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OPTIMAL VALUE BOUNDS AND POSYNOMIAL GEOMETRIC PROGRAMS. (U)

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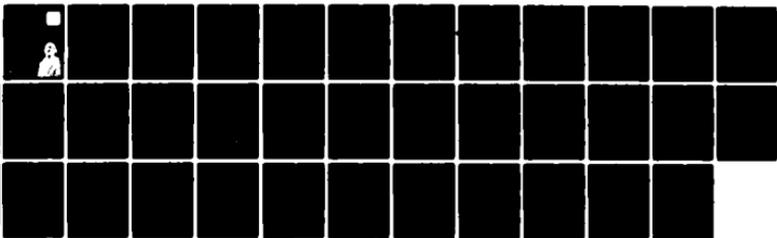
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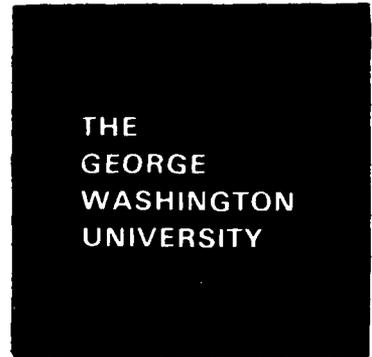
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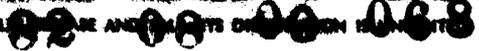
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OPTIMAL VALUE BOUNDS FOR POSYNOMIAL
GEOMETRIC PROGRAMS

by

Jerzy Kyparisis

(12)

Serial T-464
30 April 1982

The George Washington University
School of Engineering and Applied Science
Institute for Management Science and Engineering

Research Sponsored by
Program in Logistics
Contract N00014-75-C-0729
Office of Naval Research
and
Grant ECS-8114555
National Science Foundation



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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER T-464	2. GOVT ACCESSION NO. AD-A117 994	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) OPTIMAL VALUE BOUNDS AND POSYNOMIAL GEOMETRIC PROGRAMS		5. TYPE OF REPORT & PERIOD COVERED SCIENTIFIC
		6. PERFORMING ORG. REPORT NUMBER T-464
7. AUTHOR(s) JERZY KYPARISIS		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0729 Grant ECS-8114555 (NSF)
9. PERFORMING ORGANIZATION NAME AND ADDRESS George Washington University Institute for Management Science & Engineering Washington, DC 20052		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Code 411-OR Arlington, VA 22217		12. REPORT DATE 30 April 1982
		13. NUMBER OF PAGES 34
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) NONE
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) APPROVED FOR PUBLIC SALE AND RELEASE; DISTRIBUTION UNLIMITED.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) GEOMETRIC PROGRAMMING PARAMETRIC ANALYSIS SENSITIVITY ANALYSIS OPTIMAL VALUE BOUNDS		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The idea of bounds on the optimal value of a geometric programming problem has been present since the inception of geometric programming (Duffin, Peterson, and Zener), and has proved to be fruitful in applications of GP. The bounds are the immediate consequences of GP duality theory. Recently, this idea has been extended to parametric bounds on the optimal value function f^* . Woolsey and Dembo derive a lower bound on f^* using the known fact that the dual objective function at a fixed dual-feasible point. (continued)		

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20. Abstract (Cont'd)

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THE GEORGE WASHINGTON UNIVERSITY
School of Engineering and Applied Science
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Abstract
of
Serial T-464
30 April 1982

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Jerzy Kyparisis

The idea of bounds on the optimal value of a geometric programming problem has been present since the inception of geometric programming (Duffin, Peterson, and Zener), and has proved to be fruitful in applications of GP. These bounds are the immediate consequences of GP duality theory. Recently, this idea has been extended to parametric bounds on the optimal value function f^* . Woolsey and Dembo derive a lower bound on f^* using the known fact that the dual objective function at a fixed dual-feasible point underestimates f^* for all values of certain coefficients (parameters). Fiacco proposed a general approach for calculating upper and lower bounds on f^* (particularly simple whenever f^* is convex or concave), which utilizes sensitivity information as well as Wolfe's duality theory. This paper is based on similar ideas, except that GP duality theory is used instead of Wolfe's duality and the special structure of GP primal and dual problems is exploited. Several classes of perturbed GP problems are shown to possess convex or concave f^* , or at least "tight" overestimating and underestimating problems with convex or concave optimal values.

Research Sponsored by
Office of Naval Research
and
National Science Foundation

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1. INTRODUCTION

The idea of bounds on the optimal value of a geometric programming problem has been present since the inception of geometric programming (Duffin, Peterson, and Zener [5]) and has proved to be fruitful in applications of GP. These bounds are the immediate consequences of the duality theory of GP. Recently, this idea has been extended to parametric bounds on the optimal value function f^* . Woolsey (in [2]) derives a lower bound on f^* using the known fact that the dual objective function at a fixed dual-feasible point underestimates f^* for all values of coefficients (parameters) (see, e.g., Dembo [4]). He also shows how to apply this result in a practical problem. Fiacco [7] has proposed a general approach for calculating upper and lower bounds on f^* (particularly simple whenever f^* is convex or concave), which utilizes sensitivity information as well as Wolfe's duality theory. This paper is

based on similar ideas, except that GP duality theory is used instead of Wolfe's duality, and the special structure of GP primal and dual problems is exploited. Several classes of perturbed GP problems are shown to possess convex or concave f^* or at least "tight" overestimating and underestimating problems with convex f^* . The calculation of bounds is illustrated for different classes of perturbations and on a simple example problem. In this paper we are mainly concerned with posynomial GP problems. Possible extensions to general signomial GP problems using, for example, the idea of condensed programs (see, e.g., [1]) remain to be developed. Also, the topic of bounds on the primal and dual optimal solution points is not discussed here (see, e.g., [2,4]). Bounds on f^* based on a general idea in [7] were calculated for a convex equivalent of a GP model of a stream water pollution abatement system by Ghaemi [11] and by Fiacco and Kyparisis [9,10]. Bounds on f^* were also obtained by Fiacco and Ghaemi [8], using results coinciding with some results of Section 4, for a convex equivalent of a GP model of a power system energy model.

2. GENERAL PRIMAL AND DUAL BOUNDS ON THE OPTIMAL VALUE FUNCTION OF A GP PROBLEM

A posynomial primal geometric programming problem is a nonlinear programming problem of the form

$$\begin{aligned} & \min_{t \in E^m} g_0(t, c) \\ & \text{subject to } g_k(t, c) \leq 1, \quad k=1, \dots, p, \quad P_0(c) \\ & \quad t_j > 0, \quad j=1, \dots, m \end{aligned}$$

where

$$g_k(t, c) = \sum_{i \in J_k} c_i \prod_{j=1}^m t_j^{a_{ij}}, \quad k=0,1,\dots,p,$$

$$J_k = \{m_k, m_k+1, \dots, n_k\}, \quad k=0,1,\dots,p,$$

$$m_0 = 1, m_1 = n_0+1, m_2 = n_1+1, \dots, m_p = n_{p-1}+1, n_p = n,$$

$$c = (c_1, \dots, c_n).$$

The exponents a_{ij} are arbitrary real numbers and the coefficients c_i are positive. The functions g_k are posynomials.

By using the transformation $t_j = e^{x_j}$ we derive the following equivalent program

$$\begin{aligned} \min_{x \in E^m} f_0(x, c) & & P(c) \\ \text{subject to } f_k(x, c) &\leq 1, \quad k=1, \dots, p \end{aligned}$$

where

$$f_k(x, c) = \sum_{i \in J_k} c_i e^{\sum_{j=1}^m a_{ij} x_j}, \quad k=0,1,\dots,p$$

and the sets J_k are the same as in $P_0(c)$. Program $P(c)$ is called a transformed primal program and it is well known ([5]) that it is a convex programming problem (for any fixed c).

A dual geometric programming problem corresponding to $P_0(c)$ (and $P(c)$) has the form

$$\begin{aligned} \max_{\delta \in E^n} v(\delta, c) &= \prod_{i=1}^n \left(\frac{c_i}{\delta_i} \right)^{\delta_i} \prod_{k=1}^p \lambda_k(\delta)^{\lambda_k(\delta)} \\ \text{subject to } \lambda_k(\delta) &= \sum_{i \in J_k} \delta_i, \quad k=1, \dots, p, \quad D(c) \\ \sum_{i \in J_0} \delta_i &= 1, \\ \sum_{i=1}^n a_{ij} \delta_i &= 0, \quad j=1, \dots, m, \\ \delta_i &\geq 0, \quad i=1, \dots, n, \end{aligned}$$

where the sets J_k are defined in $P_0(c)$. Under the assumption

$$\exists \delta_0 \in E^n \ni \sum_{i=1}^n a_{ij} \delta_{0i} = 0, \quad j=1, \dots, m, \delta_{0i} > 0, \quad i=1, \dots, n, \quad (A)$$

Zangwill [13] showed that a program equivalent to $D(c)$, with $\log v(\delta, c)$ substituted for $v(\delta, c)$, can be obtained using Wolfe's duality theory.

