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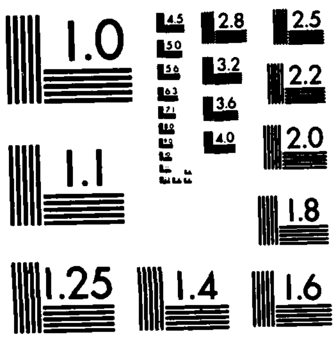
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Methods for calculating atmospheric refraction and its perturbation

Eugene A. Margerum

AUGUST 1982

U.S. ARMY CORPS OF ENGINEERS
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Equations for tracing rays through the atmosphere in spherical coordinates are developed including lateral refraction effects. The spherically symmetrical case is considered in greater detail and equations for the error in altitude angle are obtained. Perturbation analysis is also performed to obtain the deviations in the refraction error due to differences in some standard atmospheric profile and the actual atmospheric profile. In addition, numerical expressions for evaluating the integrals are given, and some simple standard atmospheric models are discussed.

PREFACE

The work reported was done under DA Project 4A161102B52C, Task A, Work Unit 00003, "Inertial–Gradiometric and Astronomic Methods for Gravity Field Determination, Point Positioning, and Subterraneous Mass Detection."

The work was performed during 1980–81. Dr. H. Baussus von Luetzow was the Team Leader, Center for Geodesy; and Mr. M. Crowell, Jr., was the Director, Research Institute.

COL Edward K. Wintz, CE was Commander and Director and Mr. Robert P. Macchia was Technical Director of the Engineer Topographic Laboratories during the report preparation.

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INTRODUCTION

In obtaining location coordinates on the Earth's surface by means of star observations, errors are introduced by the refraction of light by the atmosphere because the actual positions of the stars or other bodies used differ from their observed positions. A knowledge of the refractive index of the atmosphere as a function of position permits the calculation of the actual ray path through the atmosphere by numerical integration of a set of differential equations that may be used to extend the ray step by step through the atmosphere to obtain the true direction of the ray upon emergence.

In this report, the equations and numerical procedures for performing such a ray trace are derived for general refractive index functions including lateral refraction effects, and explicit forms are given for computation of the refraction error. Since the greater variations of refraction occur with altitude and because the detailed state of the atmosphere is usually not known at the time a given set of observations is made, it is desirable (and sufficient for many purposes) to consider the refractive index to vary only with the radial coordinate and to neglect the effects due to lateral variations. For this spherically symmetrical case, the refraction error is present only for the altitude angle and is reduced to a quadrature over the radial distance or height. The strongest part of the error can be considered to originate from average atmospheric profiles that can be approximated by simple formulas and several of these standard atmospheric representations that have commonly been used are presented. The perturbation of the refraction error due to variation of the atmosphere from the standard atmosphere leads to a further correction that is derived and given in terms of another quadrature formula.

RAY TRACING IN SPHERICAL COORDINATES

According to Fermat's principle (also referred to as the principle of least time),[1] the ray joining any two arbitrary points, P_1 and P_2, is determined by the condition that its optical length

$$V = \int_{P_1}^{P_2} n\, ds \tag{1}$$

be stationary as compared with the optical lengths of arbitrary neighboring curves joining P_1 and P_2. If the refractive index n is considered to be a given smooth continuous function of position and the location along the path is given in terms of a parameter t, then an actual ray path must furnish an extremum

$$\delta \int_{P_1}^{P_2} n(r, \theta, \phi)\, S\,(r, \theta, \dot{r}, \dot{\theta}, \dot{\phi})dt = 0 \tag{2}$$

where spherical coordinates are indicated with

$$\frac{ds}{dt} = S(r, \theta, \dot{r}, \dot{\theta}, \dot{\phi}) = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\phi}^2} \tag{3}$$

and where the dots indicate differentiation with respect to t. The partial derivatives

$$\begin{aligned}
\frac{\partial S}{\partial r} &= \frac{r(\dot{\theta}^2 + \sin^2\theta\ \dot{\phi}^2)}{S}, & \frac{\partial S}{\partial \dot{r}} &= \frac{\dot{r}}{S} \\
\frac{\partial S}{\partial \theta} &= \frac{r^2 \sin\theta \cos\theta\ \dot{\phi}^2}{S}, & \frac{\partial S}{\partial \dot{\theta}} &= \frac{r^2\dot{\theta}}{S} \\
\frac{\partial S}{\partial \phi} &= 0 , & \frac{\partial S}{\partial \dot{\phi}} &= \frac{r^2 \sin^2\theta\ \dot{\phi}}{S}
\end{aligned} \tag{4}$$

---

[1] H.A. Buchdahl, *An Introduction to Hamiltonian Optics,* Cambridge University Press, 1970.

