AD-A119 346 NAVAL POSTGRADUATE SCHOOL MONTEREY CA F/G 17/8
THE OPTIMAL SEARCH FOR A MOVING TARGET WHEN THE SEARCH PATH IS --ETC(U)
AUG 82 J N EAGLE
UNCLASSIFIED NPS55-82-020 NL

1 OF 1
AD A
119346

END
DATE
FILMED
10-82
DTIC

NPS55-82-020

# NAVAL POSTGRADUATE SCHOOL

## Monterey, California

THE OPTIMAL SEARCH FOR A
MOVING TARGET WHEN THE
SEARCH PATH IS CONSTRAINED

by

James N. Eagle

August 1982



Prepared for:
Naval Postgraduate School
Monterey, California 93940

NAVAL POSTGRADUATE SCHOOL
MONTEREY, CALIFORNIA

Rear Admiral J. J. Ekelund
Superintendent

David A. Schrady
Acting Provost

Reproduction of all or part of this report is authorized.

James N. Eagle
Associate Professor
Department of Operations Research

Reviewed by:

K. T. Marshall, Chairman
Department of Operations Research

Released by:

William M. Tolles
Dean of Research

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

| REPORT DOCUMENTATION PAGE | | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|---|---|---|
| 1. REPORT NUMBER<br>NPS55-82-020 | 2. GOVT ACCESSION NO.<br>AD-A119346 | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle)<br>THE OPTIMAL SEARCH FOR A MOVING TARGET WHEN THE SEARCH PATH IS CONSTRAINED | | 5. TYPE OF REPORT & PERIOD COVERED<br>Technical<br>6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s)<br>James N. Eagle | | 8. CONTRACT OR GRANT NUMBER(s) |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>Naval Postgraduate School<br>Monterey, CA 93940 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br>Naval Postgraduate School<br>Monterey, California 93940 | | 12. REPORT DATE<br>August 1982<br>13. NUMBER OF PAGES |
| 14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) | | 15. SECURITY CLASS. (of this report)<br>Unclassified<br>15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

search
moving target
partially observable Markov decision process (POMDP)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A search is conducted for a target moving in discrete time between a finite number of cells according to a known Markov process. The set of cells available for search in a given time period is a function of the cell searched in the previous time period. The problem is formulated and solved as a partially observable Markov decision process (POMDP). A finite time horizon POMDP solution technique is presented which is simpler than the standard linear programming methods.

DD FORM 1473, 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

# THE OPTIMAL SEARCH FOR A MOVING TARGET WHEN THE SEARCH PATH IS CONSTRAINED

James N. Eagle

Department of Operations Research
Naval Postgraduate School
Monterey, CA 93940



1 August 1982

## Abstract

A search is conducted for a target moving in discrete time between a finite number of cells according to a known Markov process. The set of cells available for search in a given time period is a function of the cell searched in the previous time period. The problem is formulated and solved as a partially observable Markov decision process (POMDP). A finite time horizon POMDP solution technique is presented which is simpler than the standard linear programming methods.

# THE OPTIMAL SEARCH FOR A MOVING TARGET WHEN THE SEARCH PATH IS CONSTRAINED

## 1. Problem Statement

A discrete time search is conducted for a target moving between a finite set of cells $C = \{1,...,N\}$ . At the beginning of each time period, one cell is searched. If cell $i$ was searched in the previous time period, the current search cell must be selected from the set $C_i \subseteq C$ . If the target is in the selected cell $k$ , it is detected with probability $q_k \in [0,1]$ . If the target is not in the cell searched, it can not be detected during the current time period. After an unsuccessful search, a target in cell $i$ moves to cell $j$ with probability $p_{ij}$ for the next time period. The transition matrix, $P = [p_{ij}]$ , is known to the searcher. The object of the search is to maximize the T-time period probability of detection.

