BAYES ADAPTIVE ESTIMATION OF THE POINT OF SHIFT TO THE WEAR-OUT PHASE OF RELIABILITY SYSTEMS

by

S. Zacks

TECHNICAL REPORT NO. 1

August 1, 1982

Prepared under Contract N00014-81-K-0407 (NR 042-276) For the Office of Naval Research

Reproduction is permitted for any use of the U.S. Government

DEPARTMENT OF MATHEMATICAL SCIENCES
STATE UNIVERSITY OF NEW YORK
BINGHAMTON, NEW YORK

82 08 05 010
BAYES ADAPTIVE ESTIMATION OF THE
POINT OF SHIFT TO THE WEAR-OUT
PHASE OF RELIABILITY SYSTEMS

by

S. Zacks

State University of New York at Binghamton

Abstract

A new family of life distributions, called the wear-out distributions, is developed on the basis of a failure rate function, which is a constant up to the change-point and strictly increasing afterwards. Properties of these wear-out distributions are derived and a Bayes adaptive procedures is developed for the estimation of the change point. Recursive formulae are given for the determination of the posterior probability that the change has occurred and of its Bayes estimator. The results of numerical simulations are given to illustrate the properties of the adaptive procedure.

Key Words: Wear-out, Reliability System, Bayes Adaptive Procedure, Stopping Times, Failure Rate Functions, Simulations
1. **Introduction**

Consider a system having several components, whose life distributions depend on the state of the system. Each component is replaced immediately after failure (corrective maintenance), or at scheduled replacement epochs (preventive maintenance). The optimal scheduling of the replacement epochs depends on various economic considerations, and on the particular life distribution of the component. This in turn depends on the state of the whole system. We recognize two phases of the system. Phase I is the phase in which the components have a constant failure rate, \( \lambda \). Phase II is the phase in which the failure rate is greater than \( \lambda \) and increasing. Phase I is a mature stable phase of the system, while Phase II is a wear-out phase. It is important to detect early the transition of the system from Phase I to Phase II. Optimal replacement scheduling and inventory management during the wear-out phase of the system are different from those appropriate to the constant failure rate phase. The present paper is devoted to the study of the detection of the shift from Phase I to Phase II, and estimating the change point, \( \tau \), where this shift occurs.

In Section 2 we derive from a specified failure rate model an appropriate life distribution for the present study. This life distribution is a composition of an exponential distribution with a Weibull. It seems to be new in the sense of not being dealt with in the literature on reliability. We shall call it the *wear-out life distribution*. Several properties of this distribution are presented in Section 2. In Section 3 we develop a Bayes adaptive procedure of estimating the change-point \( \tau \), on the basis of a sequence of the replacement times of the component. Section 4 is devoted to the derivation of some recursive formulae, which are required by the adaptive procedure. In Section 5 we provide some simulation results, which provide information of the characteristics of the detection procedure.
2. The Wear-Out Life Distribution

The wear-out life distribution is based on the following non-decreasing failure-rate function

\[
\begin{align*}
h(t; \lambda, \alpha, \tau) &= \begin{cases} 
\lambda & \text{if } 0 \leq t \leq \tau_+ \\
\lambda + \alpha(t-\tau_+)^{\alpha-1} & \text{if } \tau_+ < t
\end{cases} 
\end{align*}
\]

where \( \lambda \) is the constant failure-rate in Phase I, \( 0 < \lambda < \infty \); \( \alpha \) is a shape parameter, \( 1 \leq \alpha < \infty \), and \( \tau \) is the change point parameter \( \tau_+ = \max(0, \tau) \).

If \( \tau < 0 \) the system is already in Phase II. The present model represents a superposition of a Weibull failure-rate function (see Mann, Schafer and Singpurwalla [1; pp. 127]) on a constant failure-rate function for \( t \geq \tau_+ \). Notice also that the limiting case of \( \alpha = 1 \) corresponds to a shift at \( \tau \) from an exponential life distribution with MTBF \( \mu_1 = 1/\lambda \) to an exponential life distribution with an MTBF \( \mu_2 = 1/2\lambda \), i.e., for all \( t > \tau \),

\[
p(x|t; x_{zt}) = \exp(-2\lambda(t-\tau)).
\]

From this failure-rate model we obtain a life distribution, called the wear-out life distribution, with a c.d.f.

\[
F(x; \lambda, \alpha, \tau) = 1 - \exp\left\{-\lambda \int_0^x (1 + \alpha(t-\tau_+)^{\alpha-1}) \, dt\right\}
\]

\[
= 1 - \exp\{-\lambda x - (x - \tau_+)^{\alpha}\}, \quad x \geq 0
\]

Obviously, \( F(x; \lambda, \alpha, \tau) = 0 \) for \( x < 0 \).