will be useful in evaluating the Euler equations in the derivation that follows. Taking

$$f(r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi}) = n(r, \theta, \phi) S (r, \theta, \dot{r}, \dot{\theta}, \dot{\phi}) \tag{5}$$

in equation 2, the rays must lie along curves satisfying an Euler equation for each coordinate

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{r}}\right) - \frac{\partial f}{\partial r} &= 0 \\ \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{\theta}}\right) - \frac{\partial f}{\partial \theta} &= 0 \\ \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{\phi}}\right) - \frac{\partial f}{\partial \phi} &= 0 \end{aligned} \tag{6}$$

and by using the relations given in equations 4 and 5,

$$\begin{aligned} \frac{d}{dt}\left(n\,\frac{\dot{r}}{S}\right) - S\,\frac{\partial n}{\partial r} - n\,\frac{r(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)}{S} &= 0 \\ \frac{d}{dt}\left(n\frac{r^2\dot{\theta}}{S}\right) - S\,\frac{\partial n}{\partial \theta} - n\,\frac{r^2 \sin\theta \cos\theta\,\dot{\phi}^2}{S} &= 0 \\ \frac{d}{dt}\left(n\,\frac{r^2 \sin^2\theta\,\dot{\phi}^2}{S}\right) - S\,\frac{\partial n}{\partial \phi} &= 0 \end{aligned} \tag{7}$$

By taking the parameterization to be given in terms of arc length s along a ray

$$t = s$$

$$S = \frac{ds}{dt} = 1 \tag{8}$$

the differential system for the rays is simplified by eliminating the radicals appearing in S above.

$$\frac{d}{ds}(n\dot{r}) - \frac{\partial n}{\partial r} - nr(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) = 0$$

$$\frac{d}{ds}(nr^2\,\dot{\theta}) - \frac{\partial n}{\partial \theta} - nr^2 \sin\theta \cos\theta\,\dot{\phi}^2 = 0 \tag{9}$$

$$\frac{d}{ds}(nr^2 \sin^2\theta\,\dot{\phi}) - \frac{\partial n}{\partial \phi} = 0$$

If a canonical system of variables is introduced where

$$\begin{aligned} p_r &= n\dot{r} \\ p_\theta &= nr^2\dot{\theta} \\ p_\phi &= nr^2 \sin^2\theta\,\dot{\phi} \end{aligned} \tag{10}$$

the corresponding first-order differential system is easily put in normal form:

$$\begin{aligned} \dot{p}_r &= \frac{1}{nr^3}\left(p_\theta{}^2 + \frac{p_\phi{}^2}{\sin^2\theta}\right) + \frac{\partial n}{\partial r} \\ \dot{p}_\theta &= \frac{\cos\theta\, p_\phi{}^2}{nr^2 \sin^3\theta} + \frac{\partial n}{\partial \theta} \\ \dot{p}_\phi &= \frac{\partial n}{\partial \phi} \\ \dot{r} &= \frac{p_r}{n} \\ \dot{\theta} &= \frac{p_\theta}{nr^2} \\ \dot{\phi} &= \frac{p_\phi}{nr^2 \sin^2\theta} \end{aligned} \tag{11}$$

This system is suitable for numerical integration by many standard methods including the Runge-Kutta method. The equations are not completely independent but are inter-related by the implicit relationship from equations 3 and 8

$$\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2 \sin^2\theta\,\dot{\phi}^2 = 1 \tag{12}$$

which requires that the sum of the squares of the local direction cosines of the tangent to the ray at any point be unity. This enables the integration to be initiated from a knowledge of position coordinates and two angles sighted along a ray; for example, altitude and azimuth angles. It can also facilitate the change of independent variable from s to one of the coordinates if desired; for example, if it is desired to increment the radial distance r in fixed predetermined amounts. In such a case, the six equations given by 11 are reduced to five. For a general integration with s as the independent variable, the initial conditions consist of the coordinates r, θ, ϕ and the direction cosines α_r, α_θ, α_ϕ, related to the conjugate variables, as follows.

$$\begin{aligned} \alpha_r &= \frac{dr}{ds} & \qquad p_r &= n\alpha_r \\ \alpha_\theta &= r\,\frac{d\theta}{ds} & \qquad p_\theta &= nr\alpha_\theta \\ \alpha_\phi &= r \sin\theta\,\frac{d\phi}{ds} & \qquad p_\phi &= nr \sin\theta\,\alpha_\phi \end{aligned} \tag{13}$$

The altitude angle a and azimuth angle A are given by

$$\begin{aligned} \sin a &= \alpha_r \\ \tan A &= \pm\,\frac{\alpha_\phi}{\alpha_\theta} \end{aligned} \tag{14}$$

where the ambiguity of sign must be rectified to conform with the spherical coordinates, since various defining conventions are used for azimuth. By using the identity

$$\alpha_r^2 + \alpha_\theta^2 + \alpha_\phi^2 = 1, \tag{15}$$

it is easy to obtain the direction cosines in terms of altitude and azimuth.

