AN OPTIMIZATION TECHNIQUE FOR THE DEVELOPMENT OF TWO-DIMENSIONAL ETC(U)
AN OPTIMIZATION TECHNIQUE FOR THE DEPENDMENT OF TWO-DIMENSIONAL STEADY TURBULENT BOUNDARY LAYER MODELS

by

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**Title**: An Optimization Technique for the Development of Two-Dimensional Steady Turbulent Boundary Layer Models

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**Abstract**: A procedure for the development of a simple boundary layer turbulence model to account for different physical effects is described; the method is applied here to produce models for both pressure gradient and mainstream turbulence effects. Asymptotic theory is used to isolate the leading terms in an expansion for the mean velocity profile for high Reynolds numbers for both the inner and outer regions of a nominally steady two-dimensional boundary layer. The velocity profile in the outer layer satisfies a partial differential equation containing...
a Reynolds stress term and this term is modeled by a simple eddy viscosity function which contains two parameters. The velocity profile in the inner wall layer is modeled using an analytical expression which has been previously derived by consideration of the observed characteristics of the time-dependent flow in the wall layer and which contains a single independent parameter. For a self-similar flow, the outer layer equation becomes an ordinary differential equation; this equation is solved numerically and in conjunction with the analytical inner layer profile, a composite profile spanning the entire boundary layer is defined. This composite profile contains three parameters which may be adjusted systematically to obtain a best fit to a given set of experimental data. A computer optimization code is described in which any or all of the three profile parameters may be varied. This optimization code may be used simply to obtain a close analytical representation of a given set of data. The primary use described here, however, is to develop correlations for various physical effects from the results of the optimizations. In particular, correlations for the effects of mainstream pressure gradients and mainstream turbulence for the profile parameters are given. In principle, these correlations may then be used in a prediction method.
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ABSTRACT

A procedure for the development of a simple boundary layer turbulence model to account for different physical effects is described; the method is applied here to produce models for both pressure gradient and mainstream turbulence effects. Asymptotic theory is used to isolate the leading terms in an expansion for the mean velocity profile for high Reynolds numbers for both the inner and outer regions of a nominally steady two-dimensional boundary layer. The velocity profile in the outer layer satisfies a partial differential equation containing a Reynolds stress term and this term is modeled by a simple eddy viscosity function which contains two parameters. The velocity profile in the inner wall layer is modeled using an analytical expression which has been previously derived by consideration of the observed characteristics of the time-dependent flow in the wall layer and which contains a single independent parameter. For a self-similar flow, the outer layer equation becomes an ordinary differential equation; this equation is solved numerically and in conjunction with the analytical inner layer profile, a composite profile spanning the entire boundary layer is defined. This composite profile contains three parameters which may be adjusted systematically to obtain a best fit to a given set of experimental data.
A computer optimization code is described in which any or all of the three profile parameters may be varied. This optimization code may be used simply to obtain a close analytical representation of a given set of data. The primary use described here, however, is to develop correlations for various physical effects from the results of the optimizations. In particular, correlations for the effects of mainstream pressure gradients and mainstream turbulence for the profile parameters are given. In principle, these correlations may then be used in a prediction method.
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1. INTRODUCTION

Turbulent boundary layers occur in a wide variety of engineering applications including, for example, flows over turbine blades, airfoils, and in subsonic diffusers. In these and many other situations it is important to be able to develop the capability to accurately predict characteristics such as skin friction, lift, drag, and the onset of boundary layer separation. At present, the complex nature of turbulence seems to preclude any predictive analysis based on first principles. For a turbulent boundary layer which is steady in the time-mean sense, the classical approach to prediction has been to attempt to deal with the Reynolds time-averaged equations for the mean velocity components. These equations contain unknown functions referred to as Reynolds stresses; physically these terms are related to long time averages of products of fluctuations of the velocity components about their mean values. As a first step, the development of any prediction method requires a model to represent the behavior of the Reynolds stress terms. The objective of this study is to develop an approach which can be used to investigate the influence of various physical effects on the turbulent boundary layer and to incorporate these effects in a simple turbulence model. To this end, a general optimization code is developed in
which the parameters in the turbulence model may be varied systematically to obtain the best representation of measured mean profile data. As one example of how this code may be used, the method is applied to determine the influence of pressure gradients on the mean profile and a correlation for pressure gradient effects is developed. As a second example, the method is applied to some recent constant pressure profile data with mainstream turbulence and a correlation is obtained to reflect the effects of mainstream turbulence.

An immediate problem which arises in the modeling process is that the dynamics of turbulent boundary layers are not well understood. Previous investigators have attempted to resolve this problem by postulating the functional form for the Reynolds stress terms. These functional forms normally contain a number of unknown constants which are selected in a procedure often known as "computer optimization." The details of this procedure vary with the originator of the particular model and invariably are not well documented. However, the general approach is that particular data sets usually consisting of measured velocity profiles at numerous streamwise locations are "predicted" using various combinations of values of the "constants" associated with a given model. Some type of subjective judgement is then made as to which set of values of the constants best
"predict" as many data sets as possible. The approach adopted in this study is rather different and will now be discussed in detail.

An ideal approach to the modeling problem would be to isolate the primary features in the time-dependent turbulent flow and to pattern the turbulence model after motions which reflect the true physics of the turbulent boundary layer. Such a model would adequately replace the information lost in the time averaging and would presumably not contain or require a large number of adjustable constants. This approach is partially adopted in this study in the model used for the inner region of the turbulent boundary layer.

Since the objective is to isolate which effects are important and which are not for large Reynolds number, singular perturbation theory and the method of matched asymptotic expansions are used throughout the present study. It is worthwhile at this stage to summarize some of the main results concerning the asymptotic structure of the time mean boundary layer equations in the limit of large Reynolds numbers; these results have formally been demonstrated by Fendell (1972) and Mellor (1972). The turbulent boundary layer is a composite double layer consisting of a relatively thick outer layer having a thickness $O(u^*)$, (where $u^* = u_r/U_e(x)$ is the ratio of friction
to mainstream velocity) and a thin inner layer having a thickness $O(1/(Re u'))$, where $Re$ is the Reynolds number. In the joining region between the two layers, the velocity profile must be logarithmic for a self consistent asymptotic description. The complete mathematical results for the streamwise momentum equations are summarized in §2. One particularly important result concerns the inner layer and is that, to leading order, the convective terms are negligible in the time mean equations; consequently, if the mean profile is known in the inner layer, the Reynolds stress may be calculated and vice versa. In the present study, rather than a model for Reynolds stresses, a model for the inner region velocity profile is used; this model is based upon the observed coherent structure of the wall layer flow and will now be briefly described.

Over the past decade, it has been well documented (see, for example, Kline & Runstadler, 1959; Kline et al., 1967; Corino & Brodkey, 1969; Willmarth, 1975) that there is a considerable degree of ordered structure in the time dependent flow in the wall layer of a turbulent boundary layer. In particular, it is well known that there are two important phases associated with an observed cyclic behavior of the wall layer flow. In the first phase, if attention is focused on a fixed small area of the plate, the wall layer will be observed to
be in the quiescent state (Kline et al., 1967) for a majority of the total observation time. During this quiescent period, the wall layer streaks are observed with what appear to be longitudinal counter-rotating vortices between the streaks, the wall layer flow is relatively well ordered and no important interactions occur between the wall layer and the flow in the outer portion of the boundary layer; in this state, the wall layer may be regarded as passive. Eventually the second phase occurs which is generally known as the bursting phenomenon and which is characterized by a rapid and violent ejection from the wall layer into the outer layer. The ejection is of relatively short duration and is followed by an inrush of fluid from the outer layer; the streak structure appears again very rapidly and another quiescent period begins. Although many questions exist as to the causes and effects of these and other subsidiary events, the gross features of the cyclic behavior described above are now well established.

To incorporate such information in a prediction method for the time-mean flow, it is necessary in principle to analyze a typical event in the turbulence and then assess the contribution of this event to the time-mean quantities. To this end, Walker and Abbott (1977) argue, by consideration of the observed length and time scales in the wall layer, that during the quiescent period the equations for all three velocity components
must be linear and of the heat conduction type in the limit of large Reynolds number; in other words, the convective terms in the Navier-Stokes equations are negligible to leading order whenever the wall layer flow is in the quiescent state. Walker and Scharnhorst (1977) then go on to consider all possible similarity solutions of these equations which are compatible with theory and experiment; the similarity solutions correspond to the organized motion between streaks during the quiescent period and are radically different from the oscillatory Stokes type solutions which form the basis of the Van Driest (1956) model. Walker and Scharnhorst (1977) compute a time-average of the similarity solutions over a quiescent period and assess which solutions produce an important contribution to the time-mean profile and which do not. The final result is an analytical model which will be given in §2 for the inner region profile and which contains a single parameter S that is related to the mean period between bursts. The contribution to the mean profile during the bursting process and breakdown of the wall layer flow is neglected on the grounds that the breakdown is of short duration relative to the quiescent period; note that there are various theoretical reasons as well as a body of experimental evidence that verify that the period of breakdown must be small with respect to the quiescent period. On the
other hand, the vertical velocity (Walker & Abbott, 1977 and Walker & Scharnhorst, 1977) is so small during the quiescent period that there can be no contribution to the Reynolds stress to leading order during this period of time and the major contribution to the Reynolds stress must be made during the bursting process. Consequently, in this theoretical description of the wall layer flow, the dominant contribution to the mean profile occurs during the quiescent period, while the major contribution to the Reynolds stress occurs during the breakdown phase; both mean quantities are directly related to each through the leading order time-mean equations in the wall layer (since the convective terms are negligible to leading order).

In principle, it is desirable to develop a model for the outer layer which is also based on the observed dynamics of the time dependent flow in the outer region. However, the outer region problem is more complex; it appears necessary to model the Reynolds stress terms directly and unfortunately the dynamics of the outer layer are not well understood. In the outer region, large scale motions are observed which appear to be recirculating agglomerations of numerous smaller scale structures; these smaller scale structures have dimensions on the general magnitude of 100 wall layer units ($y^+$) and appear at times to be intensely vorticular in nature. At any stage, these small
vorticular structures are in various stages of coalescence and decay and when they pass close to the wall layer, are observed to induce eruptions of fluid. A number of authors have suggested that it is the vortex motion in the outer layer which induces the wall layer bursting (Nychas et al., 1973; Doligalski & Walker, 1978; Walker, 1978) and leads to the creation of new vortex structures in the outer layer; a regenerative mechanism for the production of new turbulence has recently been proposed by Doligalski, Smith & Walker (1981) and Doligalski (1981) on the basis of the observed unsteady effect of vortices on wall boundary layers. These studies demonstrate that for the vortex motions considered (two-dimensional vortices convected in a uniform flow and in a shear flow and ring vortices above a plane wall) that an eruption of the boundary layer flow near the wall will occur. However, these studies are as yet in an exploratory stage and, while a physical mechanism for the bursting is indicated, it is not yet possible to develop a constitutive model for the outer layer on this basis.

The bursting phenomena is a complex time-dependent viscous-inviscid interaction between the inner and outer layers which leads to the introduction of new vorticity into the outer layer and which occurs intermittently. However, during these brief periods of localized breakdown of the two layer structure, the
majority of the contribution to the outer layer (and inner layer) Reynolds stress occurs. For this reason, the simplifications which were possible in obtaining the wall layer model are not possible and it appears necessary to consider a typical burst and time average the results in order to model the $u'v'$ term. As discussed by Doligalski & Walker (1978) and Doligalski, Smith & Walker (1981) the inviscid-viscous interaction is a formidable theoretical problem at present; moreover, many other aspects of the outer layer are still not well understood and, at present, development of an outer layer model based on the characteristics of time dependent flow in the outer layer is not feasible.

For these reasons, a conventional type of eddy viscosity model is used in the present study for closure in the outer layer; this type of model is used here because of its simplicity and the good degree of success it has had in other prediction methods. The eddy viscosity hypothesis is commonly termed a first-order closure scheme; it assumes that there is a functional relation between the Reynolds stress and the mean profile and further that the Reynolds stress may be written as an eddy viscosity function times the mean velocity gradient. Here a model for the eddy viscosity which is similar to the Cebeci-Smith (1974) and Mellor & Herring models (1968) will
be adopted; this model is a monotonically increasing function of the outer variable behaving linearly with slope $\kappa$ (the von Karman "constant") near the logarithmic zone and approaching a value $K$ at the boundary layer edge. Note that the wall layer profile model contains $\kappa$ in addition to the cycle time parameter $S$. Consequently the present turbulence model contains the three parameters $(s, \kappa, K)$.

It is common practice in many turbulence models to assume constant universal values for the parameters appearing in the model. Scharnhorst (1978), for example, has attempted a prediction method with $(s = 10.495, \kappa = 0.41, K = 0.0168)$ (the latter two values are also used by Cebeci & Smith, 1974); however, it was determined (after the prediction method was compared to a number of data sets) that the velocity profiles were not well predicted, particularly in flows with pressure gradient. The objective of this study is to investigate a systematic way of determining any trends in these parameters for various effects such as pressure gradients or mainstream turbulence, with the ultimate goal of obtaining correlations of these parameters for use in a prediction method.

This will be carried out here in a somewhat different way from previous investigations. It has become common practice in recent times to determine values of "universal constants"
in turbulence models through a process known as computer optimization; this procedure is often associated with turbulence models that have a sizeable number of adjustable constants (see, for example, Murthy, 1977) and is usually carried out as follows. One data set or a number of data sets are chosen, each of which consists of a number of measured profiles at various downstream locations from some initial point; a prediction method with a preassigned set of constants in the turbulence model is started at the initial data station and the downstream data is "predicted." On the basis of comparison with either the integral parameters and/or the skin friction coefficient, a decision is made as to whether adjust the "constants" in the turbulence model to achieve better "predictions"; the prediction method is then used to obtain another "prediction" of the downstream data. This iterative method continues until some optimal set of constants is obtained.

One undesirable feature of this scheme is that the basis for adjusting the constants is usually not clearly defined and in any case is based upon comparisons with the gross properties such as the integral quantities or quantities obtained indirectly from experiment such as the skin friction. The details of how the iterative process is carried
out and what criteria are used to decide on the optimal set of constants are normally not supplied in the literature, particularly when the number of data sets involved are large; presumably the choice is based ultimately on some type of integrated subjective average.

In the present study, the trends in the turbulence model parameters and eventually the correlations are also determined by comparison to experimental data but the procedure is different and precisely defined. First, only equilibrium data sets are considered; an equilibrium turbulent boundary layer, by definition, is a boundary layer in which the mainstream velocity varies algebraically with streamwise distance. Such a boundary is expected (Clauser, 1956; Mellor, 1972; Fendell, 1972; Scharnhorst et al., 1978) to successively approach a self-similar flow at large distances from wherever the boundary layer flow is initiated. For this reason, equilibrium boundary layers have historically been of considerable interest.

It is worthwhile to remark, however, that, in order for self-similar velocity profiles to exist, equilibrium is a necessary but by no means a sufficient condition. To expand on this point, consider the case of laminar boundary layers where similar solutions satisfy the well known Falkner-Skan equation. The existence of such solutions in laminar flows
has been discussed by Brown & Stewartson (1965) who argue that similar solutions may be expected in two physical situations. The first of these is at an x station where the velocity profile is an initiator of the boundary-layer flow downstream; such a situation occurs physically at the front stagnation point of a bluff body or the leading edge of a flat plate, for example. In these cases, the Falkner-Skan profile gives the proper laminar boundary-layer solution at a point of attachment of the mainstream and provides the initial condition to initiate a boundary-layer calculation downstream. For turbulent boundary layer, there appears to be no analogue of this physical situation. In the absence of mainstream turbulence, the flow at a point of attachment of the mainstream is observed to be laminar and when the downstream boundary-layer flow is turbulent, there is a transition zone in between the laminar and the turbulent flow. Moreover, a wide variety of upstream experimental conditions can lead to transition and an eventual fully-developed turbulent boundary layer downstream for the same mainstream velocity distribution. Consequently, there would appear to be no reason to expect that the flow in the outer region of an equilibrium boundary layer will be self-similar at the initial stations of a fully developed turbulent flow.

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For laminar boundary layers, the other case discussed by Brown and Stewartson (1965) is that where a similarity solution becomes what may be described as a terminator of a more general boundary-layer flow and two cases of this behavior are considered by Brown and Stewartson (1965). A physical case where this can occur is at a point of detachment of the inviscid flow; examples of such behavior are known in magneto-hydrodynamic flow (Leibovich, 1967a,b) and in rotating flows (Walker & Stewartson, 1972) at the rear stagnation point of symmetrical and two-dimensional bluff bodies. Another case is flat plate flow and here if the initial velocity profile at any arbitrary location on the plate is not given by the Blasius solution, then the Blasius profile can only become the relevant solution at an infinite distance downstream. For turbulent boundary-layer flows, an analogous type of situation is expected; that is, similarity solutions are only anticipated as terminators and usually at large distances downstream from wherever the boundary layer is initiated.