In this paper we are concerned with bounds on the optimal value function $f^*(c)$ of $P(c)$. Denote by $v^*(c)$ the optimal value function of $D(c)$. From the duality theory of geometric programming [5] it follows that for any fixed c_0 , if \tilde{x} is any feasible point of $P(c_0)$ and $\tilde{\delta}$ is any feasible point of $D(c_0)$, then

$$f_0(\tilde{x}, c_0) \geq f^*(c_0) \geq v^*(c_0) \geq v(\tilde{\delta}, c_0). \quad (1)$$

It is also known [5] that if the feasible set of $P(c_0)$ is nonempty and condition (A) is satisfied, then there exists a global solution x_0 of $P(c_0)$ and

$$f_0(x_0, c_0) = f^*(c_0) = v^*(c_0). \quad (2)$$

Dembo [4] notes that since the feasible set of $D(c)$ is the same for all

c , any dual-feasible point $\tilde{\delta}$ gives us the lower bound on $f^*(c)$ and $v^*(c)$ based on (1);

$$f^*(c) \geq v^*(c) \geq v(\tilde{\delta}, c), \quad \forall c. \quad (3)$$

Woolsey (in Beightler and Phillips [2]) utilizes (3) to obtain a "tight" lower bound on $f^*(c)$ as follows. Suppose that δ_0 is the optimal solution of $D(c_0)$ (it is necessarily a global solution, since as shown in [5] $\log v(\delta, c)$ is concave in δ on the dual feasible set for any fixed c). Suppose also that (A) is satisfied so that (2) holds. Then from (3) and (2) we obtain

$$f^*(c) \geq v(\delta_0, c) = v(\delta_0, c_0) \prod_{i=1}^n \left(\frac{c_i}{c_{0i}} \right)^{\delta_{0i}} = f^*(c_0) \prod_{i=1}^n \left(\frac{c_i}{c_{0i}} \right)^{\delta_{0i}}, \quad \forall c. \quad (4)$$

The above bounds as well as the optimal value functions $f^*(c)$ and $v^*(c)$ are in general neither convex nor concave. In the next sections we will identify classes of problems for which either $f^*(c)$ and $v^*(c)$ or the bounds on $f^*(c)$ can be shown to be convex or concave. This is important since it simplifies the computation of the bounds on $f^*(c)$ (both upper and lower) and considerably enhances their applicability. These results will be illustrated using the following example problem [5, p. 88]:

$$\min_{t \in E^3} g_0(t, c) = c_1 t_1^{-1} t_2^{-\frac{1}{2}} t_3^{-1} + c_2 t_1 t_3 + c_3 t_1 t_2 t_3$$

$$\text{subject to } g_1(t, c) = c_4 t_1^{-2} t_2^{-2} + c_5 t_2^{\frac{1}{2}} t_3^{-1} \leq 1 \quad EP_0(c)$$

$$t_i > 0, \quad i=1,2,3$$

$$(c_i > 0, \quad i=1,2,\dots,5).$$

The transformed problem is of the form

$$\begin{aligned} \min_{x \in E^3} f_0(x, c) &= c_1 e^{-x_1 - \frac{1}{2}x_2 - x_3} + c_2 e^{x_1 + x_3} + c_3 e^{x_1 + x_2 + x_3} \\ \text{subject to } f_1(x, c) &= c_4 e^{-2x_1 - 2x_2} + c_5 e^{\frac{1}{2}x_2 - x_3} \leq 1. \end{aligned} \quad \text{EP}(c)$$

The dual problem corresponding to EP(c) is

$$\begin{aligned} \max_{\delta \in E^5} v(\delta, c) &= \left(\frac{c_1}{\delta_1}\right)^{\delta_1} \left(\frac{c_2}{\delta_2}\right)^{\delta_2} \left(\frac{c_3}{\delta_3}\right)^{\delta_3} \left(\frac{c_4}{\delta_4}\right)^{\delta_4} \left(\frac{c_5}{\delta_5}\right)^{\delta_5} (\delta_4 + \delta_5)^{(\delta_4 + \delta_5)} \\ \text{subject to } \delta_1 + \delta_2 + \delta_3 &= 1, \\ -\delta_1 + \delta_2 + \delta_3 - 2\delta_4 &= 0, \\ -\frac{1}{2}\delta_1 + \delta_3 - 2\delta_4 + \frac{1}{2}\delta_5 &= 0, \\ -\delta_1 + \delta_2 + \delta_3 - \delta_5 &= 0, \\ \delta_i &\geq 0, \quad i=1, 2, \dots, 5. \end{aligned} \quad \text{ED}(c)$$

One can check that if $c_1^{-1} c_2^{\frac{1}{2}} c_3^{\frac{1}{2}} c_4^2 c_5^2 > 2/27$, then the optimal solution of ED(c) is $\delta^*(c) = (1 - 2r(c), r(c), r(c), 2r(c) - \frac{1}{2}, 4r(c) - 1)^T$, where $r(c) = \left[2 + (4/27)c_1^{-1} c_2^{\frac{1}{2}} c_3^{\frac{1}{2}} c_4^{-1} c_5^{-2}\right]^{-1} \in (\frac{1}{4}, \frac{1}{2})$ and the optimal value function of EP(c) is $f^*(c) = v^*(c) = v(\delta^*(c), c)$. For $c_0 = (40, 20, 20, 1/3, 4/3)$, $r(c_0) = 2/5$, the optimal dual solution is $\delta^*(c_0) = (1/5, 2/5, 2/5, 3/10, 3/5)$, and one can calculate the optimal primal solution $x^*(c_0) = (0, 0, \log 2)$, and $f^*(c_0) = v^*(c_0) = v(\delta^*(c_0), c_0) = 100.0$. The lower bound (4) is thus easily applicable.

3. GP PROBLEMS WITH CONVEX OPTIMAL VALUE FUNCTION

Suppose that coefficients c_i are functions of the parameter vector $\epsilon = (\epsilon_1, \dots, \epsilon_r) \in E^r$ of the form $c_i(\epsilon) = 1/h_i(\epsilon)$, $i=1, \dots, n$, where $h_i(\epsilon)$ are positive concave functions of ϵ on a convex set E_0 (if $h_i(\epsilon) = h_{i0}$, then $c_i(\epsilon) = 1/h_{i0}$ remains unperturbed). A particularly simple class is obtained when $h_i(\epsilon) = \epsilon_i > 0$, $i=1, \dots, n$. The primal program $P(c(\epsilon))$ can be written as

$$\begin{aligned} \min_{x \in E^m} \quad & \tilde{f}_0(x, \epsilon) \\ \text{subject to} \quad & \tilde{f}_k(x, \epsilon) \leq 1, \quad k=1, \dots, p \end{aligned} \quad \tilde{P}(\epsilon)$$

where

$$\tilde{f}_k(x, \epsilon) = \sum_{i \in J_k} \frac{1}{h_i(\epsilon)} e^{\sum_{j=1}^m a_{ij} x_j}, \quad k=0, 1, \dots, p.$$

If we denote by $\tilde{f}^*(\epsilon)$ the optimal value function of $\tilde{P}(\epsilon)$, then the following result holds.

Proposition 1. $\tilde{f}^*(\epsilon)$ is convex on E_0 .

Proof. First note that $\tilde{f}_k(x, \epsilon)$ can be written as

$$\tilde{f}_k(x, \epsilon) = \sum_{i \in J_k} \exp \left(-\log h_i(\epsilon) + \sum_{j=1}^m a_{ij} x_j \right), \quad k=0, 1, \dots, p. \quad (5)$$

Now, introducing new variables s_1, \dots, s_n , we can write $\tilde{P}(\epsilon)$ in an equivalent form as

$$\begin{aligned} & \min_{(x,s) \in E^m \times E^n} \sum_{i \in J_0} e^{s_i} \\ & \text{subject to } \sum_{i \in J_k} e^{s_i} \leq 1, \quad k=1, \dots, p \quad \tilde{P}_1(\epsilon) \\ & \sum_{j=1}^m a_{ij} x_j - s_i \leq \log h_i(\epsilon), \quad i=1, \dots, n. \end{aligned}$$

Programs $\tilde{P}(\epsilon)$ and $\tilde{P}_1(\epsilon)$ have the same optimal value function $\tilde{f}^*(\epsilon)$. Since $h_i(\epsilon)$ is concave and positive for $\epsilon \in E_0$, $\log h_i(\epsilon)$ is concave for $\epsilon \in E_0$ and thus $\tilde{P}_1(\epsilon)$ is a jointly convex program in (x, s, ϵ) with the convex optimal function $\tilde{f}^*(\epsilon)$ for $\epsilon \in E_0$ [12]. Q.E.D.

Proposition 1 enables us to compute the upper and lower bounds on $\tilde{f}^*(\epsilon)$ using the following approach proposed by Fiacco [7]. Assume that we know the solutions x_0 of $\tilde{P}(\epsilon_0)$ and x_1 of $\tilde{P}(\epsilon_1)$, $\epsilon_0, \epsilon_1 \in E_0$. Assume also that the conditions of the sensitivity theorem (Fiacco [6]) are satisfied at $\epsilon = \epsilon_0$ and $\epsilon = \epsilon_1$. Then the gradients of $\tilde{f}^*(\epsilon)$ at these points exist, providing us with the lower bounds on $\tilde{f}^*(\epsilon)$, $\epsilon \in [\epsilon_0, \epsilon_1]$ of the form $L_i(\alpha) = \tilde{f}^*(\epsilon_i) + \nabla_{\epsilon} \tilde{f}^*(\epsilon_i)(\epsilon(\alpha) - \epsilon_i)$, $i=0,1$, where $\epsilon(\alpha) = (1 - \alpha)\epsilon_0 + \alpha\epsilon_1$, $\alpha \in [0,1]$. The upper bound on $\tilde{f}^*(\epsilon(\alpha))$ is given by $U(\alpha) = (1 - \alpha)\tilde{f}^*(\epsilon_0) + \alpha\tilde{f}^*(\epsilon_1)$. A better upper bound can be obtained, noting that $\tilde{f}_k(x, \epsilon)$ are jointly convex in (x, ϵ) for $\epsilon \in E_0$. This implies that $x(\alpha) = (1 - \alpha)x_0 + \alpha x_1$ is feasible for $\tilde{P}(\epsilon(\alpha))$, $\alpha \in [0,1]$, so that $\tilde{f}^*(\epsilon(\alpha)) \leq \tilde{f}_0(x(\alpha), \epsilon(\alpha)) \leq U(\alpha)$, $\forall \alpha \in [0,1]$, the last inequality following from convexity of $\tilde{f}_0(x(\alpha), \epsilon(\alpha))$. It is also possible to derive sharper lower bounds on $\tilde{f}^*(\epsilon)$ using the dual geometric programming problem $D(c(\epsilon))$. We consider this problem now.

The dual program $D(c(\epsilon))$ has the form

$$\begin{aligned} \max_{\delta \in E^n} \tilde{v}(\delta, \epsilon) &= \prod_{i=1}^n (h_i(\epsilon) \delta_i)^{-\delta_i} \prod_{k=1}^p \lambda_k(\delta)^{\lambda_k(\delta)} \\ \text{subject to } \lambda_k(\delta) &= \sum_{i \in J_k} \delta_i, \quad k=1, \dots, p, \\ \sum_{i \in J_0} \delta_i &= 1, \quad \tilde{D}(\epsilon) \\ \sum_{i=1}^n a_{ij} \delta_i &= 0, \quad j=1, \dots, m, \\ \delta_i &\geq 0, \quad i=1, \dots, n. \end{aligned}$$

Denote by $\tilde{v}^*(\epsilon)$ the optimal value function of $\tilde{D}(\epsilon)$. Under assumption (A), $\tilde{v}^*(\epsilon) = \tilde{f}^*(\epsilon)$ (provided that $\tilde{f}^*(\epsilon) < +\infty$) and this, together with Proposition 1, implies that $\tilde{v}^*(\epsilon)$ is convex on E_0 . We will prove it directly, together with convexity of a dual lower bound $\tilde{v}(\delta, \epsilon)$, using the following result.