$$\begin{aligned} \alpha_r &= \sin a \\ \alpha_\theta &= \cos a \cos A \\ \alpha_\phi &= \pm \cos a \sin A \end{aligned} \tag{16}$$

ATMOSPHERIC REFRACTION INCLUDING LATERAL REFRACTION

Assuming the quantities $\frac{\partial n}{\partial r}$. $\frac{\partial m}{\partial \theta}$ $\frac{\partial n}{\partial \phi}$ are known functions of position, one may trace a ray up through the atmosphere by using equations 11 for any starting location r_o, θ_o, ϕ_o and direction α_{r_o}, α_{θ_o}, α_{ϕ_o}. Assuming the initial altitude angle is great enough that atmospheric ducting and subsequent return of the ray does not occur, the ray eventually will emerge from the atmosphere at some location r_f, θ_f, ϕ_f with local direction coordinates α_{rf}, $\alpha_{\theta f}$, $\alpha_{\phi f}$. In order to determine the amount of bending of the ray, the transformation of the final direction coordinates back into the initial frame must be known. This transformation will now be obtained. For a general position vector $\vec{R}$ given in rectangular components but expressed in spherical coordinates

$$\vec{R} = \hat{i}\, r \sin\theta \cos\phi + \hat{j}\, r \sin\theta \sin\phi + \hat{k}\, r \cos\theta. \tag{17}$$

A local reference frame of unit vectors $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$ may be defined by

$$\hat{r} = \frac{\partial \vec{R}}{\partial r} \Big/ \left| \frac{\partial \vec{R}}{\partial r} \right| = \hat{i} \sin\theta \cos\phi + \hat{j} \sin\theta \sin\phi - \hat{k} \cos\theta$$

$$\hat{\theta} = \frac{\partial \vec{R}}{\partial \theta} \Big/ \left| \frac{\partial \vec{R}}{\partial \theta} \right| = \hat{i} \cos\theta \cos\phi + \hat{j} \cos\theta \sin\phi + \hat{k} \sin\theta \qquad (18)$$

$$\hat{\phi} = \frac{\partial \vec{R}}{\partial \phi} \Big/ \left| \frac{\partial \vec{R}}{\partial \phi} \right| = -\hat{i} \sin\phi + \hat{j} \cos\phi$$

If the unit direction vector of the ray

$$\hat{\alpha}' = \alpha_r \hat{r} + \alpha_\theta \hat{\theta} + \alpha_\phi \hat{\phi} \qquad (19)$$

is expressed in terms of the initial frame, the components are found to depend on the cosines of angles between the initial and current frame vectors

$$\begin{aligned} \hat{\alpha}' = {} & \hat{r}_o \, [(\hat{r} \cdot \hat{r}_o)\alpha_r + (\hat{\theta} \cdot \hat{r}_o)\alpha_\theta + (\hat{\phi} \cdot \hat{r}_o)\alpha_\phi] \\ & + \hat{\theta}_o \, [\hat{r} \cdot \hat{\theta}_o)\alpha_r + (\hat{\theta} \cdot \hat{\theta}_o)\alpha_\theta + (\hat{\phi} \cdot \theta_o)\alpha_\phi] \\ & + \hat{\phi}_o \, [(\hat{r} \cdot \hat{\phi}_o)\alpha_r + (\hat{\phi} \cdot \hat{\phi}_o)\alpha_\theta + (\hat{\phi} \cdot \hat{\phi}_o)\alpha_\phi] \end{aligned} \qquad (20)$$

where the direction cosines involved are readily obtained from equations 18 applied at the initial and current positions (The prime added to $\hat{\alpha}'$ is to avoid confusion with the starting direction $\hat{\alpha}_o$).

$$
\begin{aligned}
\hat{r} \cdot \hat{r}_o &= \sin\theta \sin\theta_o \cos(\phi - \phi_o) + \cos\theta \cos\theta_o \\
\hat{r} \cdot \hat{\theta}_o &= \sin\theta \cos\theta_o \cos(\phi - \phi_o) - \cos\theta \sin\theta_o \\
\hat{r} \cdot \hat{\phi}_o &= \sin\theta \sin(\phi - \phi_o) \\
\hat{\theta} \cdot \hat{r}_o &= \cos\theta \sin\theta_o \cos(\phi - \phi_o) - \sin\theta \cos\theta_o \\
\hat{\theta} \cdot \hat{\theta}_o &= \cos\theta \cos\theta_o \cos(\phi - \phi_o) + \sin\theta \sin\theta_o \\
\hat{\theta} \cdot \hat{\phi} &= \cos\theta \sin(\phi - \phi_o) \\
\hat{\phi} \cdot \hat{r}_o &= -\sin\theta_o \sin(\phi - \phi_o) \\
\hat{\phi} \cdot \hat{\theta}_o &= -\cos\theta_o \sin(\phi - \phi_o) \\
\hat{\phi} \cdot \hat{\phi}_o &= \cos(\phi - \phi_o)
\end{aligned}
\tag{21}
$$