## 2. Background

The moving target problem has received considerable attention, much of it recent. Washburn [1980] and Stone and Kadane [1981] list the important references. Pollock [1970] solved the problem addressed here for $N = 2$ and $C_1 = C_2 = C$. Washburn [1980] and Brown [1980] introduced a powerful technique giving exact solutions for the N-cell case, if all cells are available for search in each time period (i.e., $C_i = C$, $i = 1,\ldots,N$), search effort can be infinitesimally divided between the cells, and the detection function is exponential. Stewart [1980] adapted this technique to the search problem considered here by using branch-and-bound methods. As Stewart observed, however, the nonconvexity of the space of possible search plans allows this method to converge to suboptimal solutions.

Smallwood and Sondik [1973] and Monahan [1982] noted that the 2-cell problem solved by Pollock [1970] could be modelled as a partially observable Markov decision process (POMDP) and that an N-cell extension was possible. This paper makes that extension and, in addition, allows that the set of possible search cells in a given time period be a function of the search cell selected in the previous time period. This permits searches to be modelled where the searcher can travel only a limited distance between time periods. Thus, the search cell in a given time period must be within some specified neighborhood of the search cell in the previous time period.

Also reported on is a finite time horizon POMDP solution technique which is simpler than the standard linear programming techniques (e.g., Monahan [1982]), and which, initial computational experience indicates, is more quickly executed.

## 3. Mathematical Development

As is standard for many problems exploiting a Markov assumption, the solution technique used here is dynamic programming. This method requires that the process being modelled be defined in terms of a sufficient statistic (Bersekas [1976], p. 122). Following Sondik [1971], Smallwood and Sondik [1973] and Platzman [1980], we use the row vector $(\pi(k), i) \in R^{N+1}$, where $\pi_j(k) = P_r\{$the target is in cell $j$ at the beginning of time period $k$, given unsuccessful search in all previous time periods$\}$, and $i \in C$ is the cell searched in the previous time period. If the dependence on $k$ is clear from context, $\pi(k)$ will be written as $\pi$. The state space then becomes $\Pi \times C$ where

$$\Pi = \{\pi \in R^N | \pi 1 = 1 \ , \ \pi \geq 0\} \ ,$$

and $1$ and $0$ can be either vectors or scalars. The vector inequality $a \geq b$ means $a_i \geq b_i$, $\forall i$.

Following the dynamic programming convention of labelling "backwards in time", we define $V_n(\pi, i)$ to be the maximum obtainable probability of detection with $n$ time periods remaining and a current state vector $(\pi, i)$. Let $T_j(\pi) \in R^N$ be $\pi$ updated for unsuccessful search in cell $j$, using Bayes's rule. That is,

$$T_j(\pi) = (1-q_j\pi_j)^{-1}\pi P_j \ , \qquad (1)$$

where $P_j \in R^{N\times N}$ is $P$ with row $j$ multiplied by $(1-q_j)$. If $q_j\pi_j = 1$, then the search in the current time period detects the target with certainty, and (1) is not defined. We can now write $V_n(\pi,i)$ in terms of $V_{n-1}(\pi,j)$ as follows:

$$V_n(\pi,i) = \max_{j\in C_i}\left\{q_j\pi_j + (1-q_j\pi_j)V_{n-1}(T_j(\pi),j)\right\}, \tag{2}$$

with $V_0(\pi,i) = 0$ .

Equation (2) is the dynamic programming recursion that must be solved in each time period. It looks formidable, primarily because $\pi$ is real rather than discrete. We will show, however, that $V_n(\pi,i)$ may be expressed in a particularly simple form. Namely,

$$V_n(\pi,i) = \max_{a\in A(n,i)} \pi a , \tag{3}$$

where $A(n,i)$ is a finite collection of N-vectors. The dynamic programming problem then becomes one of constructing $A(n,i)$ from $A(n-1,j)$.

If $C_i = C$ , $\forall i$ , then the search problem as formulated becomes a standard POMDP and can be solved using the linear programming methods of Sondik [1971], Smallwood and Sondik [1973], or Monahan [1982]. Allowing that the action selected in the previous time period can constrain the actions available in the present time period requires an augmented state space ($\Pi \times C$ vice $\Pi$) and represents a generalization of the standard model. However, as the next theorem shows, the basic form of the POMDP solution remains the same. Specifically, $V_n(\pi,i)$ is piecewise linear and convex.