Proposition 2. (i) If R is an arbitrary set in E^m , $F(x, \epsilon)$ is concave in ϵ on a convex set $S \subset E^r$ for any fixed $x \in R$, then $F_1^*(\epsilon) = \inf_{x \in R} F(x, \epsilon)$ is concave on S .

(ii) If R and S are as in (i), $F(x, \epsilon)$ is convex in ϵ on S for any fixed $x \in R$, then $F_2^*(\epsilon) = \sup_{x \in R} F(x, \epsilon)$ is convex on S .

Proof. (i) Let $\epsilon_1, \epsilon_2 \in S$, $\lambda \in [0, 1]$. Then

$$\begin{aligned} F_1^*(\lambda \epsilon_1 + (1 - \lambda) \epsilon_2) &= \inf_{x \in R} F(x, \lambda \epsilon_1 + (1 - \lambda) \epsilon_2) \\ &\geq \inf_{x \in R} \lambda F(x, \epsilon_1) + (1 - \lambda) F(x, \epsilon_2) \\ &\geq \lambda \inf_{x \in R} F(x, \epsilon_1) + (1 - \lambda) \inf_{x \in R} F(x, \epsilon_2) \\ &= \lambda F_1^*(\epsilon_1) + (1 - \lambda) F_1^*(\epsilon_2). \end{aligned}$$

(ii) $F_2^*(\epsilon) = \sup_{x \in R} F(x, \epsilon) = -\inf_{x \in R} (-F(x, \epsilon))$. Since $-F(x, \epsilon)$ satisfies assumptions of (i), $-F_2^*(\epsilon)$ is concave on S by (i), so $F_2^*(\epsilon)$ is convex on S . Q.E.D.

Remark. Proposition 2 appears to be a variation of a well known result of convex analysis: if all $\phi \in \Phi$ are convex, Φ is an arbitrary set, then $\tilde{\phi}(x) \triangleq \sup_{\phi \in \Phi} \phi(x)$ is also convex (on a convex set).

Proposition 3. (i) $\tilde{v}(\delta, \epsilon)$ is convex in ϵ on E_0 for any fixed dual-feasible δ .

(ii) $\tilde{v}^*(\epsilon)$ is convex on E_0 .

Proof. (i) $\tilde{v}(\delta, \epsilon)$ can be written as

$$\tilde{v}(\delta, \epsilon) = \exp \left[- \sum_{i=1}^n \delta_i \log(h_i(\epsilon) \delta_i) \right] \prod_{k=1}^p \lambda_k(\delta)^{\lambda_k(\delta)}. \quad (6)$$

(We define $x \log x = 0$ for $x = 0$.) Since $h_i(\epsilon)$ is concave and positive on E_0 , $-\delta_i \log(h_i(\epsilon) \delta_i)$ is convex on E_0 for any dual-feasible δ and thus $\tilde{v}(\delta, \epsilon)$ is convex in ϵ on E_0 for any such δ .

(ii) Denote the dual feasible set by R_D . Then $\tilde{v}^*(\epsilon) = \sup_{\delta \in R_D} \tilde{v}(\delta, \epsilon)$ is convex on E_0 by (i) and Proposition 2(ii). Q.E.D.

Suppose now that condition (A) is satisfied so that $\tilde{f}^*(\epsilon) = \tilde{v}^*(\epsilon)$. Let δ_0 and δ_1 be the solutions of $\tilde{D}(\epsilon_0)$ and $\tilde{D}(\epsilon_1)$, respectively. If the assumptions of the sensitivity theorem hold for either problem $\tilde{P}(\epsilon)$ or $\tilde{D}(\epsilon)$ at $\epsilon = \epsilon_0$ and $\epsilon = \epsilon_1$, then the gradients of both $\tilde{f}^*(\epsilon)$ and $\tilde{v}^*(\epsilon)$ exist at $\epsilon = \epsilon_0, \epsilon_1$ and we have

$$\nabla_{\epsilon} \tilde{f}^*(\epsilon_i) = \nabla_{\epsilon} \tilde{v}^*(\epsilon_i) = \nabla_{\epsilon} \tilde{v}(\delta_i, \epsilon_i), \quad i=0,1. \quad (7)$$

Since also $\tilde{f}^*(\epsilon_1) = \tilde{v}^*(\epsilon_1) = \tilde{v}(\delta_1, \epsilon_1)$, $i=0,1$, convexity of $\tilde{v}(\delta_1, \epsilon)$, $i=0,1$ implies that for $i=0,1$, $\alpha \in [0,1]$,

$$L_1(\alpha) = \tilde{v}(\delta_1, \epsilon_1) + \nabla_{\epsilon} \tilde{v}(\delta_1, \epsilon_1)(\epsilon(\alpha) - \epsilon_1) \leq \tilde{v}(\delta_1, \epsilon(\alpha)), \quad (8)$$

proving that $\tilde{v}(\delta_1, \epsilon(\alpha))$, $i=0,1$, are uniformly better lower bounds on $\tilde{f}^*(\epsilon(\alpha))$ than $L_1(\alpha)$. In fact, to derive the above bounds we only need the directional derivatives of $\tilde{f}^*(\epsilon) = \tilde{v}^*(\epsilon)$ in directions $\epsilon_1 - \epsilon_0$ and $\epsilon_0 - \epsilon_1$. If the feasible set of $\tilde{D}(\epsilon)$ is compact and unique global solutions δ_0 of $\tilde{D}(\epsilon_0)$ and δ_1 of $\tilde{D}(\epsilon_1)$ exist, then from Danskin's theorem [3] we obtain the directional derivatives of $\tilde{f}^*(\epsilon)$ at $\epsilon = \epsilon_0, \epsilon_1$ in the direction z as

$$D_z \tilde{f}^*(\epsilon_1) = D_z \tilde{v}^*(\epsilon_1) = \nabla_{\epsilon} \tilde{v}(\delta_1, \epsilon_1) z, \quad i=0,1. \quad (9)$$

Consider now a more general class of perturbations of the form

$$c_1(\epsilon) = \left(\prod_{\ell=1}^{\ell_1} h_{1\ell}(\epsilon) \right)^{-1} = \prod_{\ell=1}^{\ell_1} h_{1\ell}^{-\beta_{1\ell}}(\epsilon)$$

where $h_{1\ell}(\epsilon)$ are positive concave functions of ϵ on a convex set $E_0 \subset E^r$, ℓ_1 are positive integers, and $\beta_{1\ell} \geq 0$ for $\ell=1, \dots, \ell_1$, $i=1, \dots, n$. (The previous case is obtained by setting $\ell_1 = 1$, $\beta_{11} = 1$, $i=1, \dots, n$.) All results obtained in this section extend immediately to this class (note that $\log c_1(\epsilon) = -\sum_{\ell=1}^{\ell_1} \beta_{1\ell} \log h_{1\ell}(\epsilon)$ is convex on E_0). This allows us in particular to obtain bounds on convex $\tilde{f}^*(\epsilon)$ for perturbations of the type $c_1(\epsilon) = \prod_{\ell=1}^r \epsilon_{\ell}^{-\beta_{1\ell}}$, $\beta_{1\ell} \geq 0$, $\ell=1, \dots, r$, $i=1, \dots, n$.

Dembo [4] considers a slightly more general problem than $P(c)$ of the form

$$\begin{aligned} & \min_{x \in E^m} f_0(x, c) \\ & \text{subject to } f_k(x, c) \leq r_k, \quad k=1, \dots, p \end{aligned} \quad P(c, r)$$

where r_k , $k=1, \dots, p$ are positive numbers and functions f_k , $k=0, 1, \dots, p$ are defined in $P(c)$. If we define $c_i(\epsilon, r) = c_i(\epsilon)/r_k$, $i \in J_k$, $k=1, \dots, p$, where $c_i(\epsilon)$ has the general form given above, then it is easy to show that all results of this section extend to this problem, too. This is also clear from the formulation of $P(c, r)$ and its corresponding dual $D(c, r)$ in terms of (c, r) ,

$$\max_{\delta \in R_D} v(\delta, c, r) = \prod_{i=1}^n \left(\frac{c_i}{\delta_i} \right)^{\delta_i} \prod_{k=1}^p \left(\frac{\lambda_k(\delta)}{r_k} \right)^{\lambda_k(\delta)} \quad D(c, r)$$

where R_D is the feasible set of $D(c)$ and $\lambda_k(\delta)$ are defined in $D(c)$.

As an illustration of the preceding results, consider our example problem, $EP(c)$, with the following perturbations: $c(\alpha) = (c_{01}/(1+\alpha), c_{02}, c_{03}, c_{04}, c_{05}/(1+\alpha))$, $\alpha \in [0, 1]$. The optimal value function $\tilde{f}^*(\alpha)$ of $EP(c(\alpha))$ is convex on $[0, 1]$ by Proposition 1. Since $c(0) = c_0$, $\tilde{f}^*(0) = \tilde{v}^*(0) = 100.0$, $\delta_0 = \delta^*(c(0)) = (1/5, 2/5, 2/5, 3/10, 3/5)$ and $x_0 = x^*(c(0)) = (0, 0, \log 2)$, where $\tilde{v}^*(\alpha)$ denotes the optimal value function of $ED(c(\alpha))$. Since $r(c(\alpha)) = 2/(5+\alpha)$, for $\alpha = 1$, $c_1 = c(1) = (20, 20, 20, 1/3, 2/3)$, $r(c(1)) = 1/3$, and $\delta_1 = \delta^*(c(1)) = (1/3, 1/3, 1/3, 1/6, 1/3)$. Calculations give us $x_1 = x^*(c(1)) = (0, 0, 0)$ and $\tilde{f}^*(1) = \tilde{v}^*(1) = v(\delta_1, c_1) = 60.0$. Define $\tilde{v}(\delta, \alpha) = v(\delta, c(\alpha))$. From inequality (4) we obtain convex dual lower bounds

$$\tilde{f}^*(\alpha) \geq \tilde{v}(\delta_0, \alpha) = \tilde{f}^*(0) \left(\frac{1}{1+\alpha} \right)^{\delta_{01} + \delta_{05}} = \frac{100.0}{(1+\alpha)^{.8}}$$

and

$$\tilde{f}^*(\alpha) \geq \tilde{v}(\delta_1, \alpha) = \tilde{f}^*(1) \left(\frac{2}{1+\alpha} \right)^{\delta_{11} + \delta_{15}} = \frac{2^{2/3} \cdot 60.0}{(1+\alpha)^{2/3}}, \quad \alpha \in [0, 1].$$

The linear lower bounds on $\tilde{f}^*(\alpha)$ are computed using the formulas for

$\tilde{v}(\delta_i, \alpha)$, $i=0,1$, as

$$L_0(\alpha) = \tilde{f}^*(0) + \nabla_{\alpha} \tilde{v}(\delta_0, 0)\alpha = 100.0 - 80.0\alpha$$

and

$$L_1(\alpha) = \tilde{f}^*(1) + \nabla_{\alpha} \tilde{v}(\delta_1, 1)(\alpha-1) = 60.0 - 20.0(\alpha-1) = 80.0 - 20.0\alpha .$$

The linear upper bound is computed as

$$U(\alpha) = (1-\alpha)\tilde{f}^*(0) + \alpha\tilde{f}^*(1) = (1-\alpha)100.0 + \alpha 60.0 = 100.0 - 40.0\alpha .$$

A sharper convex upper bound is given by

$$\tilde{f}_0(\alpha) = f_0(x(\alpha), c(\alpha)) ,$$

where $x(\alpha) = (1-\alpha)x_0 + \alpha x_1$, $\alpha \in [0,1]$ and calculations show that

$\tilde{f}_0(\alpha) = 40.0(1/2)^{1-\alpha}/(1+\alpha) + 40.0 \cdot 2^{1-\alpha}$. All the above bounds are depicted in Figure 1.