By applying equations 20 and 21 to the emerging ray direction $\hat{\alpha}_f$, the components referred to the initial frame can be expressed in matrix form as given by equation 22.

$$
\begin{bmatrix} \alpha'_{rf} \\ \alpha'_{\theta f} \\ \alpha'_{\phi f} \end{bmatrix}
=
\begin{bmatrix}
\sin\theta_f \sin\theta_o \cos(\phi_f - \phi_o) + \cos\theta_f \cos\theta_o & \cos\theta_f \sin\theta_o \cos(\phi_f - \phi_o) - \sin\theta_f \cos\theta_o & -\sin\theta_o \sin(\phi_f - \phi_o) \\
\sin\theta_f \cos\theta_o \cos(\phi_f - \phi_o) - \cos\theta_f \sin\theta_o & \cos\theta_f \cos\theta_o \cos(\phi_f - \phi_o) + \sin\theta_f \sin\theta_o & -\cos\theta_o \sin(\phi_f - \phi_o) \\
\sin\theta_f \sin(\phi_f - \phi_o) & \cos\theta_f \sin(\phi_f - \phi_o) & \cos(\phi_f - \phi_o)
\end{bmatrix}
\begin{bmatrix} \alpha_{rf} \\ \alpha_{\theta f} \\ \alpha_{\phi f} \end{bmatrix}
\tag{22}
$$

It was found that the transformation matrix could be factored as

$$\begin{bmatrix} \alpha'_{rf} \\ \alpha'_{\theta f} \\ \alpha'_{\phi f} \end{bmatrix} = \begin{bmatrix} \sin\theta_o & \cos\theta_o & 0 \\ \cos\theta_o & -\sin\theta_o & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi_o & \sin\phi_o & 0 \\ 0 & 0 & 1 \\ -\sin\phi_o & \cos\phi_o & 0 \end{bmatrix} \begin{bmatrix} \cos\phi_f & \sin\phi_f & 0 \\ \sin\phi_f & -\cos\phi_f & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{23}$$

$$\begin{bmatrix} \sin\theta_f & \cos\theta_f & 0 \\ 0 & 0 & -1 \\ \cos\theta_f & -\sin\theta_f & 0 \end{bmatrix} \begin{bmatrix} \alpha_{rf} \\ \alpha_{\theta f} \\ \alpha_{\phi f} \end{bmatrix}$$

or alternatively in the form given by equation 24 as probably the most convenient for computation.

$$\begin{bmatrix} \alpha'_{rf} \\ \alpha'_{\theta f} \\ \alpha'_{\phi f} \end{bmatrix} = \begin{bmatrix} \sin\theta_o & \cos\theta_o & 0 \\ \cos\theta_o & -\sin\theta_o & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\phi_f - \phi_o) & \sin(\phi_f - \phi_o) & 0 \\ 0 & 0 & 1 \\ \sin(\phi_f - \phi_o) & -\cos(\phi_f - \phi_o) & 0 \end{bmatrix} \tag{24}$$

$$\begin{bmatrix} \sin\theta_f & \cos\theta_f & 0 \\ 0 & 0 & -1 \\ \cos\theta_f & -\sin\theta_f & 0 \end{bmatrix} \begin{bmatrix} \alpha_{rf} \\ \alpha_{\theta f} \\ \alpha_{\phi f} \end{bmatrix}$$

From equation 14, the vertical refraction correction is given by

$$a_f' - a_o' = \arccos \alpha_{rf}' - \arccos \alpha_{ro} \tag{25}$$

and the lateral refraction error by

$$\begin{aligned} A_f' - A_o &= \arctan \left(\pm \frac{\alpha_{\phi f}'}{\alpha_{\theta f}'}\right) - \arctan \left(\pm \frac{\alpha_{\phi o}}{\alpha_{\theta o}}\right) \\ &= \pm \arctan \left(\frac{\alpha_{\phi f}' \alpha_{\theta o} - \alpha_{\phi o} \alpha_{\theta f}'}{\alpha_{\theta f}' \alpha_{\theta o} - \alpha_{\phi o} \alpha_{\phi f}'}\right) \end{aligned} \tag{26}$$

where the sign again depends on the convention used for azimuth.