The corresponding p.d.f. is

\[
f(x; \lambda, \alpha, \tau) = \lambda e^{-\lambda x} (1 + \alpha (x - \tau_+)^{\alpha-1}) \cdot \exp\{-\lambda^\alpha (x - \tau_+)^{\alpha}\}, \quad x \geq 0.
\]

It follows the conditional p.d.f. of \( X \) (the life length), given \( (Xzt) \), which is the p.d.f. of the residual life distribution in Phase II, is

\[
g(x - \tau_+; \lambda, \alpha) = \lambda e^{-\lambda (x-\tau_+)} (1 + \alpha (\lambda(x-\tau_+)))^{\alpha-1}) \cdot \exp\{-\lambda^\alpha (x - \tau_+)^{\alpha}\}, \quad x > \tau
\]

Notice that \( \beta = 1/\lambda \) is a scale parameter of this distribution, and that \( \tau_+ \) is a location parameter of the truncation type (see Zacks [3; pp. 140]).
$f(x; 1, 2, r) = (1 + 2(x - r)_+) \exp\{-x - (x - r)_+^2\}$

Figure 1. Density functions of the wear-out distribution for $\lambda = 1$, $\alpha = 2$ and $\tau = 0, 1$ and $\infty$ (exponential).
In Fig. 1 we present graphs of the wear-out life density (2.3) for \(a=2, \lambda=1\) and various values of \(\tau\). We remark here that, if \(\tau\) is very large compared to \(1/\lambda\), the wear-out distribution is practically an exponential distribution with mean \(\mu=1/\lambda\). This is the case when the system is far from its wear-out phase. We conclude the present section with a presentation of the formula for the moments of the wear-out distribution (2.3). Since \(1/\lambda\) is a scale parameter, we will present the moment \(\mu_r(a,\tau)\) for the case of \(\lambda=1\). If \(\lambda=1\) then

\[
E[X^r; \lambda, a, \tau] = \frac{1}{\lambda^r} \mu_r(a, \lambda \tau) .
\]

The \(r\)-th moment in the standard case is for \(\tau>0\),

\[
\mu_r(a, \tau) = \int_0^\tau x^r e^{-x} \, dx + \int_0^\tau x^r e^{-(x-\tau)}(1+a(x-\tau)^{a-1}) \, dx
\]

\[
= r!(1 - \text{Pos}(\tau | \tau)) + \int_0^\tau x^r e^{-(x-\tau)}(1+a(x-\tau)^{a-1}) \, dx
\]

where

\[
\text{Pos}(k | \lambda) = e^{-\lambda} \sum_{j=0}^k \frac{\lambda^j}{j!}
\]

is the Poisson c.d.f. and

\[
M_j(a) = \int_0^\infty y^j e^{-y} (1+\alpha \cdot y^{a-1}) \, dy , \quad j \geq 0
\]

Integration by parts yields, for every \(j \geq 1\),

\[
M_j(a) = j \int_0^\infty y^{j-1} e^{-(y+\alpha^2)} \, dy
\]

which is smaller than \(j! (j+1) = j!\) for all \(j \geq 1\). Notice that \(M_0(a)=1\) for all \(a\geq 1\). The integral in (2.8) can be evaluated numerically. In the case of \(a=2\) we obtain, for all \(j \geq 1\),

\[
M_j(2) = j e^{1/4} \int_0^\infty y^{j-1} e^{-(y+1)^2/2} \, dy
\]

\[
= j e^{1/4} \sum_{\ell=0}^{j-1} \frac{(-1)^\ell (j-1)!}{\ell!} \frac{(1/2)^{j-\ell-1}}{(j-\ell - 1)!} B(\ell; 1/2)
\]

\[
B(\ell; r) = \frac{\Gamma(\ell+1)}{\Gamma(\ell+r+1)} = \frac{(\ell!)^2}{(\ell+r)!}
\]

where \(B(\ell; r)\) is the beta function.
where, for all \( \ell \geq 2 \),

\[
B(\ell; \xi) = \int_0^\infty e^{-\frac{1}{2} z^2} \frac{1}{\xi} dz
\]

\[
= \xi^{\ell-1} \exp\left( -\frac{1}{2} \xi^2 \right) + (\ell-1) B(\ell-2; \xi)
\]

and

\[
B(0; \xi) = \sqrt{2\pi} \left( \phi(\xi) \right)
\]

\[
B(1; \xi) = \exp\left( -\frac{1}{2} \xi^2 \right)
\]

where \( \phi(\xi) \) is the standard normal c.d.f.

The functions \( B(\ell; \frac{1}{2}) \) can be determined recursively, according to (2.10) and (2.11). In Table 2.1 we provide the first four moments, the standard deviation \( \sigma \), the measures of skewness \( \gamma_1 \) and kurtosis \( \gamma_2 \), of the wear-out distribution for \( \lambda = 1 \), \( \alpha = 2 \) and several values of \( \tau \).

