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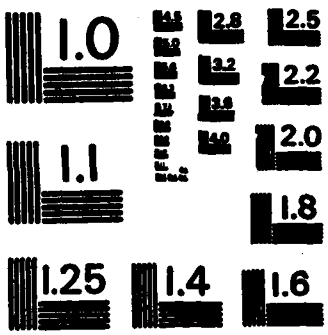

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MATHEMATICAL MODELING AND FINITE ELEMENT ANALYSIS OF ELASTIC-PLASTIC BEHAVIOR

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July 1981

Final Report for Period June - August 1979

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WRIGHT-PATTERSON AIR FORCE BASE, OHIO 45433

NOTICE

This technical report has been reviewed and is approved for publication.

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This investigation is motivated by the need to study the cyclic plasticity response of standard aerospace fasteners within the context of finite element analysis. The available literature on mathematical models of plasticity along with their implementation in finite element procedures was reviewed. The investigation was limited to models of rate-independent plasticity based on the existence of a yield surface and the incremental (rate-type) theory of plasticity.

Section II of the report introduces certain definitions and basic concepts

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) including a discussion of the typical response of metals to cyclic loading. The mathematical models of the plastic behavior of metals are reviewed in Section III Section IV covers the finite element implementation of certain models. A summary and some recommendations for future investigations are presented in Section V.

PREFACE

This report is part of research conducted at the University of Dayton Research Institute, Dayton, Ohio under contract No. F33615-76-C-3145 with the Air Force Wright Aeronautical Laboratories, Flight Dynamics Laboratory, Wright Patterson Air Force Base, Ohio.

The work reported herein was performed during the period 2 June 1979 to 29 September 1979 by Dr. Ranbir S. Sandhu, Visiting Scientist at the University of Dayton Research Institute. The report was released by the author in March 1980.

The author wishes to thank Dr. V. B. Venkayya, AFWAL/FBR and Dr. Fred K. Bogner of the University of Dayton Research Institute for their support of this effort.

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Many failures of fasteners used on aerospace structures have been attributed to progressive damage under cyclic operational loads. Study of low-cycle fatigue and failure of the fastener metal as well as analysis of the range of stress/strain the metal is subjected to during the operational life of the system have, therefore, been extensively studied. To establish the spatial and temporal variation in the stresses and strains in the fasteners, models of mechanical behavior of the metal as well as methods of solution of the boundary value problem have been developed. The finite element method has been extensively used to obtain approximate solutions and several mathematical models have been implemented in finite element schemes.

Mathematical theory of plasticity has developed quite rapidly over the last 30 years or so and a large volume of literature on the subject has accumulated. Several surveys have been attempted covering various topics. Some of these discuss phenomenological observations and mathematical modeling of behavior (Morrow,1965; Lin,1971; Sandor,1972; Knets,1972; Mroz,1973; Jhansale,1973; Krempl,1974), some address finite element solution procedures for nonlinear problems (e.g. Stricklin,1972, 1973) and still others cover the mathematical modeling as well as the solution process (e.g. Armen,1972, 1979; Katona, 1978; Dafalias,1975; Jhansale,1977a).

The present investigation was motivated by the need to study cyclic plasticity response of standard aerospace fasteners within the context of finite element analysis. Accordingly, available literature on mathematical models of plasticity along with their implementation in finite element procedures was reviewed. To limit the scope of the investigation, only models of rate-independent plasticity based on the existence of a yield surface and incremental (rate-type) theory of plasticity have been included. Undoubtedly, a more comprehensive review would also cover cyclic creep, plasticity theories without a yield surface, models of low-cycle fatigue damage, response to a random cyclic loading, thermodynamic considerations in plasticity theories, experimental investigations and other topics relevant to the response of fasteners to cyclic loads.

Section II of the report introduces certain definitions and basic concepts including a discussion of typical response of metals to cyclic loading. In Section III, mathematical models of plastic behavior of metals are reviewed and Section IV covers the finite element implementation of certain models.

SECTION II

PRELIMINARIES

In this section we introduce certain notations and definitions that are repeatedly used in the remainder of the report. These include description of deformation and stress as well as certain notions about stress and strain potentials.

1. STRAIN AT A POINT

Let the motion of a body be referred to a fixed system of rectangular Cartesian axes. Let the position of a typical material particle in a reference (undeformed) configuration be denoted by coordinates X_i, i = 1, 2, 3 and in the current configuration by x_i. We assume that x_i is a sufficiently smooth function of X_i and the time variable t. We assume further, in accordance with the principle of material impenetrability, that x_i is single-valued. Thus,

$$x_i = x_i(X_j, t) \tag{2.1}$$

and the deformation gradient F_{ij} exists such that

$$F_{ij} = x_{i,j}; \quad \det(F_{ij}) \neq 0 \tag{2.2}$$

Here $x_{i,j} = \dfrac{\partial x_i}{\partial X_j}$.

The Cauchy-Green measure of deformation c_{ij} is defined by

$$c_{ij} = x_{k,i} x_{k,j} \tag{2.3}$$

We define a symmetric strain tensor E_{ij} by

$$E_{ij} = \frac{1}{2} (c_{ij} - \delta_{ij}) \tag{2.4}$$

where δ_{ij} is Kronecker's delta. Introducing a displacement vector u_i through the definition

$$u_i = x_i - X_i \tag{2.5}$$

the components of the Green strain tensor are

$$\left.\begin{aligned} E_{ij} &= \frac{1}{2} (u_{i,j} + u_{j,i} + u_{k,i} u_{k,j}) \\ &= u_{(i,j)} + \frac{1}{2} u_{k,i} u_{k,j} \end{aligned}\right\} \tag{2.6}$$

Here parentheses around a pair of subscripts denote symmetry with respect to these subscripts. In cases where the gradients $u_{i,j} \ll 1$, the strain components may be approximated sufficiently closely by

$$\varepsilon_{ij} = u_{(i,j)} \tag{2.7}$$

Equation (2.7) applies for the linear theory or infinitesimal strain theory and Equation (2.6) for finite strains. Because of symmetry, both E_{ij} and ε_{ij} are completely defined by six components. The three invariants of strain are

$$\left.\begin{aligned} I_1 &= E_{kk} \\ I_2 &= \frac{1}{2}(E_{ij}E_{ij} - E_{kk}^2) \\ I_3 &= \frac{1}{6}(2E_{ij}E_{jk}E_{ki} - 3E_{ij}E_{ji}E_{kk} \\ &\quad + E_{ii}E_{jj}E_{kk}) \end{aligned}\right\} \quad (2.8)$$

In terms of eigenvalues E_i of the strain tensor (principal strains)

$$\left.\begin{aligned} I_1 &= E_1 + E_2 + E_3 \\ I_2 &= E_1E_2 + E_2E_3 + E_3E_1 \\ I_3 &= E_1E_2E_3 \end{aligned}\right\} \quad (2.9)$$

The strain deviation tensor e_{ij} defined by

$$e_{ij} = E_{ij} - \frac{1}{3} E_{kk} \quad (2.10)$$

is a symmetric strain tensor with invariants

$$I_1^* = 0$$

$$I_2^* = \frac{1}{2} e_{ij}e_{ij} \tag{2.11}$$

$$I_3^* = \frac{1}{3} e_{ij}e_{jk}e_{ki}$$

The invariant I_1 is a measure of volumetric deformation. Thus, Equation (2.11) will apply to the case of materials with no volumetric deformation. The strain deviation tensor is defined by five independent quantities. Often, the complete strain tensor is represented by I_1 and the strain deviation. Other measures of strain have been used. These include the use of convected coordinates and updated Lagrangian strain. Yoshimura (1962) used a generalization of the logarithmic strain as a measure of the strain history.

2. STRESS AT A POINT

The symmetric Cauchy stress tensor T_{ij} is defined by

$$t_i = T_{ij}n_j \tag{2.12}$$

where t_i, n_j are components, respectively, of the traction vector per unit area and the unit normal to surfaces in the current configuration. The Cauchy stress tensor is related to the symmetric Piola-Kirchhoff stress tensor, or the material stress tensor σ_{ij} as

$$T_{ij} = [\det(F_{mn})]^{-1} F_{ik}F_{jl}\sigma_{kl} \tag{2.13}$$

Under orthogonal transformation of the reference frame, the Cauchy stress tensor transforms as

$$T^*_{ij} = Q_{ik} T_{kl} Q_{jl} \tag{2.14}$$

where Q_{ij} is a proper orthogonal transformation, i.e.

$$Q_{ji} Q_{jk} = Q_{ij} Q_{kj} = \delta_{ik} \tag{2.15}$$

Under the same transformation, the symmetric Piola-Kirchhoff stress tensor is invariant, i.e.

$$\sigma^*_{ij} = \sigma_{ij} \tag{2.16}$$

In the case of small deformations, i.e., $u_{i,j} << 1$, the distinction between T_{ij} and σ_{ij} may be negligible.

As in the case of the strain tensor, the symmetric stress tensor is uniquely defined by six components. The three invariants of stress are

$$\left.\begin{aligned} J_1 &= \sigma_{kk} \\ J_2 &= \frac{1}{2} (\sigma_{ij}\sigma_{ij} - \sigma^2_{kk}) \\ J_3 &= \frac{1}{6} (2\sigma_{ij}\sigma_{jk}\sigma_{ki} - 3\sigma_{ij}\sigma_{ji}\sigma_{kk} \\ &\quad + \sigma_{ii}\sigma_{jj}\sigma_{kk}) \end{aligned}\right\} \tag{2.17}$$

In terms of principal stresses σ_i (eigenvalues of σ_{ij}) the invariants are

$$\left.\begin{aligned} J_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ J_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ J_3 &= \sigma_1\sigma_2\sigma_3 \end{aligned}\right\} \quad (2.18)$$

The stress deviation tensor s_{ij} is defined by

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk} \quad (2.19)$$

The invariants of the stress deviation tensor are

$$\left.\begin{aligned} J_1^* &= 0 \\ J_2^* &= \frac{1}{2} s_{ij}s_{ij} \\ J_3^* &= \frac{1}{3} s_{ij}s_{jk}s_{ki} \end{aligned}\right\} \quad (2.20)$$

i.e., the stress-deviation tensor is uniquely defined by five quantities. Often the complete stress tensor is defined by J_1 along with the stress deviation tensor s_{ij}.

3. STRESS AND STRAIN POTENTIALS

The discussion in this section summarizes some of the concepts presented by Mroz (1973). We start by assuming the existence of potentials $U = U(E_{kl})$ and $W = W(\sigma_{kl})$ such that

$$\sigma_{kl} = \frac{\partial U}{\partial E_{kl}} \tag{2.21}$$

and

$$E_{kl} = \frac{\partial W}{\partial \sigma_{kl}} \tag{2.22}$$

It is easy to show that

$$U = \sigma_{kl}E_{kl} - W \tag{2.23}$$

We assume further that U, W are convex, i.e., for states (1) and (2)

$$U\left(\lambda E_{kl}^{(1)} + (1 - \lambda)E_{kl}^{(2)}\right) \leq \lambda U\left(E_{kl}^{(1)}\right) + (1 - \lambda)U\left(E_{kl}^{(2)}\right) \tag{2.24}$$

and

$$W\left(\lambda\sigma_{kl}^{(1)} + (1 - \lambda)\sigma_{kl}^{(2)}\right) \leq \lambda W\left(\sigma_{kl}^{(1)}\right) + (1 - \lambda)W\left(\sigma_{kl}^{(2)}\right) \tag{2.25}$$

A consequence of convexity is

$$U\left(E_{kl}^{(1)}\right) + W\left(\sigma_{kl}^{(2)}\right) - \sigma_{kl}^{(2)}E_{kl}^{(1)} \geq 0 \tag{2.26}$$

Also, convexity implies for states (1) and (2)

$$\int_1^2\left(\sigma_{kl} - \sigma_{kl}^{(1)}\right)dE_{kl} \geq 0 \tag{2.27}$$

$$\int_1^2 \left(E_{kl} - E_{kl}^{(1)}\right) d\sigma_{kl} \geq 0 \tag{2.28}$$

For state (2) approaching arbitrarily close to state (1), the foregoing inequalities, along with the definition of U, W imply

$$\left(\frac{\partial^2 U}{\partial E_{kl} \partial E_{mn}}\right) \delta E_{kl} \delta E_{mn} \geq 0 \tag{2.29}$$

$$\left(\frac{\partial^2 W}{\partial \sigma_{kl} \partial \sigma_{mn}}\right) \delta \sigma_{kl} \delta \sigma_{mn} \geq 0 \tag{2.30}$$

Considering the surface $U(E_{kl}) = c$ (c a constant) in the strain space, for states $E_{kl}^{(1)}$, $E_{kl}^{(2)}$ lying in this space, convexity implies

$$\sigma_{kl}^{(1)} \left(E_{kl}^{(1)} - E_{kl}^{(2)}\right) \geq 0 \tag{2.31}$$

Similarly for states $\sigma_{kl}^{(1)}$, $\sigma_{kl}^{(2)}$ on the surface $W(\sigma_{kl})$ = constant in the stress space,

$$E_{kl}^{(1)} \left(\sigma_{kl}^{(1)} - \sigma_{kl}^{(2)}\right) \geq 0 \tag{2.32}$$

The assumption (2.21) implies $\sigma_{kl}^{(1)}$ is directed along the normal to $U(E_{kl})$ = constant. Hence, Equation (2.31) implies convexity of the interior of U = constant. Similarly, the interior of $W(\sigma_{kl})$ = constant is convex. The set of surfaces $U = c_i$, $i = 1, 2, \ldots, n$ represents non-intersecting similar surfaces each having its center at the origin. The set is ordered by c_i.

If U is homogeneous of order m, i.e., for any scalar λ

$$U(\lambda E_{kl}) = \lambda^m U(E_{kl}) \tag{2.33}$$

we have

$$E_{kl}\sigma_{kl} = E_{kl}\frac{\partial U}{\partial E_{kl}} = mU(E_{kl}) \tag{2.34}$$

For U homogeneous of order m, its conjugate W is homogeneous of order

$$k = \frac{m}{m-1} \tag{2.35}$$

The foregoing discussion applies if instead of strain measure E_{kl} and stress σ_{kl} we choose any kinematic variables q_i and corresponding force variables Q_i such that $U = U(q_i)$ and $W = W(Q_i)$ exist with

$$q_i = \frac{\partial W}{\partial Q_i} \tag{2.36}$$

and

$$Q_i = \frac{\partial U}{\partial q_i} \tag{2.37}$$

This is the basis for representation of the mechanical state of a body in the traction space or displacement space. For elastic bodies undergoing small deformation and with homogeneous strain energy, it can be shown (Mroz, 1973) that proportional loading on the boundaries of the body induces proportionally varying stresses at each point in the body. Another case of generalized forces and displacements is the use of mean stress and mean strain (Mroz, 1973) over representative volumes. For non-uniform stress states, the representative volume should be sufficiently large to represent macroscopic properties and small enough so that variation in stress can be neglected. Thus, this concept may not be applicable in regions of large stress and displacement gradients relative to grain size of the material.

SECTION III

MATHEMATICAL MODELS OF MECHANICAL BEHAVIOR OF ELASTIC-PLASTIC SOLIDS

Mathematical theories are simple models to represent experimental data so as to be readily applicable to design and analysis. Simplicity often implies inadequacy and care must be exercised in selecting a model appropriate to a problem. The actual model must reflect the actual experimentally observed material behavior to an acceptable extent and still be easy to implement. In this report we are concerned with elastic-plastic rate-independent materials. We assume that a yield surface exists. In this section we define rate-independence of materials and present alternative decompositions of the strain components. In later discussion the additive decomposition is, in general, assumed. The notions of initial and subsequent yield surfaces are reviewed in section III.3 and the constitutive equations for elastic-plastic materials in section III.4.

1. RATE INDEPENDENT SIMPLE MATERIALS

Pipkin (1965) formulated constitutive equations for rate independent materials with memory. The subject was also discussed by Owen (1970), Coleman (1970), Holsapple (1973) and White (1975). For a simple material the dependence of the stress at any point upon the entire history of deformation at that point may be represented by

$$\sigma_{ij}(t) = F_{ij}\left[\mathop{E_{kl}}_{\tau = -\infty}^{t}(\tau)\right] \qquad (3.1)$$

Pipkin (1965) described the strain history $E_{kl}(\tau)$, $\tau \in (-\infty, t]$ by specifying the strain path and the rate of traversal of the path. The length of the strain trajectory was defined as

$$s = \int_{-\infty}^{\tau} (\dot{E}_{kl}\dot{E}_{kl})^{1/2} \qquad (3.2)$$

where $\dot{E}_{kl}$ is an increment in the strain. Assuming $E_{ij} = 0$ initially, and setting $s(t) = S$, equation (3.1) may be replaced by

$$\sigma_{ij}(t) = F_{ij}\left[\mathop{E_{kl}}_{s=0}^{S}(s),\ \mathop{s(t-\tau)}_{\tau=0}^{\infty},\ t\right] \qquad (3.3)$$

For material to be rate-independent, the stress must be independent of the rate of traversal of the strain path viz. of s. Hence,

$$\sigma_{ij}(S) = F_{ij}\left[\mathop{E_{kl}}_{s=0}^{S}(s)\right] \qquad (3.4)$$

Equation (3.4) expresses the assumption that the stress of any material point at a given time depends only upon the history of deformation at that point up to that time without dependence upon rate of deformation. Theories of plasticity seek simple representation for the effects of history of deformation.

Following Pipkin(1965) the Green strain measure has been used in the above discussion. Other measures of deformation used include the strain representation in convected coordinates and the updated Lagrangian strains. Yoshimura (1962) used a generalization of logarithmic strains as a measure of the strain history. Alternatively, the formulation could be based on expressing current strain as a function of history of stress. In that case (Mroz, 1973)

$$E_{ij} = \phi_{ij}\left[\underset{\tau=-\infty}{\overset{t}{\sigma_{kl}}}(\tau)\right] = \phi_{ij}\left[\underset{\tau=0}{\overset{\infty}{\sigma_{kl}}}(t-\tau)\right] \tag{3.5}$$

Introducing

$$s = \int_{-\infty}^{t} (\dot{\sigma}_{ij}\dot{\sigma}_{ij})^{1/2} \tag{3.6}$$

we could write

$$E_{ij}(S) = \psi_{ij}\left[\underset{s=0}{\overset{S}{\sigma_{kl}}}(s)\right] \tag{3.7}$$

where S is the current value of the monotonically increasing function s.

2. DECOMPOSITION OF STRAIN COMPONENTS

The traditional approach based on Prandtl's idealization of uniaxial stress-strain curve has been to regard the strain increment as the sum of an elastic and an inelastic component, i.e.,

$$\dot{E}_{ij} = \dot{E}'_{ij} + \dot{E}''_{ij} \tag{3.8}$$

Here a single prime denotes the elastic or recoverable part and the double prime the plastic or irrecoverable part and a superposed dot denotes an increment. For rigid plastic materials $\dot{E}'_{ij} = 0$. Often it is assumed that the stress is linearly related to the elastic part of the strain, the relationship being independent of history or the mechanical state. For nonlinear elastic materials this leads to different definitions of $\dot{E}''_{ij}$ depending upon which definition is used for $\dot{E}'_{ij}$ (Mroz, 1973). Eisenberg (1977) distinguished between the hyperelastic and hypoelastic definitions of $\dot{E}'_{ij}$. In either case, the plastic strain component would be redefined in accordance with Equation (3.8). Eisenberg (1977) related these two definitions by an expression which, for the isothermal case, reduces to the form

$$\dot{E}''^{(1)}_{ij} = \left[\delta_{ip}\delta_{jq} + \rho K^{-1}_{klij} \frac{\partial \phi_{kl}}{\partial E''^{(2)}_{pq}}\right] \dot{E}''^{(2)}_{pq} \qquad (3.9)$$

Here

$$\left.\begin{aligned} \dot{E}''^{(1)}_{ij} &= \dot{E}_{ij} - \dot{E}'^{(1)}_{ij} \\ \dot{E}''^{(2)}_{ij} &= \dot{E}_{ij} - \dot{E}'^{(2)}_{ij} \\ \dot{E}'^{(1)}_{ij} &= K^{-1}_{klij}\dot{\sigma}_{ij} \end{aligned}\right\} \qquad (3.10)$$

$\dot{E}'^{(2)}_{ij}$ is such that ψ exists with the property

$$\sigma_{ij} = \rho \frac{\partial \psi}{\partial E'^{(2)}_{ij}} \qquad (3.11)$$

and

$\psi = \psi\left(E^{(2)}_{kl}, E''^{(2)}_{kl}, q_i\right)$ is the specific Helmholtz free energy, q_i, $i = 1, 2, \ldots, n$ are a finite number of measures of effects of prior deformation history, K_{ijkl} are components of an elasticity tensor and $\phi_{kl} = \phi_{kl}(E''^{(2)}_{ij}, q_i)$ is the part of $\frac{\partial \psi}{\partial E''^{(2)}_{kl}}$, which is temperature independent.

A multiplicative decomposition of the deformation gradient was proposed by Sedov (1965) and further discussed by Lee (1967, 1968, 1969, 1970). In this

$$F_{ij} = F'_{ik}F''_{kj} \tag{3.12}$$

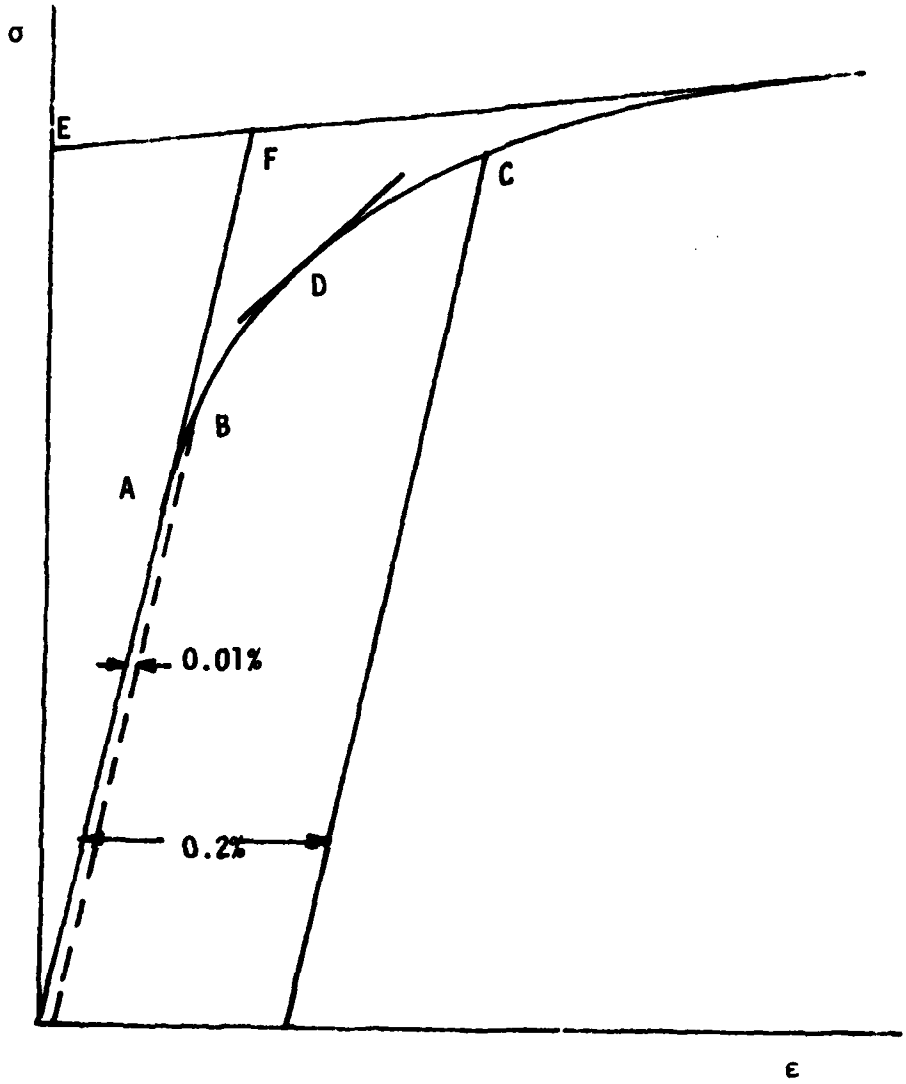
F'_{ik}, F''_{kj} are, respectively, the elastic (recoverable) and the plastic (irrecoverable) deformation gradients. If the material particle is elastically returned to the unstressed state, i.e. $(F'_{ij})^{-1}$ is applied to F_{ij}, the residual deformation gradient will be F''_{kj}. A justification for this decomposition was based on the argument that the additive decomposition is not valid when both the elastic and the plastic strains are finite. Hahn (1974) and White (1975) adopted this decomposition in development of theory of elastic-plastic solids. Freund (1970) developed constitutive equations using Lee's theory and assuming a weak thermodynamic coupling between the elastic and the plastic effects. Haddow (1971) developed a flow rule for an incompressible solid under finite strain.

Green (1971) showed that the kinematic decomposition proposed by Lee would lead to problems associated with invariance requirements under superposed rigid body motions in certain cases. It was noted that Lee had avoided these difficulties by restricting his analysis to materials which are initially isotropic and have special properties. In defense of the additive decomposition, it was contended that it was not necessary to assign a physical or kinematic interpretation to E'_{ij} in a general theory. Green showed that Lee's results can be deduced from his general theory (1965). Key (1976) showed that underlying Lee's multiplicative decomposition of the total strain is an additive decomposition of the equivalent strain rates. The multiplicative decomposition was regarded as akin to the deformation theory of plasticity in that total strains are used. Holsapple (1973) pointed out that both decompositions are imbedded in the functional theory of plasticity where the plastic strain is defined from the stress functional. Both approaches are correct because the definition of elastic strain is arbitrary.

3. THE YIELD SURFACE

a. Basic Concepts

In analyzing uniaxial (or proportional) loading test data, it is customary to define a yield point as the stress level below which the material behaves elastically. In actual tests the transition from elastic to plastic behavior is smooth and the elastic limit cannot be precisely defined. Various definitions of onset of inelastic behavior have been proposed. It has been identified with departure from linearity (e.g., Naghdi, 1957; Ivey, 1961; Phillips, 1974), measurable (usually .01 or .02 percent) nonlinear component of strain, slope of the stress-strain curve becoming a preselected multiple of the initial slope, intersection of the post-yield slope with the initial slope or the ordinate corresponding to zero strain etc. Haythornwaite (1968) compared various definitions illustrated in Figure 1. Because of the uncertainty in location of the yield point, theories of plasticity have been developed



A. Deviation from linearity

B. Small measurable offset

C. 0.2% offset

D. Slope equal to a constant x elastic slope

E. Extrapolation of post yield slope to ordinate

F. Intersection of elastic slope and definition E

Figure 1. Various Definitions of Yield (Haythornwaite, 1968; Lamba, 1976)

(e.g., Valanis, 1971, 1974; Bodner, 1975; Stouffer, 1979) which do not use the concept of a yield surface.