THE SPHERICALLY SYMMETRICAL CASE

For the case where the refractive index depends only upon r, equation 9 becomes

$$\begin{aligned} &\frac{d}{ds}(n\dot{r}) - \frac{\partial n}{\partial r} - nr(\dot{\theta}^2 + \sin^2 \theta \, \dot{\phi}^2) = 0 \\ &\frac{d}{ds}(n \, r^2 \dot{\theta}) - nr^2 \sin \theta \cos \theta \, \dot{\phi}^2 = 0 \\ &nr^2 \sin^2 \theta \, \dot{\phi} = C_1 \end{aligned} \tag{27}$$

where an integral has been found for the last equation. The coordinate system may be chosen so that initially $\frac{d\phi}{ds} = 0$. Then, $C_1 = 0$ and $\frac{d\phi}{ds}$ vanishes indentically

$$\phi = \phi_o = \text{constant} \tag{28}$$

and the problem is reduced to two dimensions. From the fact that $\dot{\phi} = 0$, the second equation of 27 becomes integrable:

$$nr^2 \frac{d\theta}{ds} = C_2 \tag{29}$$

Inserting the resultant value for $\frac{d\theta}{ds}$ into the first equation of 27 (together with $\dot{\phi} = 0$) yields the following relationship:

$$\frac{d}{ds}(n \frac{dr}{ds}) - \frac{\partial n}{\partial r} - \frac{C_2{}^2}{nr^3} = 0 \tag{30}$$

By multiplying by n and using the relationship $\frac{d}{ds} = \frac{dr}{ds} \frac{d}{dr}$,

$$n \frac{dr}{ds} \frac{d}{dr}(n \frac{dr}{ds}) - n \frac{dn}{dr} - \frac{C_2{}^2}{r^3} = 0 \tag{31}$$

and integrating yields

$$(n \frac{dr}{ds})^2 - n^2 + \frac{C_2{}^2}{r^2} = C_3 \tag{32}$$

If $\frac{C_2{}^2}{r^2}$ is replaced by $(nr \frac{d\theta}{ds})^2$ from equation 29, it is found that

$$n^2 [(\frac{dr}{ds})^2 + (r \frac{d\theta}{ds})^2 - 1] = C_3. \tag{33}$$

The bracketed quantity must vanish because arc length s is defined by $ds = \sqrt{dr^2 + r^2\ d\theta^2}$ and hence $C_3 = 0$. It then follows that equations 29 and 30 (in r and θ) are not independent. As a matter of convenience, equation 29 will be used and the geometrical relations between dr, ds and $d\theta$ will be exploited.

As demonstrated in figure 1, a star is observed at the apparent position A_1 given by angle ψ_o.

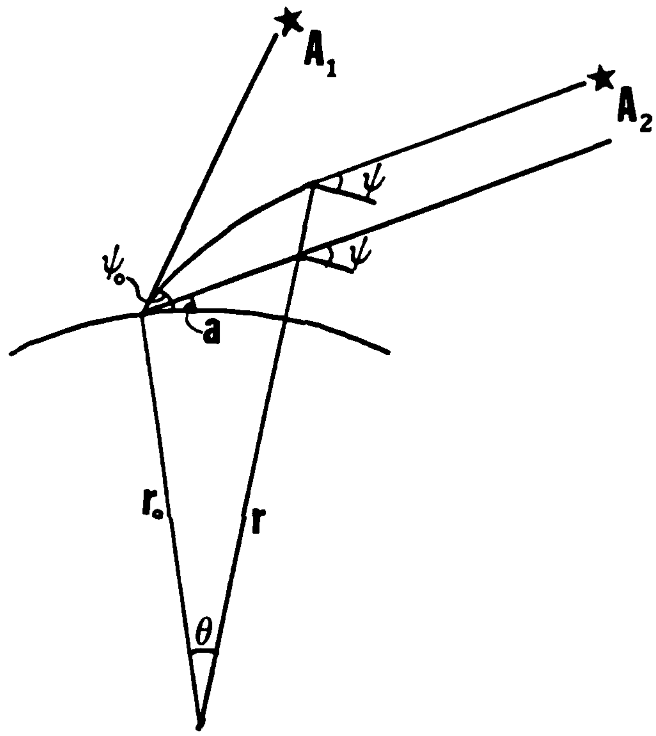

FIGURE 1. Geometrical Parameters for the Atmospheric Ray Path.

If no atmosphere were present, the actual position A_2 would coincide with A_1. For a ray traveling in the reverse direction and emanating at the surface at angle ψ_o in a medium with variable refractive index, the ray path is curved and its inclination ψ at r is given by

$$nr\left(r \frac{d\theta}{ds}\right) = nr \cos \psi = C \tag{34}$$

from equation 29 where the constant is determined from the initial values of n, r and ψ.

$$C = n_o r_o \cos \psi_o \tag{35}$$

After passing through the region of variable index, the ray will emerge at r_f, θ_f in the direction ψ_f toward A_2. By the optical Principle of Reversibility, an object at A_2 would be observed to have elevation ψ_o; whereas, if the refractive index were constant (atmosphere removed), the true elevation would be angle a. As layers of variable refractive index are added in the reversed ray system, a would change and so in this inbedded sense can be regarded as a function of r.

