ON THE SOLUTIONS IN THE LARGE OF THE TWO-DIMENSIONAL FLOW OF A NON-VISCOUS INCOMPRESSIBLE FLUID

H. Beirão da Veiga

Mathematics Research Center
University of Wisconsin-Madison
610 Walnut Street
Madison, Wisconsin 53706

September 1982

(Received April 20, 1982)
We study the Euler equations (1.1) for the motion of a non-viscous incompressible fluid in a plane domain \( Q \). Let \( E \) be the Banach space defined in (1.4), let the initial data \( v_0 \) belong to \( E \), and let the external forces \( f(t) \) belong to \( L^1_{\text{loc}}(\Omega;E) \). In theorem 1.1 we prove the strong continuity and the global boundedness of the (unique) solution \( v(t) \), and in theorem 1.2 we prove the strong-continuous dependence of \( v \) on the data \( v_0 \) and \( f \). In particular the vorticity \( \text{rot} \, v(t) \) is a continuous function in \( \Omega \), for every \( t \in \mathbb{R} \), if and only if this property holds for one value of \( t \). In theorem 1.3 we state some properties for the associated group of nonlinear operators \( S(t) \). Finally, in theorem 1.4 we give a quite general sufficient condition on the data in order to get classical solutions.

AMS (MOS) Subject Classifications: 35B30, 35F25, 35Q20

Key Words: non-viscous incompressible fluids, nonlinear evolution equations, continuous dependence on the data

Work Unit Number 1 (Applied Analysis)

*Department of Mathematics, University of Trento (Italy).

Sponsored by the United States Army under Contract No. DAAG29-64-C-0041.
In this paper we study the Euler equations (1.1) for the motion of a non-viscous incompressible fluid in a plane domain \( \Omega \).

Let \( E \) be the Banach space consisting of all divergence free vector fields in \( \Omega \), tangential to the boundary \( \Gamma \) and having a continuous curl in \( \Omega \); let the initial data \( v_0 \) belong to \( E \) and the external forces \( f(t) \) be integrable in time with values in \( E \). Under these assumptions the (unique) solution \( v(t) \) with values in \( E \) of the Euler equations is globally bounded and continuous in time (theorem 1.1). Moreover, we prove the strong continuous dependence of the solution \( v \) with respect to the data \( v_0 \) and \( f \) (theorem 1.2). In particular, curl \( v(t) \) is a continuous function in \( \Omega \), for every \( t \in \mathbb{R} \), if and only if this property holds for one value of \( t \).

In theorem 1.3 it is shown that if \( \nabla \times f \equiv 0 \), the nonlinear operators \( S(t) \), mapping the initial data \( v_0 \) to the solution at time \( t \), form a strongly continuous group of isometries. Finally, a general sufficient condition guaranteeing the existence of classical solutions is given in theorem 1.4.

The responsibility for the wording and views expressed in this descriptive summary lies with NRL, and not with the author of this report.
ON THE SOLUTIONS IN THE LARGE OF THE TWO-DIMENSIONAL
FLOW OF A NON-VISCOS INCOMPRESSIBLE FLUID

E. Beirão da Veiga

1. INTRODUCTION AND MAJOR RESULTS.

Let \( \Omega \) be an open, connected, bounded set of the plane \( \mathbb{R}^2 \) with a regular boundary \( \Gamma \), say of class \( C^2, \theta > 0. \) We denote by \( n \) the outward unit normal to \( \Gamma. \) In this paper we study the Euler equations

\[
\begin{align*}
\frac{\partial v}{\partial t} + (v \cdot \nabla) v &= f - \nabla p \quad \text{in} \quad Q = \Omega \times (0, T), \\
\text{div } v &= 0 \quad \text{in } \Omega, \\
v \cdot n &= 0 \quad \text{on } \Gamma, \\
v |_{t=0} &= v_0(x) \quad \text{in } \Omega,
\end{align*}
\]

where the velocity field \( v(t,x) \) and the pressure \( p(t,x) \) are unknowns. In (1.1) the external force field \( f(t,x) \) and the initial velocity \( v_0(x) \) are given; moreover,

\( \text{div } v_0(x) = 0 \quad \text{in } \Omega \) and \( v_0 \cdot n = 0 \quad \text{on } \Gamma. \)

Existence of local solutions of (1.1) was proved by L. Lichtenstein. Global classical solutions were studied by many authors as for instance E. Hildner, J. Leray, A. C. Schaeffer [7], and N. Wolibner [8]. More recent studies are those of V. I. Judovich [3], T. Kato [4], J. C. V. Rogers [9], and C. Bardos [1].

The aim of our paper is to prove some properties for the global solutions of equation (1.1) by setting the problem in a very natural functional framework, the Banach space

\( L^2(\Gamma) \) consisting of all divergence free vector fields \( v(x) \) which are tangential to the boundary and for which \( \text{rot } v(x) \in C(\Gamma) \). The properties of global solutions in this space can be summarized as follows:

---

Department of Mathematics, University of Trento (Italy)

Sponsored by the United States Army under Contract No. DAAG29-90-C-0041.
(i) for every initial velocity \( v_0 \in \mathcal{H}(H) \) and for every exterior force
\[ f \in L^1_{\text{loc}}(0;H(H)) \] the solution \( v(t) \) is strongly continuous, i.e., \( v \in C(0;H(H)) \) (see theorem 1.1; see also remark 2.1).

(ii) the solution \( v(t) \) depends continuously, in the norm topology, on the data
\( v_0 \) and \( f \). More precisely, if \( v_0^{(n)} \to v_0 \) in \( H(H) \) and if \( f_n \to f \) in \( L^1(I;H(H)) \) for every compact time interval \( I \), then \( v_0^{(n)}(t) \to v(t) \) in \( H(H) \), the convergence being uniform on every compact time interval \( I \) (theorem 1.2).

(iii) estimate (1.6') holds; in particular the solution is globally bounded in time
\[ f \in L^1_{\text{loc}}(0;H(H)) \]. Moreover if \( f \equiv 0 \) then \( \delta v(t) = 0 \), \( v \in C(0;H(H)) \), \( \delta v \) being the norm of \( H(H) \).

The crucial property (ii) appears not to have been proved in any Banach space. Note that continuous dependence with respect to weaker topologies can be (in many cases) trivially verified.

Property (iii) shows that \( H(H) \) might be a suitable space for the study of asymptotic properties; note that \( H(H) \) seems to be the space of the most regular functions for which property (iii) holds.

Assuming for simplicity that \( f \equiv 0 \), and combining the above results, one gets theorem 1.3, which shows that the essential properties of hyperbolic groups of operators hold for equation (1.1) in the space \( H(H) \).