The uncertainty in the definition of yield point is not too significant for monotonic loading. However, in unloading, reloading and cyclic loading these become important. Assuming yield stress to coincide with the limit of proportionality (Figure 2 Point A_0) a nonrecoverable plastic strain is associated with excursion to a stress level say B beyond this point. Upon unloading from B the process is elastic and linear up to a point say C_0 and then onwards it is nonlinear. Upon reversal from D, again the behavior is elastic upto A_0 and inelastic thereafter. During the process the size of the interval of elastic stress states may change and its mid-point shift. The change in size is termed isotropic hardening (or softening) and the translation of the center is called kinematic hardening. Figure 3 taken from Lamba (1976) illustrates some hardening theories for the case of two-dimensional stress field. If the amount of hardening is proportional to the plastic (nonrecoverable) strain, the stress strain curve beyond the yield point is linear. In general, the curve is nonlinear. Figure 2 shows the approximation of a stress-strain curve by purely isotropic and purely kinematic linear hardening curves. Evidently, neither of the two models is adequate. A combination of the two is needed, at the least, to describe the actual behavior.

Under cyclic loading, we may assume the existence of a stable state reached after a few cycles or asymptotically. Figure 4 taken from Jhansale (1977a) gives a classification of cyclic transient phenomena. Essentially, the saturation state defines a limiting value of stress (strain) amplitude for given amplitude of strain (stress) cycling along with a change of mean

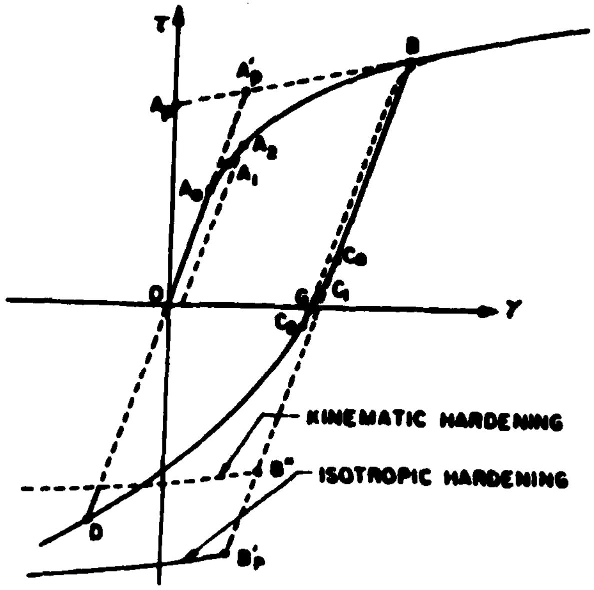


Figure 2. Linear Kinematic and Isotropic Hardening Approximation to Actual Material Behavior. (Mroz, 1973)

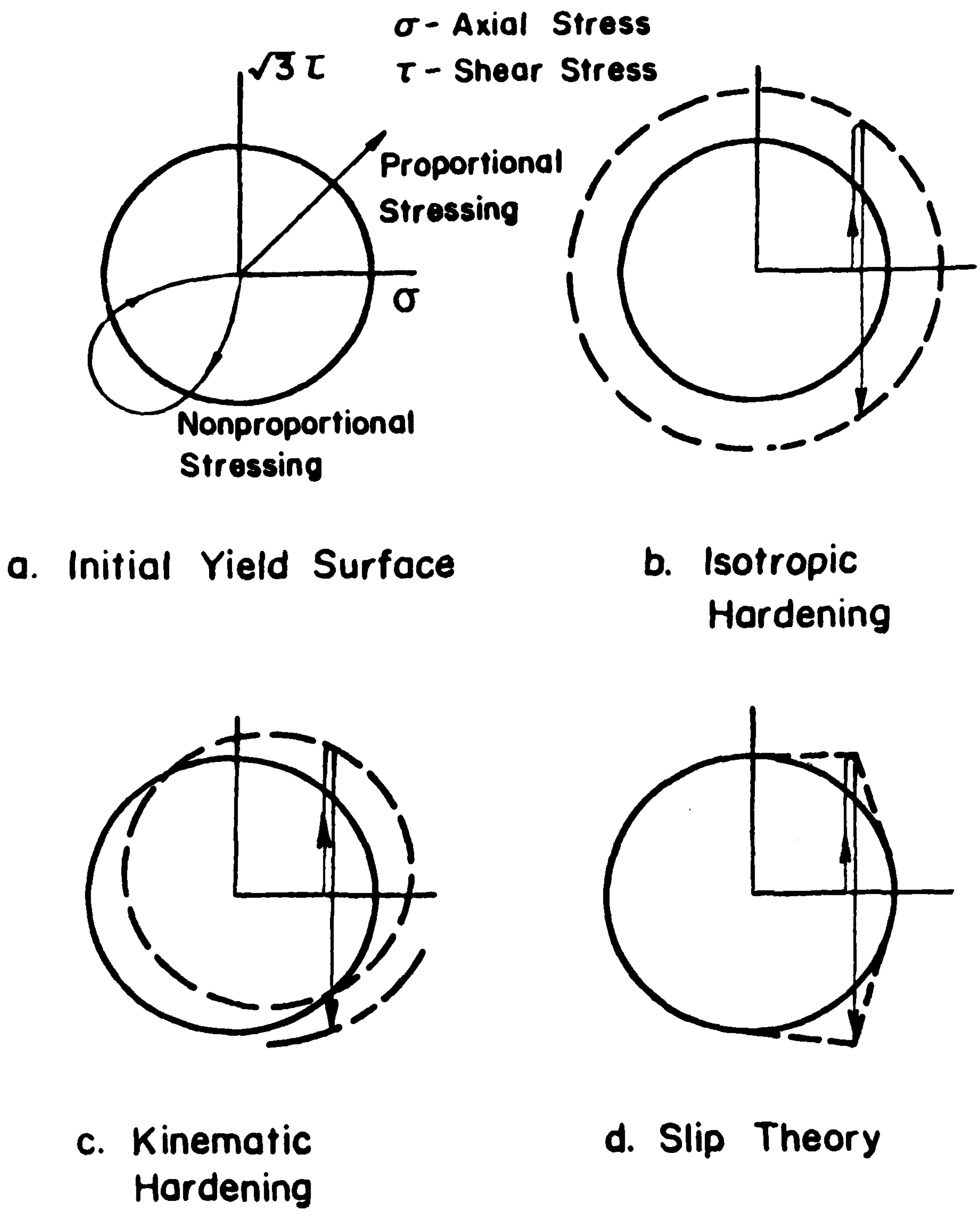


Figure 3. Some Hardening Models for Two-Dimensional Stress. (Lamba, 1976).

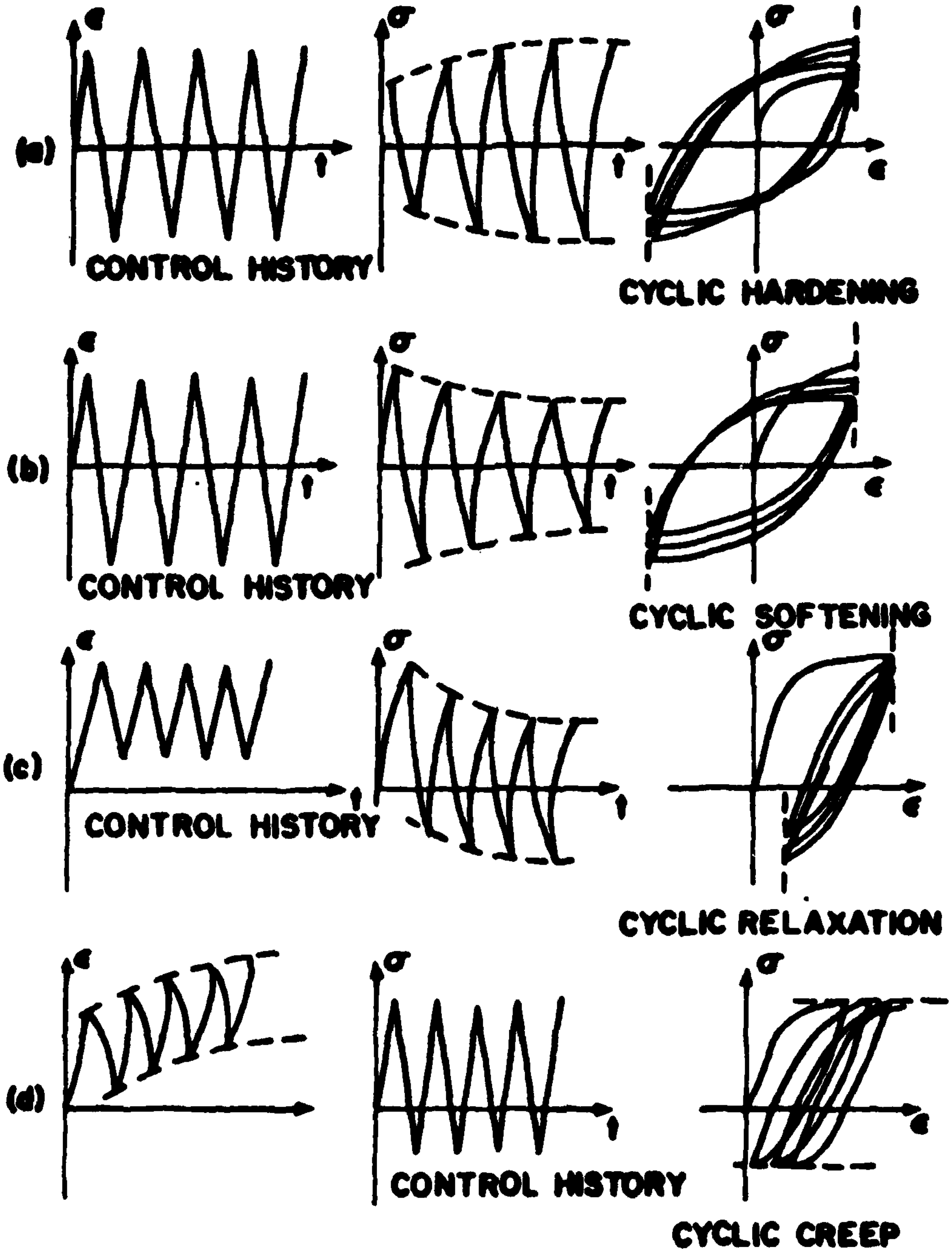


Figure 4. Classification of Cyclic Transient Phenomena (Jhansale, 1977).

stress (irrecoverable strain). Koibuchi (1971) noted that hysteresis loops are similar in shape (Figure 5). Figure 6 taken from Koibuchi (1971) shows stress-strain curve obtained in an incremental step test program. Burbach (1971) noted that an hysteresis loop was uniquely defined by the location of its center regardless of prior history (Figure 7). Figure 8 from Jhansale (1971) shows hysteretic characteristics of normalized mild steel. A description of the stable hysteresis curve is Masing's rule. According to this (Figure 9) the ascending and the descending part of the curve, i.e., B-A, and AB'B are identical but for change in sign and origin and are obtained from the initial loading curve OA by doubling the stress as well as the stress range. Jhansale (1971, 1977a) and Sharma (1977) plotted the stable hysteresis loops (Figure 10) superposed on their lower tips (Figure 11) and noticed that for A-36 steel Masing's rule did not apply. Admitting translation of hysteresis loops along the elastic slopes they matched the curve with Masing's rule (Figure 12).

In uniaxial (or proportional) loading, the set of all stress states is represented by an interval on the real line. Consequently, loading, unloading, reloading have the obvious meaning. For extension of the concept to multiaxial or non-proportional stress paths, it is necessary to define a functional on the six-dimensional stress space to order this space. In the simplest form,

$$f(\sigma_{ij}) = c \qquad (3.13)$$

where f is a functional mapping the six-dimensional stress space into

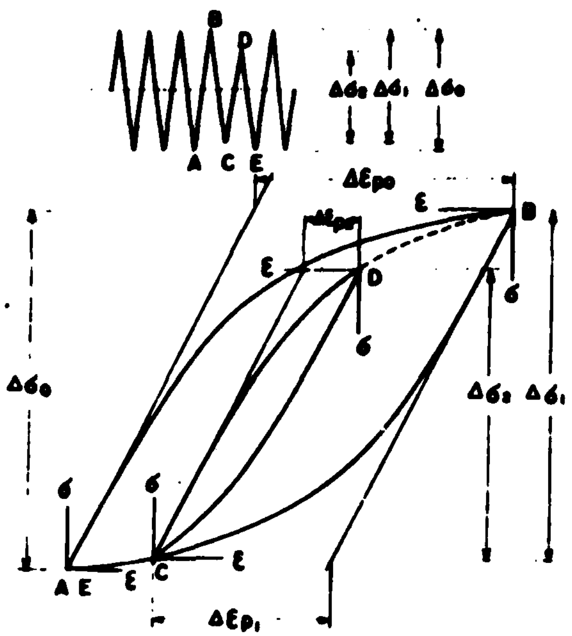


Figure 5. Similarity of Hysteresis Loops (Koibuchi, 1971).

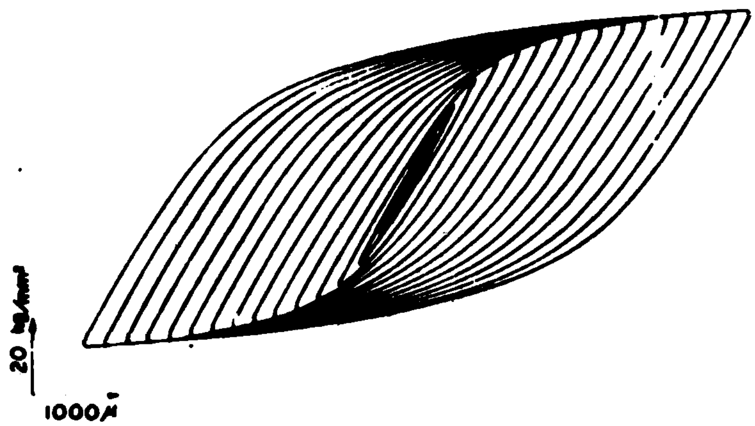


Figure 6. Stress-Strain Curves in Incremental Step Loading.
(Koibuchi, 1971)

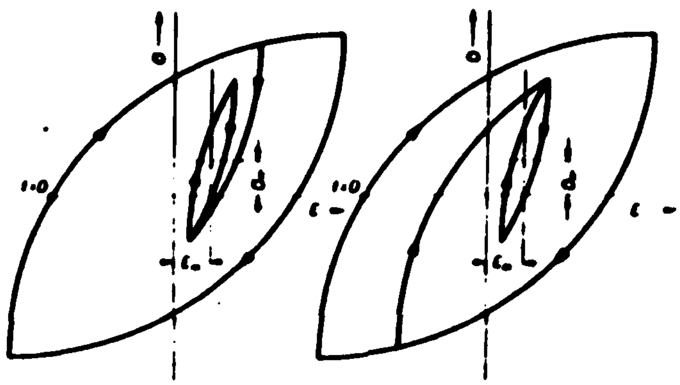


Figure 7. Hysteresis Loops of Equal Center Point Coordinates with Different States of Internal Stress. (Burbach, 1971)

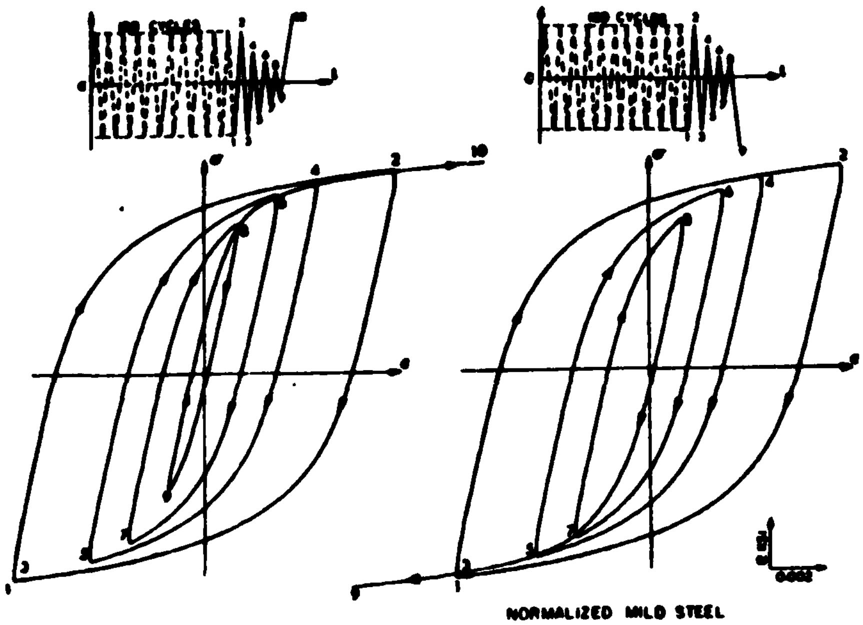


Figure 8. Hysteretic Characteristics of Normalized Mild Steel. (Jhansale, 1971)

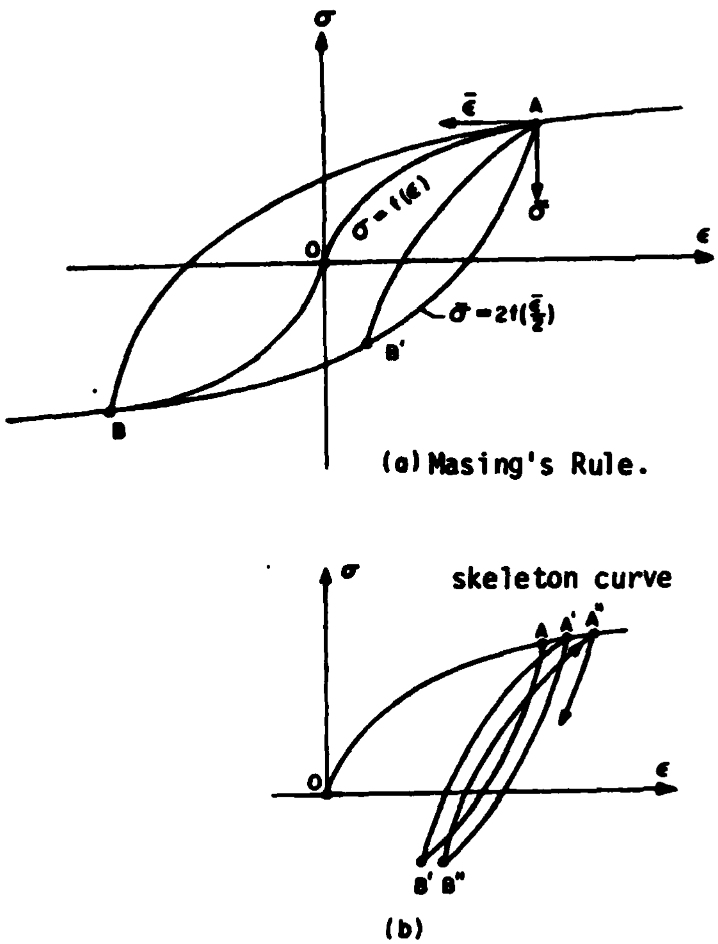


Figure 9. Cyclic Stress-Strain Curves. (Mroz, 1973)

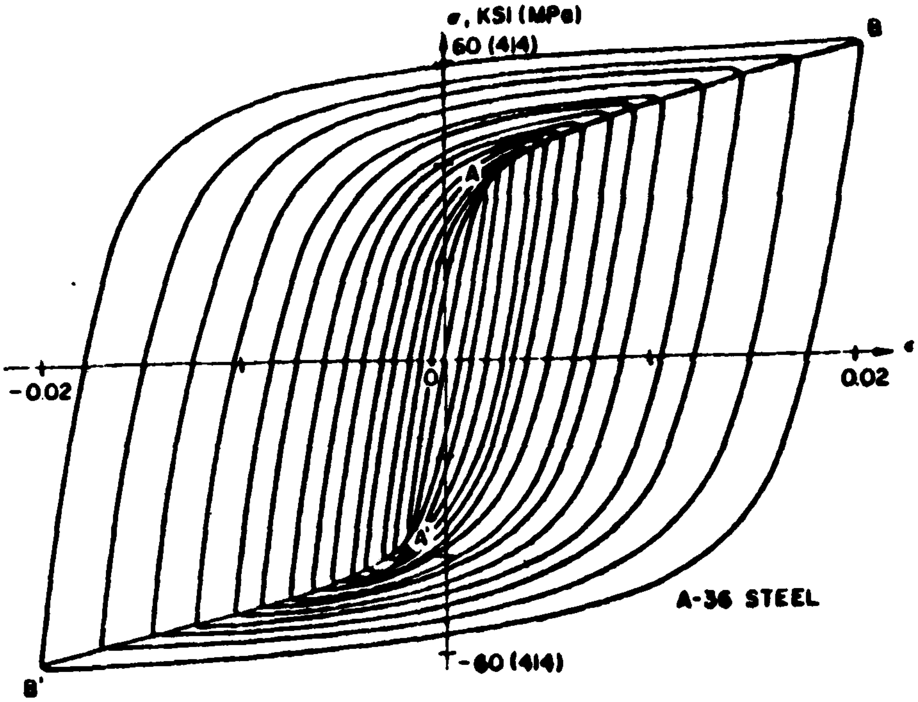


Figure 10. Stable Hysteresis Loops for A-36 Steel. (Jhansale, 1977)

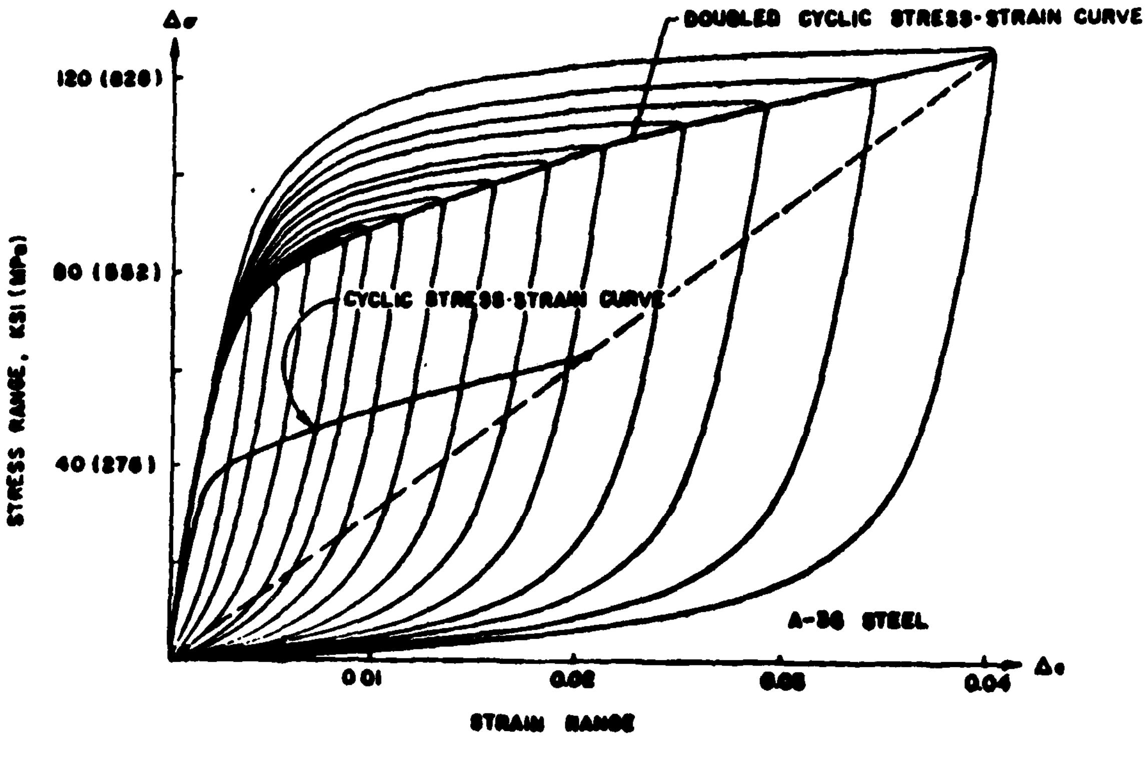


Figure 11. Stable Hysteresis Loops (of Figure 10) Superimposed on Their Lower Tips. (Jhansale, 1977)

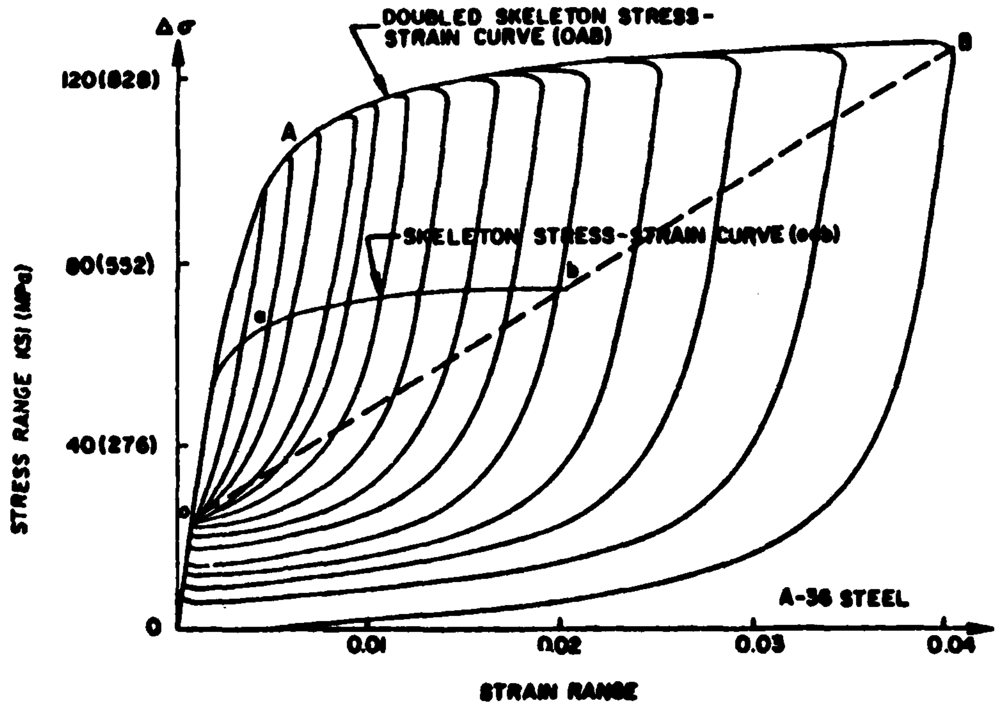


Figure 12. Translation of the Lower Ends of Hysteresis Loops Needed to Match the Doubled Stress-Strain Curve (Non-Masing Behavior). (Jhansale, 1977)

the real line. Often, the range of f is the positive interval. Equation (3.13), for a sequence of values of c defines a family of non-intersecting surfaces in the stress space. The surface

$$f(\sigma_{ij}) = \kappa \tag{3.14}$$

is called the yield surface. It encloses all possible elastic states ($f(\sigma_{ij}) < \kappa$). The constant κ is a material property chosen such that it corresponds to the yield stress in a uniaxial test. For ideal plasticity, κ is constant. Corresponding to different definitions of the yield point in the uniaxial test, different yield surfaces may be constructed for various choices of κ with the attendant difficulties in modelling behavior under cyclic stress/strain paths.

Instead of describing the state of a material particle by its stress, the strain measure could be used. This would lead to the introduction of a non-negative functional g on the strain space along with a definition of yield strain. The set of all elastic states then would be

$$\{E_{ij} | g(E_{ij}) < \mu\} \tag{3.15}$$

where μ is the yield strain.