In practice, one would expect to be able to measure turbulent boundary-layer profiles, at large distances downstream of the transition zone, which become arbitrarily close to being self-similar; however, near the transition zone, there is no reason to expect a self-similar behavior. In zero and favorable pressure gradients, measured profiles in an equilibrium flow should increasingly approach self-similarity
at subsequent data stations downstream. In adverse pressure gradient flows, there is an additional difficulty in that reversed flow and boundary layer separation may occur before similarity is achieved.

The self-similar flow is particularly attractive insofar as turbulence model development is concerned because the parabolic partial differential equation governing the turbulent boundary layer flow becomes an ordinary differential equation of the boundary value type. In the present study, the ordinary differential equation associated with the outer layer is solved numerically for a given set of the turbulence model parameters \((\kappa,K)\) to determine an outer layer velocity profile; this outer layer profile is matched with an inner layer profile containing the inner profile parameters \((s,\kappa)\) and a composite velocity profile for the entire boundary layer is defined. The composite profile is then compared directly to measured experimental velocity profile data and a root-mean-square error is defined as a criterion of how well the profile represents the data. The optimization procedure then adjusts one or more of the three profile parameters \((s,\kappa,K)\) until a "best fit" to each profile is obtained. Once this procedure has been carried out for several data stations, the results are plotted to determine any trends in the profile parameters; the
objective here is to obtain correlations for the turbulence model parameters for use in a prediction method. In the present study, this procedure is carried out for two situations and correlations are obtained for $\kappa$ and $K$ for the effects of pressure gradients and mainstream turbulence. Note that this type of procedure is only applicable to measured profiles for which the mainstream velocity is of the equilibrium type and where the data is at locations far downstream of wherever the turbulent boundary layer was initiated. The final correlations for the effects of pressure gradients and mainstream turbulence lead to excellent representations of the velocity profile data. In principle, the models developed here may be used in a prediction method for non-equilibrium flows.

The plan of this report is as follows. In §2, the basic equations and principle results of the asymptotic theory are summarized. In §3 the optimization procedure is described and in §4 and §5 the results of the method for pressure gradients and mainstream turbulence effects respectively are given. A description and test cases for the optimization code are given in Appendices C and D. Finally, the conclusions of the study are discussed in §6.
2. THE TURBULENT BOUNDARY LAYER TIME MEAN EQUATIONS

2.1 Basic Equations

The turbulent boundary layer equations governing two-dimensional, incompressible time-mean flow are the mass continuity equation and the Reynolds equation for streamwise momentum:

\[ \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 , \]  
\[ \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = \bar{U}_0 \frac{d\bar{u}}{dx} + \frac{1}{Re} \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial \sigma}{\partial y} . \]  

In these equations, \((x,y)\) represent Cartesian coordinates with corresponding mean velocities \((\bar{u},\bar{v})\). All lengths and velocity components have been made dimensionless with respect to \(L\) and \(U_0\), a reference length and velocity respectively. The Reynolds number is defined as \(Re = \frac{U_0 L}{\nu}\) and is assumed to be large; here \(\nu\) is the kinematic viscosity. The momentum equation (2.2) contains an additive stress term \(\sigma\) for turbulent flow which is related to the usual turbulent shear stress by

\[ \sigma = \frac{-\bar{u}' \bar{v}'}{U_0^2} . \]  

In equation (2.3) the primed quantities are instantaneous velocity fluctuations about the corresponding mean values and the over bar signifies a long time average. The boundary conditions
associated with the equations (2.1) and (2.2) are

\[ u(x,0) = v(x,0) = 0 , \]

\[ u(x,y) \to U_\infty(x) \text{ as } y \to \infty . \]

In addition to the above boundary conditions, the turbulent shear stress term must be chosen to satisfy

\[ \sigma(x,0) = 0, \sigma(x,y) \to 0 \text{ as } y \to \infty . \]

To completely specify the problem, an initial velocity profile

\[ u(0,y) = f(y) \text{ for } 0 \leq y < \infty \]

must be given at some x-station in the fully turbulent region of the flow which is denoted here by x=0. Note that the problem is indeterminate until a model for the Reynolds shear stress term is specified; however, it is possible through asymptotic analysis of this problem to provide useful information about the velocity components and shear stress without the introduction of any specific functional form for the Reynolds stress term. Before this is carried out, it is desirable to define an important physical parameter, \( u^* \), that will play an important role in the asymptotic analysis. The dimensional wall shear stress is given by
and is used to define the dimensional skin friction velocity

\[ u_\tau = \sqrt{\frac{\tau_w}{\rho_w}} \]

(2.8)

where \( u_w \) and \( \rho_w \) represent the dynamic viscosity and density at the wall respectively. A dimensionless skin friction velocity can be defined as

\[ u^* = \frac{u_\tau}{U_0 U_w(x)} \]

(2.9)

Note that equation (2.7) can be rewritten in an equivalent form as

\[ u^{*2} U_w^2 = \frac{1}{Re} \frac{\partial u}{\partial y} \bigg|_{y=0} \]

(2.10)

The asymptotic structure of equation (2.2) in the limit of large Reynolds number will now be considered.

2.2 Asymptotic Structure of the Time-Mean Equations

It is well known that the turbulent boundary layer is a composite double layer consisting of a thin inner layer adjacent to the wall and a relatively thick outer layer. A number of authors, including Fendell (1972) and Mellor (1972), have
considered the asymptotic structure of the time-mean momentum equations for a constant property incompressible flow; in these studies, asymptotic methods are used to isolate the leading terms in an expansion for large Reynolds number for the flow quantities in both the inner and outer region of the turbulent boundary layer. These expansions are then matched in a manner which is consistent mathematically and which is also consistent with experimental measurements of the time-mean flow quantities for both regions. Fendell (1972) also examines the conditions necessary for self-similar flow in the outer region. The asymptotic results that follow summarize the work of Fendell (1972) and are also discussed by Scharnhorst et al (1978).

In the outer region of the turbulent boundary layer, the velocity profile may be written as a small perturbation about the mainstream value as $\operatorname{Re} \to \infty$, according to,

$$
\bar{u} = U_\infty(x) \left(1 + u^*(x; \operatorname{Re}) \frac{\partial F}{\partial \eta}(x, \eta) + \ldots \right),
$$

(2.11)

Here $u^*$ is the dimensionless skin friction velocity defined by equation (2.9) and $u^* \to 0$ as $\operatorname{Re} \to \infty$. In the outer region, the turbulence term $\sigma$ may be written to leading order as

$$
\sigma = U_\infty^2(x)(u^2(x; \operatorname{Re}) \Sigma_1(\eta, x) + \ldots).
$$

(2.12)

Here $\eta$ is the scaled normal coordinate for the outer region.
defined by,
\[
n = y/\Delta_0 .
\] (2.13)

The dimensionless outer region length scale $\Delta_0$ is proportional to the boundary layer thickness and is $O(u^*)$; a convenient choice for $\Delta_0$ is made in section 2.4. It is of interest to note that the shear stress term given in (2.12) is equivalent to the result that $\overline{u'v'}$ is $O(u_{+}^{2})$ which, in general, is confirmed by experimental measurement. When the velocity profile expansion (2.11) and the turbulence term expansion (2.12) is substituted into (2.2) and terms quadratic in the perturbation $u^*$ are neglected, an equation for the velocity defect function $u_1 = \partial F_1/\partial n$ is obtained according to

\[
\frac{\partial \Sigma_1}{\partial n} = \frac{\Delta_0}{u^2 U_{\infty}} (u^* U_{\infty}^2)' \frac{\partial F_1}{\partial n} - \frac{1}{u^2 U_{\infty}} (U_{\infty} \Delta_0)' n \frac{\partial^2 F_1}{\partial n^2} + \frac{\Delta_0}{u^*} \frac{\partial^2 F_1}{\partial n \partial x} .
\] (2.14)

Here a prime denotes differentiation with respect to $x$.

This equation is subject to the boundary conditions

\[
F_1(x,n) \to 0 \text{ as } n \to 0 ,
\] (2.15)

and

\[
\frac{\partial F_1}{\partial n} (x,n) \to 0 \text{ as } n \to \infty .
\] (2.16)
In the inner layer, sometimes referred to as the wall layer, the velocity profile and turbulence stress term expansions may be written to leading order according to

\[ \tilde{u} = u^* U_\infty(x) \frac{\partial f_1}{\partial y^+}(x,y^+) + \ldots, \]  
(2.17)

and

\[ \sigma = U_\infty^2(x) u^* \sigma_1(x,y^+) + \ldots, \]  
(2.18)

where

\[ y^+ = \frac{y}{u^*} = Re u^* U_\infty(x) y, \]  
(2.19)

is the inner region variable. The matching of the leading order terms in the asymptotic expansions (2.11) and (2.17) and (2.12) and (2.18) occur in the limits \( n \to 0 \) and \( y^+ \to \infty \) for the inner and outer layers respectively. A self consistent mathematical structure which is compatible with experimental measurement may be obtained if both velocity profiles merge smoothly with a logarithmic profile behavior according to

\[ u_1 - \frac{1}{\kappa(x)} \log n + C_0(x), \text{ as } n \to 0, \]  
(2.20)

and

\[ u^+ - \frac{1}{\kappa(x)} \log y^+ + C_1(x), \text{ as } y^+ \to \infty. \]  
(2.21)

Here \( \kappa, C_0 \) and \( C_1 \) are in general functions of \( x \). Although \( \kappa \)
is analogous to the von Karman "constant" which is normally assumed to have a value of about 0.41, in general $\kappa$ could depend on local flow conditions. It is worthwhile to note that the conditions for the turbulence terms also follow from the analysis (Fendell, 1972) and are

$$\varepsilon_1 \to 1 \text{ as } \eta \to 0, \quad (2.22)$$

and

$$\sigma_1 \to 1 \text{ as } y^+ \to \infty. \quad (2.23)$$

First order matching of the inner and outer asymptotic expansions leads to the velocity match condition given by,

$$\frac{1}{u^*} = \frac{1}{\kappa(x)} \log(\Delta_0 \text{ Re } u^* u(x)) + C_1 - C_0. \quad (2.24)$$

This skin friction relation connects $u^*$ and the outer region length scale $\Delta_0$. The match condition to leading order as $\text{Re} \to \infty$ implies that

$$u^* \sim \frac{\kappa(x)}{\log \text{ Re}} \quad (2.25)$$

and since $\Delta_0$ is $O(u^*)$ the turbulent boundary layer thickness approaches zero as the inverse of a logarithm in the high Reynolds number limit.

A composite velocity profile valid to leading order across the entire boundary layer can be formulated. Van Dyke (1975)
defines a composite expansion according to

\[
[\text{composite expansion}] = [\text{inner expansion}]
+ [\text{outer expansion}] - [\text{common terms}]. 
\]

The common terms represent the behavior of the inner and outer expansions in the matching region. A composite streamwise velocity profile is defined here according to,

\[
\bar{u} = u^* U^* \left[ U^* + U_1 - \frac{1}{\kappa^2} \log n + C_0 \right]. 
\]

Alternatively, an equivalent expression could be composed by using equation (2.24) for the common terms in terms of the inner region variable \( y^+ \).

The question of determining good model profile approximations for the inner profile \( U^* \) and the outer profile \( U_1 \) will now be considered.

2.3 Similarity in the Outer Region

The model profile that will be used for the outer region flow is a self-similar profile and for this reason it is of interest to examine the conditions necessary for self-similar flow in the outer region. The terms self-similar and equilibrium are often used interchangeably in the literature but it is important to make a distinction here. The term equilibrium is understood to apply
to a turbulent boundary layer for which the mainstream velocity behaves as \( U_\infty(x) = x^\alpha \) or \( U_\infty(x) = e^{\beta x} \) where \( \alpha \) and \( \beta \) are constants; on the other hand, self similarity of the outer layer velocity profile can only occur when the boundary layer is exposed to the equilibrium outer velocity contributions for large streamwise distances. Consequently, equilibrium is a necessary but not sufficient condition for the existence of self-similar profiles.

The necessary conditions for similarity follow from equation (2.17) for \( U_1 \) and are that,

\[
\frac{\Delta_0}{u^*u_\infty} (u^*U_\infty^2)' = -2\beta = \text{constant} \tag{2.28}
\]

and

\[
\frac{1}{u^*u_\infty} (\Delta_0 U_\infty)' = \alpha = \text{constant}. \tag{2.29}
\]

To assess the magnitude of the ratio \( u^*/u_* \) which appears in equation (2.28), the match condition, given by equation (2.24), is differentiated with respect to \( x \) to obtain,

\[
- \frac{u^*}{u^*u_\infty} = -\frac{\kappa'}{\kappa} \log(\Delta_0 Re u^*U_\infty(x)) + \frac{1}{\kappa} \left( \frac{\Delta_0'}{\Delta_0} + \frac{u^*'}{u_*'} + \frac{U_\infty'}{U_\infty} \right) C_1' - C_0'. \tag{2.30}
\]

This equation can be simplified by using the match condition (2.24) to eliminate the first term on the right hand side and this procedure yields
\[
\frac{u^*'}{u^*} = \frac{\kappa'}{\kappa} - \frac{u^*}{\kappa} \left( \Delta_0' \Delta_0 + \frac{u^*'}{u^*} + \frac{U_\infty'}{U_\infty} + \kappa'(C_i-C_o') + \kappa(C_i-C_o) \right).
\]  

(2.31)

Since \( \kappa \) must approach a constant for a self-similar flow, the order of magnitude of the ratio \( u^*' / u^* \) follows from equation (2.31) and

\[
\frac{u^*'}{u^*} = O(u^*).
\]

(2.32)

Terms containing this ratio may be neglected to leading order in (2.28) and (2.29) which become

\[
q \frac{U_\infty'}{U_\infty} = -\beta
\]

(2.33)

and

\[
q' + \frac{U_\infty'}{U_\infty} q = \alpha
\]

(2.34)

respectively where \( q = \Delta_0' / u^* \). These two equations are combined to give

\[
q' = \alpha + \beta.
\]

(2.35)

As a result, there are only two types of mainstream velocity distributions which can lead to self-similar solutions in the outer layer and these are:

\[
U_\infty(x) = D_1(x-x_0)^{\alpha_1} \text{ for } \alpha + \beta \neq 0
\]

(2.36)

where
\[ a_1 = \frac{-q}{\alpha+\beta}, \quad q = \frac{\Delta_0}{u^*} = (\alpha+\beta)(x-x_0) \]  

(2.37)

and

\[ U_\infty(x) = D_2 e^{-\alpha_2 x} \quad \text{for} \quad \alpha + \beta = 0 \]  

(2.38)

where

\[ \alpha_2 = \frac{\beta}{x_0}, \quad q = \frac{\Delta_0}{u^*} = x_0. \]  

(2.39)

Here \( D_1, D_2, \) and \( x_0 \) are all arbitrary constants.

Equilibrium flows have been examined experimentally for a number of years and an equilibrium boundary layer has been defined experimentally as a flow in which the dimensionless velocity defect \((U_e - \bar{U}_e^*)/u^\tau\) expressed as a function of \(y/\delta\) becomes close to being invariant with downstream distance. After careful experimentation, Clauser (1954, 1956) concluded that the criterion for equilibrium was a constant value of the parameter \( \beta_c \) which is defined as

\[ \beta_c = -\frac{\delta^*}{u^\tau} U_e \frac{dU_e}{dx}. \]  

(2.40)

Here \( \delta^* \) represents the dimensionless displacement thickness defines as

\[ \delta^* = \int_{0}^{\infty} (1 - \frac{\bar{U}}{U_e(x)})dy. \]  

(2.41)

From the definition of \( \delta^* \) and equations (2.11) and (2.13) it
follows that,
\[ \delta^* = -\Delta_0 u*F_1(x,\infty) + \ldots \]  
(2.42)

As self-similarity conditions are approached, the x dependence must vanish and \( F_1(x,\infty) \) must approach a constant value, say \( F_1(\infty) \). Thus (2.40) becomes
\[ \beta_c = \frac{\Delta_0 F_1(\infty) dU_\infty}{u*U_\infty} \frac{d}{dx} = -\beta F_1(\infty) . \]  
(2.43)

Note that a constant value of \( \beta \) implies a constant value of \( \beta_c \) and therefore Clauser's (1954,1956) experimental results are consistent with theory.