On the other hand, we note that theorems 1.1 and 1.2 also prove the nonexistence of shocks for the curl of the velocity field; more precisely, \( \text{rot} v(t) \) is a continuous function in \( \tilde{H} \), for every \( t \in \mathbb{R} \), if and only if this property holds for one (arbitrary) value of \( t \); this statement holds even in presence of quite discontinuous (in time) external forces. Actually, \( \text{rot} v(t) \) must then be a continuous function in \( \tilde{H} \). In the remainder of this section we introduce notation and state the above results in complete form. For simplicity, we will assume that \( \tilde{U} \) is simply-connected; the reader should verify that the usual device (see [3] §9 and [4]) utilized to treat the general case also applies to our proofs; hence the results stated in our paper hold for non-simply-connected domains.
In the sequel \( \bar{\mathcal{G}} \) denotes the closure of \( \mathcal{G} \) and \( C(\bar{\mathcal{G}}) \) the space of continuous (scalar or vector valued) functions in \( \bar{\mathcal{G}} \) normed by \( \| f \|_{\bar{\mathcal{G}}}=\sup_{\bar{\mathcal{G}}}|f(x)|, \quad x \in \bar{\mathcal{G}} \). For simplicity we avoid in our notation any distinction between scalars and vectors. \( C^k(\bar{\mathcal{G}}) \) \((k, \text{ a positive integer})\) is the space of all \( k \) times continuously differentiable functions in \( \bar{\mathcal{G}} \) equipped with the usual norm \( \| \cdot \|_k \). Sometimes we will write \( D^k_{\mathcal{G}} \) to denote a generalised derivative of order \( k \). The scalar product in the Hilbert space \( L^2(\mathcal{G}) \) is denoted by \( ( , ) \).

If \( \mathcal{H} \) is a Banach space, \( L^1_{\mathcal{G}}(\mathbb{R}/\mathcal{H}) \) is the linear space of all \( \mathcal{H} \)-valued strongly measurable functions \( u(t), \quad t \in \mathbb{R} \), such that \( \| u(t) \|_{\mathcal{H}} \) is integrable on compact intervals \([-T,T], \quad T > 0 \).

Some of the above definitions will be utilized also with \( \mathcal{G} \) replaced by \( \mathcal{Q} \) or by \( \mathcal{Q}_T(0,T) = \mathcal{G} \).

If \( \theta(t,n) \) is defined in \( \mathcal{Q} \) we sometimes denote by \( \theta(t) \) the function \( \theta(t,\cdot) \) defined for \( x \in \mathcal{Q} \).

Finally \( \mathbb{N} \) denotes the set of positive integers and \( a, a_0, a_1, \ldots \) denote constants depending at most on \( \mathcal{G} \). Different constants may be denoted by the same symbol \( a \).

The following definitions are classical: for a scalar function \( \varphi(x) \) in \( \mathcal{Q} \) we define the vector \( \text{rot} \, \varphi = (\partial \varphi / \partial x_2, -\partial \varphi / \partial x_1) \) and for a vector function \( v = (v_1, v_2) \) we define the scalar \( \text{rot} \, v = \nabla \times v = \begin{vmatrix} 2v_2 & -v_1 \\ v_1 & 2v_2 \end{vmatrix} \). One has \( -\Delta \, \text{rot} \varphi \) is the rotation of the gradient \( \nabla \varphi \) by \( \pi/2 \) in the negative direction (counter clock wise). Let now \( \psi \) be the solution of

\[
\begin{cases}
-\Delta \psi = 0 \text{ in } \mathcal{G}, \\
\psi = 0 \text{ on } \Gamma.
\end{cases}
\]

By the above remarks \( v = \text{rot} \, \psi \) is the solution of

\[
\begin{cases}
\text{rot} \, v = 0 \text{ in } \mathcal{G}, \\
\text{div} \, v = 0 \text{ in } \mathcal{G}, \\
v \cdot n = 0 \text{ on } \Gamma.
\end{cases}
\]
Let us introduce the Banach space

\[(1.6) \quad B(G) \equiv (v \in C_0^\infty \cap L^1(\Omega)) : \nabla \cdot v = 0 \quad \text{in} \quad \Omega, \quad \nabla \cdot v = 0 \quad \text{on} \quad \Gamma, \quad \text{rot} \, v \in C(\overline{\Omega})\]

equipped with the norm \( \|v\|_B = \|\text{rot} \, v\|_{L^1(\Omega)} \). In the sequel (\( \Omega \) being simply-connected) we use the equivalent norm

\[\|v\|_B = \|\text{rot} \, v\|_{L^1(\Omega)}\]

Concerning the existence of solutions we prove the following statement:

**Theorem 1.1.** Let \( v_0 \in B(G) \) (or equivalently rot \( v_0 \in C(\overline{\Omega}) \), div \( v_0 = 0 \) in \( \Omega \), \( v_0 \cdot n = 0 \) on \( \Gamma \)) and let \( f \in L_\text{loc}^1(B(\Omega)) \)). Then problem (1.1) is uniquely solvable in the large, the solution \( v \) belongs to \( C(\Omega \times [0,T]) \) (or equivalently rot \( v \in C(\Omega \times [0,T]) \)) and

\[\|v(t)\|_B + \int_0^t \|\text{rot} \, f(s)\|_{L^1(\Omega)} \, ds \leq C, \quad v \in C_T[\Omega].\]

If \( \text{rot} \, f \equiv 0 \) equality holds in (1.6).

**Remark.** Instead of \( f \in L_\text{loc}^1(B(\Omega)) \) we could assume that \( f \) is a distribution in \( \Omega \) the significant condition being only \( \text{rot} \, f \in L_\text{loc}^1(B(\Omega)) \). Furthermore, in view of the decomposition formulas (5.1), our assumption on \( f \) is equivalent to \( f \in L_\text{loc}^1(B(\Omega)) \) and estimate (1.6) is equivalent to

\[\|v(t)\|_B + \int_0^t \|f(s)\|_{L^1(\Omega)} \, ds \leq C, \quad v \in C_T[\Omega].\]

**Remark.** We don't consider explicitly the regularity of \( \partial v/\partial t \) and \( \partial v \) since it follows from the regularity of \( v \) and \( f \). See appendix 2.

**Theorem 1.7.** Let \( v_0, \, v_0^{(n)} \) and \( \tilde{v}_n, \, \tilde{v}_n^{(m)} \), \( n, \, m \in \mathbb{N} \), be as in theorem 1.1, and let \( v \) and \( v_n \) be the solutions of (1.1) with data \( v_0 \), \( f \) and \( v_0^{(n)} \), \( f^{(m)} \), respectively. If

\[\text{(f, } v_0^{(n)}) \in L_\text{loc}^1(\Omega), \quad n = 1, \ldots, N \text{, if } \Omega \text{ is not simply-connected. For the definition of } v_0 \text{ see appendix 1.}\]
\[ v_0^{(n)} + v_0 \text{ in } E^1 \text{ and rot } f_0 = \text{ rot } f \text{ in } L^1_{\text{loc}}(\mathbb{R}^d) \ (2) \Rightarrow v_0(t) + v(t) \text{ in } E^1 \text{, the convergence being uniform on every compact interval, i.e. } v_n \to v \text{ in } C([-T,T]; E^1) \text{, for every } T > 0. \]

Now assume rot \( f \neq 0 \) and denote by \( S(t), \ t \in \mathbb{R} \), the nonlinear operator defined by 
\[ S(t)v_0 \in v(t), \quad v_0 \in E^1, \] 
where \( v(t) \) is the solution of problem (1.1). Put also \( J_0 \equiv 0 \). One has the following result:

**Theorem 1.2.** Under the above assumptions and definitions one has:

1. \( S(t)S(t) = S(t + T), \quad t, t + T \in \mathbb{R} \), \( S(0) = I \).
2. \( S(t) \) is "unitary" in the sense that \( S(t)0 = 0 = 0, \quad v \in E^1 \). Moreover 
   \[ S(t)^{-1} = S(-t) = S(t)^* \quad t \in \mathbb{R}. \]
3. \( S(t) \) is a strongly continuous group of operators, i.e. for every \( v \in E^1 \) the function \( S(t)v \) is a strongly continuous function in \( t \) with values in \( E^1 \).
4. For every \( t \in \mathbb{R} \) the nonlinear operator \( S(t) \) is a bijective map (in the sense topology of \( E^1 \) ) from all of \( E^1 \) onto itself. Moreover if \( v_n \to v \) the convergence \( S(t)v_n \to S(t)v \) is uniform on compact \( t \)-intervals.