**Table 2.1 The first four moments, standard deviation, and measure of skewness and kurtosis of the wear-out distribution, for \( \lambda = 1 \), \( \alpha = 2 \) and \( \tau = 0, 0.5, 1, 1.5, 2 \).**

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>0</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_1 )</td>
<td>0.54564</td>
<td>0.72442</td>
<td>0.83285</td>
<td>0.89862</td>
<td>0.93851</td>
</tr>
<tr>
<td>( \mu_2 )</td>
<td>0.45436</td>
<td>0.78694</td>
<td>1.09709</td>
<td>1.35098</td>
<td>1.54846</td>
</tr>
<tr>
<td>( \mu_3 )</td>
<td>0.47769</td>
<td>1.03764</td>
<td>1.76118</td>
<td>2.53153</td>
<td>3.25967</td>
</tr>
<tr>
<td>( \mu_4 )</td>
<td>0.59026</td>
<td>1.55837</td>
<td>3.18161</td>
<td>5.34890</td>
<td>7.86491</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>0.39577</td>
<td>0.51201</td>
<td>0.63518</td>
<td>0.73720</td>
<td>0.81490</td>
</tr>
<tr>
<td>( \gamma_1 )</td>
<td>0.94918</td>
<td>0.65373</td>
<td>0.68460</td>
<td>0.85060</td>
<td>1.04106</td>
</tr>
<tr>
<td>( \gamma_2 )</td>
<td>3.80641</td>
<td>2.95378</td>
<td>2.68424</td>
<td>2.87374</td>
<td>3.32183</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_1 )</td>
<td>0.97738</td>
<td>0.99168</td>
<td>0.99694</td>
<td>0.99887</td>
<td>0.99959</td>
</tr>
<tr>
<td>( \mu_2 )</td>
<td>1.78732</td>
<td>1.90511</td>
<td>1.95897</td>
<td>1.98265</td>
<td>1.99279</td>
</tr>
<tr>
<td>( \mu_3 )</td>
<td>4.42171</td>
<td>5.15969</td>
<td>5.57697</td>
<td>5.79371</td>
<td>5.90446</td>
</tr>
<tr>
<td>( \mu_4 )</td>
<td>12.93668</td>
<td>17.10481</td>
<td>20.00519</td>
<td>21.81282</td>
<td>22.85481</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>0.91217</td>
<td>0.96005</td>
<td>0.98239</td>
<td>0.99242</td>
<td>0.99680</td>
</tr>
<tr>
<td>( \gamma_1 )</td>
<td>1.38130</td>
<td>1.63008</td>
<td>1.79281</td>
<td>1.89034</td>
<td>1.94467</td>
</tr>
<tr>
<td>( \gamma_2 )</td>
<td>4.55947</td>
<td>5.85946</td>
<td>6.96183</td>
<td>7.77159</td>
<td>8.30424</td>
</tr>
</tbody>
</table>
The exponential distribution is the limiting case when $\tau \to \infty$. The moments of the exponential distribution with $\lambda=1$ are $\mu_1=1$, $\mu_2=2$, $\mu_3=6$ and $\mu_4=24$. We see in Table 2.1 that when $\tau=7$ the moments of the wear-out distribution are already close to those of the limiting exponential distribution.
3. Bayes Estimation of the Change Point

In the present section we develop the Bayesian framework for estimating the change point, $T$, and deciding whether the transition to Phase II has taken place.

Given a sequence $t_1, t_2, \ldots$ of replacement epochs, where $0 < t_1 < t_2 < \ldots$, we consider the random variables $X_i = t_i - t_{i-1}$ ($i = 1, 2, \ldots$), where $t_0 = 0$. $X_i$ represents the operational time length of the $i$-th sequentially installed component. Since the preventive maintenance replaces the component after $\Delta$ units of time, $X_i = \min(Y_i, \Delta)$, where $Y_1, Y_2, \ldots$ are independent random variables having the wear-out distribution, with parameters $\lambda, \alpha$ and $T_i = t_i - t_{i-1}$ ($i = 1, 2, \ldots$). $T_i$ is the length of time left for the system, after the $i$-th installment of the component till the transition to Phase II. If $T_i < 0$, the change to Phase II has taken place before $t_{i-1}$.

In the present study we assume that $\lambda$ and $\alpha$ are known. The only unknown parameter is the change point, $T$. Without loss of generality we assume that $\lambda = 1$. If $\lambda = 1$ replace $X_i$ in the formulae below by $\lambda X_i$ ($i = 1, 2, \ldots, n$).