From figure 1,

$$a = \psi - \theta \tag{36}$$

or

$$\frac{da}{dr} = \frac{d\psi}{dr} - \frac{d\theta}{dr} \tag{37}$$

and a may be determined by integrating equation 37. By rewriting equation 34 in the form

$$nr^2 \frac{dr}{ds} \frac{d\theta}{dr} = C \tag{38}$$

and using the fact that $\sin\psi = \frac{dr}{ds}$, an expression for $\frac{d\theta}{dr}$ is found.

$$\sin\psi \frac{d\theta}{dr} = \frac{C}{nr^2} \tag{39}$$

By differentiating equation 34 in the form

$$\cos\psi = \frac{C}{nr} \tag{40}$$

an expression containing $\frac{d\psi}{dr}$ is obtained.

$$\sin\psi \frac{d\psi}{dr} = \frac{C}{nr^2} + \frac{C}{n^2 r} \frac{dn}{dr} \tag{41}$$

Combining equations 39 and 41

$$\sin\psi \frac{d\psi}{dr} - \frac{d\theta}{dr} = \frac{C}{n^2 r} \frac{dn}{dr} \tag{42}$$

and using the fact that

$$\sin\psi = \sqrt{1 - \cos^2\psi} = \sqrt{1 - \left(\frac{C}{nr}\right)^2} \tag{43}$$

an expression for $\frac{da}{dr}$ is readily found.

$$\frac{da}{dr} = \frac{C\frac{dn}{dr}}{n^2 r\sqrt{1-\left(\frac{C}{nr}\right)^2}} = \frac{\frac{r}{C}\frac{dn}{dr}}{\frac{nr}{C}\sqrt{\left(\frac{nr}{C}\right)^2-1}} \tag{44}$$

If this expression is integrated by parts from r_o to r_f, the value for the observational error is found:

$$\delta a = a_f - a_o = \int_{r_o}^{r_f} \frac{\frac{r}{C}\frac{dn}{dr}}{\frac{nr}{C}\sqrt{\left(\frac{nr}{C}\right)^2 - 1}}\, dr \tag{45}$$

$$\delta a = \operatorname{arcsec}\left(\frac{n_f r_f}{C}\right) - \operatorname{arcsec}\left(\frac{n_o r_o}{C}\right) - \int_{r_o}^{r_f} \frac{dr}{r\sqrt{\left(\frac{nr}{n_o r_o}\sec\psi_o\right)^2 - 1}} \tag{46}$$

where $a_o = \psi_o$, r_o, n_o and a_f, r_f, n_f are initial and final values, and C is given by equation 35.

$$\delta a = \operatorname{arcsec}\left(\frac{n_f r_f}{n_o r_o}\sec\psi_o\right) - \psi_o - \int_{r_o}^{r_f} \frac{dr}{r\sqrt{\left(\frac{nr}{n_o r_o}\sec\psi_o\right)^2 - 1}} \tag{47}$$

For determining δa by numerical integration, equation 45 should be preferable to equation 47 by virtue of its simplicity and certainly a need to carry fewer significant figures. It can be easily evaluated with the trapezoidal rule, using a linear interpolation for $\frac{dn}{dr}$ For higher degree approximations, standard spline methods are suggested. Although it can be integrated by quadrature formulae (e.g. Newton-Cotes), equation 47 appears to offer no distinct advantage.

NUMERICAL INTEGRATION

The equation for numerical integration with the trapezoidal rule using linear .terpolation for the derivative can be copied directly from equation 45 by inspection. If the interval $[r_o, r_f]$ is broken into N sub-intervals such that

$$r_o < \ldots < r_j < r_{j+1} < \ldots < r_N = r_f \tag{48}$$

and where

$$n_j = n(r_j)$$

$$f_j = \frac{\frac{r_j}{C}}{\frac{n_j r_j}{C}\sqrt{\left[\frac{n_j r_j}{C}\right]^2 - 1}} \tag{49}$$

$$\nu_j = n_j - n_{j-1}$$

the value of the integral is given approximately by

$$\delta a = \sum_{j=1}^{N} \frac{f_j + f_{j-1}}{2} \nu_j \tag{50}$$

where $C = n_o r_o \cos \psi_o$ as previously defined. This form enables one to use unequal intervals that may be necessary for some sets of data.