We also study some questions concerning the existence of classical solutions. Our main concern will be the continuity of \( \tilde{v} \) on \( 0 \), additional conditions on \( f \) in order to get continuity for \( \tilde{v}/\theta t = \) and \( \tilde{v} \) will then be trivial. We want to characterize explicitly a Banach space \( C_0(E^1) \), the data space, such that \( v \in C(E^1) \) whenever \( \text{rot } v_0 \in C_0(E) \) and \( \text{rot } f \in L^1_{\text{loc}}(E^1) \).

We don't expect the above result if we just define \( C_0(E^1) \) as \( C(E^1) \). On the other hand, if we define \( C_0(E^1) \) as \( C^{\lambda,1}_0(E^1) \), \( \lambda > 0 \), the result holds easily; hence we want a larger space. We construct \( C_0(E^1) \) as follows: for every \( 0 \in C(E^1) \) let us denote by \( \mathcal{U}_R(0) \) the oscillation of \( 0 \) on sets of diameter less or equal to \( R \):

\begin{align*}
\mathcal{U}_R(0) = \sup_{0 < |x-y| \leq R} |0(x) - 0(y)|. 
\end{align*}

(1.9) If \( E \) is not simply-connected we also assume that \( (f_0, u_0) \to (f, u_0) \) in \( L^1_{\text{loc}}(E) \), \( n = 1, \ldots, N \).
Clearly \( u_{r}(r) = u_{0}(r), \forall r > R \) is diameter of \( R \). Let us put

\[
(1.10) \quad \mathcal{N}_{r} = \int_{\theta} u_{r}(r) \, d\theta
\]

and define \( C_{r}(\mathcal{N}) = \{ \theta \in C_{r}: \| \theta \| < \infty \} \). Then \( \mathcal{N}_{r} \in C_{r}(\mathcal{N}) \). Moreover \( C_{r}(\mathcal{N}) \) is a Banach space. Note, by the way, that \( (\theta, \mathcal{N}) \in \mathcal{N}_{r} \) where \( \mathcal{N}_{r} \) is the usual \( \mathcal{N}_{r} \)-dimension semi-norm.

We prove the following result:

**Theorem 1.4.** Let \( \nu_{r} \in C_{r}(\mathcal{N}) \) and \( f \in C_{r}(\mathcal{N}) \) with \( \nu_{r} \in L_{r}^{1}(\mathcal{N}) \). If the solution of problem (1.1) belongs to \( C(\mathcal{N}) \) then

\[
(1.1) \quad \nu(t, \mathcal{N}) \in L_{r}^{1}(\mathcal{N})
\]

where \( \nu(t) = \nu_{r}(t) + \nu_{r}(t) \). If in addition \( \nu \) is such that the terms \( g(t) \) and \( \nu_{r}(t) \) in the classical equation (1.1) are continuous in \( \mathcal{N} \) also \( \nu_{r}(t) \) and \( \nu_{r}(t) \) are continuous in \( \mathcal{N} \) (classical solution).

2. **Proof of theorem 1.1.** In the following we consider equation (1.1) in the time interval \([0,T], T > 0 \) arbitrary. Proofs apply also to intervals \([-T,0] \); alternatively one can reduce this case to the previous one by a change of variables. In fact the solution of the problem \( (\theta / \partial t) + (\nu / \partial t) = f - \nu, \quad t \in [-T,0] \), with \( \nu_{t} = \nu_{0}(t) \) is given by \( \nu(t) = -\nu(t) \) where \( (\theta / \partial t) + (\nu / \partial t) = f - \nu_{t}, \quad g(t, u) = f(t, u), \quad \nu_{t} = g(t, u), \quad \nu_{t} = -g(t, u), \quad t \in [0,T], \quad u_{t} = -\nu_{0}(t) \).

Assume the data \( \nu_{0} \) and \( f \) fixed as well as \( T > 0 \). For convenience put \( C_{0} = \text{rot } \nu_{0}, \quad f = \text{rot } f, \quad \mathcal{N} = \mathcal{N}_{r} + \int_{0}^{T} (\theta(t) \, dt \), and define

\[
(2.1) \quad \mathcal{N} = \{ \theta \in C_{r}(\mathcal{N}) : \| \theta \| < \infty \}
\]

\footnote{It suffices that \( f \in C(\mathcal{N}^{1}(\mathcal{N})) \), for some \( p > 2 \).}
where \( \phi \) denotes the norm in \( C(\overline{\Omega}_0) = C([0,\tau], C(\overline{\Omega})) \). \( \Omega \) is convex, closed and bounded in \( C(\overline{\Omega}_0) \). From now on \( \Phi \) denotes an arbitrary element of \( \Omega \). Now let \( \Phi \) be the solution of problem (1.2)

\[
\Phi(u) = \int_{\partial} q(x,y) \Phi(y) dy, \quad u \in \Omega,
\]

and let \( v = \text{det} \Phi \) be the solution of (1.3); since \( v \in \mathbb{V}^{1,p}(\Omega) \), \( \nu < \infty \), the meaning of equation (1.3) is clear. It is well known that the Green's function \( q(x,y) \) for the Laplace operator \(-\Delta\) with zero boundary condition (see for instance [8]) verifies the estimates

\[
|\partial_x q(x,y)| < \alpha |x-y|^{-1}, \quad |\partial_x^2 q(x,y)| < \alpha |x-y|^{-2}.
\]

By using classical devices in potential theory or shows that \( \nu \in \mathbb{V}^{1,p}(\Omega) \) and that

\[
|v(x) - v(y)| < \alpha \log(|x-y|) \quad \text{where } x(x) = \log(|x|), \quad \nu > 0; \quad \text{see } [4] \text{ lemma 1.4.}
\]

Hence for every \( \xi \in (0,\tau) \), \( \Phi(t) \in \mathbb{O}_0 \) and

\[
|\Phi(t,x) - \Phi(t,y)| < \alpha t|x-y|^{1} |x-y|^{1/2}, \quad \nu > 0,
\]