In metals, the yield level κ is known to depend upon temperature and previous plastic straining. Thus, in its general form

$$f = f(Q, H, \theta) \tag{3.16}$$

where Q is a set of quantities describing the stress or load state, H is the deformation history or a set of measures of deformation history and θ is the temperature. Using the symmetric Piola-Kirchhoff stress tensor σ_{ij} for Q, and assuming that the effects of history of deformation are adequately described by the current value of the irrecoverable (plastic) strain E''_{ij} along with parameters q_i,

$$f = f(\sigma_{ij}, E''_{ij}, q_i, \theta) \tag{3.17}$$

Assuming the stress σ_{ij} to be uniquely related to an elastic (recoverable) strain tensor E'_{ij}, Equation (3.17) may be rewritten as

$$f = \bar{f}(E'_{ij}, E''_{ij}, q_i, \theta) \tag{3.18}$$

If the additive decomposition of strain is used, i.e., $E'_{ij} = E_{ij} - E''_{ij}$, the functional in equation (3.18) is

$$\bar{f}(E'_{ij}, E''_{ij}, q_i, \theta) = \bar{\bar{f}}(E_{ij}, E''_{ij}, q_i, \theta) \tag{3.19}$$

This is the strain space representation of the functional (Pipkin, 1965; Naghdi, 1975a, 1975c). It should be noted here (Naghdi, 1975c) that the initial yield surfaces ($E''_{ij} = 0$, $q_i = 0$) in the stress and the strain space will differ only by a scaling factor representing the transformation from stress to strain. However, subsequent yield surfaces ($E''_{ij} \neq 0$, $q_i \neq 0$) will be shifted by E''_{ij} in addition to any scaling and rotation.

To allow for the phenomenon of overelasticity resulting in an increase of the yield limit if the state of stress is spatially non-homogeneous, Konig (1974) introduced stress and strain gradients as additional arguments in the functional. The ordering functional would thus be

$$f = f(\sigma_{ij}, E''_{ij}, q, \theta, \sigma_{ij,k}, E''_{ij,k}) \qquad (3.20)$$

For $f = \kappa$, where κ itself may be a functional of history of deformation and temperature, defining the yield surface, the set of elastic states is enclosed by this surface, i.e., for any elastic state $f < \kappa$. A stress path is designated as elastic if $f < \kappa$ throughout the path. If $f = \kappa$ to start with and $\dot{f} < 0$, then excepting the origin of the path, it lies entirely in the region $f < \kappa$. This is called unloading. For loading $f = \kappa$ and $\dot{f} = 0$, i.e., the stress point stays on the yield surface and moves with it. However, the surface itself may expand, contract, translate, rotate or distort in the space defined by the arguments. These changes are described by changes in the internal variables q_i as well as κ. Often, the yield parameter κ is included in the list of internal parameters q_i. In that case the ordering function has the implicit form and $f = 0$ is the yield surface. $f < 0$ represents the set of all elastic states.

Due to the uncertainty in definition of yield, different yield surfaces can be constructed for the same material. To overcome this difficulty, Phillips (1965) proposed a two surface theory. The limit of proportionality was assumed to define onset of yield. In addition to this yield surface, an outer surface enclosing the yield surface was defined as the loading surface. In the region between the two surfaces, an unloading path would produce no plastic strain and, therefore, the two surfaces would not change. However, during a loading path plastic strains would occur disturbing the two surfaces. Through each point in the region (called metelastic region by Krieg, 1975) between the yield and the loading surface an unique intermediate loading surface can be identified. This implies the existence of a continuum of ordered, non-intersecting, loading surfaces. This theory is a forerunner of the theories recently developed by Mroz (1967, 1971, 1973) and Dafalias (1975, 1976) on the one hand and on the other, for the limiting case of vanishing set of purely elastic states, of theories without a yield surface.

In the foregoing summary we have used the symmetric Piola-Kirchhoff stress tensor as the measure of stress. Other measures have been used. For instance, Hutchinson (1973) used convected coordinate representation and set up the yield surface using convected stress components. Freund (1970), Key (1976), Carter (1977), among others, summarized relationships between various descriptions of stress. Key compared the use of Cauchy's stress and the second (symmetric Piola-Kirchhoff stress in constitutive models for elastic-plastic materials. He concluded that the models employed in computations while appearing quite different due to the choice of coordinate systems, are, in fact, very nearly the same.

b. Convexity of the Yield Surface

In the case of uniaxial test, the set of elastic stresses, i.e., stresses below the yield point or between the yield point in direct and reversed loading is an interval on the real line. Convexity is an inherent property of such an interval. In extension of the concept of yield to multiaxial stress states, convexity may therefore be assumed as a primitive characteristic of the set of elastic states.

Naghdi (1960) used Drucker's thermodynamic postulate (1952) to prove convexity of yield surfaces. Green (1965b) showed that for the yield surface $f = \frac{1}{2} s_{kl}s_{kl} - \kappa^2$, plastic deformation without volume straining is possible only if Drucker's stability postulate holds and the plastic strain rate is normal to the yield surface. Palmer (1967) showed that convexity may exist even in unstable materials where the stress falls continuously with increasing strains. On the other hand, if the elastic response is nonlinear and is altered by plastic deformation, nonconvex yield surfaces become possible for stable as well as unstable materials. For noncoincident yield and loading surfaces, convexity of the loading surface cannot be proved (Phillips, 1965) on the basis of Drucker's postulate because the loading surface is changed before it is reached during loading from an intermediate loading surface.

Pipkin (1965) used Ilyushin's postulate, i.e., the work done in a closed cycle of strain is non-negative, to establish convexity in a strain space formulation. Naghdi (1975b) considered materials in which

$$\dot{\sigma}_{ij} = L_{ijkl}(\dot{E}_{kl} - \dot{E}''_{kl}) \qquad (3.21)$$

where L_{ijkl} may depend upon history of deformation. Then the assumption of normality of plastic strains combined with Ilyushin's postulate leads to the condition of convexity

$$\dot{\sigma}_{mn}\dot{E}''_{mn} \geq - L_{mnpq}\dot{E}''_{pq}\dot{E}''_{mn} \tag{3.22}$$

This condition is less restrictive than Drucker's viz.

$$\dot{\sigma}_{mn}\dot{E}''_{mn} \geq 0 \tag{3.23}$$

Dafalias (1977) showed that convexity does not necessarily follow from Ilyushin's postulate in case of elastic-plastic coupling. Justusson (1966) used loading surfaces enclosing the yield surface which was defined as the limit of proportionality of stress and strain. For this case it was found that Drucker's postulate of stability in the large, i.e., non-negative total work done during loading or non-negative total work in a closed stress cycle, does not imply convexity of the loading surface. The modified postulate would require that completely irreversible portions cannot exist on straight line loading paths between any two points on or in the interior of the loading surface. Indeed, this amounts to assuming convexity to exist as a primitive characteristic of states interior to loading surfaces. It should be noted that the forms of the functional dependence of yield on stress often used define norms on the shifted stress space. Convexity is an inherent property of norms.

Convexity of the region enclosed by the loading surface places restrictions on the form of the loading surface. Caulk (1978) investigated the restrictions on the coefficients in a generalized von Mises loading surface.

c. The Initial Yield Surface

The initial yield surface is the yield surface before any irreversible (plastic) deformation has taken place. At this stage $E''_{ij} = 0$, $q_i = 0$. Therefore, for the case of homogeneous stress/strain, the yield surface has the form

$$f(\sigma_{ij}, \theta, \kappa) = 0 \tag{3.24}$$

in the stress space or

$$g(E_{ij}, \theta, \mu) = 0 \tag{3.25}$$

in the strain space. If the material is isotropic, the stress (or strain) tensor may be replaced by its invariants in the list of arguments of f (or g). Thus, the yield surface is

$$f(J_i, \theta, \kappa) = 0 \tag{3.26}$$

or, in the strain space,

$$g(I_i, \theta, \kappa) = 0 \tag{3.27}$$

In many cases, the yield is unaffected by the first invariant of stress, i.e., the yield surface has the form

$$f(J_2, J_3, \theta, \kappa) = 0 \tag{3.28}$$

Specific forms of f include the Tresca, von Mises, Mohr-Coulomb and other criteria (e.g. Davis, 1978).

For anisotropic materials, Hill (1950) proposed a quadratic functional on the stress space as a generalization of von Mises yield criterion. Assuming incompressibility, Hill wrote

$$f = F(\sigma_y - \sigma_z)^2 + G(\sigma_z - \sigma_x)^2 + H(\sigma_x - \sigma_y)^2 + 2L\sigma_{yz}^2 + 2M\sigma_{zx}^2 + 2N\sigma_{xy}^2 - 1 = 0 \qquad (3.29)$$

as the functional for initial yield. For the general case we may write (Pifko, 1974)

$$f = \frac{\sigma x}{X}^2 + \frac{\sigma y}{Y}^2 + \frac{\sigma z}{Z}^2 + \frac{\sigma_{yz}^2}{R^2} + \frac{\sigma_{zx}^2}{S^2} + \frac{\sigma_{xy}^2}{T^2} - 1 = 0 \qquad (3.30)$$

Here X, Y, Z are the yield stresses in uniaxial tension and R, S, T, the yield stresses in uniaxial shear in x, y, z directions. Sawczuk (1959) considered a piecewise linear yield surface (a generalization of Tresca's criterion) in the form

$$f^{(i)} = A_{kl}^{(i)}\sigma_{kl} = \text{constant}, \; i = 1, 2, \ldots, n \qquad (3.31a)$$

For principal stress axes coinciding with axes of material symmetry n = 6, i.e., there are six planes constituting the yield surface. Goldenblat (1965) proposed a yield functional in the form

$$f = (L_{ij}\sigma_{ij})^{\alpha} + (L_{ijkl}\sigma_{ij}\sigma_{kl})^{\beta} + (L_{ijklmn}\sigma_{ij}\sigma_{kl}\sigma_{mn})^{\gamma} + \ldots \qquad (3.31b)$$

Gotoh (1977) used a yield functional of the fourth order in the form

$$f = \sum_{i,j,k} \sigma_x^i \sigma_y^j \sigma_{xy}^k, \; i + j + 2k \le 4 \qquad (3.31c)$$

where x, y are axes of material symmetries. A survey of failure theories of isotropic and anisotropic materials was prepared by Sandhu (1972). More recently, Boehler (1977) discussed yield of oriented solids.

d. The Subsequent Yield Surfaces

If the material behavior in an isothermal process is independent of history of deformation and depends solely upon the stress level, the yield condition is

$$f(\sigma_{ij}) - \kappa = 0 \qquad (3.14)$$

where κ is the yield parameter. The material is said to be ideally or perfectly plastic. The yield surface is invariant and indefinite plastic straining can occur at $f(\sigma_{ij}) - \kappa = 0$ along with $\dot{f} = 0$, i.e. the stress point staying on the yield surface. However, most materials are affected by plastic deformation. This effect may consist of expansion, contraction, translation, rotation and distortion of the yield surface.

(1) Isotropic hardening

A material is termed as isotropically hardening (or softening) if the yield parameter κ varies with plastic straining but the function $f = f(\sigma_{ij})$ is independent of history. In this case, κ in Equation (3.14) is a functional defined over the history of plastic deformation. Scalar measures most often used are the work done during plastic deformation

$$\kappa = \int_0^\infty \sigma_{ij}\dot{E}''_{ij}(s)ds \qquad (3.32a)$$

or the length of the plastic strain trajectory

$$\kappa = \int_0^\infty (\dot{E}''_{ij}\dot{E}''_{ij})^{1/2}ds \qquad (3.32b)$$

Depending upon which of the two measures is used, the material is termed workhardening or strainhardening. This formulation allows for expansion and contraction of the yield surface but not its translation, rotation or distortion. The formulation also fails to account for Bauschinger effect.

(2) Kinematic hardening

To allow for Bauschinger effect, the yield surface was assumed (Prager, 1955, 1956) to translate during plastic deformation. Thus, defining α_{ij}, the coordinates of the origin as internal variables, the yield surface is

$$f(\sigma_{ij} - \alpha_{ij}) - \kappa = 0 \tag{3.33}$$

where κ is constant. Admitting translation as well as expansion of the yield surface, an immediate extension of Equation (3.33) would treat κ as a function of plastic strain history according to Equation (3.31) or (3.32). For nonlinear kinematic hardening, Eisenberg (1968) pointed out that this description of the yield surface is not appropriate for metals insomuch as it would admit a monotonically increasing "modulus" in reversed loading. He proposed instead a form

$$f(\sigma_{ij}, E''_{ij}, \kappa_1) - \kappa_0 = 0 \tag{3.34}$$

where κ_1 is an additional internal variable and κ_0 is constant. Mroz (1967) proposed a more general form of Equation (3.33) viz.

$$f(\sigma_{ij} - \alpha_{ij}) - F(\kappa) = 0 \tag{3.35}$$

For κ to be monotonically increasing, the workhardening definition of κ (Equation 3.32b) has to be modified to

$$\kappa = \int_0^\infty (\sigma_{ij} - \alpha_{ij})\dot{E}''_{ij}(s)ds \tag{3.36}$$

The reason for this is that if Equation (3.32b) is used, for the case of the origin lying outside the yield surface the rate $\dot{\kappa}$ would be negative and hence κ would not be monotonically increasing.

(3) Anisotropic hardening

During loading, in addition to expansion and translation in the stress space or the strain space, the yield surface may also rotate and/or distort. Hodge (1956) considered the stretching of Tresca type surfaces. Berman (1959) proposed a general theory for distortion of piecewise linear yield surfaces. Anisotropy in the initial yield surface, in plastic flow and in the change of yield surfaces due to loading was considered. However, it was assumed that the principal directions of stress and strain coincide and remain fixed throughout the loading process. The only part of the yield surface translating during loading was the linear segment containing the incremental stress vector. For the stress point at an intersection of several linear segments, all of these would translate.

Baltov (1965) generalized kinematic hardening rule to admit mechanical anisotropy. The generalization accounted for the translation as well as rotation and expansion of the yield surface. For the kinematically hardening yield surface for initially isotropic von Mises solid, the yield surface was expected to be, assuming incompressibility,

$$f = \frac{1}{2}(s_{ij} - \beta_{ij})(s_{ij} - \beta_{ij}) - \kappa^2 = 0 \tag{3.37}$$

where s_{ij} is the stress deviation and β_{ij} is the deviation of the translation α_{ij} of the center. This was generalized to

$$f = \frac{1}{2} N_{ijkl}(s_{ij} - \beta_{ij})(s_{kl} - \beta_{kl}) - \kappa^2 = 0 \tag{3.38}$$

Here N_{ijkl}, β_{ij} are functions of E''_{ij}. Admitting an "initial" value C_{ijkl} for N_{ijkl}, Equation (3.38) is equivalent to Edelman's (1951) and to Shih's (1978) yield function for anisotropic materials. Baltov followed Rivlin (1955) and expressed N_{ijkl} as a polynomial in E''_{ij}. Using symmetry and incompressibility conditions, it was shown that the number of distinct components of N_{ijkl} is only 15. Moreover, N_{ijkl} was written as

$$N_{ijkl} = I_{ijkl} + A_{ijkl} \tag{3.39}$$

where I_{ijkl} is the isotropic term and A_{ijkl} represents the anisotropic part. For incompressible materials

$$I_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}) \tag{3.40}$$

and

$$A_{ijkl} = \bar{A}E''_{ij}E''_{kl} \tag{3.41}$$

where $\bar{A}$ is a scalar function of the invariants of E''_{ij}. Assuming $\bar{A}$ polynomial in E''_{ij}, noting that for incompressibility $E''_{kk} = 0$, the simplest expression for A_{ijkl} is

$$A_{ijkl} = A_o E''_{ij}E''_{kl} \tag{3.42}$$

The drawback of this formulation is that A_{ijkl} does not depend upon the strain path but only upon the current value of the plastic strain. The path dependence is expected to be taken care of entirely by the parameter κ. Also, as noted by Dafalias (1975), the formulation does not account for initial symmetries and the suggestion that addition of a constant "tensor of initial anisotropy" to N_{ijkl} will account for initial anisotropy is not entirely accurate.

Shrivastava (1973) proposed a general theory to admit expansion, translation as well as rotation of the yield surface. These would explain hardening, Bauschinger effect and mechanical anisotropy. For an initially isotropic solid the dependence of the yield function on σ_{ij}, E''_{ij} can be replaced by their invariants. To admit coupling the simultaneous invariants were used. Thus, the yield surface is described by

$$f(I_i, J_i, K_A, \kappa) = 0 \tag{3.43}$$
$$i = 1, 2, 3$$
$$A = 1, 2, 3, 4$$

Here I_i, J_i have the usual meaning, the simultaneous invariants are:

$$\left.\begin{aligned} K_1 &= \sigma_{ij}E''_{ij} \\ K_2 &= \sigma_{ij}\sigma_{jk}E''_{ki} \\ K_3 &= \sigma_{ij}E''_{jk}E''_{ki} \\ K_4 &= \sigma_{ij}\sigma_{jk}E''_{kl}E''_{li} \end{aligned}\right\} \quad (3.44)$$

and f is a polynomial in its arguments. Restrictions on the admissible arguments in Equation (3.43) were developed. For instance, in the case of initially isotropic materials, for incompressibility, a sufficient condition for normality of plastic strain increments to hold is that f be independent of I_1, J_1, J_3, K_2, K_3, K_4, i.e., the yield surface has the form

$$f(I_2, J_2, K_1, I_3) - \kappa = 0 \qquad (3.45)$$

Because I_3 changes sign whenever E''_{ij} reverses, I_3 must occur in even powers. Hence, for small strain theory, its effect would be negligible leading to

$$f(I_2, J_2, K_1) - \kappa = 0 \qquad (3.46)$$

For initial yield as $E''_{ij} = 0$, Equation (3.43) specializes to the form

$$f(J_1, \kappa) = 0 \tag{3.47}$$

in general and

$$f(J_2, \kappa) = 0 \tag{3.48}$$

for the special case of initially isotropic incompressible material undergoing small deformation. Tresca criterion arises using

$$f = a_1 J_2 + a_2 J_2^2 + a_3 J_3^2 + a_4 J_2^3 + a_5 J_3^2 J_2 - \kappa^6 = 0 \tag{3.49}$$

Writing

$$f = a_1 J_2 + a_2 I_2 + a_3 K_1 - \kappa^2 = 0 \tag{3.50}$$

leads, for appropriate selection of coefficients to

$$f = J_2 + c^2 I_2 - 2cK_1 - \kappa^2 = 0 \tag{3.51}$$

or

$$f = (s_{ij} - \alpha_{ij})(s_{ij} - \alpha_{ij}) - \kappa^2 = 0 \tag{3.52}$$

where

$$\alpha_{ij} = cE''_{ij} \tag{3.53}$$

This is the form for kinematic hardening where the initial value of α_{ij} is zero.

Using only linear and quadratic terms in J_2, I_2, K_1, for suitable choice of coefficients

$$f = (s_{ij} - \alpha_{ij})(s_{ij} - \alpha_{ij}) + A_1\alpha_{ij}\alpha_{kl}(s_{ij} - \alpha_{ij})(s_{kl} - \alpha_{kl}) - \kappa^2 = 0 \tag{3.54}$$

This is Baltov's formulation (1965) for anisotropic hardening.

Including K_4, for an appropriate choice of coefficients in the polynomial of second degree in stress and fourth degree in strain, Svensson's formulation (1965) is realized where

$$f = N_{ijkl}(s_{ij} - \alpha_{ij})(s_{kl} - \alpha_{kl}) - \kappa^2 = 0 \tag{3.55}$$

Another choice of coefficients yields Yoshimura's (1959) formulation viz.

$$f = N_{ijkl}s_{ij}s_{kl} - as_{ij}E''_{ij} - \kappa^2 = 0 \tag{3.56}$$

Second order phenomenon, e.g., axial strain accumulation in cyclic torsion can be accomodated in the formulation by either admitting slightly nonsymmetric behavior during stress reversal or by introducing non-analytical forms characterized by edges.

Stating f as a polynomial in all the ten basic and joint invariants of s_{ij} and α_{ij}, for a suitable choice of coefficients,

$$f = (s_{ij} - \alpha_{ij})(s_{ij} - \alpha_{ij}) + a(s_{ij} - \alpha_{ij})(s_{jk} - \alpha_{jk})(s_{ki} - \alpha_{ki}) - \kappa^3 = 0 \tag{3.57}$$

For $\alpha_{ij} = eE''_{ij}$, this is the same form as proposed by Freudenthal (1969). Shrivastava proposed a generalization to

$$f = [(s_{ij} - \alpha_{ij})(s_{ij} - \alpha_{ij})]^{3/2} + a(s_{ij} - \alpha_{ij})(s_{jk} - \alpha_{jk})(s_{ki} - \alpha_{ki}) - \kappa^3 = 0 \tag{3.58}$$

Phillips (1974) found that during loading, the dimension of the yield surface lateral to the stressing direction in the stress space did not change, and the distortion was such that there was no cross-effect. The distortion was more pronounced in the "forward" region of the surface during the loading process. Phillips also noted that considering loading paths in stress space, upon loading to a certain stress point followed by immediate unloading, the yield surface did not pass through the prestress point. Also, repeated loading to the prestress point resulted in the yield surface gradually approaching that point. This was ascribed to rate effects which though ignored in the mathematical theory are often present in real materials. Phillips (1975) proposed a mathematical model describing distortion of yield surfaces as a function of the history of deformation.

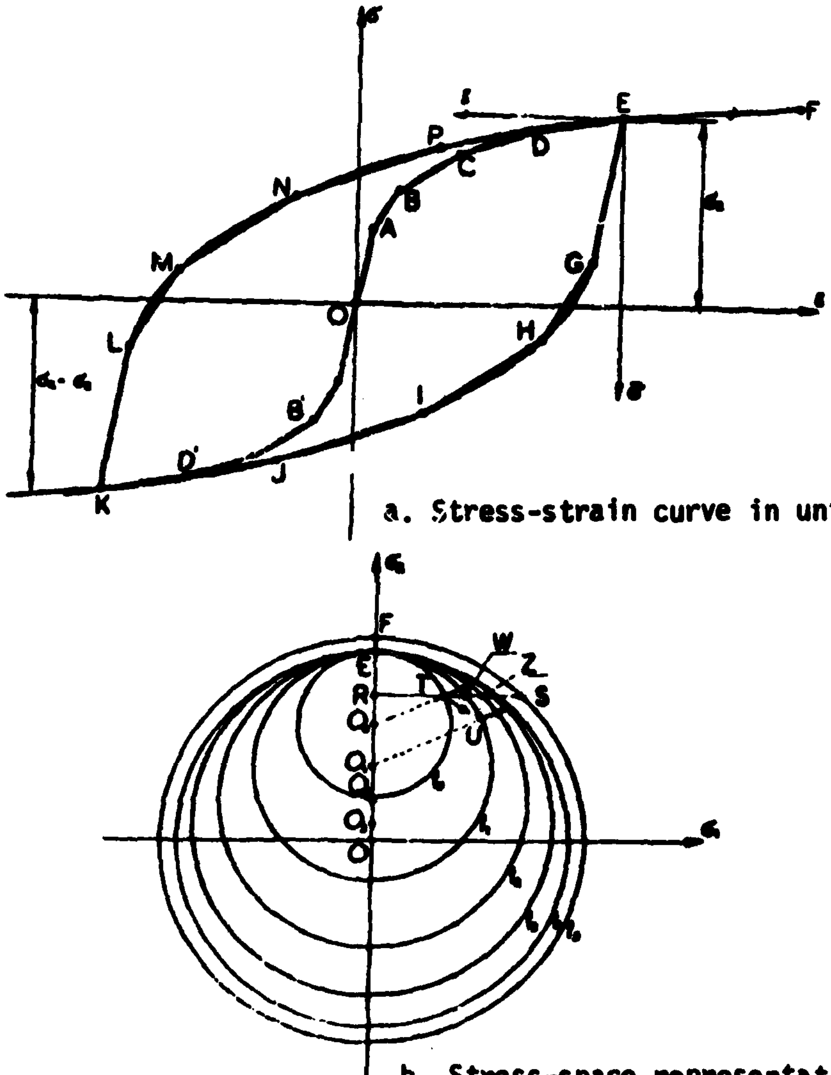
e. Cyclic Plasticity

Isotropic hardening implies that a specimen under cyclic loading will shake down to an elastic state. Linear kinematic hardening, although accounting for Bauschinger effect and mechanical anisotropy, predicts a steady state involving alternating plastic strains after the first cycle.

Actual materials reach a steady cycle of alternating plastic flow after a certain number of cycles or asymptotically. Therefore, isotropic and/ or linear kinematic hardening theories are inadequate for cases involving reversed loading, reloading and cyclic loading. In cyclic loading usually a transient hardening state is observed during which hysteresis loops change their form considerably. In addition, strain cumulation may occur.

Mroz (1967) proposed a rule of anisotropic hardening in the form of a piecewise linear approximation to the nonlinear stress-strain curve realized under proportional loading (Figure 13). This is equivalent to assuming the existence of a sequence of non-intersecting nested yield surfaces which, for initially isotropic material, are similar and concentric enclosing the stress-free state. During loading a yield surface translates, without change in form and orientation, with the stress point. As the stress point reaches another surface, all the previous surfaces along with the newly contacted surface would move with the stress point. However, at all times, the surfaces to which the stress point is interior would be unaffected. Allowing the yield surfaces to expand or contract in addition to translation, the surfaces not reached by the stress point would expand or contract uniformly without any translation.

In another study (1971) Mroz assumed the innermost surface to translate without change in size, the outermost to change in size without translation and the intermediate surfaces to translate as well as expand or contract to explain material behavior in cyclic loading. This model simulates actual observations that the history of plastic pre-straining influences material behavior but the influence of prestraining is wiped out by subsequent plastic deformation of sufficient magnitude. This



a. Stress-strain curve in uniaxial loading

b. Stress-space representation: Plane stress

Figure 13. Approximation of Stress-Strain Curve by Portions of Constant Tangent Moduli. (Mroz, 1967)

leads to the concept of "last significant event" in the deformation history such that the material has memory only as far back as that event. A bi-linear approximation of the work-hardening curve corresponds to two surfaces, the inner translating and the outer expanding isotropically.

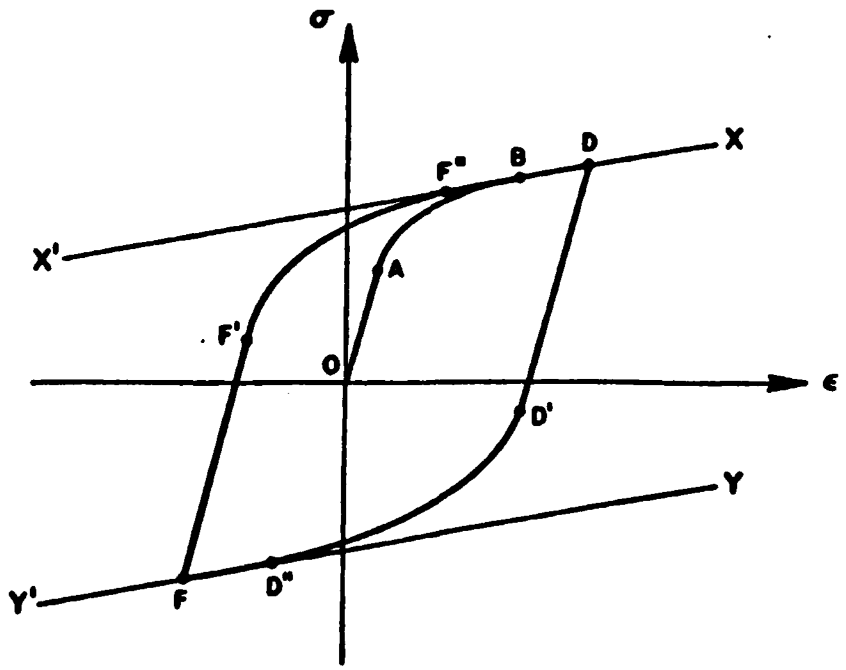
A "simplified" model proposed by Mroz (1975) for steady cyclic loading with no strain accumulation is essentially a nonlinear elastic model with memory of last stress reversal. The three principal sheering strains are expected to depend upon the corresponding principal shearing stresses. The existence of a yield surface is not required. The "skeleton" curve is described by Ramberg-Osgood equation and the hysteresis curve by Masing's rule.