ATMOSPHERIC MODELS

In order to perform an integration with equations 50 or 11, one must know the refractive index as a function of position. This is a function of temperature, pressure, and composition. Older references such as Newcomb give the refractive index in terms of density ρ by[2]

$$n^2 - 1 = 2C\rho \tag{51}$$

where the value $C = 0.22607$ gives good agreement with the well-known Poulkova tables (excerpted by Tricker[3]). For n close to 1, the expression $n^2 - 1$ may be replaced by $2(n - 1)$.

$$n = 1 + C\rho \tag{52}$$

This expression is said to give still better agreement with measured refraction.[4]

The actual values of temperature, pressure, and composition of the atmosphere vary, depending on position and time, and follow no simply expressible physical law exactly. Some tables giving detailed values of temperature and pressure as a function of altitude are available.[5] Hotine cites the following formula that was adopted by the International Association for Geodesy in 1960:[6]

$$\mu - 1 = \frac{\mu_G - 1}{1 + \alpha t}\left(\frac{p}{760}\right)\frac{55 \text{ X } 10^{-7}\, e}{1 + \alpha t} \tag{53}$$

[2]Simon Newcomb, *Compendium of Spherical Astronomy*, New York: Dover Publications. 1960.

[3]R.A.R. Tricker, *Introduction to Meteorological Optics*, Elsovier. 1971.

[4]Simon Newcomb, *Compendium of Spherical Astronomy*, New York: Dover Publications. 1960.

[5]*U.S. Standard Atmosphere, 1962*, National Aeronautics and Space Administration.

[6]Martin Hotine, *Mathematical Geodesy*, ESSA Monograph 2. 1969.

where

μ = actual refractive index

μ_G = group refractive index calculated from

$$(\mu_G - 1) \times 10^7 = 2876.04 + (3 \times 16.288)\lambda^{-2} + (5 \times 0.136)\lambda^{-4}$$

t = temperature of air in °C

p = total atmospheric pressure in mm. Hg.

e = partial pressure of water vapor content in mm. Hg. nm. Hg.

α = temperature coefficient of refractivity of air ir
(or the coefficient of thermal expansion),
(0.003661)

λ = wavelength in microns

An equivalent formula is given by Bomford.[7]

Several formulas for air density ρ as a function of altitude h are given by Newcomb.[8]

$$\rho = \rho_1 e^{-\frac{h}{h_1}} \qquad (54)$$

$$\rho = \rho_1 \left(1 - \frac{1}{2}\frac{h}{h_1}\right) \qquad (55)$$

$$\rho = \rho_1 e^{-\frac{kh}{h_1}} \qquad (56)$$

[7] G.Bomford, *Geodesy*, Oxford University Press. 1971.

[8] Simon Newcomb, *Compendium of Spherical Astronomy*, New York: Dover Publications. 1960.

$$\rho = \rho_1 \left(1 - \frac{h}{h_o}\right)^y \tag{57}$$

$$\rho = \rho_1 \ell^{-v} \; ; \; v = \frac{g}{\beta \gamma \tau_1} (e^{\beta h} - 1) - \beta h \tag{58}$$

All of the equations are based on assumptions that need not hold physically but, in some cases, are qualitatively correct or plausible.

PERTURBATION OF THE SOLUTION

The refractive index function $n(r)$ is given a variation $\epsilon m(r)$ and the new error in altitude angle is obtained from equation 45

$$J = \int_{r_o}^{r_f} \frac{\frac{r}{C} \frac{d\tilde{n}}{dr}}{\frac{\tilde{n}r}{C} \sqrt{\frac{\tilde{n}r}{C}^2 - 1}} \, dr \tag{59}$$

where

$$\tilde{n}(r) = n(r) + \epsilon m(r) \tag{60}$$

and

$$\frac{d\tilde{n}}{dr} = \frac{dn}{dr} + \epsilon \frac{dm}{dr} \tag{61}$$

By using equation 61 and the following Taylor expansion in ϵ,

$$\frac{\frac{r}{C}}{\frac{\tilde{n}r}{C} \sqrt{\left(\frac{\tilde{n}r}{C}\right)^2 - 1}} = \frac{\frac{r}{C}}{\frac{nr}{C} \sqrt{\left(\frac{nr}{C}\right)^2 - 1}} - \frac{\frac{r}{C} \frac{mr}{C} \left[2 \left(\frac{nr}{C}\right)^2 - 1\right]}{\left(\frac{nr}{C}\right)^2 \left[\left(\frac{nr}{C}\right)^2 - 1\right]^{3/2}} \epsilon + 0(\epsilon^2), \tag{62}$$

the expression given below is obtained for the perturbed (or varied) integral

$$J = \int_{r_o}^{r_f} \frac{\frac{r}{C} \frac{dn}{dr}}{\frac{nr}{C}\sqrt{\left(\frac{nr}{C}\right)^2 - 1}} dr + \epsilon \int_{r_o}^{r_f} \left\{ \frac{-\frac{r}{C}\left(\frac{mr}{C}\right)\left[2\left(\frac{nr}{C}\right)^2 - 1\right]}{\left(\frac{nr}{C}\right)^2 \left[\left(\frac{nr}{C}\right)^2 - 1\right]^{3/2}} \frac{dn}{dr} + \frac{\frac{r}{C}}{\frac{nr}{C}\sqrt{\left(\frac{nr}{C}\right)^2 - 1}} \frac{dm}{dr} \right\} dr \tag{63}$$

$$+ O(\epsilon^2).$$

In equation 63, the full variation is obtained for $\epsilon = 1$, and the conditions that the first-order term give a good representation of the corresponding variation in J are

$$\epsilon m(r) \ll n(r)$$

$$\epsilon \frac{dm}{dr} \ll \frac{dn}{dr} \tag{64}$$

where ϵ has been carried in equation 64 mainly for purposes of identification.