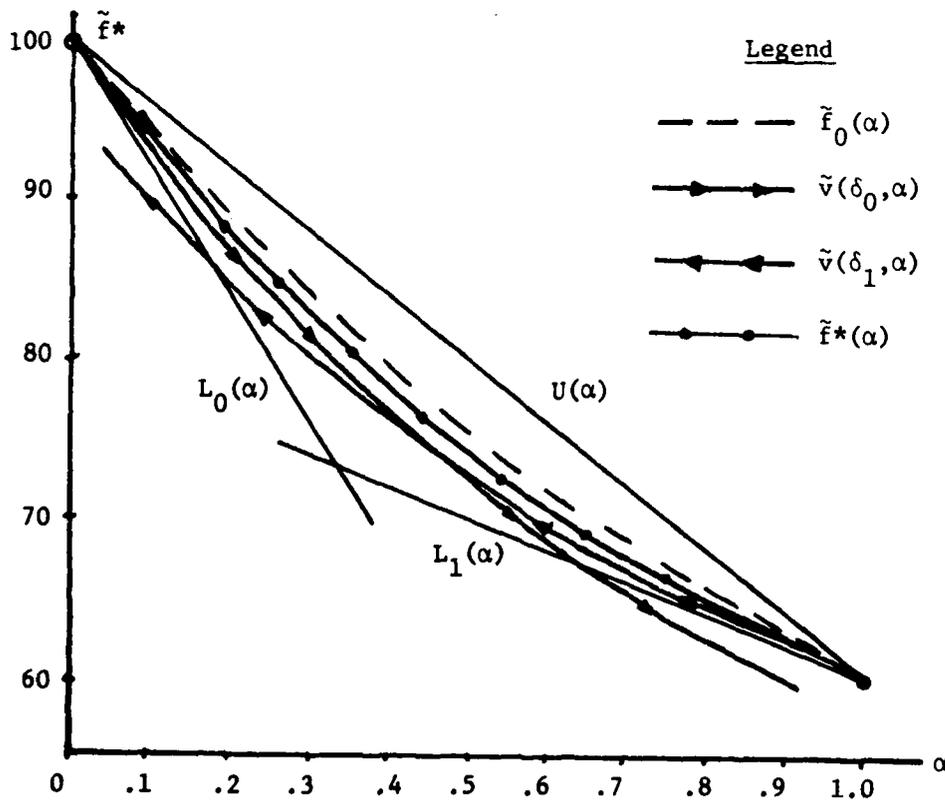


Figure 1.--Bounds on the convex $\tilde{f}^*(\alpha)$.

Consider now another class of perturbations which yields a convex optimal value function $f^*(c)$. Assume that only the coefficients of the objective function $f_0(x, \bar{c})$, $\bar{c} = (c_1, \dots, c_{n_0})$, are perturbed and define $c_i(\epsilon) = c_{0i} \gamma(\epsilon)^{\beta_i}$, $\beta_i \geq 1$, $i=1, \dots, n_0$, where $\gamma(\epsilon)$ is a positive convex function of ϵ on a convex set $E_0 \subset E^r$. Note that all the coefficients in $f_0(x, \bar{c})$ are perturbed now.

Proposition 4.

- (i) For any fixed primal-feasible x , $f_0(x, \bar{c}(\epsilon))$ is convex in $\epsilon \in E_0$.
- (ii) For any fixed dual-feasible δ and fixed $\hat{c} = (c_{n_0+1}, \dots, c_n)$, $\tilde{v}(\epsilon) = v(\delta, \bar{c}(\epsilon), \hat{c})$ is convex in $\epsilon \in E_0$.
- (iii) The optimal value function of $D(\bar{c}(\epsilon))$, $\tilde{v}^*(\epsilon)$, is convex in $\epsilon \in E_0$ and if condition (A) also holds, the optimal value function of $P(\bar{c}(\epsilon))$, $\tilde{f}^*(\epsilon)$, is convex in $\epsilon \in E_0$.

Proof.

(i) Follows immediately from the convexity of $\gamma(\epsilon)^\beta$ for any $\beta \geq 1$ and the form of $f_0(x, \bar{c})$.

(ii) Since

$$\tilde{v}(\epsilon) = \prod_{i=1}^{n_0} \left(\frac{c_{0i} \gamma(\epsilon)^{\beta_i}}{\delta_i} \right)^{\delta_i} \prod_{i=n_0+1}^n \left(\frac{c_i}{\delta_i} \right)^{\delta_i} \prod_{k=1}^p \lambda_k(\delta)^{\lambda_k(\delta)},$$

we can write

$$\tilde{v}(\epsilon) = A \prod_{i=1}^{n_0} \gamma(\epsilon)^{\beta_i \delta_i} = A \gamma(\epsilon)^{\sum_{i=1}^{n_0} \beta_i \delta_i}, \quad (10)$$

where

$$A = \prod_{i=1}^{n_0} \left(\frac{c_{0i}}{\delta_i} \right)^{\delta_i} \prod_{i=n_0+1}^n \left(\frac{c_i}{\delta_i} \right)^{\delta_i} \prod_{k=1}^p \lambda_k(\delta)^{\lambda_k(\delta)} > 0.$$

Since δ is dual-feasible, $\sum_{i=1}^{n_0} \delta_i = 1$, so that $\sum_{i=1}^{n_0} \beta_i \delta_i \geq \sum_{i=1}^{n_0} \delta_i = 1$ and the result follows from the convexity of $\gamma(\epsilon)^\beta$, $\beta \geq 1$.

(iii) The first assertion of this part follows immediately from part (ii), Proposition 2(ii), and the fact that the dual feasible set R_D is fixed. The last assertion is a consequence of the equality $\tilde{f}^*(\epsilon) = \tilde{v}^*(\epsilon)$ under condition (A). Q.E.D.

This result enables us to calculate bounds on $\tilde{f}^*(\epsilon)$ in a similar way as in the first part of this section. Using Danskin's theorem, one can find the directional derivatives of $\tilde{f}^*(\epsilon)$ using the gradients of $\tilde{v}(\epsilon)$ and obtain linear lower bounds on $\tilde{f}^*(\epsilon)$. The linear upper bounds $U(\epsilon)$ can also be obtained as before. Sharper convex bounds $\tilde{v}(\epsilon)$ are available, too. However, since $f_0(x, \bar{c})$ is in general not jointly convex in (x, \bar{c}) , the upper bound $f_0(x(\alpha), \bar{c}(\epsilon(\alpha)))$, $x(\alpha) = (1-\alpha)x_0 + \alpha x_1$, $\epsilon(\alpha) = (1-\alpha)\epsilon_0 + \alpha\epsilon_1$, $\alpha \in [0, 1]$ will not necessarily be convex and better than the linear upper bound. The convex upper bounds $f_0(x_i, \bar{c}(\epsilon(\alpha)))$, $i=0, 1$, will be better than $U(\epsilon(\alpha))$ only for α close to 0 ($i=0$), or 1 ($i=1$) in general.

This approach can be extended to include perturbations in the coefficients c_i of the constraint functions, also of the form $c_i(\epsilon) = c_{0i} \gamma(\epsilon)^{\beta_i}$, but with $\beta_i \geq 0$, $i = n_0+1, \dots, n$ (note that $\beta_i \geq 1$ for $i=1, \dots, n_0$). It can be easily shown that Proposition 4 remains true in this case. However, since the primal feasible set R_0 now depends on ϵ , at least one of the solutions x_0 of $P(\bar{c}(\epsilon_0))$ or x_1 of $P(\bar{c}(\epsilon_1))$ will no longer be feasible in general for all values of $\epsilon \in [\epsilon_0, \epsilon_1]$, reducing the availability of upper bounds $f_0(x_i, \bar{c}(\epsilon))$, $i=0, 1$. Also, in general $x(\alpha)$ will not be feasible, so that $f_0(x(\alpha), \bar{c}(\epsilon(\alpha)))$ cannot

be used as an upper bound on $\tilde{f}^*(\epsilon)$. The linear bounds on $\tilde{f}^*(\epsilon)$ and the dual lower bounds will nevertheless remain valid.

Even more generally, Proposition 4 will be valid whenever $\sum_{i=1}^n \beta_i \delta_i^*(c(\epsilon)) \geq 1$ for the considered perturbations. We utilize this fact in the following example where it can be shown that $\sum_{i=1}^{n_0} \beta_i \delta_i(c(\alpha)) = 1$, $\alpha \in [0,1]$.

