR-2, which in this respect is essentially the same as that in VECTOR-0 and VECTOR-1.

[15] From the quite similar homogeneous-force developments given in Section 7, i.e. (7.23) and its special case (7.25), the reader can see that there are other results for $p_{A_{X_iY_j}}$, analogous to those used for $p_{VA_{XY}}(t)$ in (7.21), that are based on more operationally realistic assumptions (i.e. other than assuming that the steady state has been reached) that could be used for developing

and expression for $p_{A_{X_i Y_j}}$. In particular, this availability probability could be taken to be time dependent, i.e. $p_{A_{X_i Y_j}} = p_{A_{X_i Y_j}}(t)$.

[16]Here λ_{ij}^{XY} denotes the acquisition rate of a Y_j-type observer against X_i-type targets, while λ_{ij}^{YX} denotes that of an X_j-type observer against Y_i-type targets. In our previous discussion of heterogeneous-force Lanchester attrition-rate coefficients above, e.g. see (10.12), it was not considered necessary to be absolutely precise, and for simplicity's sake we used the symbols λ_{ij}, R_{ij}, t_{ij}^{CO}, etc. without superscripts.

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APPENDIX A: Justification for Taking the Lanchester Attrition-Rate Coefficient as the Reciprocal of the Expected Time to Kill a Target.

The Lanchester attrition-rate coefficient is the rate at which a single firer kills a particular enemy target type in Lanchester-type combat. Such a single-weapon-system-type kill rate is a fundamental part of any Lanchester-type combat model, and the development of technically-sound and scientifically-valid methodology for determining numerical values for Lanchester attrition-rate coefficients is an essential prerequisite for building militarily credible Lanchester-type combat models to be used in the study of U.S. Army problems. Within the context of the basic deterministic homogeneous-force Lanchester-type paradigm for modern warfare

$$\begin{cases} \dfrac{dx}{dt} = -ay & \text{with} \quad x(0) = x_0, \\ \\ \dfrac{dy}{dt} = -bx & \text{with} \quad y(0) = y_0, \end{cases} \tag{A.1}$$

a and b are Lanchester attrition-rate coefficients, and (for example) a denotes the rate at which a single Y firer destroys X target types.

The basic construct of the Bonder/Barfoot methodology is to take a Lanchester attrition-rate coefficient as the reciprocal of the expected time for an individual firer to kill an enemy target. Within the context of the above homogeneous-force Lanchester-type paradigm (A.1), this basic construct would read, for example,

$$a = \frac{1}{E[T_{XY}]}, \tag{A.2}$$

where T_{XY} denotes a random variable (abbreviated r.v.) representing the time for an individual Y firer to kill an X target and $E[T]$ denotes the expected value of T. It is the purpose of this appendix to provide justification for taking the Lanchester attrition-rate coefficient as the reciprocal of the expected time to kill a target, e.g. to justify (A.2). It suffices to do so within the context of homogeneous forces, since all known (at least to this author) heterogeneous-force Lanchester-type paradigms assume that all firer types and target types essentially behave independently of one another (except that they are tied together with the Lanchester-type casualty-assessment equations and other combat-process models). Thus, justification of this basic principle immediately extends to heterogeneous-force Lanchester-type paradigms and will be briefly discussed in the heterogeneous-force context below. Bonder and Farrell [11] (see also [19; 57; 59]) have based their approach for determining attrition-rate coefficients for a wide spectrum of weapon-system types on this definition (A.2), and it forms the basic construct for predicting attrition-rate coefficients (and hence assessing casualties) for direct-fire maneuver-unit engagements in the Vector Research, Inc. (VRI) models. It is therefore of considerable interest to inquire as to what justification exists for basing the calculation of Lanchester attrition-rate coefficients on this basic principle, e.g. on (A.2). We will first consider a heuristic justification of (A.2), and then we will consider several more rigorous justifications (see [51, Section 5.3] for further details).

All justifications of (A.2) known to this author are ultimately based on the following basic hypothesis:

BASIC HYPOTHESIS (BH). Combat is a complex random process, but it contains enough regularity that the appropriate Lanchester-type equations are a good approximation to the mean course of combat.