Mroz's theory is essentially based on use of a number of ordered yield surfaces, each involving a limited number of internal variables associated with its translation and expansion. In application, the skeleton curve is plotted as the curve joining vertices of symmetric steady state loops. Masing hardening rule applies for reversed loading and reloading. The material has memory of the largest strain/stress amplitude. Cyclic straining of smaller magnitude cannot totally erase the anisotropy and hardening due to prestrain of larger amplitude.

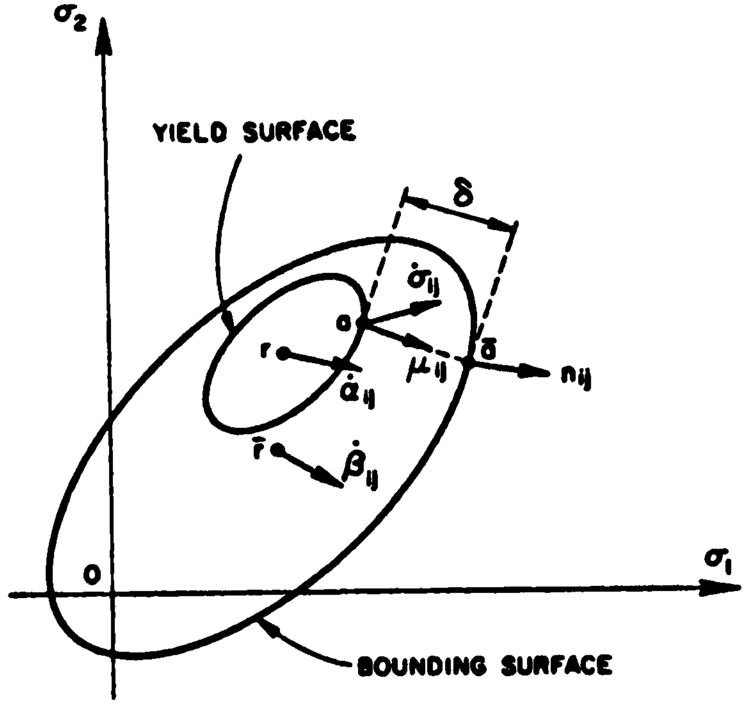
Eisenberg's theory (1976) for cyclic multiaxial loading was based on combined isotropic and kinematic hardening along with the assumptions that(i) small plastic strains do not modify the macroscopic material properties; (ii) the memory for the details of previous loading events is erased by subsequent loading events; and (iii) the effect of such prior history is described completely by the current values of the plastic strain, the length of the strain trajectory and the values of the hardening parameters. For the special case of 304 steel the isotropic component of hardening disappears after a few cycles and the hardening is purely kinematic thereafter. Linear kinematic hardening was assumed. However, the theory permits extension to more general hardening behavior, e.g., the non-homogeneous function described by Shrivastava (1973) which would include distortion of the yield surface associated with cumulation of strain. This theory is similar to Krempl's (1971) to the extent that memory is erased by subsequent flow processes. However, the details of specification of the memory and the representation differ.

Sharma's theory (1977) essentially follows the concept introduced by Mroz. However, a parameter was introduced to represent non-Masing behavior in the presaturation stage. The rate of change of this parameter (yield strength increment) was assumed to be proportional to the difference between the current and the saturation value. Jhansale's (1971, 1974) suggestion that the nonlinear part of various stress strain curves is identical to each other and that a change of elastic region is sufficient to yield a good approximation was discussed by Dafalias (1975). Dafalias observes that Jhansale's theory is applicable only to the data from fully reversed loadings and is a special case of his (Dafalias') more general theory.

Krieg (1976) and Dafalias (1975, 1976) proposed two surface theories. This was a generalization of Mroz's work in that the piecewise linear approximation of the uniaxial stress curve would be replaced by a continuous model (Figure 14). A set of internal variables associated with abrupt changes of the plastic loading process (e.g., loading direction) were



a. Hardening and Bounds on Modulus in σ-ε Space.



b. Yield and Bounding Surfaces in Two-Dimensional Stress. (Distance of Stress Point from the Bounding Surface).

Figure 14. The Concept of a Bounding Surface and Plastic Moduli Based on Distance From the Bounding Surface. (Dafalias, 1975).

introduced as influencing changes in the yield function (the hardening process). Both the surfaces are assumed to be isotropically as well as kinematically hardening. The nonlinear hardening behavior is represented by a generalized plastic modulus which is a function of the current distance from the stress point to the bouding (outer) surface and the maximum distance at the initiation of yield. Dafalias' theory includes Prager's, Ziegler's, Phillip's and Mroz's rules of kinematic hardening as specializations. A special feature of this theory is that unlike Mroz's piecewise linear representation, it provides a smooth transition from the elastic to the plastic stage for general reversed loading.

Under cyclic loading, plastic strain accumulation can occur. This cumulation might reach a limiting value after a few cycles or might continue. The steady state might represent an elastic shakedown state or steady plastic cycling. Mulcahy (1971), following Mroz, assumed that the plasticity model comprises a family of convex loading surfaces. Von Mises yield and Ziegler's linear kinematic hardening rule were used. The rate of strain accumulation was found to be dependent upon the mean stress and the stress amplitude, with no strain accumulation for zero mean.

4. CONSTITUTIVE EQUATIONS FOR PLASTIC DEFORMATION

a. Basic Concepts

We assume that a yield surface or loading surface exists in the stress space, i.e. for given E''_{ij}, q_i defining the deformation history and temperature θ

$$f(\sigma_{ij}, E''_{ij}, q_i, \theta) = 0 \tag{3.59}$$

is the yield surface such that for $f < 0$, there is no plastic deformation. Also, that changes in the yield surface occur only when E''_{ij}, q_i, θ change. Assuming f sufficiently smooth in its arguments,

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f}{\partial \theta} \dot{\theta} + \frac{\partial f}{\partial E''_{ij}} \dot{E}''_{ij} + \frac{\partial f}{\partial q_i} \dot{q}_i \tag{3.60}$$

When there is no plastic deformation $\dot{E}''_{ij}$, $\dot{q}_i$ vanish and $\dot{f}$ reduces to the loading function

$$L = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f}{\partial \theta} \dot{\theta} \tag{3.61}$$

For $f < 0$ or $f = 0$ but $L \leq 0$ there is no plastic deformation, the yield surface does not change and the stress-strain relations are elastic. For $f = 0$, $L > 0$, plastic deformation will occur.

If the strain-space representation is adopted, the yield surface is

$$g(E_{kl}, E''_{kl}, q_i, \theta) = 0 \tag{3.62}$$

and the loading function is

$$N = \frac{\partial g}{\partial E_{ij}} \dot{E}_{ij} + \frac{\partial g}{\partial \theta} \dot{\theta} \tag{3.63}$$

Constitutive relations for the plastic strain as well as the other internal variables have the form (Dafalias, 1976)

$$\dot{q}_i = \hat{q}_i(\sigma_{kl}, \theta, \dot{\sigma}_{kl}, \dot{\theta}, q_j, \xi_j)H(L) \quad \text{for } f = 0 \tag{3.64}$$

$$= 0 \text{ for } f < 0$$

Here $H(L)$ is the Heaviside step function with $H(0) = 0$ and ξ_j are the internal variables associated with abrupt changes in the loading process (e.g., change in loading direction). Using the strain-space formulation, we would have

$$\dot{q}_i = \bar{q}_i(E_{kl}, \theta, \dot{E}_{kl}, \dot{\theta}, q_j, \xi_j)H(N) \text{ for } g = 0 \qquad (3.65)$$

$$= 0 \text{ for } g < 0 \qquad (3.65)$$

In writing the Equations (3.64) and (3.65) we have included the plastic strain E''_{ij} in the set of variables q_j.

For rate independence $\hat{q}_i$ must be homogeneous of degree one in $\dot{\sigma}_{ij}$ and $\dot{\theta}$. Similarly, $\bar{q}_i$ must be homogeneous of degree one in $\dot{E}_{ij}$ and $\dot{\theta}$. Assuming linearity in $\dot{\sigma}_{ij}$ and $\dot{\theta}$, following Hill (1950), Dafalias (1976) proposed

$$\dot{q}_i = r_i L H(L) \qquad (3.66)$$

$$\dot{E}''_{ij} = \lambda \rho_{ij} L H(L) \qquad (3.67)$$

Here the factor $L = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f}{\partial \theta} \dot{\theta}$ expresses linearity in arguments $\dot{\sigma}_{ij}$ and $\dot{\theta}$, $H(L)$ assumes $\dot{q}_i$, $\dot{E}''_{ij}$ vanishing for $L \leq 0$, ρ_{ij} is a directional vector and λ, r_i are functions of σ_{ij}, θ, q_i, E''_{ij}, q_j, ξ_j. Green (1965a) started with the assumption

$$\dot{E}''_{ij} = A_{ijkl}\dot{\sigma}_{kl} + A_{ij}\dot{\theta} \qquad (3.68)$$

during loading. As $\dot{E}''_{ij} = 0$ whenever $L = 0$.

$$(A_{ijkl} - \lambda\beta_{ij} \frac{\partial f}{\partial \sigma_{kl}})\dot{\sigma}_{kl} + (A_{ij} - \lambda\beta_{ij} \frac{\partial f}{\partial \theta})\dot{\theta} = 0 \qquad (3.69)$$

where β_{ij} is a symmetric tensor function and λ is a scalar function of σ_{ij}, E''_{ij}, q_i, θ. Equation (3.69) is true for arbitrary values of $\dot{\sigma}_{kl}$ and $\dot{\theta}$. Hence

$$A_{ijkl} = \lambda\beta_{ij} \frac{\partial f}{\partial \sigma_{kl}} \qquad (3.70)$$

and

$$A_{ij} = \lambda\beta_{ij} \frac{\partial f}{\partial \theta} \qquad (3.71)$$

Hence

$$\dot{E}''_{ij} = \lambda\beta_{ij}(\frac{\partial f}{\partial \sigma_{kl}} \dot{\sigma}_{kl} + \frac{\partial f}{\partial \theta} \dot{\theta}) \qquad (3.72)$$

$$= \lambda\beta_{ij}L \qquad (3.72)$$

during loading. β_{ij} determines the direction of the plastic strain increment. If this is assumed to be normal to the yield surface (or loading surface), $\beta_{ij} = \frac{\partial f}{\partial \sigma_{ij}}$ and

$$\dot{E}''_{ij} = \lambda \frac{\partial f}{\partial \sigma_{ij}} L \qquad (3.73)$$

Often L is evaluated from Equation (3.72) as

$$L = \sqrt{\frac{\dot{E}''_{ij}\dot{E}''_{ij}}{\lambda^2 \beta_{kl}\beta_{kl}}} = c(\dot{E}''_{ij}\dot{E}''_{ij})^{1/2} \tag{3.74}$$

c is the "generalized plastic modulus" in Dafalias's theory (1976). Substituting Equation (3.74) in Equation (3.66)

$$\dot{q}_n = cr_n(\dot{E}''_{ij}\dot{E}''_{ij})^{1/2} \tag{3.75}$$

An alternative to the above procedure is to multiply both sides of Equation (3.66) by β_{ij} to get

$$\dot{q}_n \beta_{ij} = r_n LH(L)\beta_{ij} \tag{3.76}$$

$$= \frac{r_n}{\lambda} \dot{E}''_{ij} \quad \text{by Equation (3.67)} \tag{3.77}$$

Hence

$$\dot{q}_n = \frac{r_n}{\lambda} (\beta_{ij})^{-1} \dot{E}''_{ij} = A_{nij}\dot{E}''_{ij} \tag{3.78}$$

i.e., the internal variables are linear in the plastic strain increment (Equation 3.78) or in the increment of the plastic strain trajectory (Equation 3.75).

In the strain space formulation, the plastic strain rate as well as $\dot{q}_i$ vanish for $g \leq 0$ and $N \leq 0$ where

$$N = \frac{\partial g}{\partial E_{kl}} \dot{E}_{kl} + \frac{\partial g}{\partial \theta} \dot{\theta} \tag{3.63}$$

For loading ($g = 0$, $N > 0$),

$$\dot{E}''_{ij} = \bar{\lambda} \frac{\partial g}{\partial E_{ij}} N \tag{3.79}$$

It should be noted that the conditions of loading in the stress space and the strain space are different (Naghdi, 1975c).
Assuming the stresses to be uniquely related to the elastic strains

$$\dot{E}'_{ij} = E_{ij} - E''_{ij}$$

we have

$$f(\sigma_{ij}, E''_{ij}, q_i, \theta) = g(E_{ij}, E''_{ij}, q_i, \theta) \tag{3.80}$$

The loading function in the stress space

$$L = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f}{\partial \theta} \dot{\theta} = \frac{\partial g}{\partial E_{ij}} \dot{E}'_{ij} + \frac{\partial g}{\partial \theta} \dot{\theta}$$

$$= \frac{\partial g}{\partial E_{ij}} (\dot{E}_{ij} - \dot{E}''_{ij}) + \frac{\partial g}{\partial \theta} \dot{\theta}$$

$$= N - \frac{\partial g}{\partial E_{ij}} \dot{E}''_{ij} \tag{3.81}$$

As $\dot{E}''_{ij} \neq 0$ during loading, $L \neq N$ and $N > 0$ does not imply $L > 0$.

b. Consistency

During loading $f = 0$ throughout i.e., $\dot{f} = 0$. This yields

$$\frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f}{\partial \theta} \dot{\theta} + \frac{\partial f}{\partial E''_{ij}} \dot{E}''_{ij} + \frac{\partial f}{\partial q_i} \dot{q}_i = 0 \tag{3.82}$$

We shall refer to this equation as the consistency condition. Substituting for $\dot{E}''_{ij}$ and $\dot{q}_i$ from Equations (3.65) and (3.67), Equation (3.82) yields

$$L + \frac{f}{E''_{ij}} \lambda\beta_{ij}LH(L) + \frac{\partial f}{\partial q_i} r_i LH(L) = 0 \qquad (3.83)$$

As $L > 0$ during loading, the above equation implies

$$1 + \lambda \frac{\partial f}{\partial E''_{ij}} \beta_{ij} + r_i \frac{\partial f}{\partial q_i} = 0 \qquad (3.84)$$

or, equivalently

$$\lambda \frac{\partial f}{\partial E''_{ij}} \beta_{ij} + r_i \frac{\partial f}{\partial q_i} = -1 \qquad (3.85)$$

For $\lambda > 0$, we have the inequality

$$\beta_{ij} \frac{\partial f}{\partial E''_{ij}} + \frac{r_i}{\lambda} \frac{\partial f}{\partial q_i} < 0 \qquad (3.86)$$

during loading.

If the strain space formulation is used, $\dot{g} = 0$ leads to

$$\frac{\partial g}{\partial E_{ij}} \dot{E}_{ij} + \frac{\partial g}{\partial E''_{ij}} \dot{E}''_{ij} + \frac{\partial g}{\partial q_i} \dot{q}_i + \frac{\partial g}{\partial \theta} \dot{\theta} = 0 \qquad (3.87)$$

Substitution for $\dot{E}''_{ij}$ from Equation (3.79) and a similar equation for $\dot{q}_i$, during loading,

$$1 + \bar{\lambda}\bar{\beta}_{ij} \frac{\partial g}{\partial E''_{ij}} + \bar{r}_i \frac{\partial g}{\partial q_i} = 0 \qquad (3.88)$$

along with the inequality

$$\bar{\beta}_{ij} \frac{\partial g}{\partial E''_{ij}} + \frac{\bar{r}_i}{\lambda} \frac{\partial g}{\partial q_i} < 0 \qquad (3.89)$$

Inequalities (3.86) and (3.89) impose restrictions on the form of the functions f and g and on the coefficients appearing in the constitutive relations (Caulk, 1978).

c. Normality of the Plastic Strain Increment

Assuming normality and convexity in behavior of single f.c.c. crystals, Lin (1971) established normality and convexity for polycrystalline aggregates. In the mathematical theory of plasticity, normality of the plastic strain increments is often proposed as a primitive postulate. Naghdi (1960) showed normality of the incremental plastic strain to the yield surface to be a consequence of Drucker's thermodynamic postulate (1951). Palmer (1967) showed that normality and convexity do not depend upon stability in the small. Frictional materials do not exhibit normality (Drucker, 1951). Assuming that stress is derivable from a potential, Naghdi (1975a, b) showed that during loading, assuming a single internal variable κ,

$$\frac{\partial \sigma_{ij}}{\partial E''_{kl}} \dot{E}''_{kl} + \frac{\partial \sigma_{ij}}{\partial \kappa} \dot{\kappa} = - \gamma \frac{\partial f}{\partial \sigma_{kl}} \frac{\partial \sigma_{kl}}{\partial E_{ij}} , \gamma \geq 0 \qquad (3.90)$$

Further, defining elastic strain $E'_{ij} = E_{ij} - E''_{ij}$, for σ_{ij} a function of E'_{kl}, E''_{kl}, κ, Naghdi (1975b) showed that

$$\dot{E}''_{ij} - \frac{\partial E_{ij}}{\partial \sigma_{kl}} \left(\frac{\partial \sigma_{kl}}{\partial E''_{mn}} \dot{E}''_{mn} + \frac{\partial \sigma_{kl}}{\partial \kappa} \dot{\kappa} \right) = \gamma \frac{\partial f}{\partial \sigma_{ij}} \qquad (3.91)$$

If stress is uniquely related to E'_{ij} and is independent of E''_{ij} and κ, Equation (3.91) implies

$$\dot{E}''_{ij} = \gamma \frac{\partial f}{\partial \sigma_{ij}} \tag{3.92}$$

i.e., normality holds. Equation (3.92) is often referred to as the "associated" flow rule.

Mroz (1973) assumed linearity and continuity of the relationship between stress and strain increments to obtain, for n_{ij} a unit vector,

$$\dot{E}''_{ij} = G_{ijkl} n_{kl} n_{pq} \dot{\sigma}_{pq} \tag{3.93}$$

In case n_{kl} is an eigenvector of G_{ijkl},

$$G_{ijkl} n_{kl} = \lambda \delta_{ik} \delta_{jl} n_{kl} = \lambda n_{ij} \tag{3.94}$$

and we obtain an associated flow rule viz.

$$\dot{E}''_{ij} = \lambda n_{ij} n_{pq} \dot{\sigma}_{pq} \tag{3.95}$$

i.e., normality holds.

Dafalias (1977) set up the formulation in the strain-space. For this approach

$$\dot{E}''_{kl} = \bar{\lambda} \bar{\beta}_{kl} NH(N) \tag{3.79}$$

and

$$\dot{q}_i = r_i NH(N) \tag{3.96}$$

Eliminating N between Equations (3.79) and (3.96) in a manner similar to the one that led to Equation (3.78),

$$\dot{q}_i = \bar{A}_{ikl}E''_{kl} \tag{3.97}$$

Assume that potentials ψ, $\bar{\psi}$ exist such that

$$\psi = \psi(E_{ij}, \theta, E''_{ij}, q_i)$$

$$= \bar{\psi}(E'_{ij}, \theta, q_i) \tag{3.98}$$

along with

$$\sigma_{ij} = \frac{\partial\psi}{\partial E_{ij}} = \frac{\partial\bar{\psi}}{\partial E'_{ij}}$$

(3.99)

and $$\frac{\partial\bar{\psi}}{\partial q_i} = \frac{\partial\psi}{\partial q_i}, \quad \frac{\partial\bar{\psi}}{\partial E''_{kl}} = \frac{\partial\psi}{\partial E_{kl}} + \frac{\partial\psi}{\partial E''_{kl}} = \sigma_{kl} + \frac{\partial\psi}{\partial E''_{kl}}$$

Legendre's transformation yields

$$\phi = \phi(\sigma_{ij}, \theta, q_i) = \sigma_{ij}E_{ij} - \psi \tag{3.100}$$

$$\bar{\phi} = \bar{\phi}(\sigma_{ij}, \theta, q_i) = \sigma_{ij}E'_{ij} - \bar{\psi} \tag{3.101}$$

such that

$$\phi = \bar{\phi} + \sigma_{ij}E''_{ij} \tag{3.102}$$

then

$$E_{kl} = \frac{\partial\phi}{\partial\sigma_{kl}} \tag{3.103}$$

$$E'_{kl} = \frac{\partial\bar{\psi}}{\partial\sigma_{kl}} \tag{3.104}$$

and

$$\frac{\partial\phi}{\partial q_i} = -\frac{\partial\psi}{\partial q_i} \tag{3.105}$$

$$\frac{\partial \bar{\phi}}{\partial q_i} = - \frac{\partial \bar{\psi}}{\partial q_i} \tag{3.106}$$

Thermodynamic considerations assuming ψ is the Helmholtz free energy lead to

$$\sigma_{kl} = \frac{\partial \psi}{\partial E_{kl}} \tag{3.107}$$

and

$$- \frac{\partial \psi}{\partial q_i} r_i \geq 0 \tag{3.108}$$

Equation (3.108) with Equations (3.79), (3.97) and (3.99) leads to

$$- \frac{\partial \psi}{\partial q_i} A_{ikl} \bar{\lambda} \bar{\beta}_{kl} = (\sigma_{kl} - \frac{\partial \bar{\psi}}{\partial q_i} A_{ikl}) \bar{\lambda} \bar{\beta}_{kl} \geq 0 \tag{3.109}$$

or

$$(\sigma_{kl} + \frac{\partial \bar{\psi}}{\partial q_i} A_{ikl}) \bar{\lambda} \bar{\beta}_{kl} \geq 0 \tag{3.110}$$

Ilyushin's postulate that in a closed cycle of strain designated by path P, the work done

$$W = \int_P \sigma_{ij} \dot{E}_{ij} dt \geq 0 \tag{3.111}$$

leads to

$$\bar{\lambda} \bar{\beta}_{kl} = \bar{\lambda} M^{-1}_{klij} \frac{\partial \phi}{\partial E'_{ij}} , \ \lambda > 0 \tag{3.112}$$

where

$$M_{klij} = - \frac{\partial^2 \psi}{\partial E_{kl} \partial q_n} \bar{A}_{nij}$$

$$= \frac{\partial^2 \bar{\psi}}{\partial E'_{kl} \partial E'_{ij}} - \frac{\partial^2 \bar{\psi}}{\partial E'_{kl} \partial q_n} \bar{A}_{nij} \tag{3.113}$$

with

$$M_{klij} \bar{\lambda} \bar{\beta}_{ij} n_{kl} \geq 0 \tag{3.114}$$

Here n_{kl} is an arbitrary unit vector in the strain-space.

A stress-space formulation leads to

$$\lambda\beta_{kl} = \lambda Q^{-1}_{klij} \frac{\partial f}{\partial \sigma_{kl}}, \; \lambda > 0 \tag{3.115}$$

where

$$Q_{klij} = \frac{\partial^2 \phi}{\partial \sigma_{kl} \partial q_n} A_{nij} = \delta_{ki}\delta_{lj} + \frac{\partial^2 \bar{\phi}}{\partial \sigma_{kl} \partial q_n} A_{nij} \tag{3.116}$$

Naghdi's work (1975a, b) is included in Dafalias' more general theory as a specialization. The general result Dafalias obtained is

$$- \frac{\partial \psi}{\partial q_n} \bar{A}_{nij} M^{-1}_{ijkl} \frac{\partial g}{\partial E_{kl}} = \left(\frac{\partial \bar{\psi}}{\partial E^{\prime}_{ij}} - \frac{\partial \bar{\psi}}{\partial q_n} \bar{A}_{nij} \right) M^{-1}_{ijkl} \frac{\partial g}{\partial E_{kl}} \geq 0 \tag{3.117}$$

in the strain-space. Or, in the stress-space

$$\frac{\partial \phi}{\partial q_n} A_{nij} Q^{-1}_{ijkl} \frac{\partial f}{\partial \sigma_{kl}} = \left(\sigma_{ij} + \frac{\partial \bar{\phi}}{\partial q_n} A_{nij} \right) Q^{-1}_{ijkl} \frac{\partial f}{\partial \sigma_{kl}} \geq 0 \tag{3.118}$$

Here M_{ijkl}, Q_{ijkl} are curvatures of ψ and ϕ respectively. These inequalities impose restrictions on the constitutive relations involving elastic-plastic coupling. If there is no coupling, i.e.,

$$\bar{\phi} = \bar{\phi}_1(\sigma_{kl}, \theta) + \bar{\phi}_2(q_n) \tag{3.119}$$

Then

$$Q_{klij} = \delta_{ki}\delta_{lj} \tag{3.120}$$

and Equation (3.115) reduces to

$$\lambda\beta_{kl} = \lambda \frac{\partial f}{\partial \sigma_{kl}}, \; \lambda > 0 \tag{3.121}$$

which is the "normality rule." At the same time the inequality (3.118) takes the form

$$\left(\sigma_{ij} + \frac{\partial\phi_2}{\partial q_n} A_{nij}\right)\frac{\partial f}{\partial\sigma_{ij}} \geq 0 \qquad (3.122)$$

For $q_n = E''_{ij}$, the foregoing results specialize to Green's (1965) and Phillip's (1966) theories.

Conversely, if the coupling is present, convexity does not follow from Ilyushin's postulate and normality may not exist. Indeed assuming only one internal parameter q, inequality (3.118) along with the assumption of convexity and linear workhardening i.e.

$$\dot{q} = \sigma_{ij}\dot{E}''_{ij}$$

leads to

$$(1 + \frac{\partial\bar{\Phi}}{\partial q})(1 + \frac{\partial\bar{\Phi}}{\partial\sigma_{kl}\partial q}\, \sigma_{kl}) \geq 0 \qquad (3.123)$$

If normality had been assumed, we would get instead

$$1 + \frac{\partial\bar{\Phi}}{\partial q} \geq 0$$

For nonlinear flow rule proposed by Mroz (1964) the plastic strain increment depends not only upon $\frac{\partial f}{\partial\sigma_{ij}}$ but also upon the curvature of the yield surface. In this case normality does not hold in general.

Phillips (1977) investigated the behavior of tubular specimens made of pure commercial aluminum 1100-0. It was found that $\dot{E}''_{ij}$ was always normal to the yield surface and when the yield surface becomes tangent to the loading surface (in the context of a two surface theory) the plastic strain increment is normal to both the yield and the loading surface.

d. Hardening Rules

Changes in material properties during loading are described by constitutive equations for selected internal variables. These include, among others, the isotropic hardening parameter κ describing the expansion of the yield surface and α_{ij} the location of the center of the yield surface.