**Theorem:** For $n = 0,\ldots,T$, $V_n(\pi,i)$ is piecewise linear and convex in $\pi$. That is,

$$V_n(\pi,i) = \max_{a \in A(n,i)} \pi a , \tag{4}$$

where $A(n,i)$ is a finite set of $N$-vectors.

**Proof:** We proceed by induction. (4) holds trivially for $n = 0$ and $A(0,i) = 0$. For $n = 1$, it also holds, since from (2),

$$V_1(\pi,i) = \max_{j \in C_i} q_j \pi_j$$

$$= \max_{a \in A(1,i)} \pi a ,$$

where $A(1,i) = \{q_j \xi_j | j \in C_i\}$ and $\xi_j \in R^N$ is a column vector with a 1 is the jth place and 0's elsewhere.

Now assume (4) holds in time period $(n-1)$. From (2),

$$V_n(\pi,i) = \max_{j \in C_i} \left\{ q_j \pi_j + (1-q_j\pi_j) \max_{a_j \in A(n-1,j)} (T_j(\pi) a_j) \right\}$$

$$= \max_{j \in C_i} \left\{ q_j \pi_j + (1-q_j\pi_j) \max_{a_j \in A(n-1,j)} (1-q_j\pi_j)^{-1} \pi P_j a_j \right\}$$

$$= \max_{\substack{j \in C_i \\ a_j \in A(n-1,j)}} \{q_j \pi_j + \pi P_j a_j\}$$

$$= \max_{a \in A(n,i)} \pi a \tag{5}$$

where $A(n,i) = \{\hat{a} \in R^N | \hat{a} = \xi_j q_j + P_j a_j;\ j \in C_i \text{ and } a_j \in A(n-1,j)\}$. (6)

So $V_n(\pi,i)$ is of the proper form and the proof is complete.

For any finite $n$ and $i \in C$, $A(n,i)$ is a finite set. However, using (6) to generate $A(n,i)$ and assuming (for illustration purposes only) that the number of elements in $C_i$ is $M$ for all $i \in C$, the number of vectors in $A(n,i)$ is $M$ times the number in $A(n-1,i)$. Since there are $M$ vectors in $A(1,i)$, there are apparently $M^n$ vectors in $A(n,i)$. This equals the number of possible search paths for the n-time period problem that begin with cell $i$ and suggests that total enumeration of search paths might be as effective as this procedure.

Fortunately, this is not necessarily the case. Following Smallwood and Sondik [1973], we note that some of the vectors in $A(n,i)$ can be removed and the maximization of (5) left unchanged. We say that $\hat{a} \in A(n,i)$ is <u>dominated</u> if for every $\pi \in \Pi$,

$$\max_{\substack{a \in A(n,i) \\ a \neq \hat{a}}} \pi a = \max_{a \in A(n,i)} \pi a . \tag{7}$$

Dominated vectors can be removed from $A(n,i)$ and need not be used in the construction of $A(n+1,j)$.

Sondik [1971] first provided a linear programming technique to identify dominated vectors for the POMDP. Following a slight modification in Monahan [1982], we solve the following linear program to check $\hat{a} \in A(n,i)$ for dominance:

$$\min_{\pi,x} \quad x - \pi\hat{a} \tag{8}$$

$$\text{s.t.} \quad x \geq \pi a \ , \ \forall a \in A(n,i) \quad \text{but} \quad a \neq \hat{a}$$

$$\pi \in \Pi$$

Whenever the minimal value of $x - \pi\hat{a}$ is non-negative, $\hat{a}$ is dominated and can be removed from $A(n)$ . The linear programming solution technique need not necessarily continue to optimality. As soon as the objective function becomes negative, $\hat{a}$ is determined to be not dominated. (This method is similar to the branch-and-bound technique of Stewart [1980] in that both are enumerative procedures to systematically eliminate search paths which can not be optimal.)

