AN ANALYTICAL, ONE-PARAMETER FAMILY OF SELF-ADJOINT BOUNDARY OPERATORS

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AN ANALYTIC, ONE-PARAMETER FAMILY OF SELF-ADJOINT BOUNDARY CONDITIONS FOR SCHröDINGER OPERATORS ON AN INTERVAL

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ABSTRACT

A one-parameter family of real, homogeneous boundary conditions on the
interval [0,1], under which the operator $-\frac{d^2}{dx^2}$ is self-adjoint, is
constructed. The relation between such boundary conditions and Lagrangian
planes in $\mathbb{R}^4$ is used and the resulting circle of boundary conditions is seen
to include Dirichlet, Neumann, periodic, antiperiodic, and several other well-
known examples.

AMS(MOS) Subject Classifications: 34B10; 34B25

Key Words: Deformation of Boundary Conditions; Self-adjointness.

Work Unit #1 - Applied Analysis

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SIGNIFICANCE AND EXPLANATION

A more or less explicit deformation of boundary conditions for an interval is constructed, under which the operator $-\frac{d^2}{dx^2}$ remains self-adjoint. The deformation depends analytically on its parameter and includes Dirichlet, Neumann, periodic, antiperiodic, and other boundary conditions. It is hoped that this family of self-adjoint boundary conditions can be used to construct solutions in problems where one set of boundary conditions (for example, periodic or Dirichlet) leads to a significant simplification of the problem.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.
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FOR SCHröDINGER OPERATORS ON AN INTERVAL

Robert L. Sachs

Consider the operator \( L = -\frac{d^2}{dx^2} \) acting on reasonably nice functions which satisfy the pair of real linear homogeneous boundary conditions

\[
\begin{align*}
(1) & \quad a_i y(0) + b_i y'(0) + c_i y(1) + d_i y'(1) = 0, \quad i = 1, 2. \\
\end{align*}
\]

By definition, this operator is self-adjoint if and only if the bilinear form:

\[
(2) \quad B(y, z) = y(0) z'(0) - y'(0) z(0) - y(1) z'(1) + y'(1) z(1)
\]

vanishes identically for all \( u, v \) satisfying the boundary conditions (1). In terms of column vectors \( Y \equiv (y(0), y'(1), y'(0), y(1))^T \), \( Z \equiv (z(0), z'(1), z'(0), z(1))^T \), (2) is equivalent to

\[
(3) \quad Y^T J Z = 0 \quad \text{where} \quad J \quad \text{is the usual} \quad 4 \times 4 \quad \text{symplectic matrix}
\]

\[
\begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix}
\]

where 0, I are 2 \times 2 matrices. If \( Y, Z \) are in the span of

\[
\begin{pmatrix}
\alpha_1 \\
\beta_1 \\
\gamma_1 \\
\delta_1
\end{pmatrix}, \quad \begin{pmatrix}
\alpha_2 \\
\beta_2 \\
\gamma_2 \\
\delta_2
\end{pmatrix}
\]

then (3) is equivalent to \((A^T B^T)J(A) = 0\) where

\[
A \equiv \begin{pmatrix}
\alpha_1 & \alpha_2 \\
\beta_1 & \beta_2
\end{pmatrix}, \quad B \equiv \begin{pmatrix}
\gamma_1 & \gamma_2 \\
\delta_1 & \delta_2
\end{pmatrix}
\]

and this leads immediately to the condition

\[
A^T B - B^T A = 0
\]

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which is clearly no stronger than the requirement

\[ A + iB \text{ is unitary.} \]

We shall construct a one-parameter family of unitary matrices \( U(t) \) connecting the matrix representing periodic boundary conditions:

\[ y(0) = y(1); \quad y'(0) = y'(1) \]

with the matrix representing Dirichlet boundary conditions:

\[ y(0) = y(1) = 0. \]

(6) is equivalent to \( Y \in \text{span} \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} \right\} \) which leads to the unitary matrix

\[
\begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\]

while Dirichlet boundary conditions (7) are equivalent to the unitary matrix \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Thus we seek a one-parameter family \( U(t) \) of unitary matrices with

\[ U(0) = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \quad U(1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

Such a family is easily found in the form

\[ U(t) = U(0)e^{ist} \text{ where } e^{ist} = U(0)^{-1}U(1) = \begin{pmatrix} -\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & \sqrt{2} \end{pmatrix} \]

i.e.

\[ is = \log \begin{pmatrix} -\sqrt{2} & i/\sqrt{2} \\ i/\sqrt{2} & \sqrt{2} \end{pmatrix} = \log(U(0)^{-1}U(1)). \]

Now \( M = U(0)^{-1}U(1) \) has eigenvalues \( \lambda = e^{-7\pi/12}(e^{\pi i/3}) = e^{\pi i/12}, e^{-7\pi i/12} \)

and, diagonalizing \( M \), we find
\( M = P \begin{pmatrix} e^{\pi i/12} & 0 \\ 0 & e^{-7\pi i/12} \end{pmatrix} P^{-1} \) where \( P, P^{-1} \) are given by the 2 \times 2 matrices

\[
\begin{pmatrix}
\frac{\sqrt{3}-1}{2} e^{\pi i/4} & \frac{\sqrt{3}+1}{2} e^{\pi i/4} \\
\frac{1}{2} & 1
\end{pmatrix}
\]