(1) The isotropic hardening parameter κ

In line with Equations (3.75) and (3.78), the constitutive equation for the isotropic hardening parameter has the form

$$\dot{\kappa} = \dot{\kappa}(\dot{E}''_{ij}) \tag{3.124}$$

where for work hardening (Equation 3.78)

$$\dot{\kappa} = A_{kl}\dot{E}''_{kl} \tag{3.125}$$

and for strain hardening (Equation 3.75)

$$\dot{\kappa} = c(\dot{E}''_{ij}\dot{E}''_{ij})^{1/2} \tag{3.126}$$

Here A_{kl}, c are functions of σ_{ij}, E''_{ij}, q_i and θ. Caulk (1978) combined the notions of work hardening and dependence of $\dot{\kappa}$ upon E''_{ij} by assuming

$$A_{kl} = M_{klij}\sigma_{ij} + N_{klij}E''_{ij} \tag{3.127}$$

This leads to

$$\dot{\kappa} = M_{klij}\sigma_{ij}\dot{E}''_{kl} + N_{klij}E''_{ij}\dot{E}''_{kl} \tag{3.128}$$

Eisenberg (1971) regarded the yield surface as the limit of proportionality of stress and strain. The loading surface enclosing the yield surface hardens with plastic deformation which can occur for points between the two surfaces. The hardening law proposed had the form

$$\dot{\kappa} = A_{kl}\dot{E}''_{kl} + \alpha L H(L) \tag{3.129}$$

For $\alpha = 0$ this reduces to Equation (3.125). During unloading, for points outside the yield surface, $\alpha = 1$. During reloading $\alpha \leq 1$. The introduction of the parameter α would permit κ to vary even when $\dot{E}''_{kl} = 0$ i.e. the loading point can pull the loading surface with it even during unloading. One way of assigning values between zero and 1 to α is to use powers of $\bar{\alpha}$, $0 \leq \bar{\alpha} \leq 1$ and set $\alpha = \bar{\alpha}^n$. Choice of the index n governs how close the formulation is to the one-surface theory. For $n \to \infty$ the one surface theory is realized as $\alpha \to 0$.

For cyclic loading, Mroz (1969, 1971) suggested use of

$$|\Delta\sigma| = |\sigma_o - \sigma_s|$$

as a function of plastic strain history. Here, if f is homogeneous of order n in its arguments and $\kappa = \sigma_o^n$, σ_s represents the limiting value of σ_o as the steady cycle is reached. $|\Delta\sigma|$ is a decreasing function and has to be determined experimentally for the prescribed loading.

Existence of a saturation condition requires that $\dot{\kappa} = 0$ after κ has attained its saturation value. Caulk (1978) proposed a constitutive relationship of the form

$$A_{kl} = \frac{\kappa - \kappa_s}{\kappa_0 - \kappa_s} \bar{A}_{kl}(\sigma_{mn}, E''_{mn}) \tag{3.130}$$

where κ_0, κ_s are the initial and the saturation values respectively, of the parameter κ.

(2) The kinematic hardening parameter α_{ij}

Assuming the yield surface to have the form

$$f(\sigma_{ij} - \alpha_{ij}) - \kappa = 0 \tag{3.131}$$

Prager (1956) proposed a rule for the rate of translation in the form

$$\dot{\alpha}_{ij} = c\dot{E}''_{ij} \tag{3.132}$$

where c is a constant of proportionality. This is in line with Equation (3.78). The yield surface was expected to translate in the direction of the plastic strain increment. If $\alpha_{ij} = 0$ initially (at $E''_{ij} = 0$) integration yields

$$\alpha_{ij} = cE''_{ij}$$

i.e., the equation for the yield surface is

$$f(\sigma_{ij} - cE''_{ij}) - \kappa = 0 \tag{3.133}$$

This formulation was shown to be unsatisfactory (Shield, 1958) for subspaces of the stress space. Ziegler (1959) proposed

$$\dot{\alpha}_{ij} = \mu(\sigma_{ij} - \alpha_{ij}), \ \mu > 0 \tag{3.134}$$

along with

$$(\dot{\sigma}_{ij} - \bar{c}\dot{\varepsilon}''_{ij}) \frac{\partial f}{\partial \sigma_{ij}} = 0 \tag{3.135}$$

where $\bar{c}$ is a material parameter. μ is defined by the consistancy condition which for purely kinematic hardening with the yield surface defined by Equation (3.131) is

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f}{\partial \alpha_{ij}} \dot{\alpha}_{ij} = 0 \tag{3.136}$$

Noting that $\frac{\partial f}{\partial \alpha_{ij}} = - \frac{\partial f}{\partial \sigma_{ij}}$ and substituting Equation (3.134) for $\dot{\alpha}_{ij}$,

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} - \frac{\partial f}{\partial \sigma_{ij}} \mu(\sigma_{ij} - \alpha_{ij}) = 0 \tag{3.137}$$

Hence

$$\mu = \frac{\frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij}}{\frac{\partial f}{\partial \sigma_{mn}} (\sigma_{mn} - \alpha_{mn})} \tag{3.138}$$

$$\dot{\alpha}_{ij} = \frac{\frac{\partial f}{\partial \sigma_{kl}} \dot{\sigma}_{kl}}{\frac{\partial f}{\partial \sigma_{mn}} (\sigma_{mn} - \alpha_{mn})} (\sigma_{ij} - \alpha_{ij}) \tag{3.139}$$

Equation (3.139) has the form of Equation (3.66), i.e., $\dot{\alpha}_{ij}$ are proportional to L with the proportionality coefficient a function of the current yield surface configuration and the stress state.

Combining Equations (3.135) and (3.139) we have

$$\dot{\alpha}_{ij} = \frac{(\sigma_{ij} - \alpha_{ij})}{\frac{\partial f}{\partial \sigma_{mn}} (\sigma_{mn} - \alpha_{mn})} \bar{c} \frac{\partial f}{\partial \alpha_{kl}} \dot{E}''_{kl} \tag{3.140}$$

which has the same form as Equation (3.78). We note that Equation (3.135) follows from the consistency condition (Equation 3.136) for Prager's rule for translation of the yield surface (Equation 3.132). However, Equation (3.135) with the consistency equation implies

$$\bar{c}\dot{E}''_{ij} \frac{\partial f}{\partial \sigma_{ij}} = \dot{\alpha}_{ij} \frac{\partial f}{\partial \sigma_{ij}}$$

i.e., the difference between the translation rate $\dot{\alpha}_{ij}$ and the quantity $\bar{c}\dot{E}''_{ij}$ is tangential to the yield surface. The quantity $\bar{c}\dot{E}''_{ij}$ is thus the component of $\dot{\alpha}_{ij}$ along the normal to the yield surface. This relation completely defines the role of $\bar{c}$.

Admitting kinematic as well as isotropic hardening the consistency condition is

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f}{\partial \alpha_{ij}} \dot{\alpha}_{ij} - \dot{\kappa} = 0$$

By an argument similar to the one that led to Equation (3.138) we now have

$$\mu = \frac{\frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} - \dot{\kappa}}{\frac{\partial}{\partial \sigma_{mn}} (\sigma_{mn} - \alpha_{mn})} \tag{3.141}$$

Further if we assume

$$\dot{\kappa} = \gamma \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} \tag{3.142}$$

$$\mu = \frac{(1 - \gamma) \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij}}{\frac{\partial f}{\partial \sigma_{mn}} (\sigma_{mn} - \alpha_{mn})} \tag{3.143}$$

$\gamma = 0$ implies pure translation while $\gamma = 1$ yields $\mu = 0$ and isotropic hardening.

For nonlinear kinematic hardening, Kadashevitch (1959) proposed

$$c = c(E''_{ij}) \tag{3.144}$$

in the hardening rule (Equation 3.132), i.e., the translation of the yield surface was expected to be a function of the plastic deformation. Eisenberg (1968) has shown that this is inadmissible and that no function of the form

$$f(\sigma_{ij}, E''_{ij}) - \kappa = 0 \tag{3.145}$$

with κ constant or variable can satisfactorily represent stress-strain curves produced by tensile loading followed by compressive loading of metals. For such cases a form of the yield function can be

$$f(\sigma_{ij}, E''_{ij}, \kappa_1) - \kappa_0 = 0 \tag{3.146}$$

where κ_0 = constant and κ_1 is such that

$$\dot{\kappa}_1 = \dot{\kappa}_1(\dot{E}''_{kl}) \tag{3.147}$$

A specific form proposed by Eisenberg is

$$f(\sigma_{ij} - c(\kappa_1)E''_{ij}) - \kappa = 0 \tag{3.148}$$

where κ is a constant, c is the kinematic hardening function and κ_1 is the length of the plastic strain trajectory, i.e.

$$\alpha_{ij} = c(\kappa_1)E''_{ij} \tag{3.149}$$

such that

$$\dot{\alpha}_{ij} = c(\kappa_1)\dot{E}''_{ij} + \frac{dc}{d\kappa_1} E''_{ij}\dot{\kappa}_1 \tag{3.150}$$

Noting that here

$$\dot{\kappa}_1 = (\dot{E}''_{ij}\dot{E}''_{ij})^{1/2} \tag{3.151}$$

Equation (3.150) represents a combination of the notions expressed by Equations (3.75) and (3.78).

Noting that $\dot{\kappa}_1 = \dot{\kappa}_1(\dot{E}''_{kl})$, assuming $\dot{\kappa}_1$ linear in its argument, Mroz (1973) proposed

$$\dot{\alpha}_{ij} = \bar{c}(\kappa_1)\dot{E}''_{ij} \tag{3.152}$$

where

$$\bar{c}(\kappa_1) = c(\kappa_1) + \frac{dc}{d\kappa_1} E''_{ij}c_{ij} \tag{3.153}$$

and c_{ij} are the constants of proportionality between $\dot{\kappa}_1$ and $\dot{E}''_{ij}$, i.e.

$$\dot{\kappa}_1 = c_{ij}\dot{E}''_{ij} \tag{3.154}$$

To admit strain cumulation, Mroz (1976) proposed

$$\dot{\alpha}_{ij} = \bar{c}(\kappa_1)\dot{E}''_{ij} + d(\kappa_1)\dot{\kappa}_1 E''_{ij} \tag{3.155}$$

where κ_1 is defined by Equation (3.151).

In Mroz's (1967) model based on multiple loading surfaces, the inner surface has to translate with the stress points in such a way that as this surface touches the next surface, the two are tangential at the point of contact. Thus, if

$$\left.\begin{aligned} & f(\sigma_{ij} - \alpha_{ij}^{(\ell)}) - \kappa^{(\ell)} = 0 \\ \text{and} \quad & f\left(\sigma_{ij} - \alpha_{ij}^{(\ell+1)}\right) - \kappa^{(\ell+1)} = 0 \end{aligned}\right\} \tag{3.156}$$

are two neighboring loading surfaces f_ℓ and $f_{\ell+1}$ respectively, and the stress state is represented by a stress point P on f_ℓ, the surface f_ℓ will translate so that it contacts $f_{\ell+1}$ at a point R where the normal to $f_{\ell+1}$ is parallel to the normal to f_ℓ at P. This parallelism is retained throughout the translation. The point R on $f_{\ell+1}$ is not necessarily the stress point. For $\kappa^{(\ell)}$, $\kappa^{(\ell+1)}$ of degree n in stress, e.g.

$$\kappa^{(\ell)} = \left(\sigma_0^{(\ell)}\right)^n, \quad \kappa^{(\ell+1)} = \left(\sigma_0^{(\ell+1)}\right)^n$$

and f homogeneous of order n in its arguments, Mroz showed that the above relation would lead to

$$\dot{\alpha}_{ij} = \frac{\mu}{\sigma_o^{(\ell)}}\left[\left(\sigma_o^{(\ell+1)} - \sigma_o^{(\ell)}\right)\sigma_{ij}^{(\ell)} - \left(\sigma_o^{(\ell+1)}\alpha_{ij}^{(\ell)} - \alpha_{ij}^{(\ell+1)}\sigma_o^{(\ell)}\right)\right] \tag{3.157}$$

In the special case where $\alpha_{ij}^{(\ell)} = \alpha_{ij}^{(\ell+1)}$ this equation reduces to Ziegler's rule. Like Ziegler, Mroz also defined μ by the consistency condition for constant κ, i.e.

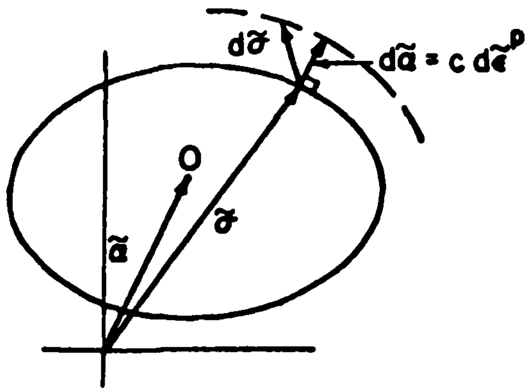
$$(\dot{\alpha}_{ij} - \dot{\sigma}_{ij})\frac{\partial f}{\partial \sigma_{ij}} = 0 \tag{3.158}$$

to get

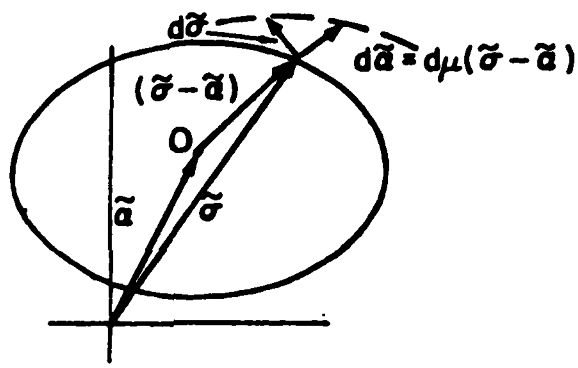
$$\mu = \frac{\frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij}}{\left(\sigma_{kl}^{(\ell+1)} - \sigma_{kl}^{(\ell)}\right)\frac{\partial f}{\partial \sigma_{kl}}} \tag{3.159}$$

Figure 15 shows a comparison of Prager's, Ziegler's and Mroz's rules for kinematic hardening. Lamba's (1976) nonproportional loading experiments on oxygen free high conductivity copper showed that Tresca yield surface translating according to Mroz' rule inside a stationary Tresca bounding surface best reflected material behavior. As a further generalization, the surfaces can be allowed to expand or contract in addition to translation. Then $\kappa^{(\ell)} = \kappa^{(\ell)}(\lambda)$ where λ is a measure of the history of plastic deformation. In this case

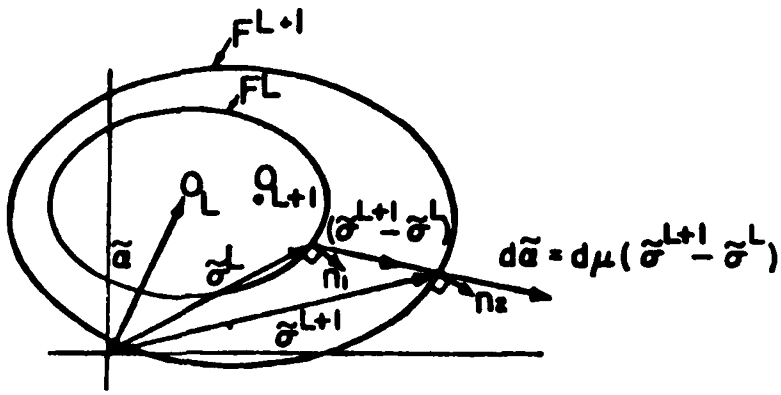
$$\mu = \frac{\dot{\sigma}_{ij}n_{ij} - \dot{\sigma}_o^{(\ell)}}{\left(\sigma_{kl}^{(\ell+1)} - \sigma_{kl}^{(\ell)}\right)n_{kl}} \tag{3.160}$$



(a) Prager Hardening Rule



(b) Ziegler Hardening Rule



(c) Mroz Hardening Rule

Figure 15. Kinematic Hardening Rules of Incremental Plasticity. (Lamba, 1976)

Dafalias (1976) introduced a scalar measure for translation. He defined

$$\dot{\alpha} = \dot{\alpha}_{ij}n_{ij} = \kappa_\alpha\left(\dot{E}''_{ij}\dot{E}''_{ij}\right)^{1/2}$$

$$= \lambda\kappa_\alpha LH(L) \qquad (3.161)$$

where κ_α is a coefficient obtained from the uniaxial test and n_{ij} is the unit normal to f at the stress point. Then

$$\dot{\alpha}_{ij} = \left(\frac{1}{\nu_{kl}n_{kl}}\right)\lambda\kappa_\alpha LH(L) \qquad (3.162)$$

where ν_{ij} is the unit vector along $\dot{\alpha}_{ij}$. Consistency condition along with Equation (3.67) leads to

$$\dot{\alpha}_{ij} = \left(\frac{1}{\nu_{kl}n_{kl}}\right)\left(1 + \frac{\left(\frac{\partial f}{\partial q_n}\right)r_n}{\left(\frac{\partial f}{\partial \sigma_{pq}}\frac{\partial f}{\partial \sigma_{pq}}\right)^{1/2}}\right)LH(L)\nu_{ij} \qquad (3.163)$$

e. The Coefficient λ in the Flow Rule.

The coefficient λ in the constitutive equation for plastic strain increment has been evaluated several different ways. The evaluation may be based on the following

i. Linear dependence of plastic strain rate upon the loading function,

ii. The normality rule,

iii. The consistency condition,

iv. Strain-space formulation, and

v. A combination of the above.

(1) λ from the constitutive equation for the plastic strain increment. Considering

$$\dot{E}''_{ij} = \lambda\beta_{ij}L \tag{3.164}$$

during loading, Dafalias (1976) took norms of both sides to get

$$L = c\left(\dot{E}''_{ij}\dot{E}''_{ij}\right)^{1/2} \tag{3.74}$$

where

$$c = \frac{1}{\lambda(\beta_{kl}\beta_{kl})^{1/2}} = \frac{1}{\lambda}$$

is the "generalized plastic modulus." Then

$$\dot{E}''_{ij} = \beta_{ij}\dot{E}''_e \tag{3.165}$$

where

$$\dot{E}''_e = \left(\dot{E}''_{ij}\dot{E}''_{ij}\right)^{1/2} \tag{3.166}$$

Assuming normality to hold,

$$\beta_{ij} = \frac{\partial f}{\partial \sigma_{ij}}$$

This formulation essentially amounts to writing (Equations 3.74 and 3.166)

$$\lambda = \frac{\dot{E}''_e}{L} \tag{3.167}$$

Introducing

$$H = \frac{\dot{\sigma}_e}{\dot{E}''_e} \qquad (3.168)$$

as the slope of the σ_e, E''_e curve, where σ_e is the "equivalent" stress defined as

$$\sigma_e = (\sigma_{ij}\sigma_{ij})^{1/2} \qquad (3.169)$$

Equation (3.167) gives

$$\lambda = \frac{\dot{\sigma}_e}{LH}$$

$$= \frac{\sigma_{ij}\dot{\sigma}_{ij}}{LH\sigma_e} \qquad (3.170)$$

This approach was also implemented by Swedlow (1966), Marcal (1968, 1969) and Yamada (1968a, b) in finite element solution procedures.

(2) λ from the normality rule.

Drucker (1951) stated the normality rule

$$\dot{E}''_{ij} = \lambda \frac{\partial f}{\partial \sigma_{ij}} \qquad (3.171)$$

and directly took the norm on both sides to get

$$\lambda = \frac{\dot{E}''_e}{\left(\frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}}\right)^{1/2}} \qquad (3.172)$$

Further, introducing the equivalent stress (Equation 3.169) and plastic modulus (Equation 3.168), we obtain

$$\lambda = \frac{1}{H} \frac{\dot{\sigma}_e}{\left(\frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}}\right)^{1/2}} \tag{3.173}$$

This formulation is the same as in Equation (3.170) for von Mises material. It was used by Swedlow (1966), Mroz (1967), Marcal (1967), Marcal (1968, 1969), among others.

Assuming normality to hold, Mroz (1967, 1973) wrote, for the isothermal process when $L = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij}$,

$$\dot{E}''_{ij} = \lambda n_{ij} n_{kl} \dot{\sigma}_{kl} \tag{3.174}$$

Multiplying both sides by $\dot{E}''_{ij}$,

$$\dot{E}''_{ij} \dot{E}''_{ij} = \lambda \dot{\sigma}_{kl} \dot{E}''_{kl} \tag{3.175}$$

Hence,

$$\lambda = \frac{\dot{E}''_{ij} \dot{E}''_{ij}}{\dot{\sigma}_{kl} \dot{E}''_{kl}} \tag{3.176}$$

In similar fashion, Haythornwaite (1968) wrote, for the case of generalized forces and displacements,

$$\lambda = \frac{\dot{u}''_i \dot{u}''_i}{\dot{Q}_j \dot{u}''_j} \tag{3.177}$$

In this formulation, λ can be evaluated at every step of the loading program from observation on incremental displacements and tractions without knowledge of the yield surface.

(3) Consistent evaluation of λ.

The consistency condition is

$$L + \frac{\partial f}{\partial E''_{ij}} \dot{E}''_{ij} + \frac{\partial f}{\partial q_i} \dot{q}_i = 0 \qquad (3.82)$$

A. Linearity and Consistency.

If $\dot{q}_i$ are linearly related to $\dot{E}''_{ij}$ and $\dot{E}''_{ij}$ in turn is linear in L, i.e.

$$\dot{q}_n = A_{nij}\dot{E}''_{ij} \qquad (3.78)$$

and

$$\dot{E}''_{ij} = \lambda\beta_{ij}LH(L) \qquad (3.67)$$

the consistency condition yields, during loading

$$\lambda = \frac{-1}{\frac{\partial f}{\partial E''_{ij}} \beta_{ij} + \frac{\partial f}{\partial q_n} A_{nij}\beta_{ij}} \qquad (3.178)$$

If the hardening rule is

$$\dot{q}_n = A_n\left(\dot{E}''_{kl}\dot{E}''_{kl}\right)^{1/2} \qquad (3.75)$$

the consistency conditions yields

$$\lambda = \frac{-1}{\frac{\partial f}{\partial E''_{ij}} \beta_{ij} + A_n \frac{\partial f}{\partial q_n} (\beta_{ij}\beta_{ij})^{1/2}} \qquad (3.179)$$

Using a combined hardening rule (e.g., strain as well as workhardening),

$$\dot{q}_n = A_{nij}\dot{E}''_{ij} + A_n\left(\dot{E}''_{ij}\dot{E}''_{ij}\right)^{1/2} \tag{3.180}$$

In that case, consistency implies

$$\lambda = \frac{-1}{\frac{\partial f}{\partial E''_{ij}}\beta_{ij} + \frac{\partial f}{\partial q_n}\left(A_{nij}\beta_{ij} + A_n(\beta_{ij}\beta_{ij})^{1/2}\right)} \tag{3.181}$$

This equation includes Equations (3.178) and (3.179) as specializations.

In Eisenberg's (1971) two surface theory the yield surface corresponds to the limit of proportionality and the loading surface encloses the yield surface. In that case, for points between the two surfaces the hardening rule is,

$$\dot{q}_n = A_{nij}\dot{E}''_{ij} + \alpha_n L \tag{3.129}$$

Use of this rule with the consistency condition yields

$$\lambda = \frac{-\left(1 + \frac{\partial f}{\partial q_n}\alpha_n\right)}{\frac{\partial f}{\partial E''_{ij}} + A_{nij}\frac{\partial f}{\partial q_n}\beta_{ij}} \tag{3.182}$$

B. Normality and consistency

Writing the equation of plastic strain increment as

$$\dot{E}''_{ij} = \lambda\frac{\partial f}{\partial \sigma_{ij}}\cdot$$

i.e., assuming normality to hold, consistency condition yields

$$L + \frac{\partial f}{\partial E''_{ij}} \lambda \frac{\partial f}{\partial \sigma_{ij}} + \frac{\partial f}{\partial q_i} \dot{q}_i = 0 \tag{3.183}$$

Hence

$$\lambda = -\frac{L + \frac{\partial f}{\partial q_i} \dot{q}_i}{\frac{\partial f}{\partial E''_{ij}} \frac{\partial f}{\partial \sigma_{ij}}} \tag{3.184}$$

For $\dot{q}_i = 0$, this is the form obtained by Prager (1949) and Naghdi (1960), among others. In the general case, writing

$$\dot{q}_n = A_{nij}\dot{E}''_{ij} + A_n\left(\dot{E}''_{ij}\dot{E}''_{ij}\right)^{1/2} + \alpha_n L \tag{3.185}$$

the consistency condition with normality gives

$$L + \lambda\left[\frac{\partial f}{\partial E''_{ij}} \frac{\partial f}{\partial \sigma_{ij}} + \frac{\partial f}{\partial q_n} A_{nij} \frac{\partial f}{\partial \sigma_{ij}} + \frac{\partial f}{\partial q_n} A_n \left(\frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}}\right)^{1/2}\right] + \frac{\partial f}{\partial q_n} \alpha_n L = 0 \tag{3.186}$$

Hence

$$\lambda = -\frac{\left(1 + \alpha_n \frac{\partial f}{\partial q_n}\right) L}{\left(\frac{\partial f}{\partial E''_{ij}} + \frac{\partial f}{\partial q_n} A_{nij}\right) \frac{\partial f}{\partial \sigma_{ij}} + \frac{\partial f}{\partial q_n} A_n \left(\frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}}\right)^{1/2}} \tag{3.187}$$

The above formulations insomuch as they express λ in terms of $\dot{\sigma}_{ij}$ ($L = \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} + \frac{\partial f}{\partial \theta} \dot{\theta}$) have been (Naghdi, 1975c) referred to as stress-space formulations. We note that these break down for perfectly plastic materials where $\dot{q}_n = 0$ and f is independent of the plastic strain.

(4) Total strain increment formulation.

Following Hill (1950), Felippa (1966) proposed a formulation designed to express λ in terms of the total strain increment $\dot{E}_{ij}$. We present here a generalization of Felippa's approach.

Assuming $\dot{E}'_{ij} = \dot{E}_{ij} - \dot{E}''_{ij}$ to be linearly related to $\dot{\sigma}_{ij}$ as

$$\dot{\sigma}_{ij} = E_{ijkl}\dot{E}'_{kl} \tag{3.188}$$

we have

$$\dot{\sigma}_{ij} = E_{ijkl}\dot{E}'_{kl} = E_{ijkl}[\dot{E}_{kl} - \dot{E}''_{kl}] \tag{3.189}$$

or

$$\frac{\partial f}{\partial \sigma_{ij}}\,\dot{\sigma}_{ij} = \frac{\partial f}{\partial \sigma_{ij}}\,E_{ijkl}[\dot{E}_{kl} - \dot{E}''_{kl}] \tag{3.190}$$

The consistency condition, for a general hardening rule viz.,

$$\dot{q}_n = A_{nij}\dot{E}''_{ij} + A_n\left(\dot{E}''_{ij}\dot{E}''_{ij}\right)^{1/2} \tag{3.191}$$

gives

$$\begin{aligned}\frac{\partial f}{\partial \sigma_{ij}}\,\dot{\sigma}_{ij} &= -\frac{\partial f}{\partial E''_{ij}}\,\dot{E}''_{ij} - \frac{\partial f}{\partial q_n}\,\dot{q}_n \\ &= -\frac{\partial f}{\partial E''_{ij}}\,\dot{E}''_{ij} - \frac{\partial f}{\partial q_n}\left[A_{nij}\dot{E}''_{ij} + A_n\left(\dot{E}''_{ij}\dot{E}''_{ij}\right)^{1/2}\right]\end{aligned} \tag{3.192}$$

Substituting Equation (3.72) for the isothermal case, i.e.,

$$\dot{E}''_{ij} = \lambda\beta_{ij}\,\frac{\partial f}{\partial \sigma_{kl}}\,\dot{\sigma}_{kl} \tag{3.193}$$

and comparing Equations (3.190) and (3.192) we have

$$\frac{\partial f}{\partial \sigma_{ij}} E_{ijkl}\dot{E}_{kl} = \left(\frac{\partial f}{\partial \sigma_{ij}} E_{ijkl} - \frac{\partial f}{\partial E''_{kl}} - \frac{\partial f}{\partial q_n} A_{nkl}\right)\dot{E}''_{kl}$$

$$- \frac{\partial f}{\partial q_n} A_n\left(\dot{E}''_{ij}\dot{E}''_{ij}\right)^{1/2}$$

$$= \lambda B \qquad (3.194)$$

where

$$B = \left[\left(\frac{\partial f}{\partial \sigma_{ij}} E_{ijkl} - \frac{\partial f}{\partial E''_{kl}} - \frac{\partial f}{\partial q_n} A_{nkl}\right)\beta_{kl} - \frac{\partial f}{\partial q_n} A_n\left(\beta_{ij}\beta_{ij}\right)^{1/2}\right]\frac{\partial f}{\partial \sigma_{mn}}\dot{\sigma}_{mn} \qquad (3.195)$$

If normality is assumed to hold, β_{ij} can be replaced by $\frac{\partial f}{\partial \sigma_{ij}}$. Equation (3.194) implies

$$\lambda = \frac{1}{B}\frac{\partial f}{\partial \sigma_{ij}} E_{ijkl}\dot{E}_{kl} \qquad (3.196)$$

Felippa's formulation is valid for perfectly plastic as well as hardening materials. Naghdi's (1975c) strain-space formulation is essentially parallel to Felippa's. This is the formulation in common use in finite element solution procedures (e.g., Pope, 1966; Zienkiewicz, 1969; Nayak, 1972; Sandhu, 1973; and Allen, 1979).