Given $n$ replacement points $t_1 < t_2 < \ldots < t_n$, let $X^{(n)} = (X_1, \ldots, X_n)$, where $X_i = t_i - t_{i-1}$, and define

$$
J_i = \begin{cases} 
1 & \text{if } X_i = \Delta \\
0 & \text{if } X_i = 0 
\end{cases} \quad (3.1)
$$

The likelihood function of $T$, given $X^{(n)}$, is

$$
L(T; X^{(n)}) = I(t = 0) \exp(- \sum_{i=1}^{n} X_i) \prod_{i=1}^{n} (1 + \alpha J_i X_i^{\alpha-1}) \\
+ \sum_{j=1}^{n-1} I(t_j < t < t_{j+1}) \exp(- \sum_{k=j+1}^{n} X_k) \prod_{k=j+1}^{n} (1 + \alpha J_k X_k^{\alpha-1}) \\
\cdot \left[ (1 + \alpha J_j (X_j + t_j - T)^{\alpha-1}) \exp(-(X_j + t_j - T)^{\alpha}) \\
+ I(t_{n-1} < t < t_n) \cdot (1 + \alpha J_n (X_n + t_n - T)^{\alpha-1}) \exp(-(X_n + t_n - T)^{\alpha}) \\
+ I(T > t_n) \right] 
$$

(3.2)

where $I(A)$ is the indicator function of the set $A$. 
Generally, let $\xi(\tau)$ denote the prior p.d.f. of $\tau$. Then, the posterior p.d.f. of $\tau$, given $\chi^{(n)}$, is

$$
\xi(\tau|\chi^{(n)}) = \frac{\xi(\tau)L(\tau;\chi^{(n)})}{D(\tau;\chi^{(n)})},
$$

(3.3)

where

$$
D(\tau;\chi^{(n)}) = \int L(\tau;\chi^{(n)})d\xi(\tau|\chi^{(n)}).
$$

(3.4)

In the present study we consider a prior distribution with p.d.f.

$$
\xi(\tau) = \begin{cases} 
(1-p)e^{-\psi(\tau-\tau_0)} & \text{if } \tau > \tau_0 \\
0 & \text{if } \tau < \tau_0
\end{cases},
$$

(3.5)

where $0 < p < 1$ and $0 < \psi < \infty$. $\tau_0$ is a time point such that, with high prior probability, $1-p$, the true wear-out point, $\tau$, exceeds it. We are interested therefore in time points greater than $\tau_0$. We shall assume accordingly, without loss of generality, that $\tau_0 = 0$. For the prior p.d.f. (3.5) under consideration, the function (3.4) assumes the form

$$
D(p,\psi;\chi^{(n)}) = p \exp\left(-\sum_{1}^{n} x_i^{\alpha} \prod_{i=1}^{n} \left[1+\alpha_j x_i^{\alpha-1}\right]\right)
$$

$$
+ (1-p) \sum_{k=1}^{n-1} \exp\left(-\sum_{j=1}^{k} x_j^{\alpha} \prod_{k=j}^{n} \left[1+\alpha_j x_k^{\alpha-1}\right]\right) \cdot e^{-\psi t_n} \int_0^{t_n} \left[1+\alpha_j (x_j-u)^{\alpha-1}\right] e^{-(x_j-u)^\alpha} du
$$

$$
+ (1-p)e^{-\psi t_n} \int_0^{t_n} \left[1+\alpha_j (x_n-u)^{\alpha-1}\right] e^{-(x_n-u)^\alpha} du
$$

$$
+ (1-p)e^{-\psi t_n}.
$$

(3.6)

The posterior probability, after observing $\chi^{(n)}$, that the transition to Phase II has already taken place (i.e., $\{\tau < t_n\}$) is $p_n = 1-q_n$, where

$$
q_n = (1-p)e^{-\psi t_n}/D(p,\psi;\chi^{(n)}).
$$

(3.7)

If $p_n$ is large we have high evidence that the change has already taken place. In addition, after observing $\chi^{(n)}$ the Bayes estimator of $\tau$, for squared error loss function, is the posterior expectation of $\tau$, given $\chi^{(n)}$. This estimator is determined by the formula
\[ \tau_n(p, \psi; X^{(n)}) = E(\tau|X^{(n)}, p, \psi) = \]

\[
\frac{n-1}{(1-p)\psi} \Pi_{k=1}^{n} \left[ 1 + \alpha J_k^X X_k^{a-1} \right] \exp \left( - \frac{n}{\Pi_{k=1}^{n} X_k^a} \right) \]

\[
\cdot e^{-\psi \int_{0}^{t} (t-j^X+u)e^{-\psi u} [1 + \alpha J_n^X (X_n-u)^{a-1}] e^{-\psi (X_n-u)^a} du}
\]

\[
+ (1-p)\psi e^{-\psi \int_{0}^{t} (t_n+u)e^{-\psi u} [1 + \alpha J_n^X (X_n-u)^{a-1}] e^{-\psi (X_n-u)^a} du}
\]

\[
+ (1-p)\psi e^{-\psi \int_{0}^{t} \psi d\psi} + D(p, \psi; X^{(n)}) \]  \hspace{1cm} (3.8)

Notice that if \( \lambda \neq 1 \) we determine \( p_n \) and \( \tau_n(p, \psi; X^{(n)}) \) by substituting in the above formula \( \lambda X_i \) for \( X_i \) and \( \psi/\lambda \) for \( \psi \).