We now consider the example EP(c), with perturbations $c(\alpha) = (c_{01}(1 - \alpha/2), c_{02}, c_{03}(1 - \alpha/2)^2, c_{04}(1 - \alpha/2), c_{05})$, $\alpha \in [0,1]$. The optimal value function $\tilde{f}^*(\epsilon)$ of EP(c(α)) is convex on [0,1] by Proposition 4(iii) (we set $\gamma(\alpha) = 1 - \alpha/2$, $\beta_1 = 1$, $\beta_2 = 0$, $\beta_3 = 2$, $\beta_4 = 1$, $\beta_5 = 0$). Since $c(0) = c_0$, $\tilde{f}^*(0) = \tilde{v}^*(0) = 100.0$, $\delta_0 = \delta^*(c(0)) = (1/5, 2/5, 2/5, 3/10, 3/5)$, and $x_0 = x^*(c(0)) = (0, 0, \log 2)$. Since $r(c(\alpha)) = (2-\alpha)/(5-2\alpha)$, for $\alpha = 1$, $c_1 = c(1) = (20, 20, 5, 1/6, 4/3)$, $r(c(1)) = 1/3$, and $\delta_1 = \delta^*(c(1)) = (1/3, 1/3, 1/3, 1/6, 1/3)$. We calculate $x_1 = x^*(c(1)) = (\log(1.05), \log 4, \log(.67))$ and $\tilde{f}^*(1) = \tilde{v}^*(1) = v(\delta_1, c_1) = 42.43$. Inequality (4) yields the following convex dual lower bounds for $\alpha \in [0,1]$.

$$\tilde{f}^*(\alpha) \geq \tilde{v}(\delta_0, \alpha) = \tilde{f}^*(0) \left(1 - \frac{\alpha}{2}\right)^{\delta_{01} + 2\delta_{03} + \delta_{04}} = 100.0 \left(1 - \frac{\alpha}{2}\right)^{1.3}$$

and

$$\tilde{f}^*(\alpha) \geq \tilde{v}(\delta_1, \alpha) = \tilde{f}^*(1) (2 - \alpha)^{\delta_{11} + 2\delta_{13} + \delta_{14}} = 42.43 (2 - \alpha)^{7/6}.$$

The linear lower bounds on $\tilde{f}^*(\alpha)$ are calculated with the help of $\tilde{v}(\delta_i, \alpha)$, $i=0,1$, as

$$L_0(\alpha) = \tilde{f}^*(0) + \nabla_{\alpha} \tilde{v}(\delta_0, 0) \alpha = 100.0 - 65.0\alpha$$

and

$$L_1(\alpha) = \tilde{f}^*(1) + \nabla_{\alpha} \tilde{v}(\delta_1, 1)(\alpha - 1) \doteq 42.43 - 22.05(\alpha - 1) \\ \doteq 64.48 - 22.05\alpha .$$

The linear upper bound is

$$U(\alpha) = (1-\alpha)\tilde{f}^*(0) + \alpha\tilde{f}^*(1) \doteq (1-\alpha)100.0 + \alpha 42.43 \\ \doteq 100.0 - 57.57\alpha .$$

Since $c_1(\alpha)$ are nonincreasing in α , the primal feasible set is increasing in α . Thus the optimal solution point x_0 of $P(c(0))$ is primal-feasible for all $\alpha \in [0, 1]$. Since the constraint $f_1(x, c)$ is binding at the optimal solution point x_1 of $P(c(1))$ for $c = c_1$, x_1 is not primal-feasible for any value of $\alpha \in (0, 1)$. Therefore, we can only use the convex upper bound

$$f_0(x_0, \bar{c}(\alpha)) = 20.0 \left[1 - \frac{\alpha}{2} \right] + 40.0 + 40.0 \left[1 - \frac{\alpha}{2} \right]^2 \\ = 10.0\alpha^2 - 50.0\alpha + 100.0 .$$

We depict the above bounds in Figure 2 ($\tilde{f}^*(\alpha)$ is not depicted, since the bounds $U(\alpha)$ and $\tilde{v}(\delta_1, \alpha)$, $i=0, 1$ are very tight).

4. GP PROBLEMS WITH CONCAVE OPTIMAL VALUE FUNCTION

Consider again the problem $P(c)$ and assume that we perturb the coefficients c_1 in the objective function $f_0(x, c)$ only. Denote the vector (c_1, \dots, c_{n_0}) by \bar{c} , the optimal value function of $P(\bar{c})$, by $f^*(\bar{c})$ and the fixed feasible set of $P(\bar{c})$ by R_0 .

Proposition 5. $f^*(\bar{c})$ is concave.

Proof. Follows immediately from Proposition 2(i) since $f_0(x, \bar{c})$ is linear in \bar{c} for any fixed x .

Q.E.D.

This result enables us to derive the upper and lower bounds on $f^*(\bar{c})$, given the solutions x_0 of $P(\bar{c}_0)$ and x_1 of $P(\bar{c}_1)$ and the gradients of $f^*(\bar{c})$ at $\bar{c} = \bar{c}_0$ and $\bar{c} = \bar{c}_1$, in the same manner as it was done in Section 3. However, the upper bound $f_0(x(\alpha), \bar{c}(\alpha))$, $x(\alpha) = (1 - \alpha)x_0 + \alpha x_1$, $\bar{c}(\alpha) = (1 - \alpha)\bar{c}_0 + \alpha\bar{c}_1$, $\alpha \in [0, 1]$, will be neither concave nor convex in general and will not necessarily

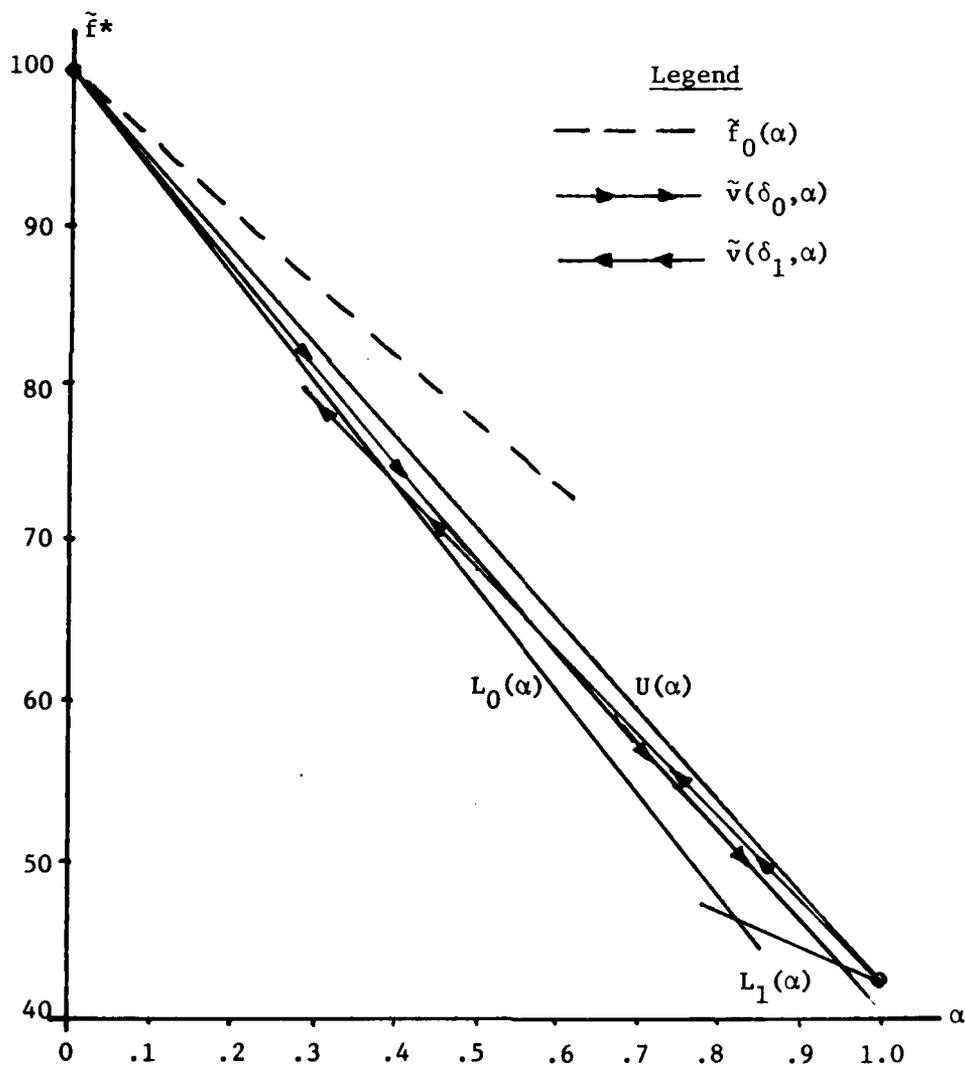


Figure 2.--Bounds on the convex $\tilde{f}^*(\alpha)$.

underestimate $U_i(\alpha)$. As mentioned before, directional derivatives of $f^*(\bar{c})$ at $\bar{c} = \bar{c}_0, \bar{c}_1$ are sufficient for our purposes. Thus, if R_0 is compact and x_0 and x_1 are the unique global solutions of $P(\bar{c}_0)$ and $P(\bar{c}_1)$, respectively, then by Danskin's theorem [3] we obtain that

$$D_z f^*(\bar{c}_i) = \nabla_{\bar{c}} f_0(x_i, \bar{c}_i) z = f_0(x_i, z), \quad i=0,1 \quad (11)$$

which means that the upper bounds on $f^*(\bar{c}(\alpha))$ are of the form $U_i(\alpha) = f_0(x_i, \bar{c}(\alpha))$, $i=0,1$, $\alpha \in [0,1]$. If the condition (A) holds, then the optimal value function of $D(\bar{c})$, $v^*(\bar{c})$, is equal to $f^*(\bar{c})$ and is thus concave by Proposition 5. Now we prove concavity of the lower bound $v(\delta, \bar{c}, \hat{c})$ in \bar{c} for any δ and $\hat{c} = (c_{n_0+1}, \dots, c_n)$ fixed.

Proposition 6. For any fixed dual-feasible δ and fixed \hat{c} , $v(\delta, \bar{c}, \hat{c})$ is concave in \bar{c} .