2.4 The Eddy Viscosity Model for the Outer Region

In order to obtain a solution of the outer layer equations, a constitutive relation for \( u'v' \) in the outer layer must be specified and for the reasons discussed in section 1, a simple eddy viscosity model will be used here. It is worthwhile to note that although it is customarily assumed that some functional relationship exists between \( u'v' \) and the mean velocity profile, no such relationship has been demonstrated either experimentally or theoretically; consequently, the eddy viscosity hypothesis should be regarded at present as a convenient approximation which is expected to be supplanted in the future by constitutive
models based on the observed coherent structure of the turbulent boundary layer.

For the outer region, the Reynolds stress is defined as

\[ -u^+v^+ = \varepsilon_0 \frac{U_0}{L} \frac{3u^+_i}{\partial y} = U_0^2 \sigma, \]

(2.44)

and from equations (2.12) and (2.13), the stress function \( \xi_1 \) becomes

\[ \xi_1 = \frac{\varepsilon_0}{\Delta_0 U_0 U_\infty(x) U_\infty^*} \frac{3U_1}{\partial n}, \]

(2.45)

where \( \varepsilon_0 \) depends upon both \( x \) and \( n \). The functional form for \( \varepsilon_0 \) is the same type as that used in the Cebeci-Smith (1969) and Mellor and Herring (1969) prediction methods. In this model, \( \varepsilon_0 \) is selected to approach a constant value for fixed \( x \) near the outer edge of the outer layer. Thus,

\[ \varepsilon_0 + K U_e(x) \delta^* L \]

(2.46)

where \( \delta^* \) is the dimensionless displacement thickness and \( K \) is an empirically determined constant. The value of \( K \) differs slightly according to the model; Cebeci and Smith (1969) take \( K = 0.0168 \) while Mellor and Herring (1968) use \( K = 0.016 \). In the present study, a universal value for \( K \) is not assumed; rather one objective is to determine if this parameter depends in any way on the pressure gradient.
Near the inner edge of the outer layer, the eddy viscosity must behave linearly to satisfy the matching condition for $\varepsilon_1$ given in equation (2.22); it is convenient to define a dimensionless eddy viscosity function $\varepsilon(n)$ according to

$$\varepsilon(n) = \frac{\varepsilon_0}{\delta^* U_0 U_\infty(x) L}, \quad (2.47)$$

whereupon equation (2.45) becomes

$$\varepsilon_1 = \frac{\delta^*}{u^* A_0} \varepsilon(n) \frac{\partial U_1}{\partial n}. \quad (2.48)$$

The function $\varepsilon(n)$ must assume the following limiting values:

$$\varepsilon + \kappa \eta \left( \frac{u^* \Delta_0}{\delta^*} \right) \text{ as } n \to 0, \quad (2.49)$$

and

$$\varepsilon \to K \text{ as } n \to \infty. \quad (2.50)$$

A particularly convenient choice for the outer region length scale $A_0$ is

$$\Delta_0 = \frac{\delta^*}{u^*}, \quad (2.51)$$

and it follows that $F_1(x, \infty) = -1$ from equation (2.42). Detailed reasons behind this choice for $\Delta_0$ are discussed by Scharnhorst et al. (1978) and Scharnhorst (1978) and Weigand (1978). Equations (2.45) and (2.49) may now be rewritten as
\[ \varepsilon_1 = \varepsilon(n) \frac{3U_1}{3n} \quad (2.52) \]

and

\[ \varepsilon + \kappa n \quad \text{as} \quad n \to \infty. \quad (2.53) \]

In general, the functional form of \( \varepsilon(n) \) must be specified. The model of Mellor and Gibson (1966) and Mellor and Herring (1969) use a simple form of two straight lines defined by

\[ \varepsilon(n) = \varepsilon_m(n) = \begin{cases} K & \text{for} \quad n \geq n_1, \\ \kappa n & \text{for} \quad n < n_1, \end{cases} \quad (2.54) \]

where \( n_1 = K/\kappa \). An awkward feature of this model lies in the discontinuous derivative at \( n = n_1 \) which may be expected to give rise to difficulties in a numerical solution of the outer layer equation; this problem is handled by Cebeci and Smith (1974) by using an artificial smoothing of the model in equation (2.54).

An alternative functional form for \( \varepsilon(n) \) must reflect the linear behavior near \( n=0 \) in equation (2.53) and the limiting constant value in equation (2.50) for large \( n \); moreover, such a function should be monotonically increasing with \( n \). A rather complex function meeting these requirements was assumed by Scharnhorst et al. (1977) who, in addition, determined that it appeared to be desirable for \( \varepsilon(n) \) to approach the linear behavior \( \kappa n \) exponentially quickly for small values of \( n \). It also appeared
important for $\varepsilon(n)$ to approach the upper bound $K$ relatively quickly for large $n$.

A particularly simple form of a function that satisfies all the desired features of $\varepsilon(n)$ is

$$\varepsilon(n) = \kappa n (1 - e^{-C/n^{N}})$$  \hspace{1cm} (2.55)

which is plotted in Figure 2.1 for integral values of $N = 1, 2, 3, 4$. Here, the term $C$ is given by

$$C = \left(\frac{K}{\kappa}\right)^{N}.$$  \hspace{1cm} (2.56)

This monotonically increasing function (2.55) meets the requirements as $n \to 0$,

$$\varepsilon(n) \to \kappa n,$$  \hspace{1cm} (2.57)

and as $n \to \infty$,

$$\varepsilon(n) \to K \{1 + O\left(\frac{1}{n^{N}}\right)\}.$$  \hspace{1cm} (2.58)

Note that this function for $N=4$ is virtually identical to the form used by Scharnhorst et al. (1977) but is a much simpler form.

The outer region eddy viscosity function given by equation (2.55) was used throughout this study with $N=4$. 

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2.5 Outer Layer Similarity Model

With the outer layer stress function defined by equation (2.52), it may be shown that the outer layer equation (2.14) becomes

\[
\frac{\partial}{\partial n} \left( \varepsilon(n) \frac{\partial U_1}{\partial n} \right) + \frac{\Delta_0 (u^* U_0^2)}{u^* U_\infty} \frac{\partial U_1}{\partial n} - \frac{\Delta_0 (u^* U_0^2)}{u^* U_\infty} U_1 = \frac{\Delta_0}{u^*} \frac{\partial U_1}{\partial x},
\]

(2.59)

where \( U_1 = \partial F_1 / \partial n \). For a self-similar flow, equation (2.59) reduces to the ordinary differential equation

\[
\frac{d}{dn} \left( \varepsilon \frac{dU_1}{dn} \right) + \left( \delta^* \frac{u^*}{u^*} - \beta_C \right) n \frac{dU_1}{dn} + 2\beta_c U_1 = 0,
\]

(2.60)

where \( \beta_C \) is the Clauser pressure gradient parameter obtained from equation (2.40) as \( \beta_C = -\delta^* U_\infty / (u^* U_\infty) \). Integration of equation (2.60) across the boundary layer yields the relation

\[
\frac{\delta^*}{u^*} = 1 + 3\beta_c,
\]

(2.61)

which can be substituted back into (2.60), to obtain

\[
\frac{d}{dn} \left( \varepsilon(n) \frac{dU_1}{dn} \right) + (1 + 2\beta_C) n \frac{dU_1}{dn} + 2\beta_c U_1 = 0.
\]

(2.62)

To find the velocity profile in the outer region, equation (2.62) must be solved subject to the boundary conditions.
\[ U_1 \sim \frac{1}{\kappa} \log \eta + C_0 \quad \text{as} \quad \eta \to 0, \quad (2.63) \]
\[ U_1 \to 0 \quad \text{as} \quad \eta \to \infty. \quad (2.64) \]

Note that the constant \( C_0 \) is unknown. This problem was solved by a combination of a series solution near \( \eta = 0 \) and a fully numerical solution for \( \eta > 0 \); the procedure is described in Appendix A.

Note that there are two parameters associated with equation (2.62) and the boundary condition (2.63); these are the eddy viscosity parameters \( K \) and the von Karman constant \( \kappa \) which appears in the eddy viscosity and in the boundary condition. In addition to the physical boundary layer quantities, the skin friction \( u_\tau \) and the displacement thickness are contained in the parameter \( \delta_c \) which is defined in equation (2.40) and which appears in equation (2.62).

2.6 Wall Layer Model

The wall layer model used in this study incorporates what are believed to be the important features of the time dependent flow in the wall layer. A complete discussion of the ideas that develop the unsteady wall layer model is given in Walker, Abbott and Scharnhorst (1976), Walker and Abbott (1977), and Walker and Scharnhorst (1977). This "unsteady wall layer model" has

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been extensively compared to experimental data by Scharnhorst et al. (1977) who demonstrate that the model closely represents measured velocity profile data in the wall layer even in flows with pressure gradient; moreover, the representation is considerably better than that obtained by a conventional Van Driest (1956) type of model and, finally, the model is simpler to use. For these reasons, this "unsteady wall layer model" was used exclusively in the present study.

The profile given by Walker and Scharnhorst (1977) is

\[ u^+ = \left[ 1 + \frac{t_0^+}{S^2} \right] \left[ R(S^2, t_0^+) Q(H) + Z(H) + P(S^2, t_0^+) W(H) \right] \]
\[ - \frac{t_0^+}{S^2} \left[ R(0, t_0^+) Q(H_0) + Z(H_0) + P(0, t_0^+) W(H_0) \right], \tag{2.65} \]

where

\[ H = \frac{y^+}{2\sqrt{S^2 + t_0^+}}, \quad H_0 = \frac{y^+}{2t_0^+}, \tag{2.66} \]

and

\[ R(S^2, t_0^+) = C_i - \frac{1}{\kappa} \left( \frac{y_0}{2} - \log 2 \right) + \frac{1}{2} p^+(S^2 + 2t_0^+) \]
\[ + \frac{1}{2\kappa} \log (S^2 + t_0^+), \tag{2.67} \]

\[ Q(H) = (2H^2 + 1) \text{erf} H + \frac{2}{\sqrt{\pi}} H e^{-H^2}, \tag{2.68} \]

\[ Z(H) = \frac{4}{\kappa \sqrt{\pi}} \left[ (2H^2 + 1) \Xi(H) + H \Xi'(H) - \frac{\sqrt{\pi}}{8} (6H^2 + 1) \text{erf} H - \frac{3}{4} H e^{-H^2} \right], \tag{2.69} \]
\[ P(S^2, t_0^+) = -\frac{2}{3} p^+(S^2 + t_0^+) \quad (2.70) \]

\[ W(H) = [H^4 + 3H^2 + \frac{3}{4}] \text{erf} H - \frac{H}{\sqrt{\pi}} \left[ H^2 + \frac{5}{2} \right] e^{-H^2} - 3H^2 \quad (2.71) \]

In the equations above, the \( \Xi \) function is defined as

\[ \Xi(\eta) = \int_0^h e^{-x^2} \int_0^x e^{y^2} \int_0^{-t^2} dt \, dy \, dx \quad (2.72) \]

A detailed description of the \( \Xi \) function and its properties is given in Appendix B. For large \( y^+ \) in the inner region, the asymptotic form of the profile (2.65) is

\[ U^+ = \frac{1}{\kappa} \log y^+ + C_i \quad (2.73) \]

Note that the model contains \( \kappa \) and \( C_i \) as parameters in addition to \( t_0^+ \) and \( S \); these last two parameters are associated with the physics of the wall layer time dependent flow and are discussed in detail by Walker and Scharnhorst (1977) and Scharnhorst (1978). The mean velocity profile of (2.65) must satisfy

\[ \frac{\partial U^+}{\partial y} \bigg|_{y^+=0} = 1 \quad (2.74) \]

and the wall compatibility condition

\[ \frac{\partial^3 U^+}{\partial y^+3} \bigg|_{y^+=0} = 0 \quad (2.75) \]
which leads to the auxiliary relations

\[
(S^2 + t_0^+)^{\frac{3}{2}} [R(S^2, t_0^+)- \frac{1}{\kappa} + P(S^2, t_0^+)]
- (t_0^+)^{\frac{1}{2}} [R(0, t_0^+)- \frac{1}{\kappa} + P(0, t_0^+)] = \sqrt{\frac{\pi}{2}} S^2,
\]

and

\[
(S^2 + t_0^+)^{-\frac{1}{2}} [R(S^2, t_0^+) + 3P(S^2, t_0^+)]
- (t_0^+)^{-\frac{1}{2}} [R(0, t_0^+) + 3P(0, t_0^+)] = 0,
\]

respectively. Note that equations (2.76) and (2.77) are two relations for the four parameters \(\kappa, C_i, S,\) and \(t_0^+\) and consequently only two of these parameters are independent. In the data comparisons that are carried out here, \(\kappa\) and \(S\) were generally assumed to be variables that were adjusted to obtain the best fit to the data; thus, at any stage in the optimization procedure for specified values of \(\kappa\) and \(S\), equations (2.76) and (2.77) were solved for \(t_0^+\) and \(C_i\).

### 2.7 Summary

A composite similarity velocity profile valid to leading order for the entire boundary layer has been developed in this section according to

\[
\tilde{u} = u^* U_{\infty} \left[ U^* + U_L \right] - \left[ \frac{1}{\kappa(x)} \log n + C_i \right].
\]

-38-
Here $U^+$ is the unsteady wall layer model given in section 2.6; this model is an analytic expression given by equation (2.65) which contains the independent parameters $\kappa$ and $S$. The outer region profile $U_l(\eta)$ must be obtained as a numerical solution of differential equation (2.62) developed in section 2.5; this numerical solution will implicitly contain the parameters $\kappa$ and $K$ which appear in the eddy viscosity model. Thus, the composite profile contains the three independent model parameters $S$, $\kappa$ and $K$ that may be adjusted in a computer optimization routine to obtain a best fit with experimental velocity profile data.
3. THE OPTIMIZATION TECHNIQUE

3.1 Background

The present turbulence velocity profile model contains three adjustable model parameters $S$, $k$, and $K$ that affect the basic shape of the analytical profile. This section addresses the development of a systematic method to determine the optimum values of the model parameters that minimize the difference between the analytical model profile and experimental data. The method described here can, in principle, be applied to any test data to attempt to determine any trends in the data with different physical effects; in particular, the procedure is applied here to data for several pressure gradient flows and mainstream turbulence levels.

Before examining the optimization method, a basis for determining a best fit must be defined. For this study a root-mean-square error $\epsilon$ was selected according to

$$
\varepsilon^2 = \frac{1}{N} \sum_{n=1}^{N} \left\{ \frac{\bar{u}_{\text{DATA}}(y_n) - \bar{u}_{\text{ANALYTICAL}}}{U_e} \right\}^2
$$

(3.1)

where

$N =$ number of data points

$\bar{u}_{\text{DATA}} =$ experimentally measured velocity at $y_n$

$\bar{u}_{\text{ANALYTICAL}} =$ analytical velocity profile at $y_n$

$U_e =$ freestream velocity.
The best analytical fit to a given experimental profile is defined as that set of model parameter values which minimizes ε as defined by (3.1). Previous prediction techniques such as those presented in Scharnhorst et al. (1977) and those used in the Stanford Conference (Coles & Hirst, 1969) utilize the standard error criterion of (3.1) as an objective basis of comparison. The particular optimization method used in this study is presented in the next subsection.

3.2 The Optimization Method - Direct Search

A direct search minimization procedure was used to optimize the composite velocity profile. This technique is simple to use since it only requires values of the objective function and not gradients to carry out the optimization search. The nature of the problem is such that gradients of the objective function cannot be computed analytically and can only readily be evaluated by numerical differentiation. Although the direct search procedure becomes very time consuming when the number of optimization variables is large, it was used here with good success since the maximum number of variables is three. The basic procedure in the direct search minimization was to vary one model parameter such that the least squares error objective function would continuously decrease while holding the other
parameters constant. Each model parameter is varied in turn so that at any stage in the optimization procedure, a one-dimensional search is made for the minimum in that direction. The method used here is somewhat similar to the direct search method of Hooke and Jeeves which is described in Himmelblau (1972) p. 142. The specific logic involved is as follows.