Clearly \( v \in C(\overline{\Omega}_0) \). Let \( U(t,u,x) \) be the solution of the system of ordinary differential equations

\[
\begin{cases}
\frac{d}{dt} U(t,u,x) = v(t,U(t,u,x)), \quad \text{for } t \in (0,\tau), \quad U(0,u,x) = u,
\end{cases}
\]

where \((t,u) \in \Omega_0\). Let us show that

\[
|U(t,u,x) - U(0,u,x)| < \alpha \tau t |x-x_1|^{1} |x-x_1|^{1/2},
\]

where \( \alpha > 0 \) and \( t \in (0,\tau) \) and \( \alpha \in (0,1). \) Put \( x_1(t) = U(t,u,x) \), \( x_1(t) = U(t,u,x_1(t)) \), \( x_1(0) = U_0(t,u,x_1) \), \( \Phi(t) = |x_1(t) - x_1(t)| \). One has \( |x_1(t)| < \alpha \Phi(t) \Phi(t) \) and \( v(t) = |x - x_1| \). On the other hand the function

\[
\phi_1(t) = c(t \cdot \frac{|x - x_1|}{\alpha})^{-}\alpha \Phi(t)
\]

-7-
is the solution of \( \Phi_1(x) = \sigma_1(x) \mathcal{L}(x_1(x)) \), \( x \in [0,T] \), with \( \Phi_1(x) = |x - x_1| \). Hence \( \Phi_1^a < \Phi_1^a \) for \( a > t \). For \( a < t \) a corresponding argument holds. Then

\[
(2.7) \quad |u(a,t,x) - u(t,x)| \leq \varepsilon \left( a^2 \right)^{1/2} |x - x_1| \leq \varepsilon |x - x_1|^{1/2} .
\]

Now one easily gets \( |u(a,t,x) - u(t,x)| \leq \varepsilon |x - x_1| \) and

\[
|u(a,t,x) - u(t,x)| \leq \varepsilon \left( a^2 \right)^{1/2} |t - x_1|^{1/2} (\text{see (4)}); \text{ estimate (2.6) follows}.
\]

Define now the map \( \zeta = \Phi_1^a \) by

\[
(2.8) \quad \zeta(t,y) = \Phi_1^a(u(t,y)), \quad \int_0^t \Phi_1^a(u(s,y))\,ds.
\]

**Theorem 2.1.** The inclusion \( \Phi(\Pi) \subset E \) holds, moreover \( \Phi(\Pi) \) is a family of equicontinuous functions in \( E \). Hence \( \Phi(\Pi) \) is a relatively compact set in \( C(E) \).

**Proof.** Obviously \( K(t,n) \subset E \). The equicontinuity of the family \( \Phi(t,n) \subset E \) follows from (2.9) and from the uniform continuity of \( \zeta \) on \( E \). Let us prove the equicontinuity of the second term on the right hand side of (2.8); clearly

\[
(2.9) \quad \int_0^t \Phi(s, u(s,t,x_1))\,ds = \int_0^t \Phi(s, u(t,x_1))\,ds.
\]

Moreover, to each \( v > 0 \) there corresponds \( \lambda_1 > 0 \) such that

\[
(2.10) \quad |t - t_1| < \lambda_1 \Rightarrow \left\| \Phi(s, u(s,t,x_1)) - \Phi(s, u(t,x_1)) \right\| \leq v .
\]

Define for every \( x > 0 \)

\[
(2.11) \quad \sigma(x) = \sup_{|y-y_1| < x} |\Phi(s, y) - \Phi(s, y_1)| .
\]

Since \( \sigma(x) < 2 \Phi(x) \) and \( \lim \sigma(x) = 0 \) for almost all \( x \in [0,T] \), it follows by the Lebesgue dominated convergence theorem that to each \( v > 0 \) there corresponds an \( \lambda_1 > 0 \) such that

\[
(2.12) \quad \sigma(x) < v .
\]
Furthermore to every $\varepsilon_0 > 0$ there corresponds a $\lambda_2 > 0$ such that
\[
\max |t - t_1|, |x - x_1| < \lambda_2 \Rightarrow |U(s,t,x) - U(s,t_1,x_1)| < \varepsilon_0
\]
uniformly with respect to $s$; this follows from (2.6). Hence
\[
\int_0^t \phi(s,U(s,t,x))\,ds - \int_0^t \phi(s,U(s,t_1,x_1))\,ds < 2\varepsilon
\]
if $\max |t - t_1|, |x - x_1| < \min(\lambda_1, \lambda_2)$. The equicontinuity of the family $\phi(\xi)$ is proved. The last statement follows from Ascoli-Arsela's compactness theorem. \hfill $\square$

**Theorem 2.2.** The map $\phi : E \times E \to E$ has a fixed point.

**Proof.** It remains to prove the continuity of the map $\phi$. Let $\theta_n \in E$, $\theta_n \to \theta$ uniformly on $\partial_G$. Denoting by $v_n$ the solution of (1.3) with data $\theta_n$ it is clear that $v_n \to v$ uniformly on $\partial_G$. Let $\varepsilon > 0$ be given and $N_\varepsilon$ be such that $|v - v_n|_{\partial_G} < \varepsilon$ whenever $n > N_\varepsilon$. Put $x(s) = U(s,t,x)$, $x_n(s) = U_n(s,t,x)$, and $p(s) = |x(s) - x_n(s)|$, where $U_n$ denotes the solution of (2.5) with $v$ replaced by $v_n$. For $n > N_\varepsilon$ one has
\[
|p'(s)| < |x'(s) - x_n'(s)| < \varepsilon + c_1(\varepsilon) x(s) - x_n(s)
\]
where $c_1(\varepsilon)$ is an increasing function on $[0,\varepsilon]$. Moreover $p(t) = 0$. Consequently $|U(s,t,x) - U_n(s,t,x)| \leq \varepsilon(1 + c_1(\varepsilon))$, $\forall s \in [0,T]$, and $U_n(s,t,x)$ is uniformly convergent to $U(s,t,x)$ on $[0,T]^2 \times \delta$, when $n \to \infty$. It follows easily from (2.8) and (2.12) that $\xi_n \to \xi$ uniformly in $\partial_G$, where $\xi_n \in \phi(\xi)$. Actually, it suffices to show the pointwise convergence of $\xi_n$ to $\xi$; uniform convergence follows from the compactness of subsets of $\phi(E)$.

**Remark 2.1.** The above method of proving strong continuity of $\xi(t)$ in $C(\delta)$ seems not to work in Holder spaces, even if $f \equiv 0$. In fact if $\xi_0 \in C^{0,\lambda}(\delta)$ we cannot prove that $\xi_0(U(t,x)) \in C\bar{M}C^{0,\lambda}(\delta)$ by using (only) regularity results for $U(t,x)$ (other arguments must eventually be added); in fact, if $\xi_0(U(t,x)) \neq \xi_0(U(t,x))$ and $U(t,x) \neq t - x$ the function $\xi(t,x) = \xi_0(U(t,x))$ verifies $|\xi(t,x) - \xi(t,y) + \xi(t,y)| = |x - y|^{1/2}$ if $x = t$, $y = t$.
The situation becomes worse with respect to the strong continuous dependence on the data.