Proof. Denote $v_0(\bar{c}) = v(\delta, \bar{c}, \hat{c}) = A(\delta, \hat{c}) \prod_{i=1}^{n_0} c_i^{\delta_i}$, where

$$A(\delta, \hat{c}) = \prod_{i=n_0+1}^n c_i^{\delta_i} \prod_{i=1}^n \delta_i^{-\delta_i} \prod_{k=1}^p \lambda_k(\delta)^{\lambda_k(\delta)} > 0$$

is fixed. Calculations give us $\nabla_{\bar{c}} v_0(\bar{c}) = [\delta_1/c_1, \dots, \delta_{n_0}/c_{n_0}] v_0(\bar{c})$

and

$$\nabla_{\bar{c}}^2 v_0(\bar{c}) = v_0(\bar{c}) \left\{ \begin{array}{l} \left[\begin{array}{c} \delta_1/c_1 \\ \vdots \\ \delta_{n_0}/c_{n_0} \end{array} \right] \left[\delta_1/c_1, \dots, \delta_{n_0}/c_{n_0} \right] - \left[\begin{array}{cc} \delta_1/c_1^2 & 0 \\ & \ddots \\ 0 & \delta_{n_0}/c_{n_0}^2 \end{array} \right] \end{array} \right\} \quad (12)$$

We want to show that $y^T \nabla_{\bar{c}}^2 v_0(\bar{c}) y \leq 0$, $\forall y \in E^{n_0}$. Since $v_0(\bar{c}) > 0$, it is enough to show that

$$\begin{aligned}
& y^T \begin{bmatrix} \delta_1/c_1 \\ \vdots \\ \delta_{n_0}/c_{n_0} \end{bmatrix} [\delta_1/c_1, \dots, \delta_{n_0}/c_{n_0}] y - y^T \begin{bmatrix} \delta_1/c_1^2 & & 0 \\ & \ddots & \\ 0 & & \delta_{n_0}/c_{n_0}^2 \end{bmatrix} y \\
&= \left(\sum_{i=1}^{n_0} \frac{\delta_i y_i}{c_i} \right)^2 - \sum_{i=1}^{n_0} \frac{\delta_i y_i^2}{c_i^2} \leq 0, \quad \forall y \in E^n \quad (13)
\end{aligned}$$

But this follows immediately from the Cauchy-Schwarz inequality [5]:

$$\left(\sum_{k=1}^n a_k b_k \right)^2 \leq \left(\sum_{k=1}^n a_k^2 \right) \left(\sum_{k=1}^n b_k^2 \right), \quad \forall a, b \in E^n \quad (14)$$

if we set $n = n_0$, $a_k = \delta_k^{1/2}$, $b_k = \delta_k^{1/2} (y_k/c_k)$, $k=1, \dots, n$ and recall that $\sum_{k=1}^{n_0} \delta_k = \sum_{k \in J_0} \delta_k = 1$. Q.E.D.

If δ_0 uniquely solves $D(\bar{c}_0)$, δ_1 uniquely solves $D(\bar{c}_1)$ and if the dual feasible set R_D is compact, then under the assumption (A), $f^*(\bar{c}) = v^*(\bar{c})$, and thus by Danskin's theorem for any z ,

$$D_z f^*(\bar{c}_i) = D_z v^*(\bar{c}_i) = \nabla_{\bar{c}} v(\delta_i, \bar{c}_i, \hat{c}) z, \quad i=0,1 \quad (15)$$

This shows that the lower bounds $v(\delta_i, \bar{c}(\alpha), \hat{c})$, $i=0,1$, are at least initially better than the linear lower bound $L(\alpha) = (1 - \alpha)f^*(\bar{c}_0) + \alpha f^*(\bar{c}_1)$ but in general they do not have to be uniformly better for all $\alpha \in [0,1]$.

There are two ways to extend the results of this section. One way is to assume that $c_i(\varepsilon) = h_i(\varepsilon)$, $i=1, \dots, n_0$, where $h_i(\varepsilon)$ are concave positive functions of the parameter vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_r)$. The case treated before is obtained when we set $h_i(\varepsilon) = \varepsilon_i$, $i=1, \dots, n_0$. One can show that Propositions 5 and 6 continue to be true for this more

general case. Upper and lower linear bounds can be obtained as before, given the solutions and directional derivatives of the optimal value function $\tilde{f}^*(\epsilon)$ at $\epsilon = \epsilon_0, \epsilon_1$. The objective function

$$\tilde{f}_0(x, \epsilon) = \sum_{i=1}^{n_0} h_i(\epsilon) e^{\sum_{j=1}^m a_{ij} x_j}$$

gives us now a sharper upper bound than the linear one, since from concavity of $\tilde{f}_0(x, \epsilon)$ in ϵ for any fixed x , fixedness of R , and equality $D_z \tilde{f}^*(\epsilon_i) = \nabla_{\epsilon} \tilde{f}_0(x_i, \epsilon_i) z$, $i=0,1$, it follows that

$$\tilde{f}^*(\epsilon(\alpha)) \leq \tilde{f}_0(x_i, \epsilon(\alpha)) \leq U_i(\alpha), \quad i=0,1 \quad (16)$$

where $\epsilon(\alpha) = (1 - \alpha)\epsilon_0 + \alpha\epsilon_1$, $\alpha \in [0,1]$ and

$$U_i(\alpha) = \tilde{f}^*(\epsilon_i) + \nabla_{\epsilon} \tilde{f}_0(x_i, \epsilon_i)(\epsilon(\alpha) - \epsilon_i), \quad i=0,1.$$

The dual lower bound, concave by the extension of Proposition 6, applies as before. Another generalization is obtained by noting that Proposition 5 is valid even if some coefficients c_i are negative and directions of inequalities defining the feasible set changed. In other words, the optimal value function $f^*(\bar{c})$ is concave also in case of the general signomial geometric program. In this case we can still utilize linear bounds on $f^*(\bar{c})$ based on sensitivity information [7], although the dual bounds may no longer be valid. In order to illustrate the above bounds, we use the same example EP(c) as before.

Define the perturbations as follows: $c(\alpha) = (c_{01}(1 - \alpha/2), c_{02}(1 - 3\alpha/4), c_{03}, c_{04}, c_{05})$, $\alpha \in [0,1]$. Since $c(\alpha)$ is linear in α , this is the basic case and the results of this section apply, so that the optimal value function $\tilde{f}^*(\alpha)$ of EP(c(α)) is concave on $[0,1]$ by

Proposition 4. Since $c(0) = c_0$, $\tilde{f}^*(0) = 100.0$, $\delta_0 = (1/5, 2/5, 2/5, 3/10, 3/5)$, and $x_0 = (0, 0, \log 2)$, as determined before. Since $r(c(\alpha)) = (2 + (1 - \alpha/2)/\sqrt{4-3\alpha})^{-1}$, for $\alpha = 1$, $c_1 = c(1) = (20, 5, 20, 1/3, 4/3)$, $r(c(1)) = 2/5$, and $\delta_1 = \delta^*(c(1)) = \delta_0$. Calculations give $x_1 = x^*(c(1)) = (\log 4, -\log 4, 0)$ and $\tilde{f}^*(1) = \tilde{v}^*(1) = v(\delta_1, c_1) = 50.0$. Define $\tilde{v}(\delta, \alpha) = v(\delta, c(\alpha))$ as before. Inequality (4) provides concave dual lower bounds for $\alpha \in [0, 1]$,

$$\tilde{f}^*(\alpha) \geq \tilde{v}(\delta_0, \alpha) = \tilde{f}^*(0) \left(1 - \frac{\alpha}{2}\right)^{\delta_{01}} \left(1 - \frac{3\alpha}{4}\right)^{\delta_{02}} = 100.0 \left(1 - \frac{\alpha}{2}\right)^{.2} \left(1 - \frac{3\alpha}{4}\right)^{.4}$$

and

$$\tilde{f}^*(\alpha) \geq \tilde{v}(\delta_1, \alpha) = \tilde{f}^*(1) \left(\frac{1 - \alpha}{\frac{1}{2}}\right)^{\delta_{11}} \left(\frac{1 - \frac{3\alpha}{4}}{\frac{1}{4}}\right)^{\delta_{12}} = 100.0 \left(1 - \frac{\alpha}{2}\right)^{.2} \left(1 - \frac{3\alpha}{4}\right)^{.4}$$

(note that $\tilde{v}(\delta_0, \alpha) = \tilde{v}(\delta_1, \alpha)$ since $\delta_0 = \delta_1$). The linear lower bound is given by

$$L(\alpha) = (1-\alpha)\tilde{f}^*(0) + \alpha\tilde{f}^*(1) = (1-\alpha)100.0 + \alpha 50.0 = 50.0 - 50.0\alpha .$$

The linear upper bounds are given by

$$U_0(\alpha) = f_0(x_0, \bar{c}(\alpha)) = 100.0 - 40.0\alpha$$

and

$$U_1(\alpha) = f_0(x_1, \bar{c}(\alpha)) = 120.0 - 70.0\alpha .$$

We can also calculate the upper bound,

$$f_0(x(\alpha), \bar{c}(\alpha)) = 20.0 \left[1 - \frac{\alpha}{2}\right] + 20.0 \left[1 - \frac{3\alpha}{4}\right] 2^{1+\alpha} + 20.0 \cdot 2^{1-\alpha} .$$

We depict the calculated bounds in Figure 3. (The upper bound

$f_0(x(\alpha), \bar{c}(\alpha))$ is not depicted since it is almost equal to $\tilde{f}^*(\alpha)$.)