We will begin with a few heuristics in a more general case for motivating justification of (A.2). Consider combat between two homogeneous forces and assume that each force's loss rate depends on only the number of opposing combatants and not time explicitly (see Fig. A.1). We may model the force-on-force attrition process with the following deterministic Lanchester-type equations

$$\begin{cases} \dfrac{dx}{dt} = -A(x,y) & \text{with} \quad x(0) = x_0, \\[2ex] \dfrac{dy}{dt} = -B(x,y) & \text{with} \quad y(0) = y_0. \end{cases} \tag{A.3}$$

Here the number of (for example) X combatants, which is actually a nonnegative integer, is represented by the real number $x(t)$, since we must take the force levels to be continuously-varying quantities in order to model their changes over time with differential equations. We will assume that there are no replacements

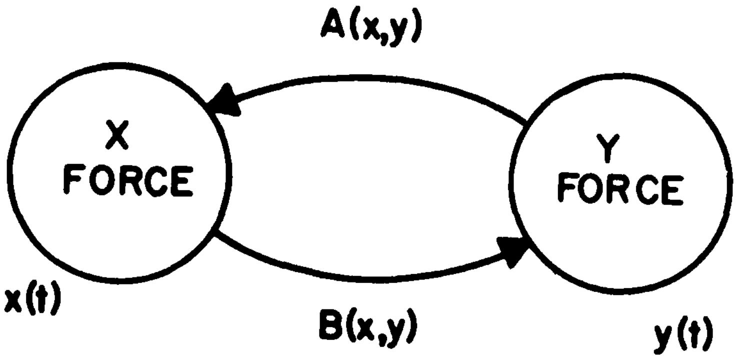

Figure A.1. Diagram of force interactions considered in heuristic justification (in more general case) of taking Lanchester attrition-rate coeffi..ient as reciprocal of expected time between casualties.

and withdrawals, and then A and B are simply the attrition rates of the X and Y forces, respectively.

Our basic hypothesis (BH) says that we may consider the Lanchester-type equations (A.3) to approximately represent the mean course of the complex random process of combat between the X and Y forces. It implies an underlying stochastic combat process. We will consider the simplest model of this stochastic attrition process: a continuous-parameter Markov chain in which time varies continuously and casualties occur discretely (see [51, Chapter 4] for a more detailed discussion of such a model and its relationship to the corresponding deterministic Lanchester-type equations). This model is equivalent to assuming that the times between casualties are exponentially distributed random variables with force-level-dependent parameters (or rates). Letting $M(t)$, a random variable, denote the integral number of X combatants alive at time t (with corresponding realization denoted as m) and $N(t)$ similarly for the Y force (see Fig. A.2), we find that the following forward Kolmogorov equations describe the probabilistic evolution of the force levels for $0 \leq m_{BP} < m \leq m_0$ and $0 \leq n_{BP} < n \leq n_0$

$$\frac{d}{dt}P(t,m,n) = P(t,m+1,n)A(m+1,n) + P(t,m,n+1)B(m,n+1) - \{A(m,n) + B(m,n)\}P(t,m,n), \tag{A.4}$$

where $P(t,m,n) = P[M(t) = m, N(t) = n | M(0) = m_0, N(0) = n_0]$, m_{BP} denotes the breakpoint (see [51, Chapter 3]) of the X force, n_{BP} denotes the breakpoint of the Y force, and for convenience

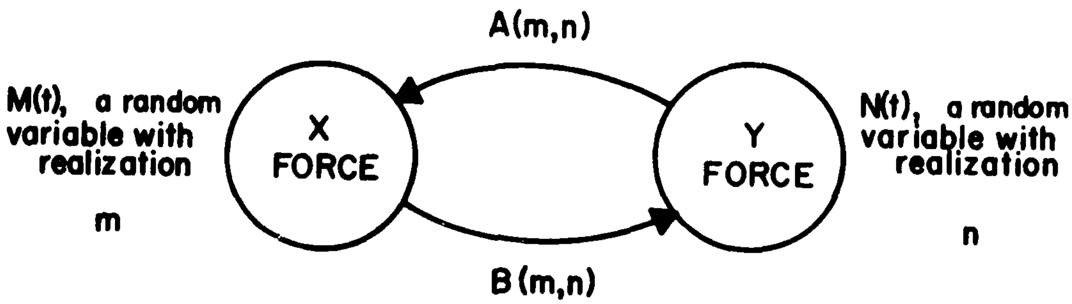

Figure A.2. Diagram of force interactions for stochastic combat model corresponding to the deterministic one depicted in Fig. A.1. Here $M(t)$, a random variable, denotes the integer number of X combatants alive at time t, and m denotes a realization of this random variable.