The optimization scheme starts with initial values for the model parameters which must be provided to the subroutine as well as a vector of initial incremental step sizes for the parameters. This initial location is established as a base point. To initiate a search, the objective function \( f^0(x) \) is evaluated at this base point and one parameter \( x_1 \) is then incremented by the specified step size \( \Delta x_1 \). Suppose first that the objective function decreases. The parameter \( x_1 \) is incremented continually according to \( x_1(i+1) = x_1(i) + \Delta x_1 \) and the objective function \( f(i+1) = f(x_1(i+1)) \) is computed. This process continues for \( i = 0, 1, 2, 3, \ldots \) as long as \( f(i) \) continually decreases. Eventually, at a certain step, say step \( k \), the objective function will increase and \( f(k+1) > f(k) \). At this stage, a local minimum in the \( x_1 \) direction has been bracketed in the range \( x_1(k-1) \leq x_1(k) \leq x_1(k+1) \). To further refine the location of this minimum, a quadratic interpolation polynomial, \( f = Ax_1^2 + Bx_1 + C \) is used, where \( A, B, \) and \( C \) are
determined by solving the set of equations:

\[ f(i) = Ax_1(i) + Bx_1(i) + C \]  \hspace{1cm} (3.2)

where

\[ i = k-1, k, \text{ and } k+1 \]

This interpolation polynomial produces a local minimum in the \( x_1 \) direction at \( x_1^* = -B/2A \). In certain cases, this interpolation scheme may produce a value of \( x_1^* \) which is not close to the midpoint of the interval \( x_1^{(k)} \) or an objective function which \( f^*(x) \) is not close to \( f^{(k)} \). When this situation occurs, then three of the values \( x_1^* \), \( x_1^{(k-1)} \), \( x_1^{(k)} \), and \( x_1^{(k+1)} \) which have the lowest objective function are relabeled as \( x_1^{(i-1)} \), \( x_1^{(i)} \), and \( x_1^{(i+1)} \) and another quadratic interpolation polynomial is fit through these values to obtain a new minimum \( x_1^* \). The interpolation scheme is repeated until a specified convergence criteria is met between the values of \( x_1^* \) and \( x_1^{(k)} \) or \( f(x) \) and \( f^{(k)} \). After convergence in the \( x_1 \) parameter, the new value of \( x_1 \) is retained as the base point; the next parameter \( x_2 \) is incremented by the specified step size \( \Delta x_2 \), and the search procedure repeats until all the independent parameters have been changed.

In the second place suppose that an increase in the objective function is realized for the initial step of a model.
parameter (for example, x) from its base point. When this situation occurs, the search direction is immediately changed to the negative direction and the program procedure continues as previously stated. In the event that the objective function increases for the first step in the negative direction as well, the program enters into the quadratic interpolating routine since the local minimum has been bracketed in the range \( x_1 - \Delta x_1 \leq x \leq x_1 + \Delta x_1 \).

After the local minimum has been refined, a one-dimensional search is carried out for the next variable.

After a one-dimensional search for each variable has been carried out by the above procedure, a new base point is established and a convergence test is performed in which the optimization function values for the last two base points are compared. If these two values differ by an amount less than a specified criterion, a minimum has been found and the program terminates. If the convergence criterion is not met, the step sizes \( \Delta x_i \) of the search are reduced ten percent and the search procedure starts anew from the current base point.

The direct search program contains two types of error flags. The first type of error flag is encountered when the number of combined step and interpolation iterations reaches a specified maximum value. This error flag then terminates the optimization routine and returns to the prediction code.
with the latest values for the optimized parameters. The second type of error flag occurs in the quadratic interpolation scheme when the value of the A coefficient of the quadratic polynomial is identically zero. This flag prevents a computation error in the interpolation process and terminates the direct search program. One variable returned by the subroutine is the error variable IER; values IER = 1 and IER = 2 indicate errors of the first and second type respectively. A value IER = 0 indicates a successful search has been completed.
4. **TURBULENT BOUNDARY LAYER PROFILE REPRESENTATION**

4.1 **Introduction**

In order to obtain a velocity profile representation for a given set of velocity profile data, one or more of the composite profile parameters $S$, $\kappa$, and $K$ can be altered in several combinations. Each such combination defines a different method of profile representation; if a parameter is not optimized, it must be assigned a universal value. In the study of Scharnhorst (1978), one method considered was to take $\kappa=0.41$ (the generally accepted universal value for the von Karman constant) and $S=10.4965$; this value of $S$ is the value which with $\kappa=0.41$ produces a value of $C_i = 5.1$ in equations (2.76) and (2.77); again $C_i = 5.1$ is a generally accepted value. Scharnhorst (1978) then carried out a one parameter optimization on $K$ over a wide range of pressure gradient data; the results of this optimization were not encouraging. The principal difficulty was that the analytical profile tended to skew through the data in the logarithmic zone. This difficulty was noted by Scharnhorst (1978) and is illustrated in Figure 4.1; in this figure the results of a one parameter optimization on $K$ (with $\kappa=0.41$, $S=10.4965$) are illustrated for three stations of the data of Anersen et al (1972). These stations are labeled 8109, 8209, and 8309 and are the last measured data stations in equilibrium flows for zero, moderate and strong
FIGURE 4.1 Velocity profile comparisons for one parameter optimization on $K$ (with $\kappa=0.41$, $S=10.4965$) for three data stations of Andersen et al. (1972). Profiles are for a zero, mild adverse and strong adverse pressure gradient flow; note procedure is less satisfactory with increasing pressure gradient.
adverse pressure gradient flows respectively. It may be observed from Figure 4.1 that the theoretical profile represents the constant pressure profile well in the overlap zone but is increasingly less satisfactory there as the pressure gradient becomes larger. It may be observed from Figure 4.1 that for stations 8209 and 8309, a larger value of $\kappa$ appears to be indicated in order to decrease the slope of the profile to conform to the data.

However, there are two main difficulties associated with the fitting problem and it is worthwhile to discuss these here before proceeding further. The first problem has been termed a "low Reynolds" number effect by Scharnhorst (1978) and is associated with a failure of the composite analytical profile to adequately delineate various regions of the boundary layer. To understand this last statement, consider profile 8309 in Figure 4.1; the solid straight line is apparently tangent to the curve in the overlap zone of the profile. The inverse of the slope of this straight line may be obtained graphically and is indicated on the figure as $\kappa=0.266$; however, the value of $\kappa$ used to obtain the profile in the figure was $\kappa=0.41$. The reason for this difficulty may be clearly observed in Figure 4.2 which illustrates composite similarity profiles for two values of the Reynolds number based upon the displacement thickness, $\delta^*$, for fixed values of $\beta_c = 0$, $K = 0.016$, and $\kappa = 0.41$. In preparing this figure,
Figure 4.2 Composite similarity profile for $S = 10.4965$, $K = 0.016$, $\kappa = 0.41$ and $\beta_c = 0$. 
a value of $\text{Re} \delta^* = 10^3$ was chosen since it is typical of practical flows that occur in engineering practice. For $\text{Re} \delta^* = 10^3$ the range of $y^+$ that exhibits logarithmic behavior is apparently relatively short and the slope of the analytical profile in this region appears to be greater than the value $1/\kappa$. The reason for this behavior is associated with the value of $\text{Re} \delta^*$ which relates the inner variable $y^+$ to the outer variable $n$ by $y^+ = \text{Re} \delta^* n$. For $\text{Re} \delta^* = 10^3$, a value of $y^+ = 100$ would correspond to an outer variable value of $n = 0.1$; for this reason, the apparent range of logarithmic behavior in Figure 4.2 for $\text{Re} \delta^* = 10^3$ is relatively short and significant portions of the inner and outer regions blend together in the overlap zone. To show the effect on the profile as the Reynolds number becomes larger, a composite similarity profile is also presented in Figure 4.2 for a value of $\text{Re} \delta^*$ arbitrarily increased to $10^4$. It may be observed that a logarithmic region emerges over a wider $y^+$ range and that the composite similarity profile corresponds on the graph more closely with the input log-law behavior.

A second difficulty associated with the fitting of the composite profile is encountered with moderate to strong pressure gradient flows where $\beta_C$ is $O(1)$. It may be observed in the series solution for the outer profile given in equation (A.4)
that for all \( \beta_c \neq 0 \), the series contains terms of the form \( a_i n^i \log n \) in addition to the purely logarithmic term \((1/\kappa)\log n\); two points about these terms are germane. In the first place, it may easily be verified that such terms arise in the series because of the pressure gradient term in the outer layer equation and are not associated with the choice of turbulence model per se. Secondly, although \( n^i \log n \to 0 \) as \( n \to 0 \) for all \( i > 0 \), such terms do give a significant contribution for \( n \neq 0 \) and the purely logarithmic behavior of \((1/\kappa)\log n\) will only be realized for very small \( n \). As previously indicated, for low Re\(_{\delta^*}\), very small values of \( n \) will correspond to moderate (but not large) values of \( y^+ \) and the logarithmic portion of the analytical profile becomes obscure. As \( \beta_c \) increases the difficulty becomes more severe since it may easily be verified from equation (A.4) that the \( a_i \) become increasingly larger as \( \beta_c \) increases. This effect further causes the composite profile to deviate from logarithmic behavior for small \( n \).

The failure of the composite profile to reflect the true input logarithmic behavior for low Reynolds number on a graph is of some concern since the values of Re\(_{\delta^*}\) which are characteristic of many experiments are \( O(10^3) \) or \( O(10^4) \). To attempt to overcome these problems, Scharnhorst (1978) suggested two approaches. In the first of these, a full three parameter
optimization of S, κ, and K was carried out. Generally this procedure produces very good representations of the data; however, as Scharnhorst (1978) points out, the parameters (S,κ,K) tend to lose physical significance in such method in two ways.

First the input value of κ used to produce the profile is not the apparent value that would be calculated from a graph of the profile; this point has been illustrated in Figure 4.1. Secondly, Scharnhorst (1978), upon plotting the values of S obtained for profiles with various values of \( \beta_c \), observed a trend for S to remain approximately constant or to actually slightly decrease with increasing \( \beta_c \). This trend is opposite to the experiments (Kline, et al. 1967) which suggest that S should increase with increasing \( \beta_c \).

The second approach was suggested by Scharnhorst (1978) as one possible way of overcoming the problems of the three parameter optimization and was attempted here. The main ideas are that the parameter K primarily influences the quality of the fit in the outer region while the parameter S mainly influences the inner region; on the other hand, κ influences the slope of the profile in the overlap zone and consequently has an important effect in both regions. The difficulties associated with the strong influence of varying κ and also with fitting this type of profile have been discussed by Scharnhorst (1978) and Weigand.
one approach found to be acceptable for heat transfer profiles (Wiegand, 1978) is to perform a one parameter optimization on $K$ holding $\kappa$ fixed; once the optimization had been complete, a new value of $\kappa$ is obtained from a graph of the data in inner region coordinates; note that in the optimization $u_*$ will vary and hence the graph of $u^+$ versus $y^+$ will change. This graphical iteration process starts with a value $\kappa=0.41$ and continually obtains new values of $\kappa$ from the slope of the data points in the logarithmic region of the velocity profile. This approach is denoted by method 1 in this study and the least square curve fit error results of the first iteration are presented in Table 4.1. First iteration curve fits for the last data station of the zero and favorable pressure gradients of Andersen, et al. (1972) are shown in Figure 4.1. Unfortunately method 1 failed for successive graphical iterations since the new values of $\kappa$ from the slope of the logarithmic region do not provide improved curve fits; in fact, the values of $\kappa$ taken from the data give a lower value of $\kappa$ than 0.41 although a larger value of $\kappa$ is suggested upon comparison of the data and the analytic profile. The lines drawn tangent to the data in the logarithmic zone are depicted in figure 4.1 as broken lines. The failure of this procedure is again due to the low Reynolds number effect for large $\beta_c$. In the next section, several other methods are discussed in an effort to obtain improved composite profile data comparisons.
4.2 Composite Profile Data Comparisons

Composite similarity profile representations were obtained using several methods in which certain model parameters were optimized using the direct search technique while holding others constant. The five methods considered in this study for profile representations are:

1. Method 1 - one parameter optimization on $K$ while holding $S = 10.4965$ and $\kappa = 0.41$ constant;

2. Method 2 - two parameter optimization on $\kappa$ and $K$ while holding $S = 10.4965$ constant;

3. Method 3 - two parameter optimization on $S$ and $K$ while holding $\kappa = 0.41$ constant;

4. Method 4 - three parameter optimization on $S$, $\kappa$, and $K$;

5. Method 5 - two parameter optimization on $S$ and $K$ while holding $\kappa = 0.46$.

To determine which method offers the best profile representations, each method was run with the non-transpired zero, mild, and strong adverse pressure gradient data of Andersen, et al. (1972) (labeled 8100, 8200, and 8300, respectively) and the favorable pressure gradient data of Herring and Norbury (labeled 2700 after Coles & Hirst, 1969). These data sets were chosen for several reasons. First, in all data sets there is a relatively large number of points in the inner layer and this is important in assessing the performance of the unsteady wall layer model. In the second
place, these data sets reflect a wide variety of pressure gradients for equilibrium mainstream velocity distributions. Thirdly, these data sets are relatively recent and are believed to be very reliable. The least squares error $\epsilon$ is presented in Table 4.1 for each of the five optimization methods. The values of $\epsilon$ in parentheses represent recalculated values which neglect the third and fourth data stations for the Herring & Norbury (Coles & Hirst, 1969) 2700 flow and the first two data stations for the Andersen, et al. (1972) 8100, 8200, and 8300 flows. These neglected data stations generally correspond to upstream stations which are suspected of not being representative of equilibrium behavior. The recalculated $\epsilon$ values are observed to have a value close to that for the average $\epsilon$ representing all data stations except for the 2700 series in which a substantial improvement is noted.

By observing the curve fit error for the mild adverse pressure gradient (8200 series) in Table 4.1, it is apparent that substantial improvements in the quality of the fits can be realized by varying more than one parameter. The best curve fits are obtained with methods 2 and 4 in the sense that the lowest least squares errors are obtained with these methods. A similar trend may be observed for the strong adverse pressure gradient (8300 series). Note that the RMS error increases in
### TABLE 4.1

ERROR CRITERIA $\varepsilon$ FOR COMPOSITE PROFILE OPTIMIZATIONS

<table>
<thead>
<tr>
<th>FLOW</th>
<th>METHOD 1</th>
<th>METHOD 2</th>
<th>METHOD 3</th>
<th>METHOD 4</th>
<th>METHOD 5</th>
</tr>
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<tbody>
<tr>
<td>2700 - Herring &amp; Norbury; Favorable Gradient</td>
<td>0.008317</td>
<td>0.005700</td>
<td>0.006993</td>
<td>0.004696</td>
<td>0.006241</td>
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<tr>
<td></td>
<td>(.006434)$^+$</td>
<td>(.004949)</td>
<td>(.005156)</td>
<td>(.004312)</td>
<td>(.005464)</td>
</tr>
<tr>
<td>8100 - Andersen, et al.</td>
<td>0.010344</td>
<td>0.008359</td>
<td>0.007356</td>
<td>0.007000</td>
<td>0.007534</td>
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<tr>
<td>Constant Pressure</td>
<td>(.009368)</td>
<td>(.008115)</td>
<td>(.007426)</td>
<td>(.007281)</td>
<td>(.007359)</td>
</tr>
<tr>
<td>8200 - Andersen, et al.</td>
<td>0.015441</td>
<td>0.008198</td>
<td>0.010854</td>
<td>0.007400</td>
<td>0.008826</td>
</tr>
<tr>
<td>Adverse Gradient</td>
<td>(.015654)</td>
<td>(.007918)</td>
<td>(.011532)</td>
<td>(.007493)</td>
<td>(.008884)</td>
</tr>
<tr>
<td>8300 - Anderson, et al.</td>
<td>0.025740</td>
<td>0.009783</td>
<td>0.017472</td>
<td>0.008447</td>
<td>0.013426</td>
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<tr>
<td>Adverse Gradient</td>
<td>(.027232)</td>
<td>(.010114)</td>
<td>(.019606)</td>
<td>(.008975)</td>
<td>(.014738)</td>
</tr>
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</table>

$^+$The numbers in parentheses represent the values of $\varepsilon$ neglecting the third & fourth stations for 2700 and the first two stations for 8100, 8200, and 8300.
the range 50-70% for methods 1, 3 and 5 in going from the mild to the strong adverse pressure gradient case; for methods 2 and 4, the percentage increase is 19 and 14 percent respectively. The 8300 series is the most difficult case for profile representation because of the low Reynolds number effects and pressure gradient effects which have increasing importance for larger $\beta_c$ as discussed in §4.1. On the basis of the results in Tables 4.1, methods 1, 3, and 5 may be ruled out as effective parameter adjusting methods.

To further assess which of the surviving methods offers the best profile representation, the complete profile optimization results for methods 2 and 4 are given in Tables 4.2 through 4.9 along with the corresponding profile representations in Figures 4.3 through 4.10. Evidently, the $S$, $\kappa$, and $K$ optimization technique of method 4 has the lowest average root-mean-square error, $\bar{e}$; however, this result is not totally unexpected since the adjustment of all three parameters offers more flexibility in curve fitting for both the inner and outer regions. Method 2 ($\kappa$, $K$ optimization with constant $S = 10.4965$) offers a close second choice in the selection of a profile representation method which suggests that the variation of the $S$ value does not drastically change the quality of the curve fits.