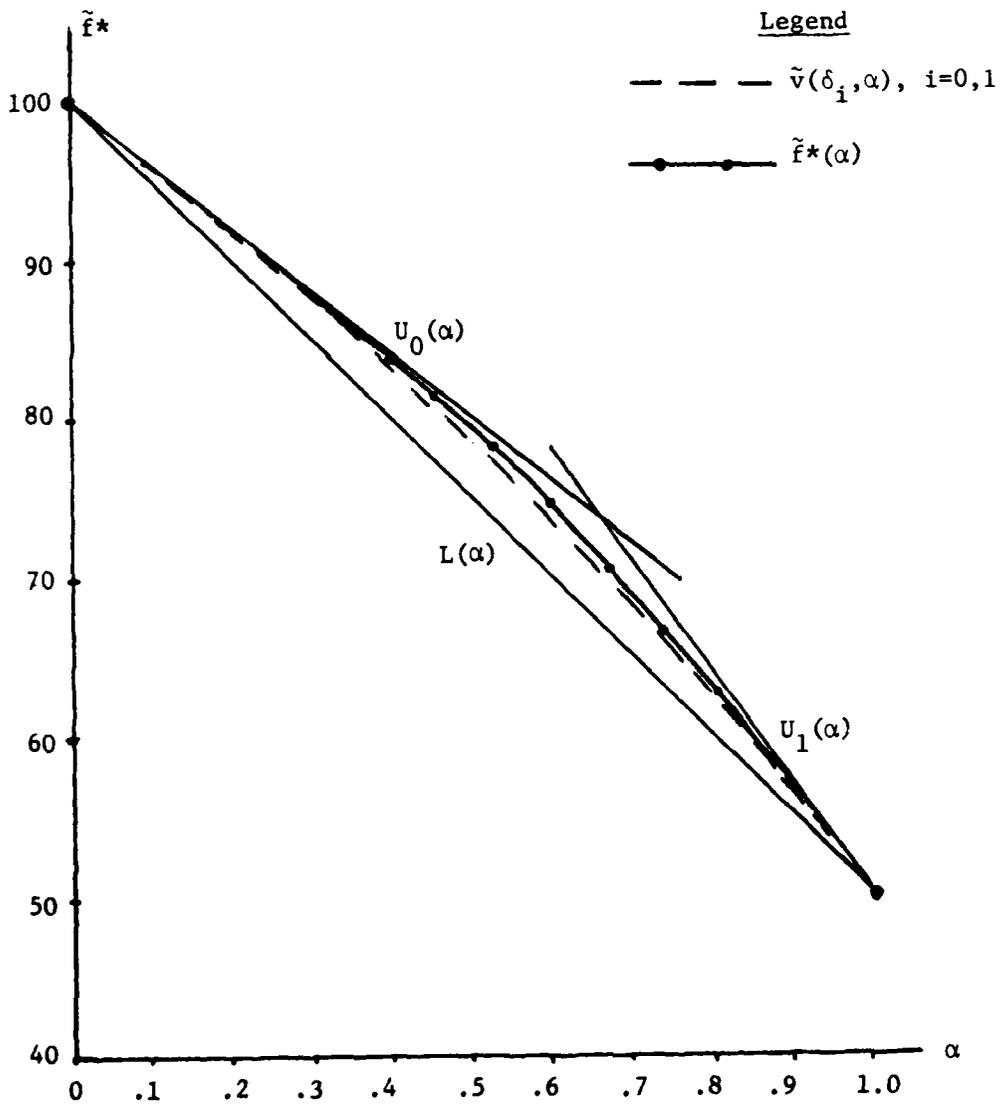


Figure 3.--Bounds on the concave $\tilde{f}^*(\alpha)$.

5. GP PROBLEMS HAVING JOINTLY CONVEX OVERESTIMATING AND UNDERESTIMATING PROBLEMS

Consider now frequently used linear perturbations of the coefficients c_i in both the objective function as well as in the constraint functions of the program $P(c)$. This "natural" class of perturbations is not covered by any case considered so far. The optimal value function $f^*(c)$ is in general neither convex nor concave in this case. However, it is possible to define "tight" overestimating and underestimating problems with convex optimal value functions to which in turn bounding techniques of Section 3 can be successfully applied. Specifically, consider perturbations of the form $c_i(\alpha) = c_{0i}(1 - d_i\alpha)$, where $d_i \geq 0$, $i=1, \dots, n$ and $\alpha \in [0, \alpha_0]$ for some $0 < \alpha_0 < \min_{1 \leq i \leq n} (1/d_i)$ (we define $1/d_i = +\infty$ if $d_i = 0$). Obviously, if $d_j = 0$ for some j , then the coefficient c_j will not be perturbed. Using linear bounds on the convex function $1/(1 - d_i\alpha)$ on the interval $[0, \alpha_0]$, one can easily derive the following inequalities:

$$\max \left\{ 1 + d_i\alpha, \frac{1 + d_i(\alpha - 2\alpha_0)}{(1 - d_i\alpha_0)^2} \right\} \leq \frac{1}{1 - d_i\alpha} \leq \frac{1 + d_i(\alpha - \alpha_0)}{1 - d_i\alpha_0} \quad (17)$$

and (substitute $+\infty$ for $(1 - d_i\alpha_0)^2 / (1 + d_i(\alpha - 2\alpha_0))$ when

$$\alpha \leq \alpha_0^2 \max_{1 \leq i \leq n} (d_i)$$

$$\frac{c_{0i}(1 - d_i\alpha_0)}{1 + d_i(\alpha - \alpha_0)} \leq c_i(\alpha) = c_{0i}(1 - d_i\alpha) \leq \min \left\{ \frac{c_{0i}}{1 + d_i\alpha}, \frac{c_{0i}(1 - d_i\alpha_0)^2}{1 + d_i(\alpha - 2\alpha_0)} \right\} \quad (18)$$

Define the following geometric programming problems:

$$\min_{x \in E^m} f_{01}^U(x, \alpha) \quad \text{s.t.} \quad f_{k1}^U(x, \alpha) \leq 1, \quad k=1, \dots, p \quad \text{PU}_1(\alpha)$$

where for $\alpha \in [0, \alpha_0]$

$$f_{k1}^U(x, \alpha) = \sum_{i \in J_k} \frac{c_{0i}}{1 + d_i \alpha} e^{\sum_{j=1}^m a_{ij} x_j}, \quad k=0, 1, \dots, p \quad (19)$$

$$\min_{x \in E^m} f_{02}^U(x, \alpha) \quad \text{s.t.} \quad f_{k2}^U(x, \alpha) \leq 1, \quad k=1, \dots, p \quad \text{PU}_2(\alpha)$$

where we define

$$f_{k2}^U(x, \alpha) = \sum_{i \in J_k} \frac{c_{0i} (1 - d_i \alpha_0)^2}{1 + d_i (\alpha - 2\alpha_0)} e^{\sum_{j=1}^m a_{ij} x_j}, \quad k=0, 1, \dots, p \quad (20)$$

for $\alpha \in [\tilde{\alpha}_0, \alpha_0]$, and $f_{k2}^U(x, \alpha) = +\infty$, $k=0, 1, \dots, p$ for $\alpha \in [0, \tilde{\alpha}_0]$,

$$\tilde{\alpha}_0 = \max\{0, 2\alpha_0 - \min_{1 \leq i \leq n} (1/d_i)\},$$

$$\min_{x \in E^m} f_0^L(x, \alpha) \quad \text{s.t.} \quad f_k^L(x, \alpha) \leq 1, \quad k=1, \dots, p \quad \text{PL}(\alpha)$$

where for $\alpha \in [0, \alpha_0]$

$$f_k^L(x, \alpha) = \sum_{i \in J_k} \frac{c_{0i} (1 - d_i \alpha_0)}{1 + d_i (\alpha - \alpha_0)} e^{\sum_{j=1}^m a_{ij} x_j}, \quad k=0, 1, \dots, p \quad (21)$$

From (18) it follows that for $\alpha \in [0, \alpha_0]$, $x \in E^m$,

$$f_k^L(x, \alpha) \leq f_k(x, c(\alpha)) \leq \min\{f_{k1}^U(x, \alpha), f_{k2}^U(x, \alpha)\}, \quad k=0, 1, \dots, p \quad (22)$$

This inequality asserts that both $\text{PU}_1(\alpha)$ and $\text{PU}_2(\alpha)$ are the overestimating problems for $P(c(\alpha))$ and that $\text{PL}(\alpha)$ is the underestimating problem for $P(c(\alpha))$, where $\alpha \in [0, \alpha_0]$.

Denote by $f_1^*{}^U(\alpha)$, $f_2^*{}^U(\alpha)$, and $f^*{}^L(\alpha)$ the optimal value functions of the programs $\text{PU}_1(\alpha)$, $\text{PU}_2(\alpha)$, and $\text{PL}(\alpha)$, respectively. (Note that $f_2^*{}^U(\alpha) = +\infty$ for $\alpha \in [0, \tilde{\alpha}_0]$.) From inequality (22) we obtain for

$\alpha \in [0, \alpha_0]$

$$f^{*L}(\alpha) \leq f^*(c(\alpha)) \leq \min\{f_1^{*U}(\alpha), f_2^{*U}(\alpha)\} \quad (23)$$

Since $f_k^L(x, 0) = f_k(x, c(0)) = f_{k1}^U(x, 0)$ and $f_k^L(x, \alpha_0) = f_k(x, c(\alpha_0)) = f_{k2}^U(x, \alpha_0)$ for $k=0, 1, \dots, p$ and $x \in E^m$, we also have that

$$f^{*L}(0) = f^*(c(0)) = f_1^{*U}(0) \quad (24)$$

and

$$f^{*L}(\alpha_0) = f^*(c(\alpha_0)) = f_2^{*U}(\alpha_0) \quad (25)$$

which proves that the bounds on $f^*(c(\alpha))$ are "tight" at $\alpha = 0$ and $\alpha = \alpha_0$.

The appealing feature of problems $PU_1(\alpha)$, $PU_2(\alpha)$, and $PL(\alpha)$ is that they are jointly convex in (x, α) ($PU_2(\alpha)$ is jointly convex in the extended sense) and therefore possess convex optimal value functions $f_1^{*U}(\alpha)$, $f_2^{*U}(\alpha)$, and $f^{*L}(\alpha)$, respectively. This is immediately seen from the fact that the coefficients in all the above problems' functions are of the form $1/h_i(\alpha)$, where $h_i(\alpha)$ are linear positive functions of α , and Proposition 1 of Section 3. Since $f^{*L}(\alpha)$ underestimates $f^*(c(\alpha))$, convex dual lower bounds as well as linear lower bounds on $f^{*L}(\alpha)$ can be calculated using results of Section 3 and then used as lower bounds on $f^*(c(\alpha))$. Similarly, convex or linear upper bounds on $f_1^{*U}(\alpha)$ and $f_2^{*U}(\alpha)$ can be computed also, using results of that section (in the case of $f_2^{*U}(\alpha)$ for α such that $f_2^{*U}(\alpha) < +\infty$), providing us with upper bounds on $f^*(c(\alpha))$ by virtue of inequality (23).

The approach described here can be extended to the case where the c_i 's are concave and, e.g., decreasing functions of a parameter α . We

will not discuss this idea in detail here, since the derivation of bounds in this case is similar to one described above.