we have adopted the convention that, for example, $A(m,n) = 0$ for $m > m_0$ or $n > n_0$.

From the above simple stochastic model in which the times between casualties are exponentially distributed random variables with force-level-dependent rates, it holds that (see [51, Chapter 4] for further details)

$$E[T_{XY}^f] = \frac{1}{A(m,n)}, \tag{A.5}$$

where T_{XY}^f, a r.v., denotes the time required for the Y force to kill an X combatant. For the case of equal total-force casualty rates that are independent of the numbers of combatants, i.e. $A(m,n) = B(m,n) = \lambda = \text{CONSTANT}$, (A.5) becomes the well-known result for casualties occurring as a Poisson stream

$$E[T] = \frac{1}{\lambda},$$

or

$$\lambda = \frac{1}{\bar{t}}, \tag{A.6}$$

where T denotes the time between the occurrences of any two consecutive casualties and $\bar{t} = E[T]$. Thus, we see in this general case that the casualty rate is equal to the reciprocal of the expected time for a force to inflict a casualty on the enemy when one has assumed that the times between casualties are exponentially distributed.

We continue on now to heuristically justify (A.2) by specializing the more general model (A.3) into (A.1). Thus, the basic deterministic homogeneous-force Lanchester-type paradigm for modern warfare (A.1) is simply the special case of (A.3) in which $A(x,y) = ay$ and $B(x,y) = bx$. Consequently, for the continuous-time-Markov-chain analogue of the basic paradigm given by constant-coefficient Lanchester-type equations of modern warfare, we have that (A.5) holds with $A(m,n) = an$, whence follows (A.2). In other words, <u>(A.2) holds exactly for the basic paradigm for exponentially-distributed times between casualties</u>. It is also true (see [51, Section 4.12]) that as long as there is "negligible" probability that either side reaches its breakpoint [a particular (but extreme) case being that a force is annihilated], then the mean course of combat (for any distributions of times between casualties) may be taken to be given by

$$\begin{cases} \dfrac{d\bar{m}}{dt} = -a\bar{n} \quad \text{with} \quad \bar{m}(0) = m_0, \\[2ex] \dfrac{d\bar{n}}{dt} = -b\bar{m} \quad \text{with} \quad \bar{n}(0) = n_0, \end{cases} \tag{A.6}$$

where $\bar{m}(t)$ denotes the average X force level at time t (i.e. $\bar{m}(t) = E[M(t)]$), and $\bar{n}(t)$ denotes the average Y force level at time t.

We now pass to discussion of the case in which the times between casualties in the stochastic process underlying (A.1) are no longer necessarily exponentially distributed. Both Bonder [7] and Barfoot [3] have essentially based their justifications of (A.2) on considering such a general stochastic attrition process corresponding to (A.1) and taking the mean course of combat to be given by

$$\begin{cases} \dfrac{d\bar{m}}{dt} = -\bar{\alpha}\,\bar{n} \quad \text{with} \quad \bar{m}(0) = m_0, \\ \\ \dfrac{d\bar{n}}{dt} = -\bar{\beta}\,\bar{m} \quad \text{with} \quad \bar{n}(0) = n_0, \end{cases} \tag{A.7}$$

where $\bar{\alpha}$ denotes the expected value of the rate at which a single Y firer kills X targets and similarly for $\bar{\beta}$. Comparison of (A.6) and (A.7) suggests defining the Lanchester attrition-rate coefficient as the expected rate at which a single firer kills enemy targets, e.g.