An important observation which can be made from the results of the three-parameter fits of method 4 is that the composite
### Experimental Values

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<tr>
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<th>QEDUX</th>
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<td>3.0000</td>
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<td>6.250</td>
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<td>4.2700</td>
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### Unsteady Wall + Similarity Model

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<td>0.3416</td>
<td>0.039555</td>
<td>0.03872</td>
<td>4.116</td>
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Mean EPSILON over 6 STATIONS = 0.005700

### Table 4.2

Results of Method 2 two parameter fits of the composite similarity profile to the favorable pressure gradient data of Herring and Norbury (Coles and Hirst, 1969).
Figure 4.3 Velocity profile comparisons for two parameter optimization on $\kappa$ and $K$ (with $S = 10.4965$) for favorable pressure gradient data of Herring and Norbury (1969).
**ANDERSEN, KAYS AND MOFFAT - ZERO P. G. - 120711-1**

**ENGLISH UNITS**

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**UNSTEADY WALL + SIMILARITY MODEL**

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**MEAN EPSILON OVER 9 STATIONS = .686359**

**TABLE 4.3** Results of Method 2 two parameter fits of the composite similarity profile to the constant pressure data of Andersen, et al. (1972).
Figure 4.4 Velocity profile comparisons for two parameter optimization on $k$ and $K$ (with $S = 10.4965$) for constant pressure data of Andersen, et al. (1972).
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**UNSTEADY WALL + SIMILARITY MODEL**

**FULL PROFILE 2 PARAMETER FIT**

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**MEAN EPSILON OVER 9 STATIONS = .088198**

**TABLE 4.4** Results of Method 2 two parameter fits of the composite similarity profile to the mild adverse pressure gradient data of Andersen, et al. (1972).
Figure 4.5 Velocity profile comparisons for two parameter optimization on $\kappa$ and $K$ (with $S = 10.4965$) for mild adverse pressure gradient of Andersen et al. (1972).
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**MEAN EPSILON OVER 9 STATIONS = .009703**

**TABLE 4.5** Results of Method 2 two parameter fits of the composite similarity profile to the strong adverse pressure gradient data of Andersen, et al. (1972).
Figure 4.6 Velocity profile comparisons for two parameter optimization on $\kappa$ and $K$ (with $S = 10.4965$) for strong adverse pressure gradient of Andersen et al. (1972).

-65-
HERRING AND NORBURY, BETA = 0.35  
(ENGLISH UNITS)

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**Unsteady Wall + Similarity Model**

- **Full Profile**
- **Parameter Fit**

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Mean Epsilon over 6 stations = .00696

**Table 4.6** Results of Method 4 three parameter fits of the composite similarity profile to the favorable pressure gradient data of Herring and Norbury (Coles and Hirst, 1969).
Figure 4.7 Velocity profile comparisons for three parameter optimization on $S$, $\kappa$ and $K$ for favorable pressure gradient data of Herring and Norbury (1969).
Table 4.7 Results of method 4 three parameter fits of the composite similarity profile to the constant pressure data of Andersen, et al. (1972).
Figure 4.8  Velocity profile comparisons for three parameter optimization on $S$, $\kappa$ and $K$ for constant pressure data of Andersen et al. (1972).
TABLE 4.8 Results of method 4 three parameter fits of the composite similarity profile to the mild adverse pressure gradient data of Andersen, et al. (1972).
Figure 4.9 Velocity profile comparisons for three parameter optimization on $S$, $\kappa$ and $K$ for mild adverse pressure gradient data of Andersen et al. (1972).
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**Unsatisfactory Wall + Similarity Model**

**Table 4.9:** Results of method 4 three parameter fits of the composite similarity profile to the strong adverse pressure gradient data of Andersen, et al. (1972).
Figure 4.10 Velocity profile comparisons for three parameter optimization on $S$, $\kappa$ and $K$ for strong adverse pressure gradient data of Andersen, et al. (1972).
similarity profile encounters difficulty in fitting velocity profiles measured in strong adverse pressure gradient flows. An increase in $\varepsilon$ with pressure gradient can be observed in Table 4.1; this is not entirely unexpected for two reasons. First, true self-similar behavior is not anticipated in situations with large adverse pressure gradients since boundary layer separation may occur before similarity is achieved; consequently, it may be that the model is not an appropriate one for large adverse pressure gradients. In the second place, there is a significant departure of the composite profile from logarithmic behavior in the overlap region for $\beta_c^0(1)$ due to the low Reynolds number effect and the $\beta_c$ effect in the series solution as discussed in section 4.1. The effects of increasing $\beta_c$ in the series solution may be offset by increasing $\kappa$, thereby reducing the magnitude of the $a_i$ and $b_i$ coefficients in equation (A.4) (see Appendix A). This cancelation effect may be observed in the results of the three-parameter fits in Tables 4.6 to 4.9 which reveal that the optimized values of $\kappa$ increase with $\beta_c$. As a result, it may be concluded that the three parameter optimization process adjusts the parameters of the composite similarity profile to correct an undesirable low Reynolds number effect or large $\beta_c$ effect in the series solution. However, the optimized value of $\kappa$ is no longer associated with the apparent inverse
slope in the logarithmic portion of the profile as taken from
a graph of the profile.

To summarize the investigation of all the optimization
methods, it was found that a significant reduction in error of
the profile fits is realized by optimizing all three parameters
of the profile, namely S, κ, and K. However, the three para-
ter optimization masks any physical interpretation originally
associated with the parameters to counteract the low Reynolds
number and large $\beta_c$ effects. Thus, the parameters S, κ, and K
become simply profile parameters which are adjusted to obtain
a close representation of measured data. The objective now is
to obtain correlations for these parameters based upon the four
equilibrium flows of 2700, 8100, 8200, and 8300 in order to
provide the basis for a prediction scheme.

4.3 Parameter Correlations

In order to develop a profile prediction method, optimized
parameter values were obtained for the three parameter optimiza-
tion of method 4 and the two parameter optimization of method 2.
Both optimization methods used the negative pressure gradient
flow of Herring and Norbury labeled 2700 and the three Andersen,
et al. (1972) flows, 8100, 8200, and 8300 which cover a range
of $\beta_c$ from about -0.4 to 1.5. The results of the optimized
data fits for the three parameter optimization of method 4 were then plotted versus the pressure gradient $\beta_c$ as shown in Figure 4.11. The first two data stations of flows 8100, 8200, and 8300, and the third and fourth data stations of flow 2700 were not included in the correlations because of the observed form of the experimental velocity profiles which did not appear to exhibit self-similar behavior; this is reflected in the fact that the results for these eight data stations showed a substantial deviation from the straight line correlations in figure 4.11. Prospective correlations which could be used in a prediction procedure were obtained by fitting a least squares straight line and quadratic curve through the optimized parameter values represented by the solid symbols. The resulting correlations for the three parameter optimization of method 4 are given in Table 4.10. The RMS curve fit error for the correlations show that the quadratic curve fits give only a very slight improvement over the linear curve fits. The three parameter optimization correlations indicate that $K$ is nearly constant; most of the variation occurs in $\kappa$ which in effect offsets the influence of $\beta_c$ in the series solution for the outer layer mode. A slight variation occurs in the $S$ correlation which is contradictory to the experimentally observed trend for the dimensionless time period between bursts. The correlation indicates that $S$ increases slightly with pressure gradient while the visual observations of Kline, et al. (1967)
FIGURE 4-11 Parameter correlations obtained from three-parameter fits of the composite similarity profile to four sets of data.
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<td>$\kappa = .41662 + .11736\beta_C - .005643\beta_C^2$ ($\cdot .10274$)</td>
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*The numbers in parenthesis are RMS errors.*
show that $S$ decreases with increasing pressure gradient. This contradiction is explained by recalling that the three parameter optimization tends to mask any physical significance that may originally have been attributed to the model parameters. Another point which concerns the value of $S$ is worthy of mention; in the wall-layer the data exhibits a rapid profile variation which may give rise to an unnatural weight in the least-squares error calculation. In addition, wall-layer data points also have been shown to exhibit velocity measurements which are dependent upon the pitot probe tip geometry (Andersen, Kays & Moffat, 1972). All of these factors contribute to the conclusion that the slight trend in $S$ with pressure gradient obtained with the three parameter optimizations may not be significant.

The correlation for $\kappa$ indicates that there is a substantial influence of pressure gradient on the value of $\kappa$; this is partly due to the influence of $\beta_c$ in the series solution which is offset in the optimization by increasing the $\kappa$ value for larger $\beta_c$. Because of this effect and the Reynolds number effect discussed previously, the values for $\kappa$ obtained from the correlations presented in Table 4.10 cannot be directly associated with the inverse slope of the logarithmic region on a graph for profiles with displacement thickness Reynolds numbers $O(10^3)$. The curve fit data for the $\kappa$ correlation of the three parameter optimization
exhibits the least scatter (lowest RMS error) of the $S, \kappa,$ and $K$ correlations which tend to strengthen its reliability for use in a profile prediction scheme.

As a point of interest, correlations based on the two parameter optimizations are also included in Table 4.10. It may be observed that these correlations show the same trends as the results based on the three parameter optimizations of method 4. Note that the dependence of $K$ on $\beta_C$ is weak while the dependence of $\kappa$ on $\beta_C$ is strong; for method 2 the RMS is larger than for the results based on method 4. Since a lower RMS is also observed in the actual profile fits using method 4, this procedure is considered somewhat superior to method 2 and the correlations in Table 4.10 based on method 4 are recommended.
5. MAINSTREAM TURBULENCE

5.1 Introduction

The influence of mainstream turbulence on a fully developed turbulent boundary layer flow with zero pressure gradient has been studied by a number of investigators. The experimental investigations of Kline et al. (1960), Kestin (1966), Huffman et al. (1972), Charnay et al. (1971) and others, indicate that mainstream turbulence affects the turbulent boundary layer velocity profile in many ways. In particular, a thickening of the boundary layer with increasing mainstream turbulence level is observed along with a progressive increase in skin friction. A general change in the shape of the non-dimensionalized mean velocity profile has also been documented in which there is a marked reduction in the "wake" component of the outer layer as the mainstream turbulence level increases. Finally, in a thermal boundary layer, the heat transfer at the wall increases progressively with increasing mainstream turbulence. To consider the possibility of including the effects of mainstream turbulence level into a boundary layer prediction method, it is appropriate to first examine the turbulent boundary layer momentum equation.

The continuity and momentum equations governing nominally steady turbulent boundary layers have been given in equations (2.1) and (2.2) in connection with the conventional type of
turbulent boundary layer; these equations still apply when the
mainstream is turbulent. Moreover, the boundary conditions in
equations (2.7) and (2.8) for the mean profile are still correct.
The single turbulence input to the equations is the Reynolds
stress term \( u'v' \); unfortunately it is not possible to incorpo-
rate the fact that mainstream turbulence is present in the Rey-
nolds stress other than through a correlation. To understand the
reason for this, consider the turbulence kinetic energy \( q^2 \)
defined as,

\[ q^2 = u'^2 + v'^2 + w'^2 \quad (5.1) \]

where \( u'^2 \), \( v'^2 \) and \( w'^2 \) are the turbulence intensities. For a
laminar mainstream flow \( q^2 \) approaches zero at the boundary
layer edge but for mainstream turbulence

\[ q^2 \rightarrow q_e^2 \text{ as } y \rightarrow \infty; \quad (5.2) \]

The mainstream turbulence level \( T_u \) is defined by

\[ T_u^2 = \frac{1}{3} \frac{q_e^2(x)}{u_e^2(x)} \quad (5.3) \]

where \( u_e \) is the local mean mainstream velocity. For simplicity,
assume that the mainstream turbulence is isotropic, viz.

\[ u'^2 = v'^2 = w'^2 \text{ as } y \rightarrow \infty; \quad (5.4) \]

consequently,
\[
\bar{u}' z, \bar{v}' z, \bar{w}' z + T_u^2, \ldots, \text{ as } y \to \infty. \quad (5.5)
\]

However, \( \bar{u}' v' \to 0 \) at the boundary layer edge just as for the normal type boundary layer.

It should be remarked that the momentum equation (2.2) does not contain the intensities \( \bar{u}' z, \bar{v}' z \) since these terms are of lower order; even if these terms were retained in equation (2.2), the dilemma is still not resolved since the intensities appear as the difference \( (\bar{u}' z - \bar{v}' z) \) which still must vanish at the boundary layer edge for isotropic mainstream turbulence.

Since it is not possible to incorporate the fact that mainstream turbulence is present in the boundary conditions for either the mean profile or the turbulence terms, the other possibility of developing a correlation for the eddy viscosity model parameters is investigated here. In particular, the eddy viscosity formula in equation (2.55) contains the parameters \( K \) and \( \kappa \) and here possible trends for these parameters with \( T_u \) will be considered. In addition, a possible trend in the inner region profile parameter \( S \) will be investigated. Again, this is carried out by a systematic adjustment of these parameters to obtain a best fit to data in an equilibrium flow but now with various levels of mainstream turbulence. The ultimate objective here is to provide correlations for these parameters which could be used in a prediction method.
AN OPTIMIZATION TECHNIQUE FOR THE DEVELOPMENT OF TWO-DIMENSIONAL ETCs

L. J. Yuhas, J. D. Walker

AFOSR-TR-82-0417
The data used for composite profile representations was obtained from the United Technologies Research Center boundary layer wind tunnel (Blair and Werle, 1980). Several turbulent boundary layer mean velocity profile data sets were available for zero pressure gradient flows with mainstream turbulence intensities ranging from 0.2% to 6.4%. These data sets have a large number of points that provide a good data base for profile fits and represent the best available test data at this time which reflects the effect of mainstream turbulence.

5.2 Composite Profile Data Comparisons

Composite similarity profile representations were obtained for two methods in which one or two model parameters were optimized using the direct search technique while holding others constant. The first profile optimization method used to represent turbulent boundary layer velocity profile data with mainstream turbulence was a one parameter optimization in $K$ with constant values $S = 11.025$ and $\kappa = 0.44789$. These constant $S$ and $\kappa$ values correspond to the values taken from the three parameter optimization correlations in §4.3 for zero pressure gradient. Results of this method are presented in Tables 5.1 through 5.3 along with the corresponding profile representations in Figures 5.1 through 5.3; note that the 18 data stations used
### UT Mainstream Turbulence Test Data

(English Units)

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### Unsteady Wall + Similarity Model

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**Mean Epsilon over 6 Stations = 0.009145**

**Table 5.1** Results of one parameter fits of the composite similarity profile to mainstream turbulence data.
FIGURE 5.1 Velocity profile comparisons for one parameter optimization on K (with $S=11.025$ and $\kappa=0.44789$) for mainstream turbulence flow.
### UT MAINSTREAM TURBULENCE TEST DATA (ENGLISH UNITS)

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#### UNSTEADY WALL + SIMILARITY MODEL

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**MEAN EPSILON OVER 6 STATIONS = 0.009642**

**TABLE 5.2** Results of one parameter fits of the composite similarity profile to mainstream turbulence data.
FIGURE 5.2 Velocity profile comparisons for one parameter optimization on K (with S=11.025 and k=0.44789) for mainstream turbulence flow.
## UT Mainstream Turbulence Test Data
(English Units)

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### Unsteady Wall + Similarity Model - Full Profile 1 Parameter Fit

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Mean Epsilon Over 6 Stations = 0.018483

**Table 5.3** Results of one parameter fits of the composite similarity profile to mainstream turbulence data.
FIGURE 5.3 Velocity profile comparisons for one parameter optimization on K (with $S=11.025$ and $\kappa=0.44789$) for mainstream turbulence flow.
in these profile comparisons have been taken from the study of Blair & Werle (1980) and have been arranged here as a 1000 series in increasing order corresponding to increasing levels of local mainstream turbulence. Consequently the data in this sequence represents a mixture of data stations from the four basic runs with different mainstream turbulence generators which were carried out by Blair & Werle (1980). By observing the least squares error $e$ tabulated in Tables 5.1 through 5.3 and the velocity profile representations presented in Figures 5.1 through 5.3, it is apparent that the quality of curve fits are acceptable and that the profile model can be used to demonstrate the effects of mainstream turbulence on turbulent velocity profiles with good success. The optimization results indicate that there is a trend in which the value of the K parameter increases with increasing turbulence level $Tu$. This trend was anticipated in the sense that it was known that increasingly larger mainstream turbulence levels result in a progressively thicker boundary layer; mathematically larger values of K in the eddy viscosity formula give rise to a thicker boundary layer.