Thus, we can compute \( p_n \) and \( \tau_n \) adaptively, after each replacement, and decide that the change has occurred at the first time \( \tau_n \leq t_n \) or \( p_n \) is sufficiently large. Shiryaev [2] studied the problem of the quickest detection of the occurrence of a change in a specified distribution. He showed that, when the distributions before and after the change are known, the optimal Bayes sequential stopping time is at the smallest \( n \geq 1 \) for which \( p_n \geq \Pi^* \). \( \Pi^* \) is a threshold that can be determined to minimize the total expected loss. Our problem is however different. First, we do not have just two known distributions with a switch from one to another at an unknown change-point, \( \tau \), but a sequence of wear-out distributions with unknown parameters \( \tau_i \), and we wish to detect the time point at which, for the first time, \( \tau_i \leq 0 \).

Moreover, we do not have a sequential stopping problem, but an adaptive decision problem. The replacement process is continuing also after the decision that \( \tau_i \leq 0 \) has been reached. The implications of this decision are on the adaptation of the scheduled replacement period, \( \Delta \), and on the inventory of spare components. In the following section we provide some recursive formulae for the determination of \( D(p, \psi; X^{(n)}) \), \( p_n \) and \( \tau_n(p, \psi; X^{(n)}) \).
4. **Recursive Formulae**

From (3.3) and (3.5) one immediately obtains that, for every $t \geq t_n$ ($n=1,2,\ldots$)
\[
P(t \geq t|X^{(n)}) = \frac{(1-p) e^{-\psi t}}{D(p,\psi;X^{(n)})}
\]  

(4.1)

Combining this result with (3.7) we obtain for all $t \geq t_n$
\[
P(t \geq t|X^{(n)}) = (1-p_n)e^{-\psi(t-t_n)}
\]

(4.2)

where $p_n$ is the posterior probability of $\{t \geq t_n\}$. This yields that, after observing $X^{(n)}$, the posterior distribution of $\tau$ for $t \geq t_n$ is of the same form as (3.1), with $p$ replaced by $p_n$, i.e., the posterior p.d.f. of $\tau$, given $X^{(n)}$ satisfies
\[
P_n(\tau) = \begin{cases} 
-\psi(t-t_n) & \text{if } t \leq t_n \\
(1-p_n) & \text{if } t > t_n
\end{cases}
\]

(4.3)

According to (3.6), the function $D_n \equiv D(p,\psi;X^{(n)})$, $n \geq 1$, can be computed by the following recursive formula
\[
D_n = D_{n-1}(1+\alpha n X_n) \exp(-X_n^\alpha) + (1-p)e^{-\psi t_{n-1}} - (1+\alpha n X_n^{\alpha-1}) \exp(-X_n^\alpha) du
\]

(4.4)

where $D_0 \equiv 1$. Notice that for all $\lambda$, $0<\lambda<\infty$, $D(p,\psi;X^{(n)}) = D(p,\psi/\lambda;X^{(n)})$.  

(4.5)

Employing (3.7) and (4.4) we obtain
\[
1-p_n = \frac{(1-p)e^{-\psi t_n}}{D_n} = \frac{(1-p)e^{-\psi t_{n-1}}}{D_{n-1}} \cdot \frac{D_{n-1}}{D_n} e^{-\psi X_n}
\]

\[
= (1-p_{n-1}) \frac{e^{-\psi X_n}}{D_{n}/D_{n-1}}, \quad n \geq 1
\]

(4.6)
Moreover,

\[
\frac{D_n}{D_{n-1}} = p_{n-1} (1+\alpha J_n x_n^{\alpha-1}) \exp\{-x_n^\alpha\} + \\
\frac{X_n}{n}
\]

\[
(1-p_{n-1}) \psi \int_0^x e^{-\psi u} (1+\alpha J_n (x_n-u)^{\alpha-1}) \exp\{-(x_n-u)^\alpha\} du \\
+ (1-p_{n-1}) e^{\psi X_n}
\]

(4.7)

Thus, define for all \(0 \leq x \leq \Delta\) the function

\[
\psi_\alpha(p,\psi, x) = pe^{-\psi x} (1+\alpha J x^{\alpha-1}) + \\
x \int_0^x e^{-\psi u} (1+\alpha J (x-u)^{\alpha-1}) \exp\{-(x-u)^\alpha\} du \\
+ (1-p) \exp\{-\psi x\}
\]