(5) Other methods.

Yamada (1969) used a rate of work equation along with normality and additive splitting of the strain increment to set up the incremental stress strain relations for a von Mises material. Reyes (1966) had previously used this approach for Mohr-Coulomb materials. These methods gave the same final equations as Felippa's approach in the cases studied but cannot be generalized to arbitrary yield surfaces.

Pifko (1974) and Sharma (1977) used the normality rule in conjunction with the consistency condition for purely kinematic hardening. The normality condition, Equation (3.171), upon multiplication by $\frac{\partial f}{\partial \sigma_{ij}}$ yields

$$\frac{\partial f}{\partial \sigma_{ij}} \dot{E}''_{ij} = \lambda \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} \tag{3.197}$$

Instead of directly evaluating λ from this equation, it was conbined with the Ziegler's condition for kinematic hardening with constant κ, Equation (3.135), to get

$$\frac{1}{\bar{c}} \frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij} = \frac{\partial f}{\partial \sigma_{ij}} \dot{E}''_{ij} = \lambda \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}}$$

Hence

$$\lambda = \frac{1}{\bar{c}} \frac{\frac{\partial f}{\partial \sigma_{ij}} \dot{\sigma}_{ij}}{\frac{\partial f}{\partial \sigma_{mn}} \frac{\partial f}{\partial \sigma_{mn}}} \tag{3.198}$$

where $\bar{c}$ is the "plastic modulus." This result is imbedded in the more general formulation, Equation (3.183), and arises as a specialization when $\frac{\partial f}{\partial E''_{ij}} = 0$ (the form of f independent of history), and a single

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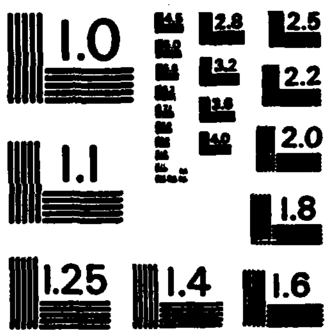

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parameter q such that $\dot{q}$ is proportional to plastic strain increment i.e.

$$\dot{q} = A_{kl}\dot{E}''_{kl} = -\bar{c}\,\frac{\partial f}{\partial \sigma_{kl}}\,\dot{E}''_{kl} = -\bar{c}\,\frac{\partial f}{\partial \sigma_{kl}}\,\lambda\,\frac{\partial f}{\partial \sigma_{kl}} \qquad (3.199)$$

f. Incremental Stress-Strain Relations

Evaluation of λ in the "flow rule" finally leads to the constitutive equation for $\dot{E}''_{ij}$ in the form

$$\dot{E}''_{ij} = G_{ijkl}\,\dot{\sigma}_{kl} \qquad (3.200)$$

or, as is the case in Felippa's formulation

$$\dot{E}''_{ij} = M_{ijkl}\dot{E}_{kl} \qquad (3.201)$$

Using Equation (3.200), we directly have

$$\begin{aligned} \dot{E}_{ij} &= \dot{E}'_{ij} + \dot{E}''_{ij} \\ &= C_{ijkl}\dot{\sigma}_{kl} + G_{ijkl}\dot{\sigma}_{kl} \\ &= L_{ijkl}\dot{\sigma}_{kl} \end{aligned} \qquad (3.202)$$

where $L_{ijkl} = C_{ijkl} + G_{ijkl}$

On the other hand, Equation (3.201) is used as follows:

$$\dot{\sigma}_{ij} = E_{ijkl}\dot{E}'_{kl} = E_{ijkl}(\dot{E}_{kl} - \dot{E}''_{kl})$$

$$= E_{ijkl}[\dot{E}_{kl} - M_{klmn}\dot{E}_{mn}]$$

$$= E_{ijkl}[\delta_{km}\delta_{ln} - M_{klmn}]\dot{E}_{mn} \qquad (3.203)$$

Mroz (1973) proposed a more general form of Equation (3.200) viz.

$$\dot{E}''_{ij} = G_{ijkl}n_{kl}n_{pq}\dot{\sigma}_{pq} \qquad (3.204)$$

from considerations of linearity and continuity. In case n_{kl} is an eigenvector of G_{ijkl}, we have

$$G_{ijkl}n_{kl} = \lambda\delta_{ik}\delta_{jl}n_{kl} = \lambda n_{ij} \qquad (3.205)$$

and

$$\dot{E}''_{ij} = \lambda n_{ij}n_{pq}\dot{\sigma}_{pq} \qquad (3.206)$$

The strain increment is along n_{ij}, the normal to f. In this case, the total strain increment $\dot{E}_{ij}$ is derivable from a potential $W = W(\dot{\sigma}_{ij})$ as

$$\dot{E}_{ij} = \frac{\partial W}{\partial \dot{\sigma}_{ij}} \qquad (3.207)$$

where

$$W = \frac{1}{2}\,\dot{\sigma}_{ij}C_{ijkl}\,\dot{\sigma}_{kl},\quad \dot{\sigma}_{ij}n_{ij} < 0 \qquad (3.208)$$

$$= \frac{1}{2}\,\dot{\sigma}_{ij}C_{ijkl}\,\dot{\sigma}_{kl} + \frac{\lambda}{2}\,(\dot{\sigma}_{ij}n_{ij})^2,\quad \dot{\sigma}_{ij}n_{ij} \geq 0$$

To express stress-rates in terms of strain-rates, a generalization of Mroz's approach may be stated as follows

$$\dot{\sigma}_{ij} = E_{ijkl}\dot{E}'_{kl} = E_{ijkl}(\dot{E}_{kl} - \dot{E}''_{kl})$$

$$= E_{ijkl}[\dot{E}_{kl} - G_{klpq}n_{pq}n_{rs}\dot{\sigma}_{rs}] \qquad (3.209)$$

Multiplying both sides by n_{ij} and solving for $\dot{\sigma}_{ij}n_{ij}$

$$\dot{\sigma}_{ij}n_{ij} = \frac{n_{ij}E_{ijkl}\dot{E}_{kl}}{1 + n_{pq}E_{pqrs}G_{rsuv}n_{uv}}$$

$$= B_{kl}\dot{E}_{kl} \qquad (3.210)$$

where

$$B_{kl} = \frac{n_{ij}E_{ijkl}}{1 + n_{pq}E_{pqrs}G_{rsuv}n_{uv}} \qquad (3.211)$$

Substituting Equation (3.210) in Equation (3.209)

$$\dot{\sigma}_{ij} = E_{ijkl}[\dot{E}_{kl} - G_{klpq}n_{pq}B_{rs}\dot{E}_{rs}]$$

$$= \dot{E}_{ijkl}[\delta_{kr}\delta_{ls} - M_{klrs}]\dot{E}_{rs} \qquad (3.212)$$

where

$$M_{klrs} = G_{klpq}n_{pq}B_{rs} \qquad (3.213)$$

Equation (3.212) is similar to Felippa's formulation. Further, if n_{pq} is an eigenvector of G_{klpq}

$$M_{klrs} = \lambda n_{kl}B_{rs} \qquad (3.214)$$

Dafalias (1975) and Sharma (1977) follow the same argument as above except that their formulation involves a scalar "plastic modulus" and the plastic strain increment is somewhat different.

Naghdi (1975a) developed relationships between plastic and total incremental strains on the assumption that stress is a function of strain history and is derivable from a potential. Thus, for the isothermal case,

$$\sigma_{ij} = \sigma_{ij}(E_{kl}, E''_{kl}, q_n) \tag{3.215}$$

Differentiation yields

$$\dot{\sigma}_{ij} = \frac{\partial \sigma_{ij}}{\partial E_{kl}} \dot{E}_{kl} + \frac{\partial \sigma_{ij}}{\partial E''_{kl}} \dot{E}''_{kl} + \frac{\partial \sigma_{ij}}{\partial q_n} \dot{q}_n \tag{3.216}$$

Assuming

$$\dot{q}_n = A_{nij}\dot{E}''_{ij}$$

and

$$\dot{E}''_{ij} = G_{ijkl}\dot{q}_{kl} \tag{3.200}$$

we have

$$\dot{E}''_{ij} = G_{ijkl}\left[\frac{\partial \sigma_{kl}}{\partial E_{mn}} \dot{E}_{mn} + \left(\frac{\partial \sigma_{kl}}{\partial E''_{mn}} + \frac{\partial \sigma_{kl}}{\partial q_r} A_{rmn}\right)\dot{E}''_{mn}\right] \tag{3.217}$$

Solving

$$\dot{E}''_{ij} = M_{ijkl}\dot{E}_{kl} \tag{3.201}$$

where

$$M_{ijkl} = B_{ijmn}D_{mnkl} \tag{3.218}$$

and

$$B_{ijmn} = \left[\delta_{im}\delta_{jn} - G_{ijkl}\left(\frac{\partial \sigma_{kl}}{\partial E''_{mn}} + \frac{\partial \sigma_{kl}}{\partial q_r} A_{rmn}\right)\right]^{-1} \tag{3.219}$$

$$D_{mnkl} = G_{mnpq} \frac{\partial \sigma_{pq}}{\partial E_{kl}} \qquad (3.220)$$

This procedure establishes a relationship between the tensors M_{ijkl} and G_{ijkl}.

g. Restrictions on the Coefficients.

The coefficients appearing in the constitutive equations cannot take on arbitrary values. Certain restrictions and interrelationships based on thermodynamic and/or phenomenological considerations have to be satisfied.

Nicholson (1975) noted that the tangent modulus of one-dimensional stress-strain curves is non-negative and decreases during loading. Also that during reversed loading, the curvature changes sign. These properties were stated in the form

$$\frac{d\sigma}{dE''} \geq 0 \qquad (3.221)$$

$$\dot{\sigma} \frac{d^2\sigma}{dE''^2} \leq 0 \qquad (3.222)$$

These were shown to imply

$$\dot{\sigma} \ddot{E}'' \geq 0 \text{ for } \ddot{\sigma} = 0 \qquad (3.223)$$

Nicholson proposed a generalization of the inequality (3.223) to multiaxial states in the form

$$\dot{\sigma}_{ij}\ddot{E}''_{ij} \geq 0 \text{ for } \ddot{\sigma}_{ij} = 0 \qquad (3.224)$$

A consequence of this inequality, along with Drucker's stability postulate ($\dot{\sigma}_{ij}\dot{E}''_{ij} \geq 0$) and a linear plastic strain rate relationship

$$\dot{E}''_{ij} = G_{ijkl}\dot{\sigma}_{kl}$$

is seen to be

$$\left[\frac{\partial G_{ijkl}}{\partial \sigma_{pq}} + \left[\frac{\partial G_{ijkl}}{\partial E''_{mn}} + \frac{\partial G_{ijkl}}{\partial q_r} A_{rmn}\right] G_{mnpq}\right]\dot{\sigma}_{pq}\dot{\sigma}_{kl}\dot{\sigma}_{ij} \geq 0 \qquad (3.225)$$

whenever $\ddot{\sigma}_{ij} = 0$. Here we have assumed $\dot{q}_r$, the increments of internal variables to be linearly related to the plastic strain rates. Inequality (3.223) restricts the choice of the form of the yield surface as well as the coefficients A_{rmn} and G_{ijkl}.

Naghdi (1975a) showed that if

$$\dot{\sigma}_{ij} = (L_{ijkl} + K_{ijkl})\dot{E}_{kl} \qquad (3.226)$$

where $L_{ijkl} = \dfrac{\partial \sigma_{ij}}{\partial E_{kl}}$ reflects the dependence of σ_{ij} upon E_{kl} and K_{ijkl} reflects the dependence of σ_{ij} upon history, i.e.,

$$\begin{aligned}
\frac{\partial \sigma_{ij}}{\partial E''_{kl}}\dot{E}''_{kl} + \frac{\partial \sigma_{ij}}{\partial q_r} A_{rkl}\dot{E}''_{kl} &= \left[\frac{\partial \sigma_{ij}}{\partial E''_{kl}} + \frac{\partial \sigma_{ij}}{\partial q_r} A_{rkl}\right] M_{klpq}\dot{E}_{pq} \\
&= B_{ijkl}M_{klpq}\dot{E}_{pq} \\
&= K_{ijpq}\dot{E}_{pq} \qquad (3.227)
\end{aligned}$$

several inequalities hold for the stress point on the yield surface. These arise as a consequence of a work inequality over a closed cycle of homogeneous deformation in the strain space. This inequality, derived by Naghdi, is

$$\int_{t_1}^{t_2} \left(E_{ij} - E_{ij}^{(o)}\right) \dot{\sigma}_{ij} \leq 0 \qquad (3.228)$$

where $E_{ij}^{(o)}$ is the origin of the path at time t_1. For m_{ij}, q_{ij} arbitrary tensors directed towards the interior and the exterior respectively, of the yield surface

$$\left.\begin{array}{l} m_{ij}\, k_{ijkl}\, q_{kl} \leq 0 \\ q_{ij}\, k_{ijkl}\, q_{kl} \leq 0 \\ m_{ij}\, \beta_{ijkl}\, \dot{E}''_{kl} \leq 0 \\ \beta_{ijkl}\, \dot{E}''_{kl}\, \dot{E}_{ij} \leq 0 \end{array}\right\} \qquad (3.229)$$

We also recall here the inequalities (3.86) and (3.89) due to requirement of consistency during loading. Assuming stress to be derivable from a potential, Naghdi (1974, 1975a,b) also established inequalities (3.90) and (3.91). These inequalities impose restrictions on the coefficients appearing in the constitutive relationships. Other phenomenological restrictions might exist, e.g. the existence of a saturation state under cyclic loading. Caulk (1978) investigated these restrictions for the case of an initially isotropic, generalized von Mises, linearly hardening material under cyclic loading to saturation. We summarize here the discussion for the strain-space formulation as illustration. Caulk assumed

$$g(E_{mn}, E''_{mn}, \kappa) = 4\mu^2 e'_{kl}e'_{kl} - 2\mu\alpha e'_{kl}e''_{kl} + \sigma e''_{kl}e''_{kl} - \kappa \tag{3.230}$$

for initial isotropy and no volumetric strains along with the hardening law

$$\dot{\kappa} = (2\mu\bar{\beta}e'_{kl} + \bar{n}e''_{kl})\dot{e}''_{kl} \tag{3.231}$$

Here μ is the elastic shear modulus and $\bar{\beta}$, $\bar{n}$ include $\dfrac{\kappa - \kappa_s}{\kappa - \kappa_a}$ as a factor to ensure existence of a saturation state (Equation (3.130)). Inequalities (3.89) and (3.90) lead to the conditions

$$\bar{\beta} + \alpha + 4\mu > 0 \tag{3.232}$$

and

$$(\bar{\beta} + \alpha + 4\mu)^2 M_{kl}M_{kl} > \left[\frac{1}{2}\,\alpha(\bar{\beta} + \alpha) + \bar{n} - 2\sigma\right]^2 e''_{kl}e''_{kl} \tag{3.233}$$

where

$$M_{kl} = 2\mu e'_{kl} - \frac{1}{2}\,\alpha e''_{kl}$$

Noting that using (3.230), during loading

$$M_{kl}M_{kl} = \kappa - (\sigma - \frac{\alpha^2}{4})e''_{kl}e''_{kl} \tag{3.234}$$

(3.233) becomes

$$(\bar{\beta} + \alpha + 4\mu)^2\kappa > [(\bar{\beta} + \alpha + 4\mu)^2(\sigma - \frac{\alpha^2}{4}) + \{\frac{1}{2}\ \alpha(\bar{\beta} + \alpha) + \bar{n} - 2\sigma\}^2]e''_{kl}e''_{kl} \tag{3.235}$$

A sufficient condition for this inequality to hold for all κ and e''_{kl} is

$$\frac{1}{2}\alpha\bar{\beta} + \bar{n} = 0 \text{ and } \alpha^2 = 4\sigma \tag{3.236}$$

Substitution of these values in Equations (3.230), (3.231) yields

$$g = M_{kl}M_{kl} - \kappa \tag{3.237}$$

and

$$\dot{\kappa} = \bar{\beta}M_{kl}\dot{e}''_{kl} \tag{3.238}$$

For anisotropic materials, restrictions on the coefficients appearing in Equation (3.29) were described by Hill (1950). Tsai (1971) listed the restrictions on the yield parameters in Equation (3.30) imposed by the requirement of stability. These are

$$\left.\begin{aligned}
&\frac{1}{x^2y^2} - \frac{1}{4}\left(\frac{1}{x^2} + \frac{1}{y^2} - \frac{1}{z^2}\right)^2 > 0\\
&\frac{1}{y^2z^2} - \frac{1}{4}\left(\frac{1}{y^2} + \frac{1}{z^2} - \frac{1}{x^2}\right)^2 > 0\\
&\frac{1}{z^2x^2} - \frac{1}{4}\left(\frac{1}{z^2} + \frac{1}{x^2} - \frac{1}{y^2}\right)^2 > 0
\end{aligned}\right\} \tag{3.239}$$

5. MECHANICAL MODELS

Mechanical models to simulate hysteretic behavior of rate-independent materials were introduced by Masing and have been developed further by several investigators. These consist of a collection of perfectly elastic and rigid-plastic or slip elements in series-parallel or parallel-series combination. Introduced by Masing, the parallel-series model was further developed by Ivlev (1963) and Prager (1966). Iwan (1967) considered both the parallel-series and the series parallel models (Fig. 16). The number of elements was assumed to be very large and the element properties distributed in some fashion. This distribution would define the hysteretic behavior. In the parallel series model, for a total of N elements, assuming n are in the elastic range and the remainder are in perfectly plastic state, the total stress in all the elements is

$$\sigma = \sum_{i=1}^{n} \frac{E_i \varepsilon}{N} + \sum_{i=n+1}^{N} \frac{\sigma_i^*}{N} \qquad (3.240)$$

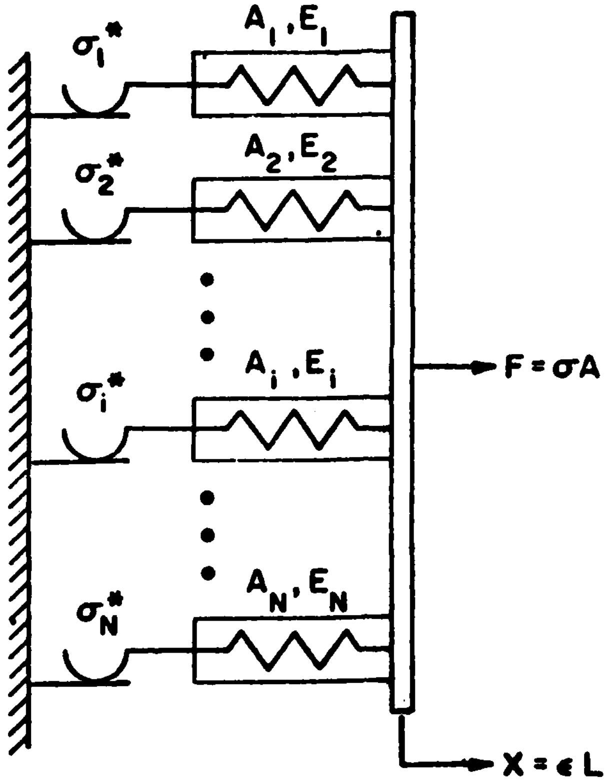
where E_i is the elastic modulus of the elastic subelement in the ith unit, ε is the common strain in all elements and σ_i^* is the limiting value of stress for the rigid-plastic subelement in the ith unit. It is assumed that all elements have the same area $= \frac{A}{N}$ where A is the total area such that force $F = \sigma A$. For a series-parallel model, assuming the ith elastic element to have area A, elastic modulus E and nominal length L_i, the total strain, allowing one element (i=0) to be purely elastic,

$$\varepsilon = \frac{L_0}{1 + L_0} \frac{\sigma}{E} + \frac{1}{1 + L_0} \sum_{i=1}^{N} \frac{\varepsilon_i}{N} \qquad (3.241)$$

where

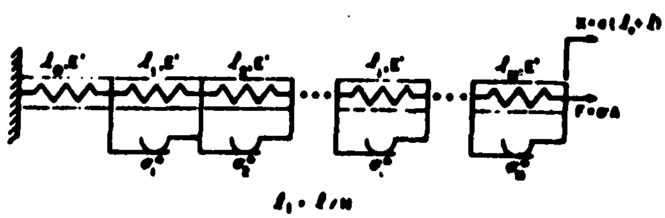
$$\varepsilon_i = 0, \; 0 \le \sigma \le \sigma_i^*$$

$$= \frac{\sigma - \sigma_i^*}{E}, \; \sigma \ge \sigma_i^*$$



$A_i = A / N$

a. Parallel-Series Model.



b. Series-Parallel Model

Figure 16. Mechanical Models. (Iwan, 1967)

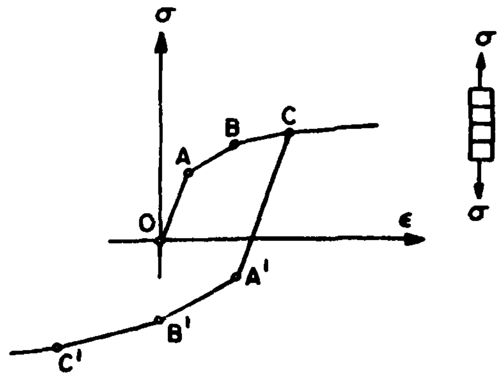
Iwan assumed $N \to \infty$ and for selecting a distribution of properties proposed that it be related to the curvature of the observed stress-strain behavior. This seems reasonable because the curvature is integrable over the stress domain, tends to zero at both extremes ($\varepsilon \to 0$ and $\varepsilon \to \infty$) and the integral changes monotonically between reversals. For stable cycling the distribution could possibly be related to the energy dissipated per cycle per unit volume. Cycling of strain/stress within fixed limits was also considered, the parallel series model for strain cycling and the series-parallel model for stress cycling. The development for uniaxial behavior was extended to multi-axial stress and strain cycling.

Mroz (1973) showed that the mechanical behavior of a finite assemblage of elastic-plastic kinematically hardening elements is essentially equivalent to his piecewise linearization of the hardening curve (Fig. 17). As the number of elements in series becomes very large, a smooth stress-strain curve is realized.

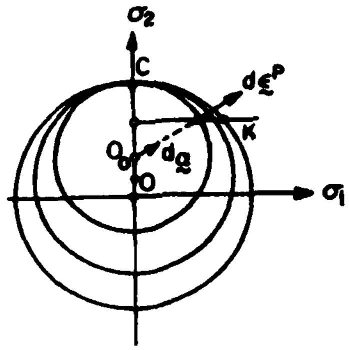
Martin (1971) allowed for relaxation of mean stress by making the stiffness of each spring dependent upon the stress at which it is activated. Cyclic hardening and softening were represented by changing the coefficients k, n in the power law

$$\Delta\varepsilon = \frac{\Delta\sigma}{E_1} + \left(\frac{\Delta\sigma}{k}\right)^{1/n} \qquad (3.242)$$

where $\Delta\varepsilon$, $\Delta\sigma$ are the strain and stress amplitudes in the hysteresis loop. A cumulative damage theory for fatigue failure was also proposed.



(a) Stress-Strain Curve



(b) A Set of Yield Surfaces in Plane Stress Case

Figure 17. Behavior of a Series Model Consisting of Elastic-Plastic Elements. (Mroz, 1973)

Sharma (1977) and Jhansale (1977) found that the mechanical models could match nonlinear behavior in uniaxial loading and model hysteresis loop but could not model hardening or softening prior to saturation. Experimental evidence cited by Jhansale indicates non-Masing behavior denoted by variation in the yield strength, the nonlinear portion of the hysteresis curve remaining practically the same. Jhansale and Sharma introduced a single stress parameter (yield strength increment) depending upon prior history to describe this departure from Masing rule. Rate of increase of this parameter was found to be proportional to the difference between its current value and the saturation value for cyclic loading. The change in radius of the hypersurface is

$$\Delta f = C(Y_s - Y) \tag{3.243}$$

where C is a constant of proportionality and Y_s, Y are the saturation and the current values of the yield strength parameter. This corresponds to Caulk's (1978) hardening rule for cyclic loading (Equation 3.130). Jhansale and Sharma's approach is essentially an extension of Mroz's internal variable theory to the presaturation stage.

Zienkiewicz (1972) and Nayak (1972) proposed an overlay model in the context of finite element analysis. It essentially consists of each finite element made up from subelements with varying yield surfaces connected in parallel. Katona (1978) proposed elastic, viscous as well as friction elements in combined series/parallel arrangements.

6. THEORIES BASED ON CONCEPT OF SLIP

These theories regard a metal as a polycrystalline aggregate and seek to explain macroscopic mechanical behavior on the basis of the behavior of single crystals. Taylor (1938) considered rigid-plastic polycrystals as aggregates of randomly oriented f.c.c. (face-centered cubic) crystals under tension. Assuming homogeneous strain, using the principle of virtual work, strain was calculated as the minimum sum of the amounts of slip for a given crystal orientation. Bishop (1951a, b) used the principle of maximum work to show that for rigid plastic crystal aggregates, among all stress states lying within the yield surface, the actual state giving $\dot{\varepsilon}''_{ij}$ is the one which lies on the yield surface and gives maximum work. Lin (1957) considered aggregates of elastic-plastic crystals under homogeneous strain and assumed that slip occurs sequentially because of the presence of an elastic component. Czyzak (1961) calculated the tensile stress-strain curve and the Bauschinger effect of a f.c.c. crystal. Batdorf's theory (1949) considered an aggregate of crystals each of which has a slip system and assumed homogeneous stress. This satisfied equilibrium but not compatibility. Lin's analysis (1971), based on virtual work principle, satisfied both equilibrium and compatibility. Lin assumed additive decomposition of strain and displacement. The yield surface for f.c.c. crystals consists of twelve pairs of yield planes. Normality of the plastic strain increment was assumed. For stress points on an edge of the polyhedron, $\dot{\varepsilon}''_{ij}$ was between the normals to the two adjacent planes and at a vertex $\dot{\varepsilon}''_{ij}$ was within the cone bounded by normals to the yield planes intersecting at the vertex (Koiter, 1953). Associated flow rule was assumed to apply, i.e., the yield surface was also a surface of constant plastic potential function. Lin found that the theoretical initial yield surface of an aggregate of crystals with the same isotropic elastic constants and the same initial shear stress coincides with Tresca's yield surface of maximum sheering stress. Normality of the incremental plastic strain was found to hold based on the principle of maximum work. The slip characteristics predict the existence of a vertex at the loading point for infinitesimal plastic strain. However, for finite incremental plastic strain the loading surface giving this strain has no vertex but the curvature at the loading point is increased. This is close to von Mises initial yield surface. The observation that measurable strains imply disappearance of vertex may be taken as the explanation for the experimental observations not having been able to directly prove or disprove the existence of vertices. Koiter (1953) showed that the slip theory is a particular case of the incremental theory based on an infinite set of plane surfaces. Sanders (1954) used the plane loading surfaces to establish stress-strain relations partially resembling those of deformation theories.