A second profile optimization method was considered in an attempt to obtain improved velocity profile fits. This method was initiated to investigate possible changes in both the inner and outer layers of the velocity profile and utilized
a two parameter optimization on $S$ and $K$ with a constant value of $\kappa = 0.44789$; again this value of $\kappa$ corresponds to the value taken from the three parameter optimization in §4.3. The results of this two parameter optimization are presented in Tables 5.4 through 5.6 along with the corresponding profile representations in Figures 5.4 through 5.6.

An observation which can be made concerning the results of the two parameter optimization is that the least squares error is generally smaller than for the one parameter optimization; but that the improvement in the quality of the curve fits is not dramatic. The reason for this behavior may be explained by close examination of the nature of the curve fits near the wall; in this region there is a greater curve fit error for the one parameter $K$ optimization than for the error associated with the two parameter $S$ and $K$ optimization. The remaining sections of the model velocity profiles for the one and two parameter optimizations are similar and exhibit almost the same curve fit error. In the two parameter fit the value of $S$ adjusts to minimize this error for the data points nearest the wall. Unfortunately these data points are usually the most uncertain; it is also important to note that the $S$ parameter values obtained from the two parameter optimization exhibit a great deal of scatter and any attempt to derive a trend from this
### UT Mainstream Turbulence Test Data

#### (English Units)

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#### Unsteady Wall + Similarity Model

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Mean Epsilon Over 6 Stations = .005385

#### Table 5.4

Results of two parameter fits of the composite similarity profile to mainstream turbulence data.
FIGURE 5.4  Velocity profile comparisons for two parameter optimization on S and K (with $\kappa=0.44789$) for mainstream turbulence flow.
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#### Unsteady Wall + Similarity Model

- Full Profile 2 Parameter Fit

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Mean Epsilon over 6 stations = 0.006052

Table 5.5 Results of two parameter fits of the composite similarity profile to mainstream turbulence data.
FIGURE 5.5 Velocity profile comparisons for two parameter optimization on S and K (with $k=0.44789$) for mainstream turbulence flow.
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(English Units)

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#### Unsteady Wall + Similarity Model

- Full Profile 2 Parameter Fit

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Mean Epsilon over 6 Stations = 0.007357

**Table 5.6** Results of two parameter fits of the composite similarity profile to mainstream turbulence data.
FIGURE 5.6 Velocity profile comparisons for two parameter optimization on S and K (with κ=0.44789) for mainstream turbulence flow.
information was considered to be potentially misleading. Finally, the magnitude of the S parameter for the low turbulence intensity cases did not match the value of $S = 11.025$ which was obtained from the S correlation of §4.3. All of the above observations suggest that the one parameter K optimization with constant values $S = 11.025$ and $\kappa = 0.44789$ provides the most realistic method for developing velocity profile predictions. The objective now is to obtain a correlation for the K parameter based upon the results of this section.

5.3 Parameter Correlations

In order to develop a profile prediction method optimized parameter values were obtained from the one parameter optimization of K with constant values of $S = 11.025$ and $\kappa = 0.44789$. This optimization method examined eighteen zero pressure gradient data sets with mainstream turbulence levels which cover a range of Tu from 0.002 to 0.0640. The results of the optimized data fits for the K parameter were then plotted versus the turbulence level Tu as shown in Figure 5.7. Prospective correlations which could be used in a prediction procedure were obtained by fitting a least squares straight line and quadratic curve through the optimized parameter values represented by the symbols. The resulting correlations for the one parameter
FIGURE 5.7 K parameter correlation obtained from one parameter fits of the composite similarity profile to constant pressure test data with mainstream turbulence.
optimization on $K$ are

$$K = 0.01489 + 0.39285 \, Tu \quad (.0008519) \quad (5.10)$$

and

$$K = 0.01567 + 0.30941 \, Tu + 1.35909 \, Tu^2 \quad (.0008462) \quad (5.11)$$

for the linear and quadratic fits respectively. Both correlations represent increasing functions of mainstream turbulence intensity as expected and discussed in section 5.1. The RMS curve fit errors for the correlations are given in parenthesis following the equations. These RMS values indicate that the quadratic curve fit gives only a very slight improvement over the linear fit; however, one feature of the quadratic correlation is that the value of its intercept is close to the value 0.01591 obtained from the three parameter correlation of §4.3 for turbulent boundary layers affected by pressure gradient. This indicates that the prediction methods for pressure gradient effects outlined in §4 and the method for mainstream turbulence are compatible. This compatibility adds to the credibility of the correlations for use in profile prediction. To obtain a correlation that incorporates the zero pressure gradient intercept value of $K = 0.01591$ from §4.3, the quadratic curve fit of equation (5.11) was rerun with the intercept held at 0.01591. The resulting correlation for the one parameter optimization
on $K$ was

$$K = 0.01591 + 0.292727Tu + 1.58500Tu^2 \quad (0.0008465) \quad (5.12)$$

where the RMS curve fit error is given in parenthesis. The correlation of equation (5.12) is recommended for prediction purposes to insure complete compatibility.
6. SUMMARY AND CONCLUSIONS

In the present study, a profile model for the mean velocity in a nominally steady two-dimensional turbulent boundary layer has been developed. To obtain this profile, the leading terms in an asymptotic expansion for high Reynolds number for the mean velocity were derived for both the inner and outer layers of the turbulent boundary layer. A self-similar behavior in the velocity was assumed. In the outer layer, a simple eddy viscosity formula was assumed containing two parameters \( \kappa \) and \( K \); the outer layer self-similar profile satisfies an ordinary differential equation which was solved numerically. For the inner layer, an analytical profile model which is based on the observed coherent structure of the time-dependent flow in the wall layer was used; this model contains the parameters \( \kappa \) and \( S \). A composite profile which spans the entire boundary layer was defined and this profile is in general a function of the three parameters \( (K, \kappa, S) \). A computer code was developed for which any or all of the three profile parameters may be varied to obtain a best fit to a given set of experimentally measured profile data; this code is described in Appendix C. A test case for the code is presented in Appendix D.

There are two potential uses for this optimization procedure. In the first of these, a very close representation of a given
set of velocity data may be obtained by carrying out a three parameter optimization using the code described in Appendix C even in situations where the flow does not exhibit self similar behavior. The second use is rather more significant and is associated with the development of the basic model for a boundary-layer prediction method. By carrying out a series of optimization studies for given sets of data, it is demonstrated in §4 and §5 that trends in the optimized values of \((s, \kappa, K)\) may be observed; from these trends correlations for the model parameters for a physical effect may be obtained. In the present report, two such studies have been carried out and correlations have been developed for the effects of both pressure gradients and main-stream turbulence.


APPENDIX A
OUTER LAYER SIMILARITY SOLUTION

A.1 INTRODUCTION

In this section, the outer layer similarity equation

\[ \frac{d}{dn} \left( \varepsilon(n) \frac{dU_1}{dn} \right) + (1 + 2 \beta_c) n \frac{dU_1}{dn} + 2 \beta_c U_1 = 0, \quad (A.1) \]

is solved to obtain the outer layer velocity profile. Because of the irregular logarithmic behavior near \( n = 0 \), two types of solution methods are used and these consist of a series solution for small \( n \) and a numerical solution for large \( n \). The matching point where the series and numerical solutions are required to merge smoothly is taken to be \( n_m = K/2\kappa \).

A.2 SERIES SOLUTION FOR SMALL \( n \)

The eddy viscosity function \( \varepsilon(n) \) was selected such that \( \varepsilon(n) \) approaches a linear variation \( \kappa n \) exponentially quickly as \( n \) approaches zero. Consequently for small \( n \), the outer layer similarity equation (A.1) becomes

\[ \kappa n \frac{d^2 U_1}{dn^2} + (\kappa + (1 + 2 \beta_c)n) \frac{dU_1}{dn} + 2 \beta_c U_1 = 0. \quad (A.2) \]

For equation (A.2), a series solution of the form
\[ u_1 = n^\alpha \sum_{i=0}^{\infty} a_n n^i, \]  

(A.3)

is assumed and the indicial equation yields a double root \( \alpha = 0 \).

The solution thus takes the form of a regular power series plus a logarithmic term of the form,

\[ U_1 = \sum_{n=0}^{\infty} b_n n^n + \{ \sum_{n=0}^{\infty} a_n n^n\} \log n, \]  

(A.4)

with derivatives

\[ \frac{dU_1}{dn} = \sum_{n=1}^{\infty} h b_n n^{n-1} + \{ \sum_{n=1}^{\infty} n a_n n^{n-1}\} \log n + \sum_{n=0}^{\infty} a_n n^{n-1}, \]  

(A.5)

and

\[ \frac{d^2 U_1}{dn^2} = \sum_{n=2}^{\infty} n(n-1)b_n n^{n-2} + \{ \sum_{n=2}^{\infty} n(n-1)a_n n^{n-2}\} \log n \]

\[ + \sum_{n=1}^{\infty} n a_n n^{n-2} - \frac{a_0}{n^2} + \sum_{n=2}^{\infty} a_n(n-1)n^{n-2}. \]  

(A.6)

As \( n \to 0 \), the required behavior for \( U_1 \) is given by equation (2.63) and is

\[ U_1 \sim \frac{1}{\kappa} \log n + C_0; \]  

(A.7)

therefore

\[ -110 - \]
Recursion relations for $a_n$ and $b_n$ are obtained by substitution of $U_1$ and its derivatives from equations (A.4), (A.5), and (A.6) into equation (A.2). It may easily be shown that

\begin{equation}
\frac{a_{n+1}}{a_n} = \frac{-2\beta_c (n+1) + n(1+2\beta_c)}{\kappa (n+1)^2}
\end{equation}

and

\begin{equation}
\frac{b_{n+1}}{b_n} = \frac{-1}{\kappa (n+1)^2} \left[ \frac{a_n}{n+1} \left[ 1 - (2\beta_c + n(1+2\beta_c)) \right] + (2\beta_c + n(1+2\beta_c)) \right].
\end{equation}

A.3 NUMERICAL SOLUTION OF OUTER REGION FOR LARGE $n$

A numerical solution to the outer layer similarity equation (A.1) is calculated in the range $n_m < n < n_0$ where $n_m$ is the matching point to the series solution and $n_0$ represents the outer bound of the numerical mesh which is chosen large enough to ensure no significant change in the solution. In order to obtain an accurate numerical solution, a small mesh size is needed near the wall whereas far from the wall, the solution decays rapidly and a larger mesh may be used. For this reason, a numerical method developed by Kellor (1969) which permits a
non-uniform mesh was used in this study; the details of this method will be described here.

The ordinary differential equation to be solved is of the form

$$\frac{d^2U}{dn^2} + P(n) \frac{dU}{dn} + q(n)U = 0,$$  \hspace{1cm} (A.11)

where

$$P(n) = \frac{1}{\epsilon(n)} \left\{ \frac{de}{dn} + (1+2\beta_n)\eta \right\},$$  \hspace{1cm} (A.12)

$$q(n) = \frac{2\beta_c}{\epsilon(n)}.$$  \hspace{1cm} (A.13)

Equation (A.11) is reduced to two first order differential equations by the introduction of an auxiliary variable $v$ defined as

$$\frac{du}{dn} = v,$$  \hspace{1cm} (A.14)

which transforms (A.11) to

$$\frac{dv}{dn} = -p(n)v - q(n)u.$$  \hspace{1cm} (A.15)

A non-uniform mesh is defined by

$$n_j = n_{j-1} + h_{j-1}$$  \hspace{1cm} (A.16)
where \( h_{j+1} \) is the variable mesh length. Equations (A.14) and (A.15) are approximated at the midpoint of \( n_j \) and \( n_{j+1} \) using a central difference for the first derivative and average values for \( u \) and \( v \). As a result, equations (A.14) and (A.15) become,

\[
\frac{u_{j+1} - u_j}{h_j} = \frac{1}{2} (v_{j+1} + v_j), \quad (A.17)
\]

and

\[
\frac{v_{j+1} - v_j}{h_j} = \frac{-p_{j+\frac{1}{2}}}{2} (v_{j+1} + v_j) - \frac{q_{j+\frac{1}{2}}}{2} (w_{j+1} + w_j), \quad (A.18)
\]

where

\[
p_{j+\frac{1}{2}} = P(n_j + \frac{h_j}{2}) \quad \text{and} \quad q_{j+\frac{1}{2}} = q(n_j + \frac{h_j}{2}).
\]

The auxiliary variable \( v_{j+1} \) is eliminated using equation (A.17) to obtain

\[
-2v_j + \frac{2}{h_j} (u_{j+1} - u_j) = -p_{j+\frac{1}{2}} (u_{j+1} - u_j) - \frac{h_j q_{j+\frac{1}{2}}}{2} (u_{j+1} + u_j). \quad (A.19)
\]

A similar procedure is used to approximate (A.14) and (A.15) at the midpoint of \( n_{j-1} \) and \( n_j \). Eliminating the variable \( v_{j-1} \) in a similar fashion reveals
By adding (A.19) and (A.20) to eliminate \( v_j \), it may be shown that the finite difference equations reduce to the general form

\[
2v_j - \frac{2}{h_j} (u_j - u_{j-1}) = -P_{j-1}(u_j - u_{j-1}) - \frac{h_j}{2} \frac{q_{j-\frac{1}{2}}}{h_{j-1}} (u_j + u_{j-1}).
\]  

(A.20)

By adding (A.19) and (A.20) to eliminate \( v_j \), it may be shown that the finite difference equations reduce to the general form

\[
b_j u_{j+1} + a_j u_j + c_j u_{j-1} = d_j,
\]

(A.21)

where

\[
a_j = -1 - \frac{h_j}{2} (P_{j+\frac{1}{2}} - P_{j-\frac{1}{2}}) + \frac{h_j^2}{4} (q_{j+\frac{1}{2}} - \frac{1}{\gamma} q_{j-\frac{1}{2}}),
\]

\[
b_j = 1 + \frac{h_j}{2} P_{j+\frac{1}{2}} + \frac{h_j^2}{4} q_{j+\frac{1}{2}},
\]

\[
c_j = \frac{1}{\gamma} - \frac{h_j}{2} P_{j-\frac{1}{2}} + \frac{h_j^2}{4\gamma} q_{j-\frac{1}{2}},
\]

\[
d_j = 0,
\]

\[
\gamma = \frac{h_j}{h_{j-1}}.
\]

This general form can be used to generate a system of \( N-1 \) finite difference equations for a mesh with \( j=1,2,3,\ldots,N \) number of grid points. The system of equations forms a tri-diagonal matrix which can be solved in a direct and efficient manner by using the Thomas algorithm; this
algorithm defines two functions $\delta_j$ and $F_j$ in which the system (A.21) may be solved as,

$$u_j = F_j u_{j+1} + \delta_j,$$  \hspace{1cm} (A.22)

where

$$F_1 = 0, \hspace{0.5cm} F_j = \frac{-b_j}{a_j + c_j F_{j-1}},$$  \hspace{1cm} (A.23)

$$\delta_1 = u_1, \hspace{0.5cm} \delta_j = \frac{d_j - C_j \delta_{j-1}}{a_j + c_j F_{j-1}},$$

for $j=1,2,3,...,N-1$. After the arrays $F_j$ and $\delta_j$ are calculated, the solution $u_j$ is obtained from (A.22) by back substitution from $j=N-1$ to $j=1$.

A.4 SOLUTION PROCEDURE

The procedure used in this study to obtain a smooth match between the series and numerical solutions at the match point $\eta_m$ is outlined below. Let the solution to equation (A.1) be represented by

$$u = \begin{cases} 
  u_s, & 0 < \eta \leq \eta_m \\
  u_n, & \eta_m \leq \eta < \eta_0 
\end{cases}$$

where $u_s$ and $u_n$ denote the series and numerical solutions respectively. First, an arbitrary value for the constant $C_0$
in equation (A.7) is guessed, say \( C_0^{(1)} \). From this value, a series solution may now be calculated for \( 0 < n < n_m \), say \( u_s^{(1)} \). In particular, the series solution is used at the match point to obtain \( u_s^{(1)}(n_m) \) and its derivative \( du_s^{(1)}(n_m)/dn \). Using \( u_s^{(1)}(n_m) \), the numerical solution for \( n_m < n < n_o \) is calculated, say \( u_n^{(1)}(n) \); the derivative \( du_1^{(1)}(n_m)/dn \) is calculated numerically using a six point forward difference formula (Abramowitz & Stegum 1972, p. 914). It is worthwhile to note that at this step the resulting solution will not have a continuous derivative at \( n = n_m \). A second arbitrary value for \( C_0 \) is guessed, say \( C_0^{(2)} \) from which \( u_s^{(2)} \) and \( u_n^{(2)} \) are calculated using a similar procedure as that stated above. Again, the resulting solution will not have a continuous derivative at \( n = n_m \).