Consider once more our example problem EP(c), with perturbations $c(\alpha) = (c_{01}(1 - (5/8)\alpha), c_{02}, c_{03}, c_{04}(1 - \alpha/4), c_{05}(1 - \alpha/2))$. Since $d_1 = 5/8$, $d_2 = 0$, $d_3 = 0$, $d_4 = 1/4$, $d_5 = 1/2$, $\min_{1 \leq i \leq 5} (1/d_i) = 8/5$, and we choose $\alpha_0 = 1$. Since

$$r(c(\alpha)) = \left[2 + \frac{1}{2} \frac{1 - \frac{5}{8}\alpha}{\left(1 - \frac{\alpha}{4}\right)\left(1 - \frac{\alpha}{2}\right)^2} \right]^{-1}, \quad \text{for } \alpha = 1$$

$c_1 = c(1) = (15, 20, 20, 1/4, 2/3)$, $r(c(1)) = 1/3$, and $\delta_1 = \delta^*(c(1)) = (1/3, 1/3, 1/3, 1/6, 1/3)$. From calculations we obtain $x_1 = x^*(c(1)) = (\log(.86), 0, 0)$ and $\tilde{f}^*(1) = \tilde{v}^*(1) = v(\delta_1, c_1) = 51.96$. The overestimating problems are of the form

$$\min_{x \in E^3} f_{01}^U(x, \alpha) = \frac{c_{01}}{1 + \frac{5}{8}\alpha} e^{-x_1 - \frac{1}{2}x_2 - x_3} + c_{02} e^{x_1 + x_3} + c_{03} e^{x_1 + x_2 + x_3}$$

$$\text{subject to } f_{11}^U(x, \alpha) = \frac{c_{04}}{1 + \frac{\alpha}{4}} e^{-2x_1 - 2x_2} + \frac{c_{05}}{1 + \frac{\alpha}{2}} e^{\frac{1}{2}x_2 - x_3} \leq 1 \quad \text{EPU}_1(\alpha)$$

for $\alpha \in [0, 1]$ and

$$\min_{x \in E^3} f_{02}^U(x, \alpha) = \frac{\frac{9}{64} c_{01}}{1 + \frac{5}{8}(\alpha - 2)} e^{-x_1 - \frac{1}{2}x_2 - x_3} + c_{02} e^{x_1 + x_3} + c_{03} e^{x_1 + x_2 + x_3}$$

$$\text{subject to } f_{12}^U(x, \alpha) = \frac{\frac{9}{16} c_{04}}{1 + \frac{1}{4}(\alpha - 2)} e^{-2x_1 - 2x_2} + \frac{\frac{1}{4} c_{05}}{1 + \frac{1}{2}(\alpha - 2)} e^{\frac{1}{2}x_2 - x_3} \leq 1$$

EPU₂(α)

for $\alpha \in (2/5, 1]$ (for $\alpha \in [0, 2/5]$ we define $f_{k2}^U(x, \alpha) = +\infty$ for

$k=0,1$, all $x \in E^3$). The underestimating problem is defined for $\alpha \in [0,1]$ as

$$\min_{x \in E^3} f_0^L(x, \alpha) = \frac{\frac{3}{8} c_{01}}{1 + \frac{5}{8} (\alpha-1)} e^{-x_1 - \frac{1}{2}x_2 - x_3} + c_{02} e^{x_1 + x_3} + c_{03} e^{x_1 + x_2 + x_3}$$

$$\text{subject to } f_1^L(x, \alpha) = \frac{\frac{3}{4} c_{04}}{1 + \frac{1}{4} (\alpha-1)} e^{-2x_1 - 2x_2} + \frac{\frac{1}{2} c_{05}}{1 + \frac{1}{2} (\alpha-1)} e^{\frac{1}{2}x_2 - x_3} \leq 1$$

EPL(α)

The above three problems have convex optimal value functions $f_1^U(\alpha)$, $f_2^U(\alpha)$, and $f^L(\alpha)$, respectively, for $\alpha \in [0,1]$.

Recalling from Section 2 that for $\alpha = 0$, $c_0 = (40, 20, 20, 1/3, 4/3)$, $\delta_0 = \delta^*(c_0) = (1/5, 2/5, 2/5, 3/10, 3/5)$, $x_0 = x^*(c_0) = (0, 0, \log 2)$, and $\tilde{f}^*(0) = \tilde{v}^*(0) = 100.0$, application of inequality (4) gives us convex lower bounds on $\tilde{f}^*(\alpha)$, $\alpha \in [0,1]$, as follows:

$$\tilde{f}^*(\alpha) \geq \tilde{v}(\delta_0, \alpha) = \tilde{f}^*(0) \left[\frac{3/8}{1 + \frac{5}{8} (\alpha-1)} \right]^{\delta_{01}} \left[\frac{3/4}{1 + \frac{1}{4} (\alpha-1)} \right]^{\delta_{04}} \left[\frac{1/2}{1 + \frac{1}{2} (\alpha-1)} \right]^{\delta_{05}}$$

$$= \frac{100.0}{\left(1 + \frac{5}{3} \alpha\right)^{.2} \left(1 + \frac{1}{3} \alpha\right)^{.3} (1 + \alpha)^{.6}}$$

$$\tilde{f}^*(\alpha) \geq \tilde{v}(\delta_1, \alpha) = \tilde{f}^*(1) \left[\frac{1}{1 + \frac{5}{8} (\alpha-1)} \right]^{\delta_{11}} \left[\frac{1}{1 + \frac{1}{4} (\alpha-1)} \right]^{\delta_{14}} \left[\frac{1}{1 + \frac{1}{2} (\alpha-1)} \right]^{\delta_{15}}$$

$$= \frac{95.24}{\left(1 + \frac{5}{3} \alpha\right)^{1/3} \left(1 + \frac{1}{3} \alpha\right)^{1/6} (1 + \alpha)^{1/3}}$$

The linear lower bounds on $\tilde{f}^*(\alpha)$ can be easily computed using the formulas

$$L_0(\alpha) = \tilde{f}^*(0) + \nabla_{\alpha} \tilde{v}(\delta_0, 0)\alpha = 100.0 - 86.67\alpha$$

and

$$L_1(\alpha) = \tilde{f}^*(1) + \nabla_{\alpha} \tilde{v}(\delta_1, 1)(\alpha-1) = 51.96 - 21.65(\alpha-1) = 73.61 - 21.65\alpha$$

In order to obtain an upper bound we solve the dual of the problem $EPU_1(\alpha)$ for $\alpha = 1$, obtaining $c_1^U = (320/13, 20, 20, 4/15, 8/9)$, $r(1) = 52/149$, and $\delta_1^U = \delta^*(c_1) = (45/149, 52/149, 52/149, 59/298, 59/149)$. From this we calculate $f_1^{*U}(1) = v(\delta_1, c_1) \doteq 68.34$ and $x_1^U = x^*(c_1) = (\log(.9), 0, \log(1.325))$. Since $f_1^{*U}(0) = \tilde{f}^*(0) = 100.0$, the linear upper bound on $\tilde{f}^*(\alpha)$ for $\alpha \in [0, 1]$ can be obtained from the formula

$$\begin{aligned} U(\alpha) &= (1-\alpha)f_1^{*U}(0) + \alpha f_1^{*U}(1) \doteq (1-\alpha)100.0 + \alpha 68.34 \\ &= 100.0 - 31.66\alpha \end{aligned}$$

A better convex upper bound can be obtained in the form $f_{01}^U(x^U(\alpha), \alpha)$, $x^U(\alpha) = (1-\alpha)x_0 + \alpha x_1^U$, $\alpha \in [0, 1]$. The above bounds are depicted in Figure 4.

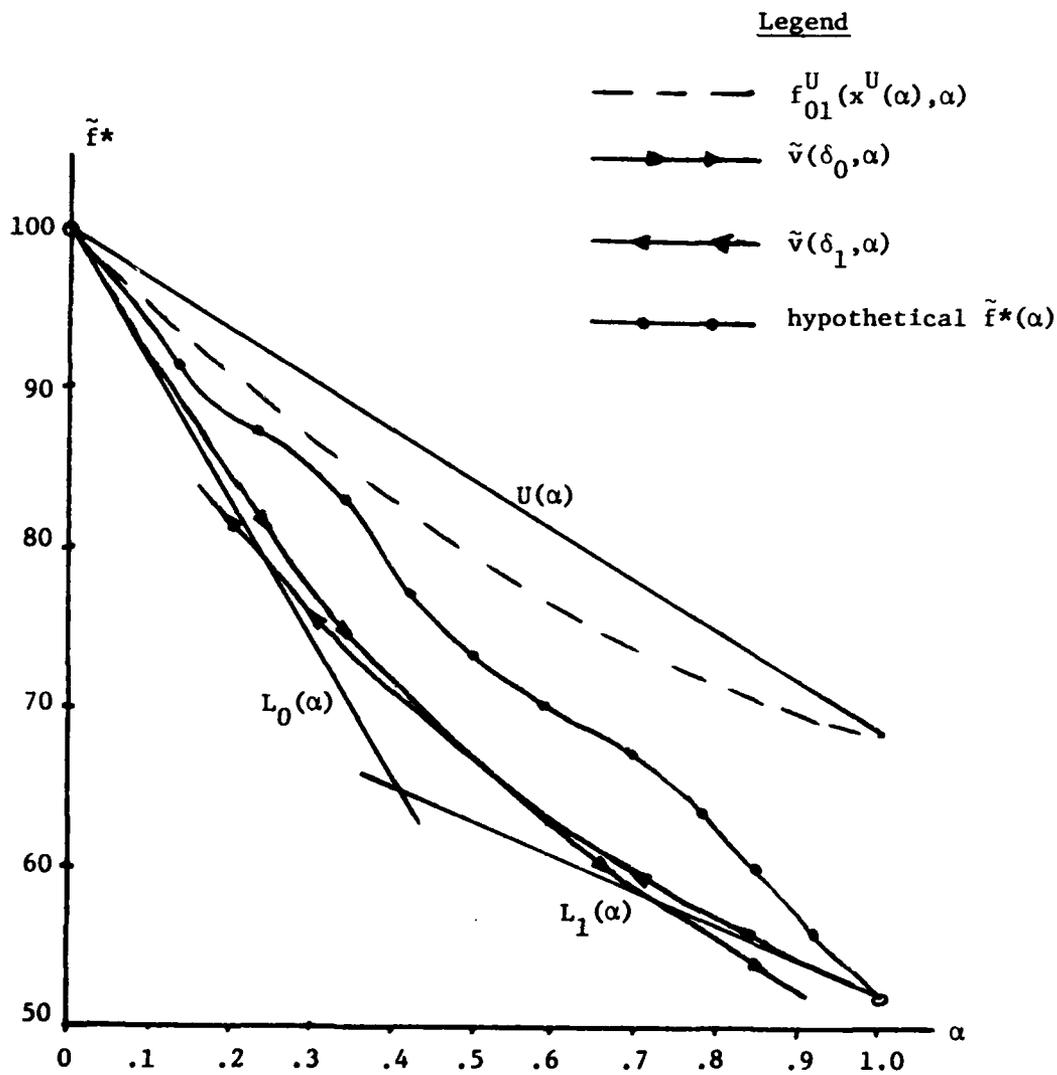


Figure 4.--Bounds on $\tilde{f}^*(\alpha)$.

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