Yoshimura (1962) explained Bauschinger effect as difference in patterns of dislocations when viewed from the loading and from the opposite direction. Work-hardening was described as a consequence of change in density of dislocations and plastic anisotropy was regarded as directional deviation of the way of grouping of the dislocations.

Kelley (1973) considered material behavior under constant strain amplitude cycling. Low initial dislocation density was associated with isotropic softening and high initial dislocation density resulted in isotropic hardening. Bauschinger effect was explained as unlocking of dislocations in stress reversal. The model proposed was:

$$v = \alpha\sigma^n \tag{3.244}$$

where σ is the effective stress, v is the velocity of mobile dislocations, α is a constant and is a material parameter.

SECTION IV

FINITE ELEMENT ANALYSIS OF ELASTIC-PLASTIC SOLIDS

The finite element method is well documented in the literature (e.g., texts by Zienkiewicz, 1976; Gallagher, 1975; Desai, 1972; Oden, 1972, 1976 among others). It has been applied to elastic-plastic solids including problems in cyclic plasticity. Among several review papers that have recently appeared we note the contributions by Armen (1972, 1979), Stricklin (1972, 1973) and Bergan (1978) where references to other work may be found. The text by Zienkiewicz also contains a good summary.

In this report we shall describe application of the finite element method to plasticity in sections dealing separately with variational formulation, material behavior models, finite element modeling and solution procedures.

1. VARIATIONAL FORMULATIONS FOR FINITE ELEMENT ELASTIC-PLASTIC ANALYSIS.

The equations for small deformation theory of plasticity are:

$$\left.\begin{array}{lc} \text{Kinematics} & \varepsilon_{ij} = u_{(i,j)} \\ \text{Constitutive relations} & \dot{\varepsilon}_{ij} = L_{ijkl}\dot{\sigma}_{kl} \\ \text{or} & \dot{\sigma}_{ij} = K_{ijkl}\dot{\varepsilon}_{kl} \\ \text{Equilibrium} & \sigma_{ij,j} = f_i \end{array}\right\} \text{ on } R \qquad (4.1)$$

Here f_i are components of the body force vector and R is the spatial region of interest. The boundary conditions for the problem are:

$$\sigma_{ij}n_j = \hat{t}_i \text{ over } s_1 \tag{4.2}$$

$$u_i = \hat{u}_i \text{ over } s_2 \tag{4.3}$$

where s_1, s_2 are complementary subsets of ∂R, the boundary of R and n_j are components of the outward normal vector on s_1. Because the stress-strain relationship is in incremental form, the strain-displacement equation also has to be written in incremental form. For small strain theory

$$\dot{\varepsilon}_{ij} = \dot{u}_{(i,j)} \tag{4.4}$$

would replace Equation $(4.1)_1$. To make the stress terms in the equilibrium equation and the constitutive relations correspond, the equilibrium equation can be stated in incremental form as

$$\dot{\sigma}_{ij,j} = \dot{f}_i \tag{4.5}$$

Similarly, the boundary condition, Equation (4.2), in incremental form is

$$\dot{\sigma}_{ij}n_j = \hat{\dot{t}}_i \tag{4.6}$$

Alternatively, introducing $\bar{\sigma}_{ij}$ as the stress at the beginning of an increment, we have the constitutive relation as

$$\sigma_{ij} = \bar{\sigma}_{ij} + K_{ijkl}\dot{\varepsilon}_{kl} \tag{4.7}$$

Using the incremental form of the equations, Lee (1970) introduced the functional

$$\Omega = \frac{1}{2}\int_R \dot{\sigma}_{ij}\dot{\varepsilon}_{ij}dR - \int_{s_1} \hat{\dot{t}}_i\dot{u}_i ds \tag{4.8}$$

for the case $f_i = 0$ and $\dot{u}_i$, $\dot{\varepsilon}_{ij}$, $\dot{\sigma}_{ij}$ identically satisfy Equations $(4.1)_3$, (4.3) and (4.4). This functional is analogous to the potential energy formulation for elasticity. Stationarity of Ω is equivalent to Equations (4.5) and (4.6).

Sandhu (1973) wrote the field equations in the symmetric form

$$\begin{bmatrix} 0 & 0 & -\frac{1}{2}\left(\delta_{ik}\frac{\partial}{\partial j} + \delta_{jk}\frac{\partial}{\partial l}\right) \\ 0 & K_{ijkl} & -1 \\ \frac{1}{2}\left(\delta_{kl}\frac{\partial}{\partial l} + \delta_{li}\frac{\partial}{\partial k}\right) & -1 & 0 \end{bmatrix} \begin{Bmatrix} \dot{u}_i \\ \dot{\varepsilon}_{kl} \\ \sigma_{ij} \end{Bmatrix} = \begin{Bmatrix} f_k \\ -\bar{\sigma}_{ij} \\ 0 \end{Bmatrix} \tag{4.9}$$

Here $\bar{\sigma}_{ij}$ is the stress at the beginning of the increment. This leads, with usual assumptions, to the potential energy type formulation

$$\Omega = \int_R \left[\frac{1}{2}\dot{\varepsilon}_{ij}K_{ijkl}\dot{\varepsilon}_{kl} - \dot{u}_i f_i + \dot{u}_{i,j}\bar{\sigma}_{ij}\right]dR - \int_{s_1}\dot{u}_i\hat{t}_i ds \tag{4.10}$$

if the material behavior is assumed to be constant over the increment. Other formulations of equilibrium or mixed type have been proposed. In the context of finite element approximations these are imbedded in the general treatment of linear problems given by Sandhu (1975, 1976).

Admitting interelement continuity constraints

$$(\sigma_{ij}n_j)' = 0 \text{ over } s_1^i \tag{4.11}$$

$$(\dot{u}_j n_j)' = 0 \text{ over } s_2^i \tag{4.12}$$

where s_1^i, s_2^i are disjoint surfaces in the interior of R such that they are imbedded in the union of interelement boundaries, a general functional for finite element representation of R and equivalent to Equation (4.9) through (4.12) along with the boundary conditions, Equation (4.2) and the incremental form of Equation (4.3) is:

$$\Omega = \sum_{e=1}^{M}\left[\int_{R_e} \{\dot{u}_i(-\sigma_{ij,j} - 2f_i) + \dot{\varepsilon}_{ij}(K_{ijkl}\dot{\varepsilon}_{kl} - \sigma_{ij} + 2\bar{\sigma}_{ij}) + \sigma_{ij}(\dot{u}_{i,j} - \dot{\varepsilon}_{ij})\}dR_e + \int_{s_1\cap\partial R_e} \dot{u}_i(\sigma_{ij}n_j - 2\hat{t}_i)ds - \int_{s_2\cap\partial R_e} \sigma_{ij}n_j(\dot{u}_i - 2\hat{\dot{u}}_i)ds\right] + \int_{s_1^i} (\sigma_{ij}n_j)'u_i ds - \int_{s_2^i} (\dot{u}_j n_i)'\sigma_{ij} ds \tag{4.13}$$

Here M is the number of elements, R_e the region occupied by the element e and ∂R_e its boundary. Noting that over any element e

$$\int_{R_e} \dot{u}_i \sigma_{ij,j} dR_e = -\int_{R_e} \dot{u}_{i,j}\sigma_{ij} dR_e + \int_{\partial R_e} \dot{u}_i \sigma_{ij} n_j ds \tag{4.14}$$

the governing functional may be modified to eliminate either $\dot{u}_{i,j}$ or $\sigma_{ij,j}$ terms. This would yield two alternative formulations

$$\Omega_1 = \sum_{e=1}^{m} \left[\int_{R_e} \left\{ -2\dot{u}_i f_i + \dot{\varepsilon}_{ij}(K_{ijkl}\dot{\varepsilon}_{kl} + 2\bar{\sigma}_{ij}) + 2\sigma_{ij}(\dot{u}_{i,j} - \dot{\varepsilon}_{ij}) \right\} dR_e \right.$$

$$\left. - 2\int_{s_1 \cap \partial R_e} \dot{u}_i \hat{t}_i ds - 2\int_{s_2 \cap \partial R_e} \sigma_{ij} n_j (\dot{u}_i - \hat{\dot{u}}_i) ds \right] - 2\int_{s_2^i} (\dot{u}_j n_i)' \sigma_{ij} ds \tag{4.15}$$

and

$$\Omega_2 = \sum_{e=1}^{m} \left[\int_{R_e} \left\{ 2\dot{u}_i(-\sigma_{ij,j} - f_i) + \dot{\varepsilon}_{ij}(K_{ijkl}\dot{\varepsilon}_{kl} + 2\bar{\sigma}_{ij}) - 2\sigma_{ij}\dot{\varepsilon}_{ij} \right\} dR_e \right.$$

$$\left. + 2\int_{s_1 \cap \partial R_e} \dot{u}_i(\sigma_{ij} n_j - \hat{t}_i) ds + 2\int_{s_2 \cap \partial R_e} \sigma_{ij} n_j \hat{\dot{u}}_i ds \right] + 2\int_{s_1^i} (\sigma_{ij} n_j)' \dot{u}_i ds \tag{4.16}$$

Specialization to the case that displacements are continuous across inter-element boundaries, satisfy the boundary conditions on s_2 and strain increments are derived from displacement increments, Ω_1 reduces to

$$\Omega_3 = \sum_{e=1}^{m} \left[\int_{R_e} \left\{ -2\dot{u}_i f_i + \dot{\varepsilon}_{ij}\left(K_{ijkl}\dot{\varepsilon}_{kl} + 2\bar{\sigma}_{ij}\right) \right\} dR_e - 2\int_{s_1 \cap \partial R_e} \dot{u}_i \hat{t}_i ds \right] \tag{4.17}$$

which is the discretization of the functional in Equation (4.10).

On the other hand, if stresses satisfy equilibrium, tractions are continuous across element boundaries, and the traction boundary conditions are satisfied, and the strain increments are derivable from stress increments, Ω_2 reduces to

$$\Omega_4 = \sum_{e=1}^{m} \left[\int_{R_e} \left\{ \dot{\sigma}_{ij} L_{ijkl} \dot{\sigma}_{kl} - 2\bar{\sigma}_{ij} L_{ijkl} \dot{\sigma}_{kl} \right\} dR_e - 2 \int_{s_2 \cap \partial R_e} \sigma_{ij} n_j \hat{\dot{u}}_i ds \right] \tag{4.18}$$

This is the complementary type formulation.

To admit material and geometric nonlinearity, several different approaches have been used. The difference between various formulations is essentially in the derivation of the incremental form of the virtual work equation. Early investigators (e.g. Turner, 1960; Martin, 1965; Felippa, 1966; Hofmeister, 1971) used the incremental moving coordinate system (Stricklin, 1972). In this, the reference coordinates were updated after each load increment and the stresses referred to the new configuration. The incremental strain was defined as

$$\dot{E}_{ij} = \dot{u}_{(i,j)} + \frac{1}{2} \dot{u}_{k,i} \dot{u}_{k,j} \tag{4.19}$$

Here the derivatives are with reference to the configuration at the end of the last increment and the reference frame is assumed to stay orthogonal. Felippa (1966) and Murray (1969) assumed the state at the beginning of an increment to be in equilibrium and wrote the incremental virtual work equation in the form

$$\int_V [\bar{\sigma}_{ij} \delta(\frac{1}{2} \dot{u}_{k,i} \dot{u}_{k,j}) + \dot{\sigma}_{ij} \delta(\dot{u}_{(i,j)})] dV = \int_{\partial R} \hat{\dot{t}}_i \delta(\dot{u}_i) ds \tag{4.20}$$

Here $\bar{\sigma}_{ij}$ is the initial stress referred to unit areas at the beginning of the increment. Allowing for the lack of balance at the initial state, Hofmeister (1971) added the virtual work by the unbalanced forces to get

$$\int_V [\bar{\sigma}_{ij}\delta(\tfrac{1}{2}\,\dot{u}_{k,i}\dot{u}_{k,j}) + \dot{\sigma}_{ij}\delta(\dot{u}_{(i,j)})]dV =$$

$$\int_{\partial R} \hat{\dot{t}}_i\delta(\dot{u}_i)ds - \int_V \bar{\sigma}_{ij}\delta(\dot{u}_{(i,j)})dV - \int_{\partial R} \bar{t}_i\delta(\dot{u}_i)ds \qquad (4.21)$$

In this scheme the strain increment is not derived from the Green strain tensor referred to the undeformed configuration and the theory is applicable only to the case of small strains. In a correct theory, the incremental strain must be derived from the Green strain tensor (e.g., Mallett, 1968; Haisler, 1970; Stricklin, 1972). The principle of virtual work may be stated as (Hibbit, 1970; Hutchinson, 1973)

$$\int_R \sigma^{ij}\dot{E}_{ij}dR = \int_{\partial R} \hat{t}^i\dot{u}_i ds \qquad (4.22)$$

Here ∂R is the boundary of the region R and the superscript over a quantity denotes its contravariant components. Hutchinson wrote the incremental virtual work equations as

$$\int_R [\dot{\sigma}^{ij}\dot{E}_{ij} + \dot{\sigma}^{ij}u^k_{,j}\dot{u}_{k,i}]\partial R = \int_{\partial R} \hat{\dot{t}}^i\dot{u}_i ds \qquad (4.23)$$

Recently, Brockman (1979) following Oden (1972), stated the incremental virtual work equations in a form which, for no inertial forces, reduces to

$$\int_{R_o} [\dot{\pi}_{ij}\dot{E}_{ij} + \frac{1}{2}\pi_{ij}\overline{u_{k,i}u_{k,j}}^{\bullet} - \dot{f}_i\dot{u}_i]dR = \int_{\partial R_o} \hat{\dot{t}}_i\dot{u}_i ds \quad (4.24)$$

Here R_o, ∂R_o denote the configuration at the beginning of the increment and π_{ij} is the pseudostress equal to the components of the symmetric Piola-Kirchhoff stress tensor referred to unit areas in the deformed configuration.

2. MODELS OF MATERIAL BEHAVIOR.

For monotonic uniaxial or proportional loading where the stress state can be defined by a single parameter, the nonlinear stress strain behavior can be approximated by suitable functions. For generalization to arbitrary stress paths scalar measures of stress and strain have often been introduced as effective stress and effective strain. The available test data are interpreted in terms of these effective measures to set up stress or strain-dependent "moduli." This approach is attractive because of its simplicity but fails to account properly for path dependence of mechanical behavior of materials.

In using the mathematical theory of plasticity to represent material behavior, we need to determine the elastic properties as well as the initial yield function, the parameter λ in the flow rule and the internal variables, A_{nij}.

In early work on the application of the finite element method to plasticity, following Drucker (1951), the coefficient in the flow rule was determined considering normality only (e.g., Swedlow, 1966; Marcal, 1967; Marcal, 1968, 1969; Yamada, 1968a, b; Mroz, 1969; Lee, 1970; among others). This formulation failed to satisfy consistency. Mroz (1967, 1973) following Hill (1950) also proposed a formulation based on normality and linearity (Equation 3.176) admitting "non-associated" flow rule. Yamada (1969) used rate of work equations to develop a consistent formulation for von Mises materials. Reyes (1966) had previously used this approach for Mohr-Coulomb materials under plane strain. Felippa (1966) developed a consistent approach allowing for normality and linearity as well. This is identical to Prager (1949) for hardening materials but is valid for perfect plasticity as well. Zienkiewicz (1969), Eisenberg (1976) and Allen (1979) developed formulations essentially similar to Felippa's. Pifko's (1974) and Sharma's (1977) formulation for kinematic hardening can also be seen as a specialization of Felippa's formulation to the case of purely translational hardening. It would not be applicable to cases when E''_{ij} appears explicitly in the expression for f. Naghdi's (1975c) strain-space formulation is also the same as Felippa's. Nayak (1972) extended the formulation to admit non-associated behavior.

We note that the coefficients defining the increments in internal variables also appear in the consistency Equation (3.187). Thus, the rate of hardening influences the stress-strain relations. The quantities required to define stress-strain behavior are the elastic properties, the dependence of the yield function upon σ_{ij}, E''_{ij}, q_n and the coefficients A_{nij} which relate the increments of internal parameters to plastic strain history. The available test data should be interpreted in the light of this requirement. For instance, for isotropic hardening, the uniaxial test data would define the quantity $\dot{\kappa}$ as a function of equivalent plastic strain or the plastic work depending upon whether strain hardening or work hardening approach is adopted. (For von Mises materials, these are equivalent).

Isakson (1967, 1969) and Armen (1970) incorporated kinematic hardening with Ziegler's hardening rule in a finite element computer program. This was further developed by Pifko (1974) to admit anisotropy. Eisenberg (1976) allowed for both isotropic (strain hardening) and kinematic hardening (Prager's rule) along with a consistent formulation for λ for initially isotropic von Mises material. Nayak (1972) considered an "overlay" model to represent piecewise linear hardening. Hunsaker (1973) compared four models of elastic-plastic behavior namely, isotropic hardening, kinematic hardening (both Prager and Ziegler's), Mroz' piecewise linear model and the mechanical overlay model. He found the last two most appropriate for reversed loading.

The approaches using a plastic modulus involve a relationship between scalar measures of stress and plastic or total strain called the effective stress and the effective strain. Often these curves are directly interpreted in terms of isotropic hardening. Many attempts have been made to obtain simple mathematical functions to represent experimental data. Wilson (1965) used a bilinear approximation. Admitting nonlinear hardening, Jensen (1965) and Lansing (1966) used Ramberg-Osgood formula. Salmon (1970) included Wilson's bilinear approximation, the Ramberg-Osgood formula and Richard-Goldberg (1965) equation in a single computer program. For large strain ranges, Pifkc (1974) preferred a power law. Krempl (1972), Liu (1976) and Cernocky (1978) considered the essentials of curve fitting for stress-strain diagram. They noted that typically the slope of the stress-strain curves is positive, decreases monotonically with increasing strain, and is constant at the two ends (strain equal to zero and strain exceeding a certain limiting value). Thus the slope is constant at the two ends and the curvature of the diagram is negative. Liu (1976) proposed an exponential law. Cernocky (1978) described construction of non-linear monotonic functions. In Mroz's theory (1967) the hardening curve is approximated by piecewise linear segments leading to discontinuous moduli. Dafalias (1976) introduced a continuously varying field of plastic moduli. This is similar to the theory proposed by Krempl and his coworkers except for the fact that their formulation was for total strain. A special feature of Dafalias' formulation is the dependence of the plastic modulus upon the distance between the stress point and the bounding surface. This ensures that the "modulus" has the limiting constant values at the two extremes of the stress point being upon the inner yield surface or the outer, bounding surface (Figure 14). Desai (1971) proposed use of spline functions to describe the stress-strain curve. However, he did not enforce the criteria of monotonicity of the modulus. For cyclic plasticity Pifko (1974) used Ramberg-Osgood law referred to the point of last reversal of stress as the origin. Liu (1976) used an exponential law in terms of the initial and ultimate values of the modulus.

3. FINITE ELEMENT DISCRETIZATION

Procedures for finite element spatial discretization of boundary value problems are well-known. Triangular and tetrahedral elements with linear interpolation of the displacement field over each element were the earliest to be used for finite element analysis of elastic-plastic media (i.g. Pope, 1965; Argyris, 1966; Reyes, 1966; Yamada, 1968a, 1968b, 1969; Zienkiewicz, 1969; Hofmeister, 1971). Quadrilateral elements have been extensively used. Generally, for two dimensional problems, these are formed as assemblages of constant strain triangles (e.g. Wilson, 1965; Lee, 1970; Sandhu, 1973; Hodge, 1975) or are four point isoparametric quadrilaterals (e.g. Zienkiewicz, 1968). Lee (1970) used a four point isoparametric element with an additional local mode. For thick tubes, Chen (1972) used interpolation which would give the exact solution for the elastic case. In early work, based on linear interpolation of displacement, the state of stress within an element was assumed to be uniform. Thus the entire element had to be fully plastic or fully elastic. Felippa (1966) used higher order interpolation and admitted partial yielding of elements. Nayak (1972) used isoparametric elements with biquadratic interpolation. It is customary to evaluate stresses at each Gauss integration point within the element to set up the average properties of the element. As Gauss integration points are all interior to the element this device does not notice onset of yeild in the element till it reaches well into the interior. Some investigators therefore use Simpson's integration scheme which involves points on the element boundaries.

Matrix formulation of the problem follows from insertion of finite element discretization into the variational formulation. General form of the equations for an increment is

$$[K]\{\dot{q}\} = \{\dot{P}\} \tag{4.25}$$

where [K] is the stiffness matrix which is dependent upon the history of deformation and $\{\dot{q}\}$, $\{\dot{P}\}$ are, respectively, the increments in the generalized displacements and loads. Often, the stiffness matrix is written as the sum of a linear and a nonlinear component

$$[K] = [K]_L + [K]_{NL} \tag{4.26}$$

If the stresses and forces at the beginning of an increment are not in equilibrium, an additional load term has to be introduced (Hofmeister, 1971). In that case,

$$([K]_L + [K]_{NL})\{\dot{q}\} = \{\dot{P}\} + \{E\} \tag{4.27}$$

Here {E} represents the initial equilibrium error term.

In developing a matrix formulation using the finite element method in conjunction with the principle of virtual work, Stricklin (1972) obtained

$$[K]\{q\} = \{P\} + \{Q\}_I + \{Q\}_{NL} \tag{4.28}$$

This formulation is based on additive decomposition of the strain tensor into elastic and plastic components as well as into linear and nonlinear components viz.

$$E_{ij} = E'_{ij} + E''_{ij} = E^L_{ij} + E^{NL}_{ij} \tag{4.29}$$

The stiffness matrix [K] is based on the linear part E^L_{ij} of the strain tensor. E''_{ij} are the components of the plastic strain tensor and E^{NL}_{ij} are the components due to geometric nonlinearity, {P} are the external loads and $\{Q\}_I$, $\{Q\}_{NL}$ are pseudoforces reflecting the effects of E''_{ij} and E^{NL}_{ij} respectively. Assuming a linear relationship between the symmetric Piola-Kirchhoff stress tensor and the elastic part of the strain

$$\sigma_{ij} = K_{ijkl}E'_{kl} \tag{4.30}$$

The component of $\{Q\}_I$ corresponding to q_i is, using the reduced form for strain components

$$(Q_i)_I = \int_{V_o} E_{k,i}D_{kl}E''_l dV \tag{4.31}$$

Similarly

$$(Q_i)_{NL} = \int_{V_o} (E^L_{k,i}D_{kl}E^{NL}_l + E^{NL}_{k,i}D_{kl}E_l)dV \tag{4.32}$$

Here the differentiation is with respect to q_i and V_o is the volume of the initial body. The incremental form of Equation (4.25) is (Stricklin, 1972)

$$([K] + [K]_I + [I]_{NL})\{\dot{q}\} = \{\dot{P}\} \tag{4.33}$$

where

$$(K_{ij})_I = - \int_{V_o} (E_{k,ij}D_{kl}E''_l + E_{k,i}D_{kl}E''_{l,j})dV \tag{4.34}$$

and

$$(K_{ij})_{NL} = \int_{V_o} (E^L_{k,i}D_{kl}E^{NL}_{l,j} + E^{NL}_{k,i}D_{kL}E^L_{l,j} + E^{NL}_{k,i}D_{kl}E^{NL}_{l,j} + E^{NL}_{k,ij}D_{kl}E_l)dV \tag{4.35}$$

4. SOLUTION PROCEDURES

Procedures for solving the nonlinear equations arising after finite element discretization may be divided into two groups viz.,

i. Direct methods

ii. Initial-value methods

a. Direct Methods

Direct methods of solving nonlinear equations generally use the Newton-Raphson Method or one of its variants. Almroth (1979) has reviewed some aspects of this approach. Writing the iteration scheme in the form

$$x^{(n+1)} = Gx^{(n)} \qquad (4.36)$$

the search is for the fixed point of G. Various procedures differ in the choice of G. In the basic Newton-Raphson method

$$Gx^{(n)} = x^{(n)} - (K^{(n)})^{-1} r^{(n)} \qquad (4.37)$$

where $r^{(n)}$ is the force residual for the nth approximation, i.e.

$$r^{(n)} = (K^{(n-1)})x^{(n)} - P \qquad (4.38)$$

where P is the forcing vector in the nonlinear problem

$$Kq = P \qquad (4.39)$$

Evaluating the quantity $\Delta x^{(n)} = (K^{(n)})^{-1}r^{(n)}$ represents solution of a system of equations at each step. For this reason the method can become expensive. Also $K^{(n)}$ may be ill-conditioned. The modified Newton-Raphson Method uses a constant reference matrix K_o and may be written as

$$Gx^{(n)} = x^{(n)} - (K_o)^{-1}r^{(n)} \tag{4.40}$$

The reference matrix K_o may be the initial stiffness as shown in Figure 18 from Haisler (1970) or some other matrix with suitable properties. For example, to improve the conditioning of the matrix its diagonal terms may be suitably increased, i.e.,

$$Gx^{(n)} = x^{(n)} - (K_o + \mu I)^{-1}r^{(u)} \tag{4.41}$$

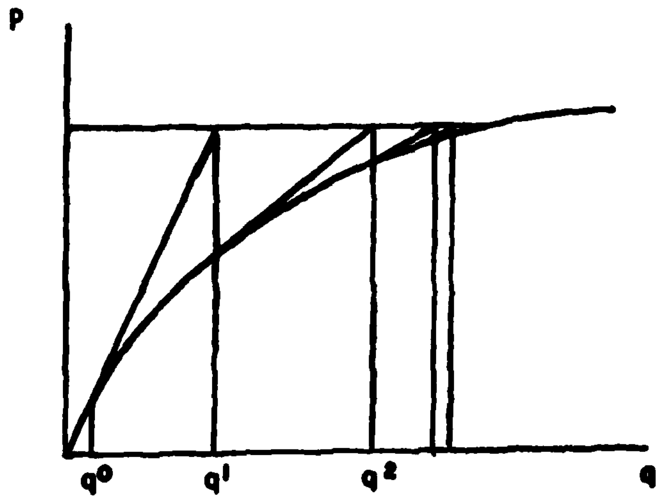
Here I is the identity matrix and μ is a scalar suitably chosen for well-conditioning. Felippa (1976) proposed a formulation

$$Gx^{(n)} = x^{(n)} - w^{(n)}(K_o + \mu w^{(n)})^{-1}r^{(n)} \tag{4.42}$$

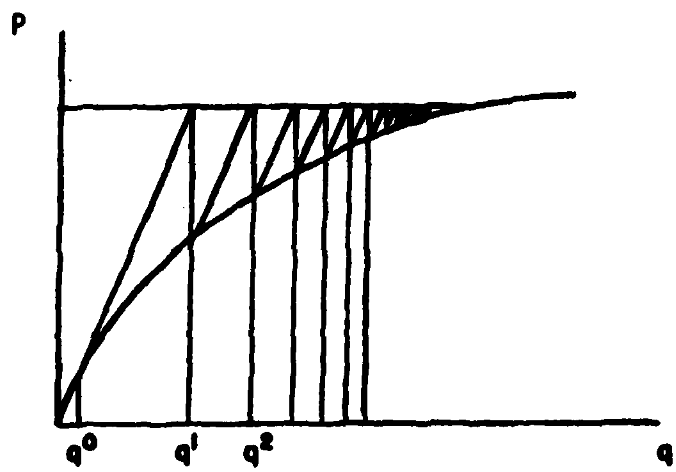
where $w^{(n)}$ is chosen automatically for optimal efficiency. The residual is assumed to be a linear function of $w^{(n)}$ i.e.

$$r^{(n)} = w^{(n)}r_1^{(n)} + (1 - w^{(n)})r_o^{(n)} \tag{4.43}$$

where $r_1^{(n)}$, $r_o^{(n)}$ are residuals corresponding to $w^{(n)} = 1$ and $w^{(n)} = 0$ respectively. Minimization of $||r^{(n)}||$ with respect to $w^{(n)}$ yields



a. Newton-Raphson Method



b. Modified Newton-Raphson Method

Figure 18. Newton and Modified Newton Method. (Haisler, 1970).

$$w^{(n)} = \frac{1 - \psi}{1 - 2\psi + \rho^2} \tag{4.44}$$

where

$$\psi = \frac{\frac{1}{2}(r_1^{(n)} \cdot r_o^{(n)})}{||r_o^{(n)}||^2}$$

$$\rho = \frac{||r_1^{(n)}||^2}{||r_o^{(n)}||^2}$$

Automatic correction is time-consuming. For this reason Schmidt (1977) suggested choice of $w^{(n)}$ as a function of condition number of K.