Now we verify that the function $v$ corresponding to the fixed point $\zeta = 0$ is a solution of (1.1); see also [4].

We start by showing that for fixed $(s, t)$ the map $x \mapsto U(s, t, x)$ is measure preserving in $\Omega$. Let $\theta \in \mathbb{R}$, $\theta_0 \in C([0, T], C^1(\Omega))$, $\theta_0 \equiv 0$ uniformly on $\partial_\Omega$. If $v_0$ is the solution of (1.3) with data $\theta_0$, one has $v_0 \in C([0, T], C^1(\Omega))$ and $\text{div} \, v_0 = 0$. Hence $x \mapsto U_0(s, t, x)$ is measure preserving. On the other hand we know from the proof of theorem 2.1 that $U_0 + U$ uniformly on $[0, T]^2 \times \hat{\Omega}$. It follows that $U$ is measure preserving.

For, define $\tau_t = U(s, t, x)$, $\tau_{n, t} = U_n(s, t, x)$, $n \in \mathbb{N}$, and let $E$ be a compact subset of $\hat{\Omega}$ and $A$ an arbitrary open set verifying $\tau(E) \subset A \subset \hat{\Omega}$. Recalling that $\tau_{n, t} \rightarrow \tau_t$ uniformly and that $\tau(E)$ is compact one shows that there exists an integer $n_0$ such that $\tau_{n_0}(E) \subset A$, hence $|\tau_{n_0}(E)| = |E| \leq |A| \leq |\tau(E)|$, where $|\cdot|$ denotes Lebesgue measure. An analogous property holds for the map $\tau^{-1}y = U(t, s, y)$, hence the measure preserving property holds.

Lemma 2.3. Let $\zeta = 0$ be the fixed point constructed above. Then

$$\frac{\partial}{\partial t} \Phi(t) = -\text{div}(\Phi(t)) + \Phi(t), \quad \Phi \in C_0^\infty(\mathbb{R})$$

Proof. We show that

$$(2.16) \quad \frac{d}{dt} \langle \psi, \psi \rangle = \langle \psi \psi, \psi \rangle + \langle \psi, \Phi \rangle, \quad \psi \in C_0(\mathbb{R})$$

Denoting by $C_2(t, x)$ the second term in the right hand side of (2.9) and taking into account the measure preserving property one gets, by the change of variable $y = U(s, t, x)$,

$$C_2(t) = \int_0^t ds \int_0^y \Phi(s, y)U(t, s, y)dy$$

Hence

$$\frac{d}{dt} \langle C_2(t) \rangle = \int_0^y \Phi(t, y)U(t, s, y)dy + \int_0^y ds \int_0^y \Phi(s, y)\Phi(t, y)U(t, s, y)dy$$

and returning to the variable $x = U(t, s, y)$ in the last integral one gets (2.16) for $C_2$.

One argues similarly with the first term on the right hand side of (2.9).
Lemma 2.4. Let $v \in W^{1,2}(\Omega)$, $\text{div } v = 0$ in $\Omega$ and $v \cdot n = 0$ on $\Gamma$. Put
\[ \text{rot } v = \xi. \]
Then $\text{rot}(v \cdot n v) = \text{div}(\xi v)$ in the sense of distributions in $\Omega$, i.e.
\[ \langle (v \cdot n v) \cdot \text{rot } v, (\xi v) \rangle = \langle v \cdot \text{rot } v, (\xi v) \rangle, \quad v \in C_0^\infty(\Omega). \]

Proof. A direct computation shows that for a regular $v$, say $v \in C^2(\Omega)$, the above equation holds pointwise. For a general $v$ consider a sequence of regular $\xi_n$ such that $\xi_n \to \xi$ in $L^2(\Omega)$. Denoting by $\Phi_n$ the solution of (1.2) with data $\xi_n$ and defining $v_n = \text{rot } \Phi_n$ it follows that $v_n \to v$ in $W^{1,2}(\Omega)$. This allows us to pass to the limit when $n \to \infty$ in the above weak form.

Now we verify that $v$ is a solution of (1.1). Clearly $D_v \Phi \in C([0,T];L^p(\Omega))$, $v \cdot n < \infty$. Moreover, $(v \in C([0,T];L^2(\Omega)))$ hence from lemma 2.3 one gets $\xi \in L^p((0,T; u^{-1/2}(\Omega)))$. Recalling that $\Phi = \xi$ equation (1.2) yields
\[ \Phi_t + \Phi = 0 \text{ in } \Omega, \quad \Phi \cdot n = 0 \text{ on } \Gamma. \]
Consequently $\Phi_t \in L^1((0,T; H^{-1/2}(\Omega))$ and $\Phi \cdot n = 0$. In particular $(\Phi_t + \Phi)v - \xi \in L^1((0,T; L^2(\Omega))$. Moreover, $\text{rot}(\partial \Phi_t + \partial \Phi v - \xi) = 0$ in the distributions sense, by lemmas 2.4 and 2.3. Consequently there exists $v \in L^1((0,T; u^{-1/2}(\Omega))$ such that (1.1) holds. On the other hand $\xi_{|t=0} = \xi_0$ i.e. $\text{rot } v_{|t=0} = \text{rot } v_0$ in $\Omega$, $\text{div } v_{|t=0} = \text{div } v_0 = 0$ in $\Omega$, and $v_{|t=0} = v_0$. Finally the uniqueness of the solution $v$ follows as in Bardos [1] theorem 2 since for every $p \in [2, \infty]$ the estimate
\[ \langle v \cdot n, \Phi \rangle = c \Phi_0 \xi_0 \]
holds; this follows from (1.2) and from well known estimates $L^p(\Omega)$ for elliptic partial differential equations in $L^p$ spaces (see for instance [3], theorem 2.1).

3. Proof of theorem 1.2. In this section we write $\xi = \Phi_1(0,\xi_0,\phi)$ instead of $\xi = \Phi(0)$ since $\xi_0$ and $\phi$ are variable. For convenience we denote by $\Phi_1, \Phi_2, \Phi_3$ respectively the maps $v = \Phi_1(0)$ defined by (1.3), $u = \Phi_2(v)$ defined by (2.5) and $\xi = \Phi_3(0,\xi_0,\phi)$ defined by (2.8). Hence $\Phi_1(0,\xi_0,\phi) = \Phi_3(\Phi_2(\Phi_1(0)), \xi_0, \phi)$. The map $\Phi_1$ is defined for every $(0,\xi_0,\phi) \in C(\tilde{\Omega}_0) = C^0(\tilde{\Omega}) = L^1((0,T; C(\tilde{\Omega})))$. Note that $v$ is the solution of problem (1.1) if and only if $v = \Phi_1(\xi)$ for a $\xi$ verifying $\xi = \Phi_1(\xi_0, \phi)$.
Theorem 3.1. Let $K_1$ be a relatively compact set in $C(\Delta)$, $K_2$ a relatively compact set in $L^1(0,T;C(\Delta))$ and $K$ a bounded set in $C(\Delta)$. Then the set $\theta_1(K \times K_1 \times K_2)$ is relatively compact in $C(\Delta)$.