$$a = \bar{\alpha} = E\begin{bmatrix}\text{rate at which a single Y firer kills} \\ \text{X targets}\end{bmatrix}, \tag{A.8}$$

and (as stressed by Bonder [7-8]) implies an underlying distribution for the attrition-rate coefficient. Bonder [6-7] originally took the Lanchester attrition-rate coefficient to be given by, for example, $a = \bar{\alpha} = E[1/T_{XY}]$, which is the arithmetic mean for a set of attrition rates and (unfortunately) does not lead to an explicit result for a. Barfoot [3] later argued that the harmonic mean $\bar{\alpha} = 1/E[T_{XY}]$ is more appropriate, since Bonder's [6-7] probability-distribution function for α represented the fraction of targets killed for which each rate is used, and the harmonic mean of these rates is the appropriate measure of average attrition. It should be noted that Barfoot's justification of (A.2) does not involve any assumption about the distributions of the times between casualties.

In the spirit of Bonder and Farrell [11], we will now give a more rigorous justification† of (A.2) that is not based on assuming that the times between casualties are exponentially distributed. We again consider combat in which the initial numbers of X and Y combatants, denoted as m_0 and n_0, are sufficiently large to insure that there is a "negligible" probability that the battle is terminated (i.e. one side or the other first reaches its breakpoint) during our examination of the battlefield. Let us now focus on a single Y weapon system. We will make no specific assumption about the distribution of times between kills, but we will assume that each individual Y weapon system kills enemy targets according to an attrition process in which the times between kills are independent and identically distributed random variables (so-called i.i.d. random variables). In the parlance of the theory of stochastic processes, such an attrition process is called a renewal process [43, Chapter 5]. Let $N_c^X(t)$ be a r.v. denoting the number of X casualties produced by a single Y weapon system in $[0,t]$, and let $\bar{n}_c^X(t)$ denote its expected value, i.e. $\bar{n}_c^X(t) = E[N_c^X(t)]$ denotes the expected number of X casualties produced by a single Y firer in the time interval $[0,t]$. Let us now introduce $\Delta\bar{n}_c^X(\Delta t,t)$ defined by

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†The reader should be cautioned that this justification is not universally accepted and is apparently somewhat controversial (see [50, p. 47] for further details). However, the author is aware of no more widely accepted such justification.

$$\Delta n_c^X(\Delta t,t) = \bar{n}_c^X(t+\Delta t) - \bar{n}_c^X(t), \tag{A.9}$$

which is the expected number of X casualties produced by a single Y weapon system in the time interval $(t,t+\Delta t]$. For exponentially distributed times between kills, it follows that (e.g. see [43, p. 177])

$$\Delta \bar{n}_c^X(\Delta t,t) = \frac{\Delta t}{\mu_T}, \tag{A.10}$$

where μ_T denotes the average time for a single Y firer to kill an X target, i.e. $\mu_T = E[T_{XY}]$. For any other distribution for the times between kills, (A.10) holds only asymptotically in the sense that

$$\lim_{t\to+\infty} \Delta \bar{n}_c^X(\Delta t,t) = \frac{\Delta t}{\mu_T}, \tag{A.11}$$

which is usually known as Blackwell's theorem (e.g. see [43, p. 183]). If we assume now that each Y firer acts independently and identically, it follows that for the entire Y force

$$E\begin{bmatrix}\text{number of kills by the} \\ \text{entire Y force in } (t,t+\Delta t]\end{bmatrix} = \frac{\bar{n}\,\Delta t}{\mu_T}, \tag{A.12}$$

which holds exactly for exponentially distributed times between kills and only asymptotically in the sense of (A.11) for any other distribution. Consideration of the basic Lanchester-type paradigm for modern warfare (A.1) with "large enough" numbers of combatants suggests that [cf. (A.6)]

$$-\Delta\bar{m} = E\begin{bmatrix}\text{number of kills by the} \\ \text{entire } Y \text{ force in } (t,t+\Delta t]\end{bmatrix} = a\,\bar{n}\,\Delta t. \qquad (A.13)$$

Comparison of (A.12) and (A.13) suggests taking the Lanchester attrition-rate coefficient to be the reciprocal of the average time for an individual firer to kill an enemy target, i.e. (A.2) has been justified.