\( P^{-1} = \begin{pmatrix} e^{-\pi i/4} & \sqrt{3}+1 \\
\frac{\sqrt{6}}{2} & 2/3 \\
\frac{e^{-\pi i/4}}{\sqrt{6}} & \frac{\sqrt{3}-1}{2} \end{pmatrix} \)

Thus \( e^{ist} = P \begin{pmatrix} e^{\pi i/12t} & 0 \\ 0 & e^{-7\pi i/12t} \end{pmatrix} P^{-1} \)

and the desired path is

\( U(t) = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\
1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} e^{ist} \)

which, by a tedious computation, is precisely the matrix

\( U(t) =
\begin{pmatrix}
\frac{e^{-\pi i/4t}}{\sqrt{2}} (\cos(\pi/3t) - \frac{\sin(\pi/3t)}{\sqrt{3}}) & \frac{e^{-\pi i/4t}}{\sqrt{2}} (i \cos \frac{\pi}{3} t + (\frac{2+1}{\sqrt{3}}) \sin \frac{\pi}{3} t) \\
\frac{e^{-\pi i/4t}}{\sqrt{2}} (i \cos(\pi/3t) + (\frac{2+1}{\sqrt{3}}) \sin(\pi/3t)) & \frac{e^{-\pi i/4t}}{\sqrt{2}} (\cos(\pi/3t - \frac{1}{\sqrt{2}} \sin(\pi/3t)))
\end{pmatrix} \)
We see easily that $U$ has period 24 and that $U(t + 12) = -U(t)$, indeed
$U(t + 6) = iU(t)$ so that, in terms of the corresponding boundary conditions, we need only consider $U(t), 0 < t < 12$. Listing $U(j), j = 0, \cdots, 11$ and the corresponding boundary conditions, we have

\[
U(0) = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \\
U(1) = \begin{bmatrix} 0 & +i \\ 1 & 0 \end{bmatrix}
\]

(15)

\[
U(2) = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \\
U(3) = \begin{bmatrix} \frac{1+i}{2} & -\frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1+i}{2} \end{bmatrix}
\]

\[
U(4) = \begin{bmatrix} 0 & -\frac{1+i}{\sqrt{2}} \\ \frac{1+i}{\sqrt{2}} & 0 \end{bmatrix}, \\
U(5) = \begin{bmatrix} \frac{-1+i}{2} & -\frac{1+i}{2} \\ \frac{1+i}{2} & -\frac{1+i}{2} \end{bmatrix}
\]

\[
U(6) = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \\
U(7) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}
\]

\[
U(8) = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}, \\
U(9) = \begin{bmatrix} \frac{-1+i}{2} & \frac{-1-i}{2} \\ \frac{-1+i}{2} & \frac{-1-i}{2} \end{bmatrix}
\]

\[
U(10) = \begin{bmatrix} 0 & -\frac{-1-i}{\sqrt{2}} \\ \frac{-1+i}{\sqrt{2}} & 0 \end{bmatrix}, \\
U(11) = \begin{bmatrix} \frac{-1-i}{2} & \frac{-1-i}{2} \\ \frac{1+i}{2} & \frac{-1-i}{2} \end{bmatrix}
\]
The corresponding boundary conditions, found by reversing the procedure above, are as follows:

<table>
<thead>
<tr>
<th>t</th>
<th>Boundary conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( y(0) = y(1) ) , ( y'(0) = y'(1) )</td>
</tr>
<tr>
<td>1</td>
<td>( y(0) = 0 ) , ( y(1) = 0 )</td>
</tr>
<tr>
<td>2</td>
<td>( y(0) = 0 ) , ( y'(1) = 0 )</td>
</tr>
<tr>
<td>3</td>
<td>( y(0) = -y'(1) ) , ( y'(0) = y(1) )</td>
</tr>
<tr>
<td>4</td>
<td>( y(0) = -y'(0) ) , ( y'(1) = y(1) )</td>
</tr>
<tr>
<td>5</td>
<td>( y(0) = -y'(0) ) , ( y'(1) = -y(1) )</td>
</tr>
<tr>
<td>6</td>
<td>( y(0) = -y(1) ) , ( y'(0) = -y'(1) )</td>
</tr>
<tr>
<td>7</td>
<td>( y'(0) = 0 ) , ( y'(1) = 0 )</td>
</tr>
<tr>
<td>8</td>
<td>( y'(0) = 0 ) , ( y(1) = 0 )</td>
</tr>
<tr>
<td>9</td>
<td>( y(0) = y'(1) ) , ( y'(0) = -y(1) )</td>
</tr>
<tr>
<td>10</td>
<td>( y(0) = y'(0) ) , ( y'(1) = -y(1) )</td>
</tr>
<tr>
<td>11</td>
<td>( y(0) = y'(0) ) , ( y'(1) = y(1) )</td>
</tr>
</tbody>
</table>

Thus our one parameter family in fact includes periodic, Dirichlet, antiperiodic, Neumann, and several other well-known boundary conditions.

RLS/db
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**Author:** Robert L. Sachs

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