Once the reduced vector sets $A(n,i)$ have been generated for all $i \in C$ and $n = (0,\ldots,T)$ , the maximum probability of detection and the optimal T-time period search plan can be determined for any initial target distribution $\pi$ . Assume that before the search begins, the searcher is in cell $i$ , and thus the initial search cell must be in $C_i$ . Then the maximum obtainable T-time period probability of detection is

$$\max_{a \in A(T,i)} \pi a \tag{9}$$

(If the searcher's starting cell, $i$ , can be any element in $C$ , (9) is maximized over all $i \in C$ to find the maximum probability of detection.) The cell searched in time period $T$ is that $j \in C_i$ used in (6) to construct the argmax of (9). If cell $j$ is searched in time period $T$ and the target is not detected, then (9) is resolved for time period $T-1$ with $T_j(\pi)$ replacing $\pi$ and $A(n-1,j)$ replacing $A(n,i)$.

Alternatively (and perhaps more simply), one can note that each $a \in A(T,i)$ has associated with it, not just the cell searched in time period $T$, but a series of $T$ cells, built up by the sequential application of (6). When a particular $a \in A(T,i)$ maximizes (9), the sequence of cells associated with the vector $a$ is the optimal search path.

## 4. The Dual Definition of Dominance and a Geometric Interpretation

The linear programming dual of (8) is

$$\begin{aligned} \max \quad & v \\ \text{s.t.} \quad & \sum_{i=1}^{k} \lambda_i a_i - v \geq \hat{a} \\ & \lambda 1 = 1 \\ & \lambda \geq 0 \end{aligned} \tag{10}$$

where $i = (1,\ldots,k)$ indexes all vectors in $A(n,i)$ except $\hat{a}$. The duality theorem of linear programming (Dantzig [1963], p. 125 or Luenburger [1973], p. 72) states that the primal has a finite optimal solution iff the dual has a finite optimal solution; and when feasible optimal solutions exist, the two optimal objective functions are equal.

We know that $\hat{a} \in A(n,i)$ is dominated when the minimal value of the objective function of (8) is non-negative. In this case, the duality theorem requires that (10) is feasible and that the optimal value of $v$ is also non-negative. Thus, from the constraints of (10), there exists a linear combination of elements in $A(n,i)$ except $\hat{a}$ which (in a vector sense) is greater than or equal to $\hat{a}$. And the strength of the duality theorem allows the implication to hold in the other direction as well. That is, if such a linear combination of vectors in $A(n,i)$ exists, then $\hat{a}$ is dominated.

The dual characterization of dominance allows a simple geometric interpretation. If $B$ is the convex hull of all vectors in $A(n,i)$ except $\hat{a}$, then $\hat{a}$ is dominated iff $\exists\, b \in B$ such that $b \geq \hat{a}$.

## 5. Alternative Solution Techniques

The POMDP solution procedure described above requires extensive calcualtions. To reduce $A(n,i)$ to its minimal size, each $a \in A(n,i)$ must be checked for dominance by solving a potentially large linear program. The question naturally arises as to whether a simpler or more quickly executed procedure could be found, even if such a procedure did not necessarily reduce $A(n,i)$ to its minimal size.

What is possibly the simplest such reduction scheme is to compare each $a_j$ and $a_k$ $(a_j \neq a_k)$ in $A(n,i)$, and to discard $a_j$ if $a_j \leq a_k$ or $a_k$ if $a_k \leq a_j$. The vectors remaining can then be further reduced using linear programming methods, or the larger-than-minimal $A(n,i)$ can be used directly to construct $A(n+1,j)$ by (6). Both of these procedures were coded for the IBM 3033 at the Naval Postgraduate School, and, for the search problems examined, the latter method, using no linear programming at all, generated optimal solutions more quickly and required less computer storage. Both methods appeared preferable to using only linear programming methods to check for dominance.