More generally, Bonder and Farrell [11] take an attrition-rate coefficient for a specific range r between firers and targets in heterogeneous-force combat to be given by, for example,

$$A_{ij}(r) = \frac{1}{E[T_{X_iY_j}|r]}, \qquad (A.14)$$

where $E[T_{X_iY_j}|r]$ denotes the expected time for a single Y firer of type j to kill an enemy target of type i, given that the range between firers and targets is r. (A.14) is the basic construct for predicting numerical values for attrition-rate coefficients for direct-fire engagements between maneuver units in the VRI models. It may be justified by the same basic type of renewal-theoretic argument just given above, since all firer types and target types are essentially assumed to behave independently in the heterogeneous-force version of the basic paradigm (A.1).

APPENDIX B: Particulars in General Case for Use of Barlow's Theorem to Develop an Expression for the Expected Time to Kill a Target $E[T]$ as First-Passage Time in Semi-Markov Process.

In this appendix we will indicate how in general Barlow's [4] theorem (i.e. Theorem 5.1 in the main text above) on the mean state-recurrence time in a continuous-time semi-Markov process may be used to develop an expression for the expected time to kill a target $E[T]$. This material is essential for understanding how the expressions for attrition-rate coefficients for maneuver-unit combat in VECTOR-2 and rates of observations by information-collection resources in TFECS (i.e. C^3I capabilities) are developed.

In general, this approach based on Barlow's theorem may be used to develop an expression for the expected time to kill a target $E[T]$ in any firing process with a set of J distinguishable states $S_1, S_2, \ldots, S_J$ as long as the following assumptions hold:

(A1) the process makes transitions at distinct points in time,

(A2) given that one is in state S_i, the probability of transition to state S_j does not depend on any history of the process; we let p_{ij} denote the probability of transition to state S_j from state S_i, i.e.

$$p_{ij} = P\left[\begin{array}{l|l} \text{system in state} & \text{system in state} \\ S_j \text{ after transition} & S_i \text{ before transition} \end{array}\right],$$

(A3) given that one is in state S_i, the mean wait before a transition to state S_j depends on only the specification of these two states; we let μ_{ij} denote the mean wait in state S_i before a transition to state S_j,

(A4) no matter where the system starts, every state has some probability of eventually occurring,

and (A5) the states are so defined that the expected time interval between successive entries into state S_1 corresponds to the expected time between casualties.

In essence, this approach may be applied to any target-destruction process that can be modelled as a semi-Markov process†. Let us now introduce the ratio r_j defined by

$$r_j = \frac{\pi_j}{\pi_1} \quad \text{for} \quad j = 2,3,\ldots,J. \tag{B.1}$$

The expected time to kill a target $E[T]$ is then simply the expected time between the occurrences of two successive casualties ℓ_{11} and is given by

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†So far our discussion has more or less paralleled that given by Farrell [11, pp. 136-137]. We now will depart from Farrell's development by expressing results in terms of the ratios of stationary probabilities $r_j = \pi_j/\pi_1$ for $j = 2,3,\ldots,J$.

$$E[T] = \mu_1 + \sum_{j=2}^{J} r_j \mu_j, \tag{B.2}$$

where $r_2, r_3, \ldots, r_J$ are determined by the linear system of equations

$$\sum_{i=2}^{J} (p_{ij} - \delta_{ij}) r_i = -p_{1j} \quad \text{for } j = 2,3,\ldots,J, \tag{B.3}$$

and δ_{ij} denotes the Kronecker delta defined by

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Here we should note that the assumption (A4) guarantees that we can always solve the linear system of equations (B.3) (e.g. see Feller [22, pp. 356-362] or Parzen [42, p. 265]). If the μ_j are not directly available, they may be obtained from the μ_{ij} by using (5.14), i.e.

$$\mu_j = \sum_{k=1}^{J} p_{jk} \mu_{jk} \quad \text{for } j = 1,2,\ldots,J. \tag{B.4}$$

APPENDIX C: Derivation of Expression for Total-Force Kill Rate in Case of Parallel Acquirer and "Billard-Table" Terrain.