Proof. Let $K_1$, $K_2$ and $K$ be contained in balls with center in the origin and radius $k_1$, $k_2$ and $K$ respectively. The set of functions $\zeta_0(U(0,t,x))$, for $0 \in K$ and $\zeta_0 \in K_1$, is bounded in $C(\Delta)$ by $k_1$. By the necessary condition of Ascoli-Arselà's theorem the functions $\zeta_0 \in K_1$ are equicontinuous in $\Delta$. By (2.6) the functions $U(0,t,x)$ are equicontinuous in $\Delta$. Hence the family $\zeta_0(U(0,t,x))$ is equicontinuous in $\Delta$ and by Ascoli-Arselà's theorem, constitutes a relatively compact set in $C(\Delta)$.

Analogously the family

\[(3.1)\quad \zeta_2(t,x) = \int_0^t \phi(s,U(s,t,x))ds, \quad 0 \in K, \phi \in K_2,\]

is bounded by $k_2$ in $C(\Delta)$. We want to prove that every sequence $\zeta_2^{(m)}(t,x)$ contains a convergent subsequence in $C(\Delta)$. This proves compactness for the family (3.1).

Let $\theta_m \in K$ and $\phi_m \in K_2$ be arbitrary sequences and consider

\[(3.2)\quad \zeta_2^{(m)}(t,x) = \int_0^t \phi_m(s,U_m(s,t,x))ds.\]

By the compactness of $K_2$ there exists a subsequence of $\phi_m$ and a function

\[\phi \in L^1(0,T;C(\Delta))\]

such that $\phi_m \to \phi$ in $L^1(0,T;C(\Delta))$ (4). Moreover a well known theorem ensures the existence of a subsequence such that

\[(3.3)\quad \phi_m(s, \cdot) + \phi(s, \cdot) \text{ in } C(\Delta), \text{ for almost all } s \in [0,T).\]

Denote by $w_m(s,\varepsilon)$ the modulus of continuity of $\phi_m(s,\varepsilon)$ in $\Delta$ (see (2.11)) and define $w(s,\varepsilon) = \sup_{\|x\|} w_m(s,\varepsilon)$. From (3.4) and from Ascoli-Arselà's theorem it follows that

\[(3.5)\quad \lim_{\varepsilon \to 0} w(s,\varepsilon) = 0, \text{ for almost all } s \in [0,T).\]

(4) For convenience we use the same index $m$ for sequences and for subsequences.
Now let \( (a_k)_{k=1}^\infty \) be a sequence of real positive numbers such that \( \sum_{k=1}^\infty a_k < \infty \). Since \( \phi_n \to \phi \) in \( L^1(0,T;C(\bar{U})) \) there exists a subsequence \( \phi_{k_n} \) such that
\[
\int_0^T |\phi_n(s) - \phi_{k_n}(s)| ds < a_k, \quad \forall k \in \mathbb{N}.
\]
Define \( b_n(s) = \sum_{k=1}^\infty |\phi_n(s) - \phi_{k_n}(s)|, \quad s \in [0,T] \); clearly \( b_n \) is integrable over \([0,T]\).
Moreover \( \omega_k(s) < \varepsilon \phi_k(s) < \varepsilon \phi(s) + 2\varepsilon b(s) \) hence \( \omega(s) < 2b(s) \) where \( \omega \) is defined respect to the subsequence \( \omega_k \) and \( b(s) \) is integrable. By using (3.5) and Lebesgue's dominated convergence theorem it follows that to every \( \varepsilon > 0 \) there corresponds an \( \varepsilon_0 > 0 \) such that
\[
\int_0^T \omega_k(s) ds < \varepsilon, \quad \forall k \in \mathbb{N}.
\]
Equation (3.6) generalizes (2.12) in the proof of theorem 2.1.

On the other hand, by the boundedness of \( \Omega \), the functions \( v_k \) and \( U_k \) verify (2.4) and (2.6) uniformly with respect to \( k \). Hence (2.13) holds for every \( U_k \) with \( \lambda_2 = \lambda_2(\varepsilon_0) \) independent of \( k \). We now proceed as in the proof of theorem 2.1 and we show the equicontinuity of the set of functions \( \xi_2^{(k)}(t,x) \) in \( \tilde{\Omega} \) (note that (2.10) holds uniformly with respect to \( k \), since \( \|\phi_k(s)\| < b(s) \)). From the equicontinuity follows the existence of a subsequence convergent in \( C(\tilde{\Omega}) \). \( \square \)

**Theorem 3.2.** The map \( \Theta : C(\tilde{\Omega}) \times C(\tilde{\Omega}) \to L^1(0,T;C(\bar{U})) \times C(\tilde{\Omega}) \) is continuous.

**Proofs.** Let \((\theta_0, \xi_0) \to (\theta_1, \xi_1)\). Arguing as in the proof of the continuity of the map \( \Theta \) in theorem 2.2 one shows that \( v_n \to \theta_1(\theta_0) + v \to \theta_1(\theta) \) uniformly on \( \tilde{\Omega} \), consequently \( U_n \to \xi_2(v_n) + U \to \xi_2(v) \) uniformly in \( [0,T]^2 \times \tilde{\Omega} \). Now one easily verifies that \( \xi_2 \Phi_2(U_n, \xi_0) \to \xi_2 \Phi_2(U, \xi_0, \theta) \) pointwise in \( \tilde{\Omega} \) since
\[
\int_0^T \|\phi_m(s, \phi(x,t,x)) - \phi(s, \phi(x,t,x))\| ds < \int_0^T |\phi_m(s) - \phi(s)| ds + \int_0^T |\phi(s, \phi(x,t,x)) - \phi(s, \phi(x,t,x))| ds.
\]

---

-13-
By using theorem 3.1 with $K = (\theta_m)$, $K_1 = (\xi_{0,0}^{(m)})$ and $K_2 = (\phi_m)$ it follows that the convergence of $\xi_m$ to $\xi$ is uniform in $\tilde{D}_2$ (this can be shown without resort to theorem 3.1).

**Proof of theorem 1.2.** Assume the hypothesis of theorem 1.2 and put $\xi_0 \equiv \text{rot } v_0$.

Define $K = (\xi_0)$, $K_1 = (\xi_{0,0}^{(m)})$, $K_2 = (\phi_m)$. From (2.8) it follows that a set $\Phi_3(\xi_0, \xi_0, \xi_0)$ is bounded whenever $\xi_0$ and $\xi_0$ are bounded, independently of the particular set $S$. Consequently $K$ is bounded because $\xi_m = \Phi_3(\xi_0, \xi_0, \xi_{0}^{(m)}, \phi_m)$, $\forall m \in \mathbb{N}$. Now theorem 3.1 shows that $\Phi_3(K, K_1, K_2)$ is a relatively compact set in $C(\tilde{D}_2)$ hence $K \subset \Phi_3(K, K_1, K_2)$ verifies the same property.