The final solution with a continuous first derivative at \( n = n_m \) is a linear combination of the solutions previously found. Thus,

\[
 u_s = B_1 u_s^{(1)} + B_2 u_s^{(2)} \quad \text{for} \quad 0 < n < n_m, \quad (A.24)
\]

and

\[
 u_n = B_1 u_n^{(1)} + B_2 u_n^{(2)} \quad \text{for} \quad n_m < n < n_o, \quad (A.25)
\]

where \( B_1 \) and \( B_2 \) are constants. One condition imposed on \( B_1 \)
and $B_2$ is

$$B_1 + B_2 = 1,$$

(A.26)

in order that $u$ satisfies

$$u \sim \frac{1}{K} \log n + C_0 \text{ as } n \to 0.$$

A second condition relating $B_1$ and $B_2$ is that the first derivatives of equations (A.24) and (A.25) must be equal at the match point according to

$$B_1 \left( \frac{du_s}{dn} \right)_{n=n_m} + B_2 \left( \frac{du_n}{dn} \right)_{n=n_m} = 0.$$

(A.27)

Equations (A.26) and (A.27) can be solved to yield the values of $B_1$ and $B_2$. The true value of $C_0$ is thus

$$C_0 = B_1 C_0^{(1)} + B_2 C_0^{(2)}.$$

(A.28)

In this manner, a solution to equation (A.1) is obtained consisting of a series solution for $n < n_m$ and a numerical solution for $n > n_m$; this solution and its first derivative are continuous at the matching point $n=n_m$. Since equation (A.1) is a second order equation all derivatives are therefore continuous at $n=n_m$. 

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APPENDIX B

THE Ξ FUNCTION

The Ξ function is defined as

\[ \Xi(n) = \int_{0}^{n} e^{-x^2} \int_{0}^{x} e^{y^2} \int_{0}^{y} e^{-t^2} dt \, dy \, dx , \quad (B.1) \]

which satisfies the differential equation

\[ \Xi'' + 2n \Xi = \frac{\sqrt{\pi}}{2} \text{erf}(n) . \quad (B.2) \]

The following expansion is readily obtained

\[ \Xi(n) = e^{-n^2} \sum_{j=1}^{\infty} \frac{2^j \alpha(j)n^{2j}}{(2j+1)!!} , \quad (B.3) \]

where

\[ \alpha(j) = \alpha(j-1) + \frac{1}{j} , \quad \alpha(1) = 1 , \quad (B.4) \]

which is uniformly convergent for all \( n \). An asymptotic expansion used to evaluate \( \Xi(n) \) as \( n \to \infty \) is

\[ \Xi(n) - \frac{\sqrt{\pi}}{4} \left\{ \log n + \frac{\gamma_0}{2} - \frac{1}{2} \sum_{j=1}^{\infty} \frac{(2j-1)!!}{j^2 n^{2j}} \right\} , \quad (B.5) \]

where \( \gamma_0 \) is Euler's constant equal to 0.57721566... .

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APPENDIX C

THE OPTIMIZATION CODE
PROGRAM MAIN(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)
COMMON/VAR/IVAR(3),X(13)
EXTERNAL F

C**** THIS IS THE MAIN PROGRAM WHICH INITIATES THE DATA FITTING
C**** BY CALLING SUBROUTINE PROFIT.
C**** NPAR - NUMBER OF PARAMETERS DESIRED FOR THE PROFILE FIT.
C**** REFERENCE: AN OPTIMIZATION TECHNIQUE FOR THE DEVELOPMENT
C**** OF A TWO-DIMENSIONAL TURBULENT BOUNDARY LAYER
C**** MODEL, YUHAS, L.J. MASTER'S THESIS
C**** ITERN - MAXIMUM NUMBER OF CYCLES TO BE PERFORMED IN DRSRCH
C**** (NORMALLY 50)
C**** IPRINT=1 IF PRINTING OF INTERMEDIATE RESULTS IN DRSRCH IS
C**** DESIRED DURING THE COURSE OF THE OPTIMIZATION.
C**** =0 IF NO PRINTING IN DRSRCH IS DESIRED.
C**** EPS - THE EXIT TOLERANCE IN DRSRCH IF, SAY, EPS=1.E-5 THE
C**** ITERATION WILL CONTINUE IN DRSRCH UNTIL THE PARAMETERS
C**** BEING OPTIMIZED HAVE CONVERGED TO 5 SIGNIFICANT
C**** FIGURES
C
C THE PARAMETERS C APK, S, KAPPA TO BE OPTIMIZED ARE INITIALIZED
C IN THE INPUT
C IVAR(I)= 0 NOT OPTIMIZED
C = 1 OPTIMIZED
C WHERE I = 1 CAPK
C = 2 S
C = 3 KAPPA

READ(5,10)(IVAR(I),I=1,3)
NPAR=0
DO 30 I=1,3
30 NPAR=NPAR+IVAR(I)
READ(5,20)ITERM,IPRINT,EPS,NA
CALL PROFIT(NPAR,ITERM,IPRINT,EPS,NA)
10 FORMAT(312)
20 FORMAT(2I4,E6.1,I4)
STOP
END
SUBROUTINE PROFIT(NPAX, ITERM, IPRINT, EPS, NA)
DIMENSION ITITLE(5), LUNIT(4), XF(3)
DIMENSION LSYS(2), IDEN(15), X(15), UTAU(15), DUEDX(15), PLUS(1+5), DELTA(15), DELSTP(15), NSHAPE(15), EPSI(15), DELT(15)
DIMENSION PPLUS(15), ETAUE(15), ETAU(15), TPLUS(15), C(15), BETE(15)
DIMENSION CF(15), CFE(15), CI(15), CAPK(15), XKAP(15), BT(15), TUT(15)
DIMENSION L1(3), DL(3), DXERF(3), DXF(3), XT(15)
COMMON/NPAR.NDP+XNU.UI.UAUXXYD(90), UD(90), DLST, UA(90), YDFP(1800), UAPL(1800), NPPY
COMMON/VAR/IVAR(3), XT(3)
COMMON/UOUT/NO, ETA(400), Delta(400), UI(400), ETAH, ETAH, COUT, CIA, 15, WN0, TDEL, DELOUT, USTAR, BETAC
EXTERNAL F
DATA LUNIT/2HFT, 2H, N, 2MIN, 2MCML, LSYS/7MEG, 7H METRIC, /, EBAR/8./
DATA NGHT/5.1/
DATA NPP/400/
NO=NA
C      SUBROUTINE PROFIT READS IN THE EXPERIMENTAL DATA AND
C      INITIATES THE OPTIMIZATION AT EACH DATA STATION BY
C      CALLING SUBROUTINE DRSRCH
C      INPUT IDENTIFICATION FOR DATA RUN
C      ID - A FOUR DIGIT NUMBER (USER SUPPLIED) TO IDENTIFY DATA
C      UNIT - ZERO FOR ENGLISH UNITS, ONE FOR METRIC UNITS
C      TITLE - TITLE OF DATA SET MAXIMUM 150 CHARACTERS
C      READ(5,100)ID,UNIT,(ITITLE(I),I=1,5)
C      IF(IUNIT.EQ.1)CON=100.
C      NPAR=NPAX
C      XNU - KINEMATIC VISCOSITY
C      NSTA - NUMBER OF DATA STATIONS
C      NPAR=NPAX
C      NSTA
C      INPUT PARAMETERS FOR PLOTTING
C      NCYC - NO. OF X CYCLES
C      NPLTPP - NO. OF ZERO PLOT TICKS ON Y AXIS
C      XLC - LENGTH OF EACH CYCLE ON X AXIS
C      INCPL - INCREMENT LABELS ON Y AXIS
C      NL - NO. OF NON-ZERO TICKS ON Y-AXIS
C      XL - LENGTH BETWEEN TICKS ON Y-AXIS
C      READ(5,98)NCYC,NPLTPP,XLC,ML,INCPX
C      DO 10 M=1,NSTA
C      WRITE(10,103)ITITLE(I),I=1,5,NO
C      IF(IUNIT.EQ.1)CON=100.
C      WRITE(10,103)ITITLE(I),I=1,5,NO
C      WRITE(ID,103)NPPY,NPPY
C      WRITE(ID,103)NPPY
C      X - LOCAL VALUE OF X-LOCATION ON THE WALL
C      UE - LOCAL MAINSTREAM VELOCITY
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**C**

**UTAUE** - ESTIMATE OR EXPERIMENTALLY QUOTED VALUE OF UTAU

**DELTA** - EXPERIMENTAL VALUE OF BOUNDARY LAYER THICKNESS

**DELSTR** - EXPERIMENTAL VALUE OF DISPLACEMENT THICKNESS

**THETA** - EXPERIMENTAL VALUE OF MOMENTUM THICKNESS

**DUEOX** - LOCAL VALUE OF THE MAINSTREAM VELOCITY GRADIENT

**CFE** - EXPERIMENTAL VALUE OF THE SKIN FRICTION COEFFICIENT

**SHAPE-Factor**

**C**

**IF** AN ESTIMATE OF UTAU IS NOT AVAILABLE, THE VALUES OF

**UTAU** ARE TO BE READ IN AS ZERO AND AN ESTIMATE FOR

**C**

**NOTE** THAT THIS IS ONLY USED AS A STARTING ESTIMATE

**FOR** UTAU IN THE OPTIMIZATION PROCESS.

**C**

**PPLUSE** - EXPERIMENTAL VALUE OF THE INNER REGION PRESSURE

**Gradient Parameter**

**C**

**YD** - Y LOCATIONS OF THE EXPERIMENTAL DATA POINTS

**UD** - EXPERIMENTALLY MEASURED MEAN VELOCITY AT YD

**SET UP AXIS FOR PLOTTING**

**C**

**READ**

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READ(5,99) X0(I),X0(I),X0(3)
FORMAT(3F10.8)
READ(5,107) Y0(I),I=1,NBP
READ(5,107) UO(I),I=1,NBP
WRITE(6,108) NBP,LUNIT,IUP3
WRITE(6,108) Y0(I),UO(I),I=1,NBP
DO 20 I=1,NBP
20 Y0(I)=Y0(I)/CONV
DELTA(N)=DELTA(N)/CONV
DLSTR=DLSTR(N)/CONV
USTAR=USTAE(N)/UE(N)
UI=UE(N)
DJIDX=DUEDX(N)

INITIALIZE STEP SIZES FOR DSRCH

DX(1)=.001
DX(2)=.9
DX(3)=.01
DXERR(1)=0.00001
DXERR(2)=0.0001
DXERR(3)=0.0001

ARRANGE VARIABLES BEING OPTIMIZED
IN ORDERED FORM FOR OPTIMIZATION IN DSRCH.

JC=0
DO 30 I=1,3
IF(IVAR(I).EQ.0)GO TO 30
JC=JC+1
XF(JC)=X0(I)
DXF(JC)=DX(I)
DXERR(JC)=DXERR(I)
30 CONTINUE

INITIATE OPTIMIZATION IN DSRCH

CALL DSRCH(NP,L,3,F,FF,FS,XF,DXF,DXERR,ITERM,EPST,ITER,
IPRINT,IER)

OPTIMIZATION COMPLETE. REARRANGE VARIABLES IN CORRECT ORDER.

JC=1
DO 40 I=1,3
IF(IVAR(I).EQ.0)GO TO 40
X0(I)=XF(JC)
DX(I)=DXF(JC)
DXERR(I)=DXERR(JC)
JC=JC+1
40 CONTINUE
EPST(N)=SQRT(FF)
EBAR=EBAR+PSI(N)
DELTA(N)=DELTA(N)+CONV
CAPK(N)=X0(I)
ES(N)=X0(2)
KKAP(N)=X0(3)
USTAU(N)=USTAR*UE(N)
CF(N)=2.*(UTAU(N)/UE(N))**2
CI(N)=CINF(ESIN),TOPLUS(N),X0(3),0.1)
BT(N)=BETAC
WRITE(6,110) ITAP1, EPSI(N), NPAR
WRITE(6,111) (X0(J), J=1, 3), UTAU(N)
WRITE(6,112) CI(N), TOPLUS(N)
WRITE(6,113) BETAC
C MATCH CONDITION CRITERIA FOR INNER AND OUTER REGION VELOCITY PROFILES
WRITE(6,114)
WRITE(6,115) ETAP, N0, ETA0
C INNER AND OUTER LENGTH SCALES
WRITE(6,200)
WRITE(6,116)
DO 50 I=1,N0
YP=Y0(I)/DELIN
UDP=UD(I)/USTAR
UAP=UA(I)/USTAR
UDPHUAP=UDP-UAP
50 WRITE(6,117) YP,UDP,UA,UDPHUAP
C SET UP FOR PLOTTING
C COMPUTE ENOUGH PLOT POINTS FOR A SMOOTH CURVE
C FOR ETA LESS THAN ETA0
C
NNPTS=400
ETAP=ETAM-1.*DELIN/DELOUT
DEL=ETAP/FLOAT(NNPTS)
YDPL(I)=1.*DELIN
DO 55 I=1,NNPTS
IF(I.EQ.1) GO TO 54
YP(YOPL(I))=YDPL(I-1)+DEL*DELOUT
54 YDLP=YDPL(I)/DELIN
ETAD=YDLP/DELOUT
CALL USER(XKAP(N),BETAC,COUT,ETAD,UL,U10,EPS)
MAKE=U1-ALOG(ETAD)/XKAP(N)-COUT
UAPL(I)=USTAR*(UP(YOPL(I)),T0TP,CIN,XKAP(N),0.)*MAKE
55 CONTINUE
DO 56 I=1,NNPTS
UPL(I)=(FLOAT(NCYC)*XLC)/ALOG(10.**FLOAT(NCYC))**(ALOG(YDPL(I))/DEL1)
56 CONTINUE
C FOR ETA GREATER THAN ETA MATCH, USE THE NUMERICAL
C SOLUTION ALREADY CALCULATED
NUA=NNPTS
NEX=NNPTS+1
DO 59 I=NEX, NUA
YDPL(I)=ETA(I-NNPTS)*DELOUT
YDLP=DOPL(I)/DELIN
ETA0=ETA(I-NNPTS)
MAKE=U1(NNPTS)-ALOG(ETA0)/XKAP(N)-COUT
UAPL(I)=USTAR*(UP(YDLP,I),T0TP,CIN,XKAP(N),0.)*MAKE
YDPL(I)=(FLOAT(NCYC)*XLC)/ALOG(10.**FLOAT(NCYC))**(ALOG(YDPL(I))/DEL1)
UAPL(I)=XL*NL*UAPL(I)/(NL*INCL*USTAR)
      END
CONTINUE
CALL PLOT(YDPL(I), UAPL(I), 3)
DO 60 I=2, NDP
CALL SYMBOL((FLOAT(NYC)) * XLC)/ALOG(10. ** FLOAT(NYC)) * IALOG(YD(I))/DELIM
10!
DO 60 I=1, NUA
IF(YDPL(I).GT.XLIM) GO TO 66
CALL PLOT(YDPL(I), UAPL(I), 2)
66 CONTINUE
CALL PLOT(0., 0., 3)
XLim=(FLOAT(NYC) * XLC)/ALOG(10. ** FLOAT(NYC)) * IALOG(YD(INP))/DELIM
+1
DO 65 I=1, NUA
NUAL=1
IF(YDPL(I).GT.XLIM) GO TO 66
CALL PLOT(YDPL(I), UAPL(I), 2)
65 CONTINUE
XIO=YDPL(NUAL) - RIGHT
YIO=UAPL(NUAL) - HGT/2.
CALL NUMBER(XIO, YIO, THETA(I), TUT1)
DO 80 I=1, NTA
WRITE(6, 110) (TITLE(I), I=1, 5), LSYS(UPL)
70 FINAL=FINAL+1
TESL(I) THETA(I), DELTA(I),
+ DELTA(I), THETA(I), TUT1
DO 80 I=1, NTA
WRITE(6, 120) NPAR
80 WRITE(6, 121) IDEN(I), BT(I), ETAU(I), EPSI(I), TOPLUS(I), CI(I)
WRITE(6, 122) NTA, EBAR
100 FORMAT(214, 5A10)
101 FORMAT(E10.3)
102 FORMAT(I4)
103 FORMAT(1H1, 5X, 5A10, 5X, *)
104 FORMAT(7F10.0, F10.2)
105 FORMAT(5X, # =, F6.3, 1X, A2, 4X, # =, F7.2, 1X, A2, 4X, SEC#)
106 FORMAT(14X, 6X, EXPERIMENTAL VALUES# =, F7.3, # 1/SEC#)
107 FORMAT(14X, 6X, EXPERIMENTAL VALUES# =, F7.3, # 1/SEC#)
108 FORMAT(14X, 6X, EXPERIMENTAL VALUES# =, F7.3, # 1/SEC#)
109 FORMAT(14X, 6X, EXPERIMENTAL VALUES# =, F7.3, # 1/SEC#)
110 FORMAT(14X, 6X, EXPERIMENTAL VALUES# =, F7.3, # 1/SEC#)
111 FORMAT(14X, 6X, EXPERIMENTAL VALUES# =, F7.3, # 1/SEC#)
112 FORMAT(14X, 6X, EXPERIMENTAL VALUES# =, F7.3, # 1/SEC#)
113 FORMAT(14X, 6X, EXPERIMENTAL VALUES# =, F7.3, # 1/SEC#)
114 FORMAT(14X, 6X, EXPERIMENTAL VALUES# =, F7.3, # 1/SEC#)
115 FORMAT(14X, 6X, EXPERIMENTAL VALUES# =, F7.3, # 1/SEC#)
116 FORMAT(14X, 6X, EXPERIMENTAL VALUES# =, F7.3, # 1/SEC#)
117 FORMAT(14X, 6X, EXPERIMENTAL VALUES# =, F7.3, # 1/SEC#)
118 FORMAT(14X, 6X, EXPERIMENTAL VALUES# =, F7.3, # 1/SEC#)
+ 14X, 6X, EXPERIMENTAL VALUES# =, F7.3, # 1/SEC#
+urtleau, 8x, nu, 6x, beta, 6x, delta, 4x, delta, 3x, theta, 5x, 
+turtleau
119 format(10x,i4,3x,f6.3,2x,f6.2,3x,f6.4,3x,f6.7,3x,f6.3,4(2x,f7.4))
120 format(///,15x,unsteady wall + similarity model - full prof 
+ilex,i4,2x,parameter fit,///,11x, 2hid,5x, 4beta,5x, 4mutau, 
+6x, 1hs,8x, 1kr,8x, shkap,2x, thepsn,4x, 3mt0,8x, 
+2hci)
121 format(10x,i4,1x,f8.4,2x,f6.3,2x,f7.3,2x,f8.6,3x,f7.4,2(1x,f8.6), f 
13x,f6.3)
122 format(///,20x,mean epsilon over#,i4,2x,#stations =#,f8.6)
130 format(f10.0)
200 format(///,5x,#length scales#)
201 format(5x,#delin = #,f7.5,#del3ut = #,f7.5)
return
end
SUBROUTINE ORSRCM (NNOIMFF4, FS, X, EPSI, MAX, EPSIF, ITER, IPRINT, 1IER)
EXTERNAL F
DIMENSION FS(NDIM), X(NDIM), DX(NDIM), EPSI(NDIM)