In this appendix we will show how equation (6.4) of the main text may be derived. This equation should be taken to model the force-on-force attrition of the X force under the following conditions:

(CC1) each member of the Y force uses "aimed fire" against enemy targets,

(CC2) line of sight (LOS) exists continuously between all pairs of opposing combatants,

(CC3) parallel acquisition of X targets by Y observers,

(CC4) all firers and targets behave stochastically independently of one another,

and (CC5) the process by which each Y firer acquires X targets may be modelled as a nonhomogeneous two-state continuous-time Markov chain.

Our terminology for stating (CC5) is that of Parzen [43]. We will now show how the above model (6.4) arises from these conditions (CC1) through (CC5).

To develop the above acquisition-attrition-process model (6.4), we assume that (CC1) holds and start with the following basic Lanchester-type paradigm for such aimed fire (cf. Section 3)

$$\left(-\frac{dx}{dt}\right) = a(t,x)y, \tag{C.1}$$

where $a(t,x)$ denotes the single-weapon-system-type kill rate for a typical Y firer against X targets. Assuming (CC3), i.e. parallel acquisition of targets, we have that

$$\begin{pmatrix}\text{rate at which}\\ \text{one } Y \text{ firer}\\ \text{kills } X \text{ targets}\end{pmatrix} = \begin{pmatrix}\text{probability that } Y \text{ firer}\\ \text{has an acquired } X \text{ target}\\ \text{available at which to fire}\end{pmatrix} \cdot \begin{pmatrix}\text{rate at which}\\ \text{one } Y \text{ firer kills}\\ \text{acquired } X \text{ targets}\end{pmatrix}, \tag{C.2}$$

or, in mathematical terms,

$$a(t,x) = p^{P}_{A_{XY}}(t,x)\, \alpha(t), \tag{C.3}$$

where $p^{P}_{A_{XY}}(t,x)$ denotes the probability that a Y firer (who is also a parallel acquirer of targets) has available one or more acquired X targets at which to fire. It will now be shown that

$$p^{P}_{A_{XY}}(t,x) = 1 - \exp\{-x \int_0^t \lambda_{XY}(s)ds\}, \tag{C.4}$$

whence follows (6.4) from combining (C.1), (C.3), and (C.4) and letting $f_{XY} = p^{P}_{A_{XY}}$. To develop (C.4), we assume (CC5) and observe that

$$p^{P}_{A_{XY}}(t,x) = \begin{pmatrix}\text{probability that } Y\\ \text{firer has acquired}\\ \text{one or more } X \text{ targets}\end{pmatrix} = 1 - \{p_{N_{XY}}(t)\}^{x}, \tag{C.5}$$

where $p_{N_{XY}}(t)$ denotes the probability that a Y firer has not detected an X target by time t when there is a single X target present and [by (CC2)] it is continuously visible. Assuming (CC2) and (CC5), i.e. independence of acquisition in short time intervals and other standard Markovian assumptions (e.g. see Parzen [43] or Kleinrock [35]), we have

$$p_{N_{XY}}(t) = \exp\{-\int_0^t \lambda_{XY}(s)\,ds\}, \tag{C.6}$$

whence follows (C.4).

APPENDIX D: Derivation of Expression for Expected Time to Kill a Target $E[T]$ by Use of Barlow's Theorem for a Serial Acquirer and Stochastic LOS.

In this appendix we will show how equation (7.16) of the main text may be derived. We will also derive the results that were used to obtain (7.17) from (7.16).

We recall (see Fig. 6 of the main text) that the following two system states have been defined:

S_1 = target-engaged-until-killed state (which lasts from the end of the engagement of the previous target until the present target is killed before LOS to it is lost),

and S_2 = target-engaged-until-LOS-lost state (which lasts from the end of the engagement of the previous target until LOS to the present target is lost without it being killed),

with the transition probabilities for the imbedded Markov chain being given by

$$p_{11} = p_{21} = p \quad \text{and} \quad p_{12} = p_{22} = 1 - p, \tag{D.1}$$

where

$$p = \text{Prob[target killed before LOS lost]}. \tag{D.2}$$

Furthermore, the expected wait in each state is independent of the next state visited and is given by

$$\mu_1 = \mu_2 = \bar{\mu} = E[T_a] + E[T_{ea}], \tag{D.3}$$

where

T_a = the time (a r.v.) required to acquire a target,

and T_{ea} = the time (a r.v.) to engage an acquired target until either the target is killed or LOS lost.