By differentiation and by a considerable amount of algebraic manipulation, the following identity may be obtained:

$$\frac{\frac{r}{C} \frac{dm}{dr}}{\frac{nr}{C}\sqrt{\left(\frac{nr}{C}\right)^2 - 1}} = \frac{d}{dr}\left[\frac{\frac{mr}{C}}{\frac{nr}{C}\sqrt{\left(\frac{nr}{C}\right)^2 - 1}}\right] + \frac{\frac{m}{C}\left(\frac{nr}{C}\right)}{\left[\left(\frac{nr}{C}\right)^2 - 1\right]^{3/2}} \tag{65}$$

$$+ \frac{\frac{r}{C}\left(\frac{mr}{C}\right)\left[2\left(\frac{nr}{C}\right)^2 - 1\right]}{\left(\frac{nr}{C}\right)^2 \left[\left(\frac{nr}{C}\right)^2 - 1\right]^{3/2}} \frac{dn}{dr}$$

and this is useful in further simplifying the form of the second integral of equation 63.

$$J = \int_{r_o}^{r_f} \frac{\frac{r}{C}\,\frac{dn}{dr}}{\frac{nr}{C}\sqrt{\left(\frac{nr}{C}\right)^2 - 1}}\,dr + \epsilon \int_{r_o}^{r_f} \frac{\frac{m}{C}\left(\frac{nr}{C}\right)}{\left[\left(\frac{nr}{C}\right)^2 - 1\right]^{3/2}}\,dr$$

$$+ \epsilon \left[\frac{\frac{mr}{C}}{\frac{nr}{C}\sqrt{\left(\frac{nr}{C}\right)^2 - 1}}\right]\Bigg|_{r_o}^{r_f} + 0(\epsilon^2) \tag{66}$$

By assuming the value of m to vanish at the endpoints,

$$m(r_o) = m(r_f) = 0 \tag{67}$$

the quantity bracketed in equation 66 will also vanish. For the upper endpoint, this is a reasonable assumption since the refractive index should assume the value for vacuum and variation or perturbation is not reasonable. For the lower endpoint, it is necessary on practical grounds, since any variation of refractive index will disturb the value of C (initial condition) used throughout the entire range of integration.

The first term of equation 66 is the unperturbed error. The integral in the second term is known as the variational or functional derivative of J of first order. Taking $\epsilon = 1$ and ignoring higher order terms yields the first order or linear perturbation of J:

$$\delta J \cong \int_{r_o}^{r_f} \frac{\frac{m}{C}\left(\frac{nr}{C}\right)}{\left[\left(\frac{nr}{C}\right)^2 - 1\right]^{3/2}}\,dr \tag{68}$$

If the value of the integrand is known for values of r given by

$$r_o < \ldots < r_j < r_j + 1 < \ldots < r_N = r_f, \tag{69}$$

the integral can be evaluated numerically using the trapezoidal rule

$$\delta J \cong \sum_{j=1}^{N-1} \beta_j \; \delta\rho_j \; \Delta r_j \tag{70}$$

where

$$\beta_j = C_o \frac{\frac{1}{C}\left(\frac{n_j r_j}{C}\right)}{\left[\left(\frac{n_j r_j}{C}\right)^2 - 1\right]^{3/2}} \tag{71}$$

$$\delta\rho_j = \frac{m(r_j)}{C_o} \tag{72}$$

$$\Delta r_j = \frac{r_{j+1} - r_{j-1}}{2} \tag{73}$$

and where C_o is given by equation 52 in order to express the variation of refractive index in terms of the variation in density. The end values of j do not appear in equation 70 because the variation in refractive index is taken to be zero. Explicit retention of the factors Δr_j enables one to use unequal intervals that may be necessitated by available data.

CONCLUSIONS

By using Fermat's Principle and the Euler equations, a system of differential equations for ray tracing in spherical coordinates was derived for media with continuously variable refractive index. This system was reduced to first order in a form suitable for numerical integration. The transformation from the final reference frame to the initial frame was derived and expressions for the true altitude and azimuth angles were derived. The spherically symmetrical case was discussed and the integral for the error in altitude angle was obtained along with the corresponding numerical formulas. Some standard atmospheric profiles and relationships between refractive index and density given in the literature were described. A perturbation equation was derived for treating the error in altitude angle due to departures of the actual refractive index profile from that of a standard reference atmosphere. The corresponding numerical integration formulas were also obtained.

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