Let $\xi_0$ be any convergent subsequence of $\xi_m$ and put for convenience $\tilde{\xi} = \lim_{m \to \infty} \xi_m$. From the identity $\xi_0 = \Phi_3(\xi_k, \xi_0, \phi_k)$ and from theorem 3.2 it follows that $\tilde{\xi} = \Phi_3(\xi, \xi_0, \phi)$. Consequently $\tilde{\xi} = \Phi_3(\xi)$ is a solution of (1.1) hence $\tilde{\xi} = \xi$ and $\tilde{\xi} = \xi$. It follows that all the sequence $\xi_m$ converges to $\xi$ uniformly in $C(\tilde{D}_2)$ i.e. $\xi_0 \equiv \xi$ in $C([0, T] \times \tilde{D}_2))$.

**Remark 3.1.** In theorem 1.2 convergence of $\xi^{(m)}$ to $\xi$ is not requested since $\xi$ is determined by system (4.2). Convergence of $\xi^{(m)}$ to $\xi$ in $L^2_{\text{loc}}(\mathbb{R}^2 \times [0, T])$ would imply the additional convergence $\nabla \xi^{(m)} \equiv \nabla \xi$ in $L^2_{\text{loc}}(\mathbb{R}^2 \times [0, T])$.

4. **Proof of theorem 1.4.** We start by proving that composition of $C^0$-functions with H"older continuous functions yields $C^0$-functions.

**Lemma 4.1.** Let $a \in C^0(\mathbb{R})$ and $u \in C^{0,\delta}(\mathbb{R}^2)$, $0 < \delta < 1$. Then $a \circ u \in C^0(\mathbb{R})$

Moreover

\begin{equation}
(a \circ u)_+ \leq K \int_0^T \frac{|u|^\delta}{r} \, dr.
\end{equation}

In particular
\begin{equation}
\frac{a \cdot u}{u} < \frac{1}{2} \frac{\|a\|}{u} + \frac{2}{\log \left( \frac{\|a\|}{u} \right)} \text{Vol} R^n,
\end{equation}

where \( R^n \) is a disk of radius \( R \) and the second term on the right-hand side of (4.3) is dropped if \( \frac{\|a\|}{u} \leq 1 \).

**Proof.** Put \( \varphi = u, \|a\| = 1 \). One easily verifies that

\[ u_t(x) < u_0(x) \varphi(x), \quad \forall x > 0, \]

consequently

\[ \|u\| \leq \int_0^R u(x) \varphi(x) \text{d}x. \]

By using the change of variables \( \rho = \text{d}x \) one has \( \text{d}x/\rho = \text{d}r/r \); hence

\[ \|u\| < \frac{1}{2} \int_0^R u(x) \varphi(x) \text{d}x, \]

\[ \frac{1}{2} \int_0^R u(x) \varphi(x) \text{d}x < \frac{1}{2} \int_0^R u(x) \varphi(x) \text{d}x, \]

\[ \frac{1}{2} \int_0^R u(x) \varphi(x) \text{d}x. \]

**Lemma 4.3.** Let \( \varphi : (0,\infty)^2 \times \overline{\Omega} \times \overline{\Omega} \) be a continuous map verifying

\begin{equation}
|\varphi(s,t,x) - \varphi(s,t,y)| \leq K(x - y)^{1/2}, \quad \forall (s,t,x) \in (0,\infty)^2, \overline{\Omega}(s,t,x) \in \overline{\Omega},
\end{equation}

where \( 0 < \delta < 1 \). Let \( \phi \in L^2((0,\infty); C_{0}(\overline{\Omega})) \) and define

\begin{equation}
\xi_2(t,x) = \int_0^t \phi(s,\varphi(s,t,x)) \text{d}s.
\end{equation}

Then \( \xi_2 \in C((0,\infty); C(\overline{\Omega})) \) is a vector

\begin{equation}
[\xi_2(t,x)]_0 < \frac{1}{2} \left[ \frac{\|\phi\|}{L^2((0,\infty); C(\overline{\Omega}))} + 2 \log \left( \frac{1}{\|\phi\|} \right) \right] L^2((0,\infty); C(\overline{\Omega}))
\end{equation}

where \( \|\phi\| \phi(s,\varphi(s,t,x)) \text{d}s. \]

**Proof.** With straightforward calculations one shows that

\begin{equation}
[\xi_2(t,x)]_0 < \int_0^t \text{d}s \int_0^\infty \sup_{x \in \Omega} \left| \phi(s,\varphi(s,t,x)) - \phi(s,\varphi(s,t,y)) \right| \text{d}x < \int_0^\infty \sup_{x \in \Omega} \left| \phi(s,\varphi(s,t,x)) - \phi(s,\varphi(s,t,y)) \right| \text{d}x.
\end{equation}
hence

$$\zeta(t) = \int_0^t [\phi(s) + U_{t_0,s}] \, ds$$

where \( \phi(s) \equiv \phi(s,*) \) and \( U_{t_0,s} \equiv U(s,t_0^*) \). By using (4.2) one gets

$$\zeta(t) \leq \int_0^t [\frac{1}{2} \rho(\phi(s)) + \frac{\rho}{R} \log(\frac{R}{r})] \rho(\phi(s)) \, ds,$$

i.e. equation (4.5).

We now prove the continuity statement. Assume for instance \( t_0 < t \). From definition (4.4) one gets

$$\zeta(t) - \zeta(t_0) < \int_0^t \frac{\rho}{R} \log(\frac{R}{r}) \rho(\phi(s)) \, ds + \int_0^t \frac{1}{2} \rho(\phi(s)) \, ds,$$

As for (4.6) we show that \( B_1 \) is bounded by the right hand side of (4.7) with the interval \((0,t)\) replaced by \((t_0,t)\); hence \( B_1 \) goes to zero when \( |t - t_0| \) goes to zero. We now prove that \( B_2 \to 0 \) when \( t \to t_0 \). Assumption (4.3) yields

$$F(t_0,t,s,r) < 2^r \| \phi(s) \| (X_t^R),$$

where \( F(t_0,t,s,r) \) is the integrand in \( B_2 \). The above function is integrable over \([0,T] \times [0,R]\) since for almost all \( s \in [0,T] \) one has

$$\int_0^T \| \phi(s) \| X_t^R \, ds < \frac{1}{2} [\rho(\phi(s)) + 2 \log(\frac{R}{r})] \rho(\phi(s)),$$

as one shows by arguing as in the proof of lemma 4.1. Moreover for every \( s \in [0,T] \) for which \( \phi(s,*) \in C(\bar{U}) \), and for every \( r \in [0,R] \), one has \( \lim_{t \to t_0} F(t_0,t,s,r) = 0 \). An application of Lebesgue's dominated convergence theorem proves that \( B_2 \to 0 \) if \( t \to t_0 \).
Lemma 4.3. Let $U$ verify the assumptions of the preceding lemma, let $\zeta_0 \in C_0(\bar{\Omega})$ and define $\zeta_1(t, x) = \zeta_0(0, t, x)$. Then $\zeta_1 \in C(0, T; C_0(\bar{\Omega}))$ moreover

\[ (4.9) \quad (\zeta_1(t))_t \leq \frac{1}{2} (\zeta_0)_t + \frac{3}{2} \log(\frac{\pi R}{r}) 1\zeta_0 R, \quad \forall t \in [0, T]. \]

Proof. Estimate (4.9) follows from lemma 4.1. The continuity statement follows as in the preceding lemma (with many simplifications).