## 6. An Example Problem

A simple 5-cell search problem is described by the following parameters.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & .75 & .25 & 0 & 0 \\ 0 & .25 & .5 & .25 & 0 \\ 0 & 0 & .25 & .5 & .25 \\ 0 & 0 & 0 & .25 & .75 \end{bmatrix}$$

$C = \{1,2,3,4,5\}$

$C_1 = 2$

$C_2 = \{2,3\}$

$C_3 = \{2,3,4\}$

$C_4 = \{3,4,5\}$

$C_5 = \{4,5\}$

$q_i = q, \forall i$

searcher's starting cell: 1

$T = 7$

$\pi(7) = (0,0,0,0,1)$

The target starts in cell 5 and the searcher in cell 1. Since $C_1 = 2$, the initial cell searched is 2. After the initial search, cell 1 is inaccessible to both the searcher and the target.

The optimal search path and the maximum obtainable probability of detection ($P_d$) are given in Table 1 for $q$ of .2, .4, .6, .8, and 1. Using the simplest reduction method (i.e., no linear programming), the number of vectors in A(7,1) increased from 3 for $q = 1$ to 187 for $q = .2$. The CPU time required to obtain the optimal solution increased from 24 seconds for $q = 1$ to 536 seconds for $q = .2$.

| q | optimal search path | $P_d$ | # vectors in A(7,1) | CPU time (sec) |
|---|---|---|---|---|
| .2 | 2 3 4 5 5 5 5 | .357 | 187 | 536 |
| .4 | 2 3 4 5 5 4 5 | .594 | 89 | 280 |
| .6 | 2 3 4 5 4 5 4 | .757 | 49 | 179 |
| .8 | 2 3 4 5 4 5 4 | .867 | 26 | 169 |
| 1.0 | 2 3 4 5 4 3 2 | .934 | 3 | 24 |

Table 1. Example Problem Results

## REFERENCES


BERTSEKAS, D. 1976. Dynamic Programming and Stochastic Control. Academic Press, New York.

BROWN, S. S. 1980. Optimal Search for a Moving Target in Discrete Time and Space. Operations Research, Vol. 28, 1275-1289.

DANTZIG, G. B. 1963. Linear Programming and Extensions. Princeton University Press, Princeton, NJ.

LUENBERGER, D. G. 1973. Introduction to Linear and Nonlinear Programming. Addison-Wesley Publishing Co., Inc., Reading, MA.

MONAHAN, G. E. 1982. A Survey of Partially Observable Markov Decision Processes: Theory, Models, and Algorithms. 1982. Management Science, Vol. 28, No. 1, 1-16.

PLATZMAN, L. K. 1980. Optimal Infinite-Horizon Undiscounted Control of Finite Probabilistic Systems. Society for Industrial and Applied Mathematics Journal, Vol. 18, No. 4, 362-380.

POLLOCK, S. M. 1970. A Simple Model of Search for a Moving Target. Operations Research, Vol. 18, 883-903.

SMALLWOOD, R. D. and SONDIK, E. J. 1973. The Optimal Control of Partially Observable Markov Processes over a Finite Horizon. Operations Research, Vol. 21, 1071-1088.

SONDIK, E. J. 1971. The Optimal Control of Partially-Observable Markov Processes. Ph.D. thesis, Stanford University, Stanford, CA.

STEWART, T. J. 1980. Experience with a Branch-and-Bound Algorithm for Constrained Searcher Motion in Search Theory and Applications, K. B. Haley and L. D. Stone, editors. Plenum Press, New York.

STONE, L. D. and KADANE, J. B. 1981. Optimal Whereabouts Search for a Moving Target. Operations Research, Vol. 29, No. 6, 1154-1166.

WASHBURN, A. R. 1980. On a Search for a Moving Target. Naval Research Logistics Quarterly, Vol. 27, 315-321.

## DISTRIBUTION LIST

| | NO. OF COPIES |
|---|---|
| Library, Code 0142<br>Naval Postgraduate School<br>Monterey, CA 93940 | 4 |
| Dean of Research<br>Code 012A<br>Naval Postgraduate School<br>Monterey, CA 93940 | 1 |
| Library, Code 55<br>Naval Postgraduate School<br>Monterey, CA 93940 | 1 |
| Assoc. Prof. J. N. Eagle<br>Code 55Er<br>Naval Postgraduate School<br>Monterey, CA 93940 | 70 |

ATE

LMED

-8