Success of the iterative procedures depends upon the rate of convergence. Yamamoto (1973), among others, investigated convergence of iterative solutions for elastic-plastic continua and proposed a scheme to accelerate convergence. Convergence is assured if the mapping G is contractive, i.e. if $\alpha < 1$ exists such that

$$||Gx - Gy|| < \alpha \; ||x - y|| \tag{4.45}$$

in some norm, the system $x^{(n+1)} = Gx^{(n)}$ converges. To check if a given scheme is indeed convergent one method, used by Sandhu (1974) for time-domain solutions, is to evaluate

$$\alpha = \frac{||x^{(n+1)} - x^{(n)}||}{||x^{(n)} - x^{(n-1)}||} \tag{4.46}$$

If the correct solution is x, G contractive implies

$$||x^{(n)} - x|| < \alpha \, ||x^{(n-1)} - x||$$

Hence,

$$(1 - \alpha)||x^{(n)} - x|| < \alpha \left[||x^{(n-1)} - x|| - ||x^{(n)} - x|| \right]$$
$$\leq \alpha \, ||x^{(n-1)} - x^{(n)}|| \quad (4.47)$$

Hence, $\alpha < .5$ is sufficient for convergence. In case α exceeds .5, the increment is reduced (halving) and if α is extremely small, it is alright to increase the increment (doubling) in the next step. Thus, the size of the increment is automatically designed to ensure convergence. Almroth (1979) reported an automatic procedure in which the increment size is determined from the iterations required for convergence in the preceding increment. In situations of rapidly changing behavior, this may not be fully effective. Sandhu (1974) would use the information from the preceding step only as the first estimate to be checked, after three solutions are available and, if necessary, reduced.

If the system is known to be convergent, the process can be accelerated. One such procedure was proposed by Boyle (1973) based on Jenning's (1971) modification of Aitken's δ^2-method.

Another iterative procedure (e.g. Sandhu, 1973) is analogous to modified Euler Method. It is based on the existence of a mean stiffness K_m for the increment such that

$$K_m q = P \quad (4.48)$$

gives the exact solution for the incremental displacements. It is assumed that

$$K_m = \alpha K_o + (1 - \alpha)K_f \qquad (4.49)$$

where K_o, K_f represent the stiffness at the beginning and the end of the increment and α [0, 1]. For $\alpha = 1$ we have the purely incremental (explicit) technique and for $\alpha = 0$ the scheme is totally implicit. Generally, α is assumed to be 0.5 though it is possible to optimize it following Yamamoto's (1973) method. An alternative is to assume K to be the stiffness corresponding to the mid-increment values of the displacements (Argyris, 1966; Felippa, 1966; Akyuz, 1968; Sandhu, 1973). The method is illustrated in Figure 19. For parabolic variation in the solution, the midincrement stiffness is identical to the mean of the slopes at the ends of the increment. Thomas (1973) proposed use of the average value of the solution from these two procedures. The error in the solutions was shown to be

$$\varepsilon = \frac{1}{6} \max_i \left| \frac{q_i'' - q_i'}{q_i(\text{mean})} \right| \qquad (4.50)$$

where {q'}, {q"} are the solutions from the two procedures and $\{q_{mean}\}$ the average of the two. For material nonlinearity, to allow for an element going from the elastic to the plastic state during an increment, Marcal (1969) proposed choosing α as the proportion of the load increment needed to the onset of yield.

To economize on computational effort, Sandhu (1973) proposed a two-level iteration scheme. For each iteration, a local iterative process

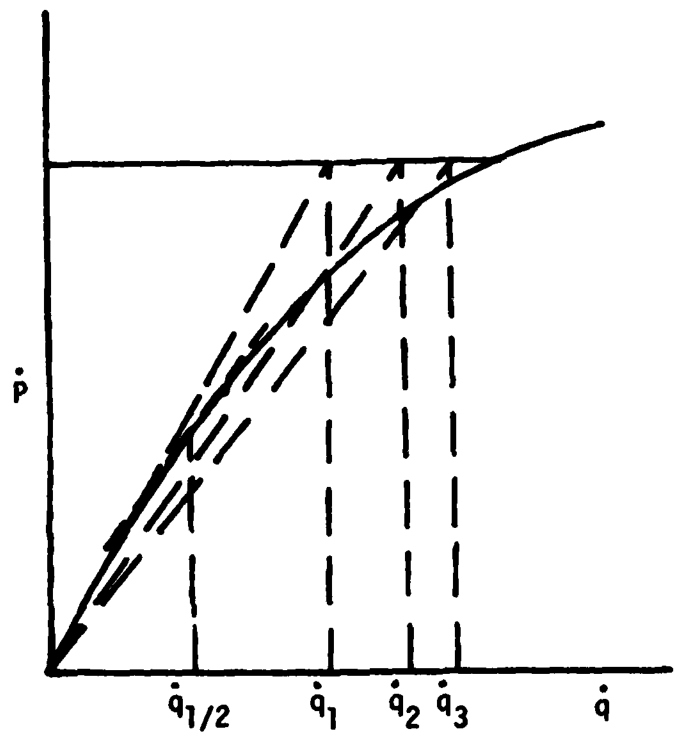


Figure 19. The Mid-Increment Stiffness Method

for each iteration was carried out to convergence assuming no changes in the displacement increment solution. The unbalanced element stresses were included as additional loads in the next iteration for the system (Figure 20). Singh (1975) proposed an improvement on this procedure. For cases where the local iteration converged slowly, or not at all due to very low stiffness, the strain in the element would be scaled down and the element assumed to deform incrementally to the calculated value of strain. For problems where plasticity is confined to a local region, the procedure has been found to be very successful.

Admitting variable increment size, each increment can be designed to correspond to an element passing from the elastic to the plastic stage and/or to meet the requirements of convergence discussed earlier. A load increment can be "scaled back" for this purpose (Zienkiewicz, 1969; Stricklin, 1971; Sandhu, 1973). If for any element, $f_1 = f(\bar{\sigma}_{ij}) < 0$ and $f_2 = f(\bar{\sigma}_{ij} + \dot{\sigma}_{ij}) > 0$, let r be such that

$$f(\bar{\sigma}_{ij} + r\dot{\sigma}_{ij}) = 0 \tag{4.51}$$

Assuming linear variation in f, a first estimate is

$$r_1 = \frac{f_1}{f_2 - f_1} \tag{4.52}$$

However, to allow for nonlinearity of f in r, Nayak (1972) proposed

$$r = r_1 - \frac{f_2}{\frac{\partial f}{\partial \sigma_{ij}} E_{ijkl}\dot{E}'_{kl}} \tag{4.53}$$

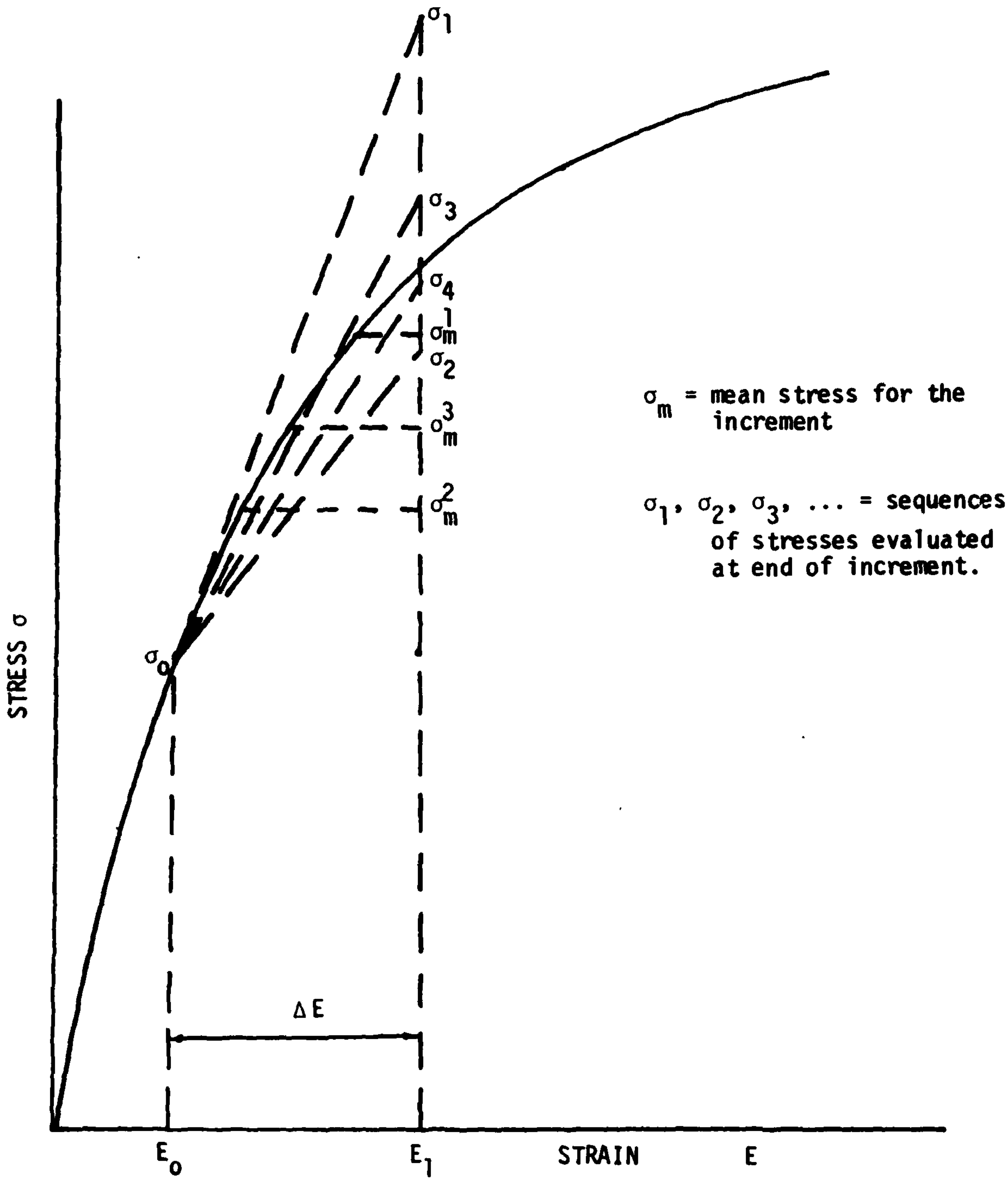


Figure 20. Local Iteration with Stiffness Based on Midincrement Stresses. Unbalanced Stress is Added to the Next Load Increment. (Sandhu, 1973)

Here $\dot{E}'_{kl}$ is the strain increment assuming elastic behavior during the increment. Sandhu (1973) would solve Equation (4.51) for the smallest positive value of r less than one. The least value of r among all elements having $f_1 < 0$ and $f_2 > 0$ is the ratio governing the scaling back of the load increment. Using variable increment concept, Sandhu (1973) would apply all the "remaining" load at each incremental step and then scale back as necessary. This incorporates an automatic equilibrium check at each step and automatically uses the largest possible increment consistent with convergence and other constraints. Tracey (1979) determined the increment size by a process of iteration requiring convergence of a scheme based on Equation (4.49) and a constraint on the magnitude of the incremental displacement solution.

b. Initial Value Methods

The problem of gradual loading of a structural system is an initial value problem. Both single-step and multistep methods have been used. The single step methods are often similar to Euler's method. The "tangent stiffness", i.e. stiffness based on the state of the body at the beginning of a load increment, is assumed to apply throughout the load increment. The incremental solution tends to drift away from the correct solution due to error accumulation (Figure 21). A sequence of solutions with decreasing size of the load increment is needed to establish the correct solution. The number of load increments has to be quite large and as at each step the stiffness changes, the solution process is generally expensive. The incremental equation is

$$K\dot{q} = \dot{P} \qquad (4.25)$$

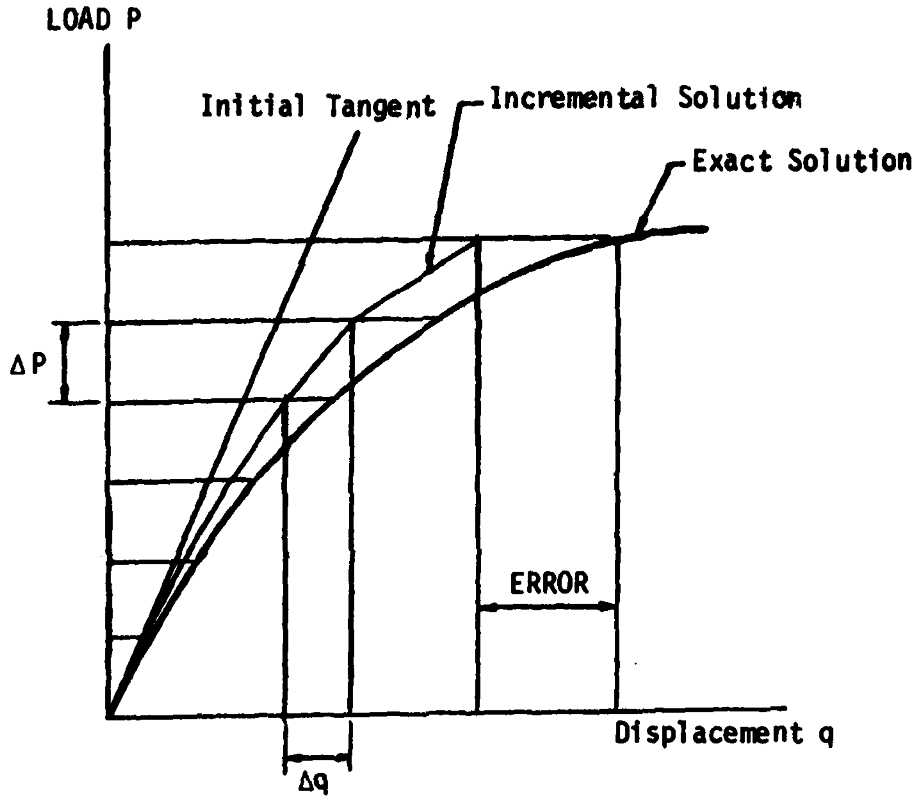


Figure 21. Drifting in Incremental Stiffness Procedure (Haisler, 1970)

Separating K into constant and displacement-dependent components K', K" respectively

$$K = K' + K'' \tag{4.54}$$

Then, Equation (4.25) may be written as

$$K'\dot{q} = \dot{P} - K''\dot{q} \tag{4.55}$$

In early work, it was customary to use $K''\dot{q}$ on the right hand side as the load equivalent to initial strain, i.e. corresponding to the plastic strain from the previous increment. This procedure was used, among others, by Gallagher (1962), Argyris (1966), Isakson (1967) and Armen (1970). The process was slow to converge with reduction in size of the load increment. Isakson (1967) suggested use of an extrapolated value for the initial strain. An immediate improvement is the use of an iterative procedure in which $K''\dot{q}$ is based on a mean value of the plastic strain effect during the load increment. Baker (1969) used this approach which essentially amounts to a Modified-Euler method for the increment. Even with the use of acceleration techniques, convergence was very slow and uncertain. Use of variable increment techniques based on convergence needs would be much preferable.

Hofmeister (1971) introduced incremental equilibrium check into the formulation (Equations (4.21), (4.27)). Figure 22 from Hofmeister shows the effect of the check. Without this correction the displacement solution would be $\bar{q}_2$. With the correction the solution q_2 is close to $q_2^{(T)}$ the correct

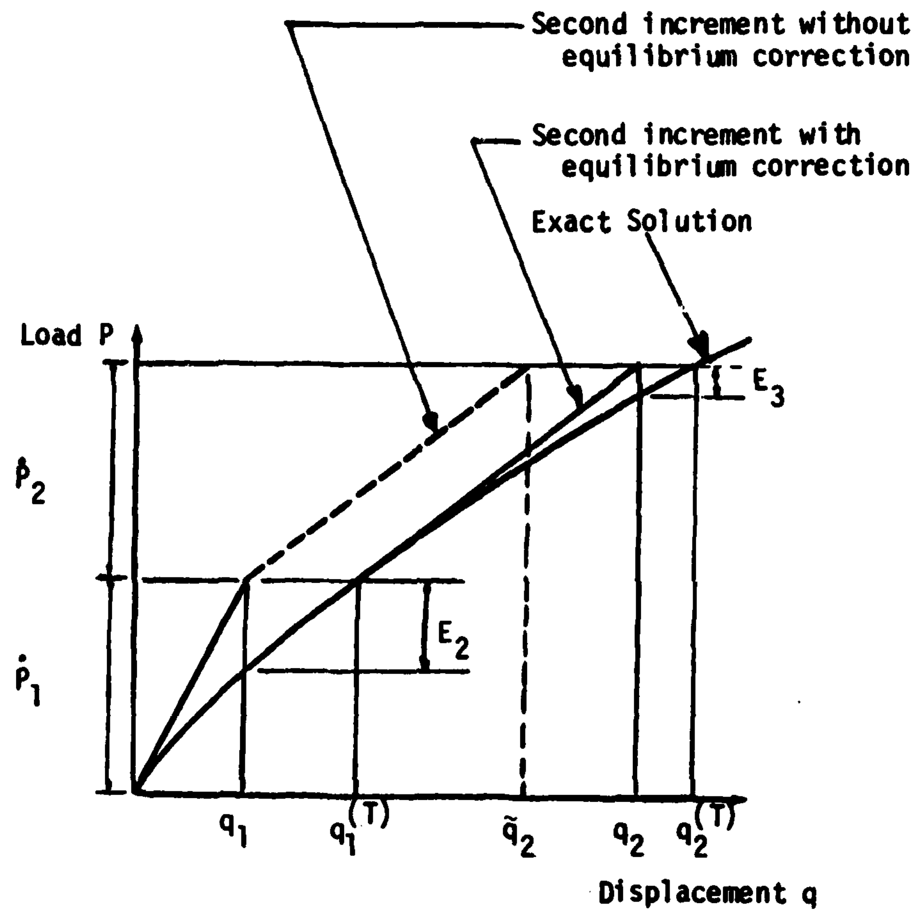


Figure 22. Incremental Solution Procedure With and Without Equilibrium Correction. (Hofmeister, 1971)

solution. The procedure represents single step Newton-Raphson method at each increment with the error included in the remaining load. A multistep Newton-Raphson within each load increment, would further improve the accuracy.

Multistep incremental procedures have been used. Richard (1969) implemented fourth order Runge-Kutta method.

Oden (1973) wrote the nonlinear equation in the form

$$f(q, p) = 0 \tag{4.56}$$

In incremental form:

$$\dot{f} = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p} = 0 \tag{4.57}$$

Haisler (1970), and Stricklin (1971) introduced two forms of self-correcting initial value formulations as follows. Consider the incremental equation

$$K'\dot{q} = \dot{P} - K''\dot{q} = \dot{P} - \dot{Q} \tag{4.58}$$

along with the imbalance of force

$$f = P - Q - K'q \tag{4.59}$$

Combining the two, for μ a scalar

$$(K' + K'')\dot{q} + \mu K'\dot{q} = \dot{P} + \mu P - \mu Q \tag{4.60}$$

Or,

$$K'(\dot{q} + \mu q) = \dot{P} + \mu P - (\dot{Q} + \mu Q) \qquad (4.61)$$

For $\mu = 1$ the formulation reduces to Hofmeister (1971) and Stricklin (1970). Differentiation of Equation (4.58) and adding linear combination of f and $\dot{f}$ leads to the second formulation

$$K'\ddot{q} = -\ddot{Q} + \mu f + w\dot{f} \qquad (4.62)$$

This is an equation for damped motion and can be solved by standard methods, e.g. Houbolt method with four point backward difference form for $\ddot{q}$ and three points backward difference for $\dot{q}$. Both the formulations are self-correcting i.e. incorporate equilibrium check and any error will decay. Success of the scheme depends largely on the choice of the damping coefficient (Oden (1973)).

c. Displacement Incrementation

As the nonlinear system approaches instability, the ratio for scaling down on the size of the increment from convergence considerations will decrease till, theoretically, no matter how small the increment there is no solution. This stage corresponds to singularity of the stiffness matrix. Bergan (1977, 1978) related the increment size to a "current stiffness parameter." This essentially is a rough measure of the conditioning of the stiffness matrix and would automatically control the increment size. Near instability, displacement incrementation must replace load incrementation till the system is again well conditioned.

The displacement incrementation must, obviously, be in the space of the eigenvector(s) corresponding to the vanishing eigenvalue(s). Argyris (1966) and Pian (1971) proposed incrementing displacement components as independent variables to determine the load increment size. Zienkiewicz (1971) admitted a pattern of applied loads in the scheme assuming the increment to be small enough to permit superposition. Haisler (1977) described a procedure for displacement as well as load incrementation based on his self-correcting procedure. The incremental equation is

$$K\dot{q} = \dot{P} + E \tag{4.27}$$

Assume increment $\dot{q}_i$ is to be specified. Then, subscript 1 denoting the parts of K, $\dot{q}$, P, E after deletion of ith components,

$$K_{11}\dot{q}_1 = \dot{P}_1 + E_1 - K_{1i}\dot{q}_i \tag{4.63}$$

and

$$K_{i1}\dot{q}_1 = \dot{P}_i + E_i - K_{ii}\dot{q}_i \tag{4.64}$$

Solving (4.63)

$$\dot{q}_1 = A + \dot{\lambda} B \tag{4.65}$$

where $\dot{\lambda}$ is such that $\dot{P}_1 = \dot{\lambda}P_1$ and

$$K_{11}A = E_1 - K_{1i}\dot{q}_i \tag{4.66}$$

and

$$K_{11}B = P_1 \tag{4.67}$$

Substituting Equation (4.65) in (4.64) and solving for $\dot{\lambda}$

$$\dot{\lambda} = \frac{E_i - K_{i1}\dot{q}_i - K_{i1}A}{K_{i1}B - 1} \tag{4.68}$$

Equation (4.65) now defines $\dot{q}_i$. This procedure avoids the solution of non-symmetric equations. Bergan (1978), using his current stiffness parameter to define the stage at which displacement incrementation would replace load incrementation, proposed the increment to be proportional to the incremental solution for the preceding step. This, for sufficiently small steps. would be quite close to the eigenvector corresponding to the smallest (in absolute magnitude) eigenvalue. To control drift, the size of the increment must be kept small. Evidently, iteration for equilibrium has no meaning in the vicinity of a stationary point.

SECTION V

SUMMARY AND RECOMMENDATIONS

The finite element method is a powerful tool for approximate solution of engineering problems. Several investigators have applied it to the problem of cyclic plasticity with varying degrees of success. We note here specially the recent work by Armen, Dafalias, Eisenberg, Jhansale, Pifko, and Sharma. A satisfactory procedure must be based on a correct definition of strain displacement relationships and simulate the stress-strain-yield behavior of the metal under cyclic and non-porportional loading. Also the numerical procedure must be economical to use and yield results of sufficient accuracy.

In this report, all the three components of the analytical procedure have been reviewed. It appears that the incremental formulation for finite strain must be based on the total Lagrangian formulation and not on the incremental Lagrangian formulation except in cases of infinitesimal deformation when the distinction between various incremental formulations disappears.

Cyclic plasticity tests indicate that a saturation state is achieved after a few cycles. Also, models based on purely kinematic or purely isotropic hardening are inadequate. For satisfactory modeling a combination of isotropic and kinematic hardening is <u>at least</u> required. Theories of rate-independent materials cannot model frequency-dependence of cyclic hardening in a low-cycle fatigue test. This emphasizes the need for viscoplasticity theories admitting rate effects.

Using rate-independent incremental theory of plasticity, we note that convexity of the set of elastic states is a reasonable postulate and the form of the initial and subsequent yield surfaces must satisfy this requirement. Fairly general mathematical models currently exist. To describe nonlinear hardening, the mechanical element overlay and Mroz's piecewise linearization with fields of hardening moduli have been found to be successful. Jhansale and Dafalias introduced continuously varying moduli. Lamba's experimental work on non-proportional loading supports this approach.

In setting up constitutive relations for incremental plasticity, there has been some confusion due to the fact that the factor λ can be determined several different ways. Here we note that the formulations using the consistency condition in addition to normality or linearity would be preferable. A strain-space formulation of the type proposed by Felippa is now in common use. The incremental formulation is referred to the last load reversal and the state variables are defined by the last significant event. This is in line with the experimental evidence that the influence of prior history is wiped out by a plastic deformation of sufficient magnitude.

The finite element solution process may be either incremental application of Newton-Raphson or other iterative (implicit) procedure or an initial value technique with equilibrium correction. Some investigators believe that the Newton-Raphson method is inadequate to allow unloading (load reversal). However, when used with variable increment procedure as well as displacement incrementation where necessary, the Newton-Raphson or other implicit methods ought to be satisfactory. Bergan's method of displacement incrementation appears to be the best candidate.

Survey of existing finite element models of cyclic plasticity response has shown that none of the currently available models incorporates all the desirable features and avoid the errors and shortcomings described above. To develop the technology for prediction of strength-failure of fasteners under cyclic plasticity environment, further work is indicated in three distinct yet inter-related areas. Evidently, there is the need for collecting more information on metal response under non-proportional cyclic loading and on its mathematical modeling. Secondly, it appears necessary to implement the available knowledge of material behavior and efficient solution techniques in a suitable finite element analytical procedure capable of handling plastic deformation in two- and three-dimensional situations. Lastly, using the finite element method to establish local plasticity histories under cyclic loads, cumulative damage criteria need to be established for prediction of local fatigue damage and its influence on fastener life.

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