(4.8)

where \(J = I\{x<\Delta\}\). It follows from (4.7) and (4.8) that, for every \(n \geq 1\)

\[
D(p,\psi; Z_n) = D(p,\psi; Z_{n-1}) \psi_\alpha(p_{n-1},\psi, X_n)
\]

(4.9)

and

\[
p_n = 1 - \frac{(1-p_{n-1}) \exp\{-\psi X_n\}}{\psi_\alpha(p_{n-1},\psi, X_n)}
\]

(4.10)

The integral in (4.8) can be evaluated numerically, or according to the formula

\[
(1-p) \int_0^x e^{-\psi u} (1+\alpha J (x-u)^{\alpha-1}) \exp\{-(x-u)^\alpha\} du = \\
(1-p) \sum_{j=0}^{\infty} \int_0^x \frac{e^{-\psi x}}{\psi^j} \frac{j!}{j!} \Gamma(x^\alpha; \frac{1+j}{a}) + \alpha J \Gamma(x^\alpha; \frac{1}{a} + 1) \]

(4.11)

where \(\Gamma(x; \nu) = \int_0^x z^{\nu-1} e^{-z} dz\) is the incomplete gamma function \((\nu>0)\).

In the case of \(\alpha=2\) we obtain the explicit formula
\[ \psi_2(p, \psi, x) = p e^{-x^2}(1+2Jx) + (1-p)e^{-\psi^x \psi^2/4} \sqrt{\pi} \left[ \left( \frac{\psi}{\sqrt{2}} \right) - \phi \left( \frac{\psi}{\sqrt{2}} \right) \right] (1+J\psi) \]

\[ +J(1-p)e^{-\psi^x \psi^2/4} \sqrt{2\pi} \left[ \phi \left( \frac{\psi}{\sqrt{2}} \right) - \phi \left( \frac{\psi}{\sqrt{2}} - \sqrt{2x} \right) \right] \]

\[ + (1-p)e^{-\psi^x} \]  \hspace{1cm} (4.12)

where \( \phi(z) \) and \( \phi(z) \) are, respectively, the p.d.f. and the c.d.f. of the standard normal distribution.

We discuss now the recursive determination of the Bayes estimator of \( \tau \). According to (3.8)

\[ \hat{\tau}_n(p, \psi; X(n)) = E_n + (1-p_n) \left( \tau_n + \frac{1}{\psi} \right) \]  \hspace{1cm} (4.13)

where

\[ E_n = \left\{ (1-p) \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} [1+\alpha J \chi^{\alpha-1}] \exp \left\{ - \sum_{k=j+1}^{n} \chi_k \right\} \right\} \]

\[ \exp \left\{ - \psi \sum_{j=1}^{n-1} X_j \right\} \]

\[ e^{-\psi \sum_{j=1}^{n-1} X_j} \int_0^{(\tau_{n-1}+u)} e^{-\psi u} [1+\alpha J (X_n-u)^{\alpha-1}] \exp \left\{ -(X_n-u)^{\alpha} \right\} du \]

\[ + (1-p) \exp \left\{ -\psi \sum_{j=1}^{n-1} X_j \right\} \int_0^{(\tau_{n-1}+u)} e^{-\psi u} [1+\alpha J (X_n-u)^{\alpha-1}] \exp \left\{ -(X_n-u)^{\alpha} \right\} du \}

\[ + D_n \]  \hspace{1cm} (4.14)

Define the function

\[ \eta_\alpha(p, \psi, t, x) = (1-p) e^{-\psi t} \int_0^x (1+\alpha J y^{\alpha-1}) e^{\psi y - y^\alpha} dy \]  \hspace{1cm} (4.15)

Then,

\[ E_1 = \eta_\alpha(p, \psi, t_1, X_1)/D_1 \]  \hspace{1cm} (4.16)

and, for every \( n \geq 1 \)

\[ E_{n+1} = \left[ E_n D_n (1+\alpha J \chi^{\alpha-1}) \exp \left\{ -X_{n+1}^{\alpha} \right\} \right] + \eta_\alpha(p, \psi, t_{n+1}, X_{n+1}) D_{n+1} \]  \hspace{1cm} (4.17)
The function \( n_2(p, \psi, t, x) \) can also be written in terms of the p.d.f. and c.d.f. of the standard normal distribution as

\[
\begin{aligned}
\eta_2(p, \psi, t, x) &= (1-p)\phi e^{-\psi t + \psi^2 / 4} \sqrt{\pi} \\
&= \left[ \phi(\sqrt{2}(x - \frac{\psi}{2})) + \phi(\frac{\psi}{\sqrt{2}}) - 1 \right] (t - \frac{\psi}{2} + J(t\psi - \frac{\psi^2}{2} - 1)) + \phi(\frac{\psi}{\sqrt{2}}) (t\sqrt{2} - \frac{1}{\sqrt{2}} - J \frac{\psi}{\sqrt{2}}) - \phi(\frac{\psi}{\sqrt{2}} - \sqrt{2}x) (t\sqrt{2} - \frac{1}{\sqrt{2}}) - J(\frac{\psi}{\sqrt{2}} + \sqrt{2}x)) \\
\end{aligned}
\]
5. Some Numerical Simulations