Equations (2.6), (2.7), (2.8), definition of $\phi$ and the two preceding lemmas give the following result:

Lemma 4.4. Assume that hypothesis of theorem 1.4 hold and let $\zeta = \text{rot } v$, 
$\phi = \text{rot } f$, $\zeta_0 = \text{rot } v_0$. Then $\zeta \in C(\Omega; C_0(\bar{\Omega}))$, moreover for every $t \in \mathbb{R}$

\[ (4.10) \quad \zeta(t) \leq \phi \cdot \left[ (\zeta_0)_t + \phi \cdot \left( \frac{3}{2} \log(\frac{\pi R}{r}) 1\zeta_0 R \right) \right]_{L^1(0, t; C_0(\bar{\Omega}))} + 31\zeta_0 + 31\phi \quad \forall t \in \mathbb{R}. \]

where $\phi = \left( \zeta_0 \zeta \right)_{L^1(0, t; C(\bar{\Omega}))}$.

The following theorem is crucial for our proof.

Theorem 4.5. Let $\phi \in C_0(\bar{\Omega})$ and let $\phi$ be the solution of problem (1.2). Then $\phi \in C^2(\Omega)$, moreover

\[ (4.11) \quad \phi \leq \zeta_0 \phi_0, \quad \forall \phi \in C_0(\bar{\Omega}). \]

This result seems well known even if an exact reference is not available to us (see [2], chapter 4, problem 4.2), we are able to prove it for a uniformly elliptic second order equation $\phi = 0$ in $\Omega$, $\phi = 0$ on $\Gamma$, at least if $L$ has smooth coefficients and the boundary operator $B$ is regular (for instance Dirichlet or Neumann boundary value problem). This result doesn't depend on the dimension $n > 2$.

The main statement in theorem 1.4 follows immediately from $v = \text{Rot } \phi$ and from (4.10), (4.11); recall that $\phi = \zeta$. Moreover if $g$ and $\nabla \phi$ are continuous in $\bar{\Omega}$, it follows from (5.3) that $\nabla \phi$ is continuous, from $\nabla \phi = \nabla \phi + \nabla \phi$ that $\nabla \phi$ is continuous and from (1.1) or (5.2) that $\nabla \phi$ is continuous. \[ \square \]
Appendix I. We recall some well known facts about vector fields defined in non simply-connected domains. Let $\Omega$ be an $(N + 1)$-times connected bounded region, the boundary of which consists of simple closed curves $\Gamma_0, \Gamma_1, ..., \Gamma_N$, the curve $\Gamma_0$ containing the others. In that case the kernel of the linear system $\text{rot} \, v = 0$ in $\Omega$, $\text{div} \, v = 0$ in $\Omega$, $v \cdot n = 0$ on $\Gamma$ has finite dimension $N$. Let us fix a basis $u_1, ..., u_N$ and assume for convenience that $(u_1, u_k) = \delta_{ik}$, $i, k = 1, ..., N$. Any tangential flow (vector field verifying $\text{div} \, v = 0$ in $\Omega$, $v \cdot n = 0$ on $\Gamma$) is uniquely determined by the field $\text{rot} \, v$ in $\Omega$ and by the real numbers $(v, u_k)$, $k = 1, ..., N$. The quantity

$$\|v\| = \text{rot} \, v + \sum_{k=1}^{N} \| (v, u_k) \|$$

is a norm in $H(\Omega)$, equivalent to the norm

$$\|v\| = \text{rot} \, v + \|v\|_{L^2(\Omega)}.$$ 

Let now $f$ be an arbitrary vector field in $\Omega$. Solve the problem $-\text{rot} \, \phi = \text{rot} \, f$ in $\Omega$, $\phi = 0$ on $\Gamma$ and put $q = \text{rot} \, \phi$. Clearly $\text{rot} \, q = \text{rot} \, f$, $\text{div} \, q = 0$ and $q \cdot n = 0$ on $\Gamma$. If $g = q + \sum \lambda_k u_k$, where $\lambda_k \in (f, u_k)$, it follows that $g$ is a tangential flow, moreover $\text{rot} \, (f - g) = 0$ in $\Omega$, $(f - g, u_k) = 0$, $k = 1, ..., N$. Hence there exists a scalar field $F$ such that $f - g = VF$ in $\Omega$, i.e. the vector field $g$ is the tangential flow in the canonical decomposition

$$f = g + VF.$$ 

Note that $g$ depends only on $\text{rot} \, f$ and on the $N$ real numbers $(f, u_k)$. 

-19-
Appendix B. Let us decompose the external force \( f \) in equation (1.1) as indicated in (5.1) and let us consider the auxiliary problem

\[
\begin{align*}
\frac{\partial v}{\partial t} + (u \cdot \nabla)v &= g - Vv, & \text{in } \mathcal{G}, \\
\text{div } v &= 0 & \text{in } \mathcal{G}, \\
v \cdot n &= 0 & \text{on } \Gamma, \\
v|_{t=0} &= v_0 & \text{in } \mathcal{G}.
\end{align*}
\]

The solution of (1.1) consists of the same velocity field \( v \) as in (5.2) and on the pressure term \( Vv = Vv_1 + Vf \). Moreover, from (5.2) it follows that

\[
\begin{align*}
-\Delta v_1 &= \sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial^2 v_j}{\partial x_i^2} \frac{\partial v_j}{\partial x_i}, \\
\frac{\partial v_1}{\partial n} &= \sum_{i=1}^{2} \frac{\partial v_i}{\partial x_i} n_i v_j.
\end{align*}
\]

Assume that the regularity of \( Vv(t) \) is known. Then the elliptic boundary value problem (5.3) gives the regularity of \( Vv_1 \) and (5.2) gives the regularity of \( \partial v/\partial t \). In particular various regularity results for \( \partial v/\partial t \) (and for \( Vv \)) are trivially obtained by assuming different conditions on \( f \). Hence the regularity of \( Vv(t) \) is the basic one.

Note by the way that \( Vv \) is the only term depending fully on \( f \). The other terms considered above depend only on \( \text{rot } f \) and on \( (f, w_k), \ k = 1, \ldots, N \).
REFERENCES


We study the Euler equations (1.1) for the motion of a non-viscous incompressible fluid in a plane domain $\mathcal{D}$. Let $E$ be the Banach space defined in (1.4), let the initial data $\mathcal{U}_0$ belong to $E$, and let the external forces $f(t)$ belong to $L^1_d(\mathcal{D}; E)$. In theorem 1.1 we prove the strong...
continuity and the global boundedness of the (unique) solution \( v(t) \), and in

theorem 1.2 we prove the strong-continuous dependence of \( v \) on the data \( v_0 \)
and \( f \). In particular the vorticity \( \text{rot } v(t) \) is a continuous function in
\( \mathcal{S} \), for every \( t \in \mathbb{R} \), if and only if this property holds for one value of
\( t \). In theorem 1.3 we state some properties for the associated group of
nonlinear operators \( S(t) \). Finally, in theorem 1.4 we give a quite general
sufficient condition on the data in order to get classical solutions.