We will now calculate the stationary probabilities π_1 and π_2, which are determined by the system of equations

$$\begin{cases} \pi_1 = \pi_1 p_{11} + \pi_2 p_{21}, \\ \pi_2 = \pi_1 p_{12} + \pi_2 p_{22}. \end{cases} \tag{D.4}$$

It follows from (D.4) that

$$\pi_1 = p \quad \text{and} \quad \pi_2 = 1 - p. \tag{D.5}$$

We are now ready to derive (7.16). Invoking Barlow's theorem (Theorem 5.1 of the main text) for the two-state semi-Markov process described above, we find that

$$E[T] = \ell_{11} = \frac{1}{\pi_1}(\pi_1\mu_1 + \pi_2\mu_2),$$

or

$$E[T] = \frac{\bar{\mu}}{p}, \tag{D.6}$$

which may also be written as

$$E[T] = \frac{1}{p}\{E[T_a] + E[T_{ea}]\}. \tag{D.7}$$

Equation (D.7) appears in the main text as (7.16).

It remains to show that

$$p = \frac{\alpha}{\alpha + \mu}, \tag{D.8}$$

and

$$E[T_{ea}] = \frac{1}{\alpha + \mu}. \tag{D.9}$$

To develop (D.8), we observe that (D.2) may be written as

$$p = \text{Prob}[T_{ka} < T_{VA}], \tag{D.10}$$

where

T_{ka} = the time (a r.v.) for a firer to kill an acquired target (given that the target is continuously visible),

T_{VA} = the length of time (a r.v.) that a target remains in the state of being visible and acquired (see Fig. 7 of the main text).

Both the random variables T_{ka} and T_{VA} are assumed to be exponentially distributed, and hence for a Y firer engaging an X target

$$F_{T_{ka_{XY}}}(t) = 1 - e^{-\alpha t}, \tag{D.11}$$

and

$$F_{T_{VA}}(t) = 1 - e^{-\mu t}, \tag{D.12}$$

where $F_T(t)$ denotes the distribution function of the random variable T, and $T_{ka_{XY}}$ denotes the time (a r.v.) for a Y firer to kill an acquired X target (given that the target is continuously visible). From (D.10), it follows that (see [51, Appendix B)

$$p = \int_0^\infty \bar{F}_{T_{VA}}(t)\, dF_{T_{ka}}(t), \tag{D.13}$$

or

$$p = \int_0^\infty e^{-\mu t} \cdot \alpha e^{-\alpha t} dt,$$

whence follows (D.8).

To derive (D.9), it suffices to consider the conditional expectation $E[T_{ea}|S_1]$, which is the expected time to engage a target given that the engagement ends with the target being killed. Now

$$\text{Prob}\begin{bmatrix}\text{engagement ends}\\ \text{between } t\\ \text{and } t+dt\end{bmatrix} S_1 = \left\{\text{Prob}\begin{bmatrix}\text{target not}\\ \text{killed}\\ \text{by } t\end{bmatrix}\cdot\text{Prob}\begin{bmatrix}\text{target still}\\ \text{visible}\\ \text{by } t\end{bmatrix}\right\}\text{Prob}\begin{bmatrix}\text{target}\\ \text{killed}\\ \text{in } dt\end{bmatrix},$$

or

$$\text{Prob}\left[\begin{matrix}\text{engagement ends}\\ \text{between } t\\ \text{and } t+dt\end{matrix}\,\middle|\, S_1\right] = e^{-\alpha t}e^{-\mu t}\alpha dt,$$

or

$$\text{Prob}\left[\begin{matrix}\text{engagement ends}\\ \text{between } t\\ \text{and } t+dt\end{matrix}\,\middle|\, S_1\right] = \alpha e^{-(\alpha+\mu)t}dt. \tag{D.14}$$

Normalizing this probability (D.14) to obtain a probability density function (p.d.f.), we find that

$$f_{T_{ea|S_1}}(t) = (\alpha+\mu)e^{-(\alpha+\mu)t}, \tag{D.15}$$

where $f_T(t)$ denotes the p.d.f. of the random variable T. Equation (D.9) follows from (D.15), since $f_{T_{ea|S_1}}(t) = f_{T_{ea}}(t)$. Finally, it should be noted that (D.8) and (D.9) are used to obtain (7.17) from (7.16).