We present here results of some simulations, which illustrate numerically the characteristics of the procedure developed in the previous sections. In order to simulate a random variable $Y$, having a wear-out distribution with parameters $\lambda, \alpha$ and $\tau$, proceed as follows. First, simulate a random variable $U$, having a uniform distribution on $(0,1)$. Then, solve for $Y$ the equation

$$\exp\left(-\lambda Y - \lambda^\alpha (Y-\tau)^{\frac{\alpha}{\lambda}}\right) = U$$  \hspace{1cm} (5.1)

Finally, determine $X = \min (Y, \Delta)$.

In the present study we restrict attention to the case of $\alpha = 2$.

Accordingly,

$$Y = \begin{cases} 
\frac{1}{\lambda} \ln U, & \text{if } U < e^{-\lambda \tau} \\
\tau - \frac{1}{2\lambda} + \frac{1}{\lambda} \left( \frac{1}{4} - \ln U - \lambda \tau \right)^{1/2}, & \text{if } U \geq e^{-\lambda \tau} \end{cases}$$ \hspace{1cm} (5.2)

In the following examples we consider a system with $\lambda^{-1} = 200$ [hr], $\Delta = 225$ [hr], $\tau = 750$ [hr], $\alpha = 2$ and the prior probability that $\{\tau < 0\}$ is $p = .2$. For the simulation of the random variable $X_n$, $(n=1,2,\ldots)$ we apply a value of $\tau_n = (\tau - \tau_{n-1}^+)$. The values of the posterior probabilities, $p_n$, and those of the Bayes estimators, $\tau_n$, are computed adaptively after each stage. In Table 5.1 we present the results of such a simulation run.

Table 5.1. A Simulation Run with Parameters

<table>
<thead>
<tr>
<th>n</th>
<th>$X_n$</th>
<th>$J_n$</th>
<th>$\tau_n$</th>
<th>$p_n$</th>
<th>$\tau_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>41.49</td>
<td>1</td>
<td>41.</td>
<td>0.288</td>
<td>742.71</td>
</tr>
<tr>
<td>2</td>
<td>156.56</td>
<td>1</td>
<td>198.</td>
<td>0.473</td>
<td>646.85</td>
</tr>
<tr>
<td>3</td>
<td>108.39</td>
<td>1</td>
<td>306.</td>
<td>0.631</td>
<td>514.63</td>
</tr>
<tr>
<td>4</td>
<td>225.00</td>
<td>0</td>
<td>531.</td>
<td>0.437</td>
<td>921.39</td>
</tr>
<tr>
<td>5</td>
<td>225.00</td>
<td>0</td>
<td>756.</td>
<td>0.309</td>
<td>1317.66</td>
</tr>
<tr>
<td>6</td>
<td>31.19</td>
<td>1</td>
<td>788.</td>
<td>0.385</td>
<td>1237.91</td>
</tr>
<tr>
<td>7</td>
<td>23.03</td>
<td>1</td>
<td>811.</td>
<td>0.446</td>
<td>1170.46</td>
</tr>
<tr>
<td>8</td>
<td>115.07</td>
<td>1</td>
<td>926.</td>
<td>0.610</td>
<td>1011.46</td>
</tr>
<tr>
<td>9</td>
<td>114.04</td>
<td>1</td>
<td>1040.</td>
<td>0.742</td>
<td>866.27</td>
</tr>
<tr>
<td>10</td>
<td>20.53</td>
<td>1</td>
<td>1060.</td>
<td>0.779</td>
<td>815.35</td>
</tr>
<tr>
<td>11</td>
<td>27.09</td>
<td>1</td>
<td>1087.</td>
<td>0.820</td>
<td>758.42</td>
</tr>
</tbody>
</table>
We see in the above example that the first value of $t_n$ larger than $\hat{t}_n$ is $t_n^* = 1040 \text{ [hr]}$. This is greater than $\tau$ by 290 [hr], which is a little over than 1 MTBF. The corresponding value of $p_n$ is $p_n^* = 0.742$. The estimate of $\tau$ is $\hat{t}_n = 866.27$. We see also that if we defer the decision until $p_n \geq 0.8$ then $t_n^* - \tau = 337 \text{ [hr]}$ and $\hat{t}_n = 758.42$. Thus, the second decision (stopping) time provided a more accurate estimate of $\tau$. In Table 5.2 we present frequency distributions and the exact means and standard deviations of $W_n = t_n^* - \tau$ as obtained by $M=100$ independent simulation runs, for each one of the prior parameters $\psi = 1/500$, 1/750 and 1/1000. The decision time in these simulations is the first value of $t_n$ greater than $\hat{t}_n$, i.e., $t_n^* = \text{least n} \geq 1$, such that $t_n^* \geq \tau$ (5.3)