APPENDIX E: Derivation of Expression for Expected Time to Acquire a Target for Stochastic LOS When Acquisition of a Single Target Is a Markov Process and All Targets Behave Independently.

In this appendix we will show how equation (7.18) of the main text may be derived. To this end, we make the following assumptions:

(AE1) the LOS process may be modelled as a time-homogeneous two-state continuous-time Markov chain as depicted in Fig. 5 of the main text with parameters η and μ,

(AE2) the process by which each firer acquires enemy targets may be modelled as a time-homogeneous two-state continuous-time Markov chain with parameter λ,

(AE3) there are N targets within the acquisition range of any observer and all these targets behave stochastically independently of one another,

(AE4) the LOS process is stochastically independent for all observer-target pairs,

(AE5) the LOS process has been operating much longer than the time that the observer has been looking for any target.

We let λ denote the rate at which an observer acquires a particular type of target when there is a single target present and it is continuously visible. Also, η denotes the rate at which an invisible target becomes visible, i.e. $1/\eta$ is the expected time that the target spends in the invisible state each time that it enters this state, and μ denotes the rate at which a visible target becomes invisible. We will now show that

$$E[T_a] = \frac{\eta + \mu}{\eta \lambda N}, \tag{E.1}$$

where

T_a = the time (a r.v.) required to acquire a target.

Equation (E.1) may be derived as follows. Consider a single target and let $F_{T_a}(t)$ denote the distribution function for the time to acquire a target T_a. Then

$$f_{T_a}(t)dt = \text{Prob}\begin{bmatrix}\text{first acquire} \\ \text{target between} \\ t \text{ and } t+dt\end{bmatrix} = \frac{dF_{T_a}}{dt}dt. \tag{E.2}$$

Now by assumptions (AE1), (AE2), and (AE4)

$$\text{Prob}\begin{bmatrix}\text{first acquire} \\ \text{target between} \\ t \text{ and } t+dt\end{bmatrix} = \text{Prob}\begin{bmatrix}\text{target not} \\ \text{acquired} \\ \text{by } t\end{bmatrix}\cdot\text{Prob}\begin{bmatrix}\text{target} \\ \text{visible}\end{bmatrix}\cdot\text{Prob}\begin{bmatrix}\text{target} \\ \text{acquired} \\ \text{in } dt\end{bmatrix},$$

or

$$\frac{dF_{T_a}}{dt}\,dt = \{1 - F_{T_a}(t)\}p_V(t)\,dt, \tag{E.3}$$

where $p_V(t)$ denotes the probability that the target is visible at time t. By assumption (AE5), i.e. the LOS process has been operating much longer than the time that the observer has been looking for the target, it follows that we may take $p_V(t)$ to be given by the equilibrium (or steady-state) probability of the target being visible $p_V(\infty) = \eta/(\eta+\mu)$, which was developed in Section 7 of the main text, i.e.

$$p_V(t) = p_V(\infty) = \frac{\eta}{\eta + \mu}. \tag{E.4}$$

Substituting (E.4) into (E.3), we obtain

$$\frac{dF_{T_a}}{dt} = \frac{\lambda\eta}{\eta + \mu}\{1 - F_{T_a}(t)\}, \tag{E.5}$$

with initial condition

$$F_{T_a}(0) = 0.$$

Thus, for a <u>single target</u>

$$F_{T_a}(t) = 1 - \exp\left\{- \left(\frac{\eta\lambda}{\eta + \mu}\right)t \right\}. \tag{E.6}$$

Finally, by assumption (AE3), it follows that for <u>N targets</u>

$$F_{T_a}(t) = 1 - \exp\left\{- \left(\frac{\eta \ \lambda \ N}{\eta + \mu}\right)t \right\}, \tag{E.7}$$

whence follows (E.1), which appears in the main text as (7.18).

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