### Table 5.2. Frequency Distributions, Means and Standard Deviations of $W_n$ in M=100 Simulation Runs

<table>
<thead>
<tr>
<th>Mid Point Interval</th>
<th>$1/\psi = 500$</th>
<th>$1/\psi = 750$</th>
<th>$1/\psi = 1000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-500</td>
<td>10</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-400</td>
<td>35</td>
<td>28</td>
<td>15</td>
</tr>
<tr>
<td>-300</td>
<td>17</td>
<td>37</td>
<td>22</td>
</tr>
<tr>
<td>-200</td>
<td>21</td>
<td>7</td>
<td>19</td>
</tr>
<tr>
<td>-100</td>
<td>8</td>
<td>11</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>100</td>
<td>6</td>
<td>7</td>
<td>10</td>
</tr>
<tr>
<td>200</td>
<td>3</td>
<td>9</td>
<td>5</td>
</tr>
<tr>
<td>300</td>
<td>0</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>400</td>
<td>0</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td><strong>Mean</strong></td>
<td>-278.15</td>
<td>-218.44</td>
<td>-117.72</td>
</tr>
<tr>
<td><strong>St. Dev.</strong></td>
<td>167.44</td>
<td>201.88</td>
<td>239.82</td>
</tr>
</tbody>
</table>

We see in Table 5.2 that the decision time $t_n^*$ tends to yield too many early decision points. If $1/\psi = 1,000 \text{ [hr]}$ the results are significantly better than in the case of $1/\psi = 500$ or 750 [hr]. The situation seems to be better in the case of the decision times defined by
\( t^{**} = \text{least } n \text{ s.t. } p_n \geq 0.9 \) \hspace{1cm} (5.4)

In Table 5.3 we present the frequency distributions of \( W^{**}_n = t^{**}_n - \tau \), obtained in independent simulation runs.

**Table 5.3. Frequency Distributions, Means and Standard Deviations of \( W^{**}_n \) in \( M=100 \) Simulation Runs**

<table>
<thead>
<tr>
<th>Mid Point of Intervals</th>
<th>Frequencies</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1/( \psi = 500 )</td>
</tr>
<tr>
<td>-300</td>
<td>15</td>
</tr>
<tr>
<td>-200</td>
<td>17</td>
</tr>
<tr>
<td>-100</td>
<td>17</td>
</tr>
<tr>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>100</td>
<td>6</td>
</tr>
<tr>
<td>200</td>
<td>4</td>
</tr>
<tr>
<td>300</td>
<td>8</td>
</tr>
<tr>
<td>400</td>
<td>2</td>
</tr>
<tr>
<td>500</td>
<td>12</td>
</tr>
<tr>
<td>600</td>
<td>9</td>
</tr>
<tr>
<td>700</td>
<td>0</td>
</tr>
<tr>
<td>Mean</td>
<td>65.60</td>
</tr>
<tr>
<td>St. Dev.</td>
<td>311.78</td>
</tr>
</tbody>
</table>

It seems that the Bayes decision times \( t^{**}_n \) based on the prior distribution (3.5), with a proper choice of the \( \psi \) value (not too small), provide good results in the problem of detecting a shift to the wear-out phase.
References


BAYES ADAPTIVE ESTIMATION OF THE POINT OF SHIFT TO THE WEAR-OUT PHASE OF RELIABILITY SYSTEMS

PERFORMING ORGANIZATION NAME AND ADDRESS
Department of Mathematical Sciences
SUNY-Binghamton
Binghamton, NY 13901

CONTRIBUTING OFFICE NAME AND ADDRESS
Office of Naval Research
Arlington, VA 22217

MONITORING AGENCY NAME AND ADDRESS (if different than Reporting Organization)
Office of Naval Research
Arlington, VA 22217

REPORT DATE
August 2, 1982

CONTRACT OR GRANT NUMBER(S)
N00014-81-K-0407

PERFORMING ORGANIZATION REPORT NUMBER

PROJECTTASK
UNIT NUMBER
NR 042-276

DISTRIBUTION STATEMENT (of this report)
APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED.

DISTRIBUTION STATEMENT (of the data encoded in this report)
non-classified

DECLASSIFICATION/DOWNGRADING SCHEDULE

A new family of life distributions, called the wear-out distributions, is developed on the basis of a failure rate function, which is a constant up to the change-point and strictly increasing afterwards. Properties of these wear-out distributions are derived and a Bayes adaptive procedure is developed for the estimation of the change point. Recursive formulae are given for the determination of the posterior probability that the change has occurred and of its Bayes estimator. The results of numerical simulations are given to illustrate the properties of the adaptive procedure.