ORSRCM - DIRECT SEARCH ROUTINE

VARIABLES
N - NUMBER OF PARAMETERS
NDIM - DIMENSION OF ARRAYS
F - FUNCTION NAME
F6 - FINAL VALUE OF F
FS - VECTOR OF INTERMEDIATE F VALUES
X - VECTOR OF INITIAL AND FINAL PARAMETER VALUES
DX - VECTOR OF INITIAL STEP SIZES
EPSI - VECTOR OF CONVERGENCE CRITERIA
MAX - MAXIMUM NUMBER OF ITERATIONS
EPSIF - FINAL CONVERGENCE CRITERIA
ITER - NUMBER OF ITERATIONS
IPRINT - PRINT CONTROL
IER - ERROR FLAG

IER=0
ITER=IER
IF (IPRINT.EQ.1) WRITE (6,120)
F6=F(I)
101 I=0
102 I=I+1
NSTEP=0
X3=X(I)
F3=F6
IF (IPRINT.EQ.1) WRITE (6,121) ITER,F3,(X(J),DX(J),J=1,N)
103 ITER=ITER+1
IF (ITER.GT.MAX) GO TO 118
NSTEP=NSTEP+1
X2=X3
F2=F3
X(I)=X(I)+DX(I)
X3=X(I)
F3=F(X)
IF (IPRINT.EQ.1) WRITE (6,121) ITER,F3,(X(J),DX(J),J=1,N)
IF (F3-F2) 104,104,105
104 X1=X2
F1=F2
GO TO 103
105 IF (NSTEP=1) 106,106,107
106 X1=X3
F1=F3
DX(I)=DX(I)
X(I)=X(I)+DX(I)
X3=X2
F3=F2
GO TO 103
107 ITER=ITER+1
IF (ITER.GT.MAX) GO TO 118
X2=X2*X2
X3=X3*X3
F2MF3=F2-F3
AFAC=F1*(X2-X3)-X1*F2MF3+F2*X3-F3*X2
IF (AFAC.EQ.0.) GO TO 119
BFAC=X1*X1*F2MF3-F1*(X22-X32)+X22*F3-X32*F2
X4=-BFAC/(2.*AFAC)
X(I)=X4
F4=F4(X)
IF (IPRINT.EQ.1) WRITE (6,122) ITER,F4,(X(J),DX(J),J=1,N)
DF=F4-F2
IF (ABS(.2*DF/(F4+F2)).LT.EPSIF) GO TO 114
DELX=X4-X2
IF (ABS(DELX).LT.EPSI(I)) GO TO 114
IF (DX(I).LT.0.) DELX=-DELX
IF (DF) 108,108,111
108 IF (DELX) 109,109,110
109 X3=X2
F3=F2
X2=X4
F2=F4
GO TO 107
110 X1=X2
F1=F2
X2=X4
F2=F4
GO TO 107
111 IF (DELX) 112,112,113
112 X1=X4
F1=F4
GO TO 107
113 X3=X4
F3=F4
GO TO 107
114 IF (N.EQ.1) RETURN
FS(I)=F4
IF (I-I) 102,102,115
115 IF (I=N) 102,102,116
116 IF (ABS(2.*FS(I)-FS(I-1))/FS(I)+FS(I-1)).LT.EPSIF) RETURN
GO TO 117
117 DX(J)=DX(J)/1.1
GO TO 101
118 IER=1
WRITE(6,124) MAX
RETURN
119 IER=2
WRITE (6,123) X1,F1,X2,F2,X3,F3
RETURN
C
120 FORMAT (1H1,3X, 5MCYCLE,8X, 1MF,13X, 4HX(I),9X, 9HDX(I))
121 FORMAT ( 5H E , I3,3X,E13.6,6(2X,E13.6))
122 FORMAT ( 5H I ,I3,3X,E13.6,6(2X,E13.6))
123 FORMAT (5X, 4HX= ,E13.6, 6H F1= ,E13.6, 6H X2= ,E13.6, 6H F3= ,E13.6)
124 FORMAT(5X,#DRSrch FAILED TO CONVERSE IN #,I4,#ITERATIONS#)
C
END
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FUNCTION F(X)
DIMENSION X(3)
COMMON/VAR/IVAR(3),XO(3)
COMMON/DATA/NPAR,NOP,XI,UI,DUIDX,Y3(90),UD(90),DLST,UA(90),
+YDPL(1000),UDPL(1000),NPP
COMMON/UOUT/N0,ETA(400),DETA(400),J(400),ETAM,ETAO,COUT,CIN,
IS,TNTP,DELIN,DEOUT,USTAR,BETAC
DATA EPS/1.E-10/
FUNCTION F COMPUTES THE ROOT-MEAN-SQUARE ERROR
BETWEEN THE MEASURED EXPERIMENTAL VELOCITY PROFILE DATA
AND THE THEORETICAL VELOCITY PROFILE. THE THEORETICAL
PROFILE CONSISTS OF THE UNSTEADY WALL LAYER MODEL
FOR THE INNER LAYER AND A SELF-SIMILAR PROFILE MODEL
FOR THE OUTER REGION.
ARRANGE VARIABLES INCOMING FROM DSRCX IN CORRECT ORDER
JC=1
DO 10 I=1,3
IF(IVAR(I),EQ,0)GO TO 10
X(I)=X(JC)
JC=JC+1
10 CONTINUE
CALL UTAUF TO DETERMINE UTAU
XKAP=XO(3)
S=XO(2)
CAPK=XO(1)
CALL UTAUF(CAPK,XKAP)
COMPUTE RMS ERROR
JLOC=2
F=0.
UA(I)=0.000
DO 20 I=2,NOP
YDP=YD(I)/DELIN
ETAO=YD(I)/DELOUT
20 CONTINUE
COMPUTE THE DEFECT PROFILE AT THE DATA POINT EITHER
FROM THE SERIES SOLUTION FOR ETA LESS THAN ETA
OR BY INTERPOLATION OF THE NUMERICAL SOLUTION FOR
ETA GREATER THAN ETA
IF(ETAD.GT.ETAM)GO TO 30
CALL UISER(XKAP,BETAC,COUT,ETAD,UI,UID,EPS)
WAKE=UI
GO TO 40
30 DO 50 J=JLOC,N0
X1=ETA(J)-ETAO
X2=ETA((J-1)-ETAO
X3=X1*X2
IF(X3.LT.0.)GO TO 60
50 CONTINUE
WAKE=0.
GO TO 40
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JLOC=J
WAKE=U(J)-U(J-1)*X1/DETA(J-1)
WAKE=WAKE·ALOG(ETA0)/XKAP·COUT
UA(I)=USTAR*(UP(YDP,S,TNTP,CIN,XAP,G)+WAKE)
FT=UD(I)=USTAR*(UP(YDP,S,TNTP,CIN,XKAP,G)+WAKE)
F=F+FT*FT
RETURN
END
FUNCTION EDYVIS(XKAP, CAPK, ETA)
DATA N, XN/4.0, 29/

C FUNCTION EDYVIS EVALUATES THE EDDY VISCOSITY FUNCTION
C OR ITS DERIVATIVE FOR THE OUTER LAYER OF A TURBULENT
C BOUNDARY LAYER. THE FUNCTION APPROACHES KAPPA*ETA
C FOR SMALL ETA, EXPONENTIALLY QUICKLY. FOR LARGE ETA
C THE EDDY VISCOSITY FUNCTION APPROACHES THE OUTER CONSTANT
C K ALGEBRAICLY
C
C INPUT PARAMETERS
C XKAP = VON KARMA# CONSTANT
C CAPK = OUTER REGION CONSTANT K
C ETA = SCALED OUTER REGION COORDINATE

I=0
GO TO 10
ENTRY EDYVIS
I=1
10 C=(CAPK/XKAP )**N
X1=EXP(-C/ETA**N)
X2=1.0-X1
IF(I.EQ.0) EDYVIS=CAPK
IF(I.EQ.1) EDYVIS=0.00
IF(ETA.GT.10.0) GO TO 20
IF(I.EQ.0) EDYVIS=XKAP*ETA*X2**XN
IF(I.EQ.1) EDYVIS=XKAP*X2**XN*(1.0-X1/X2**ETA**N))
20 RETURN
END
SUBROUTINE $\text{UOUTER} \left( \text{XXAP}, \text{CAPK} \right)$

*DIMENSION UC(400,2), CO(2), BA(2), F(400), DEL(400), A(400), B(400), IC(400)*

*COMMON/OUT/NO, ETA(400), DETA(400), UI(400), ETAM, ETA0, COUT, CIN, OU/\text{NOTP}, \text{DELIN}, \text{DEOUT}, \text{USTAR}, \text{BETAC}*

*DATA EPSO, ALPHAEPS, $1.0 \times 10^{-10}, 1.0 \times 10^{1}$*

SUBROUTINE $\text{UOUTER}$ COMPUTES A NUMERICAL SOLUTION FOR THE OUTER REGION VELOCITY DEFECT PROFILE GIVEN THE INPUT VARIABLES $\text{XXAP}$ - VON KARMAN CONSTANT $\text{CAPK}$ - OUTER REGION EDDY VISCOSITY CONSTANT $\text{BETAC}$ - CLAUSER PRESSURE GRADIENT PARAMETER $\text{NO}$ - NUMBER OF MESH POINTS BETWEEN THE MATCH WITH THE SERIES SOLUTION (FOR SMALL ETA) AND THE BOUNDARY LAYER EDGE.

ON INPUT

$\text{ETA}(I)$ - MESH POINTS BETWEEN ETAM AND ETA0

$\text{DETA}(I)$ - VARIABLE STEP SIZE (INITIALLY UNIFORM AND EQUAL TO H FOR FIRST 6 MESH POINTS, THEN SUCCESSIVELY INCREASING BY A FACTOR ALPHAEPS)

$\text{UI}(I)$ - NUMERICAL SOLUTION FOR VELOCITY DEFECT AT ETA(I)

$\text{ETAM}$ - MATCH POINT BETWEEN SERIES AND NUMERICAL SOLUTION $\text{ETA0}$ - LARGEST VALUE OF ETA(I)

COUT - OUTER REGION LOG-LAD CONSTANT

FIX THE MATCH POINT

ETAM = 0.5*CAPK/XXAP

FIX THE OUTER VALUE OF ETA

$X1 = 1.0 + 2.0 \times \text{BETAC}$

$X2 = X1 - 1.0$

$X3 = (2.0 \times \text{BETAC} - 1.0) / (2.0 \times \text{BETAC})$

IF $X1 \cdot GT. 0.0 \cdot \text{ETAO} = \sqrt{\text{EPO} / (-2.0 \times \text{CAPK} \cdot \text{ALOG(EPSO)} / X1)}$

IF $X1 \cdot EQ. 0.0 \cdot \text{ETAO} = -\sqrt{\text{ABS(CAPK} / X2)} \cdot \text{ALOG(EPSO)}$

IF $X1 \cdot LT. 0.0 \cdot \text{ETAO} = \sqrt{\text{ABS(CAPK} / X1)} \cdot \text{EPSO}^{(3)}$

DEFINE THE MESH

$N1 = NO - 1$

$\text{DETA}(1) = \text{ETAM}$

$H = (\text{ETAO} - \text{ETAM}) / (\alpha \times (1.0 - \text{ALPHA}^{2}(NO - 1)) / (1.0 - \text{ALPHA}))$

DO 10 I = 1, N1

IF (I.LT.6) DETA(I) = H

IF (I.GE.6) DETA(I) = ALPHAEPS*DETA(I-1)

EITA(I+1) = ETA(I) + DETA(I)

10 CONTINUE

CALCULATE THE ELEMENTS OF THE TETDIAGONAL MATRIX FOR THE NUMERICAL SOLUTION FOR THE DEFECT PROFILE

$XP = \text{ETA}(I) + 0.5 \times H$

$P = EPO / (XXAP \cdot \text{CAPK} \cdot XP)$

$PJP = EPO / (X2 \cdot XP) / EP$

QJP = $X2 / EP$

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DO 20 J=2,N1
GAM=1.
IF(J.GE.61 GANL-ALPHA
X3=0.25*DETA(J)*DETA(J)
X4=0.5*DETA(J)
XN=XN
XP=ETA(J)*X4
EP=EDYVIS(XKAP,CAKP,XP)
EPD=EDYVISD(XKAP,CAKP,XP)
PJP=PJP
PJP=(EPD*X3*XP)/EP
QJM=QJM/GAN
QJP=X2/EP
A(JJ)=1.-GAN*XP*(PJP-PJM)*X3*(QJM+QJP)
B(JII)=1.*XP*PJP+X3*QJP
20 C(JJ)=GAN*XP*PJM+X3*QJM
C
DEFINE TWO INITIAL GUESSES FOR COUT
C
CO(1)=1.
CO(2)=0.75*CO(1)
C
CALCULATE TWO NUMERICAL SOLUTIONS FOR U FOR ETA GREATER THAN ETAM FOR GUESSED VALUES OF COUTER
C
DO 30 K=1,2
CALL USER(KAP,BETAC,CO(K),ETAM,JC(1,K),UCDS,EPS)
F(1)=0.
DEL(1)=UC(1,K)
DO 40 J=2,N1
X1=A(JI)+C(JI)*F(JJ-1)
F(J)=B(JJ)/X1
40 DEL(J)=C(JI)*DEL(J-1)/X1
C
BACK SUBSTITION
C
UC(N0,K)=0.
DO 50 J=2,N1
J1=NO-J+1
UC(J1,K)=F(J1)*UC(J1+1,K)+DEL(J1)
50 CONTINUE
C
CALCULATE DERIVATIVE OF NUMERICAL SOLUTION AT ETA=ETAM WITH SLOPING DIFFERENCE FORMULA
C
UCDM=(-274.*UC(1,K)+680.*UC(2,K)-580.*UC(3,K)
1+680.*UC(4,K)-158.*UC(5,K)+26.*UC(6,K))/(120.*H)
C
CALCULATE DIFFERENCE BETWEEN NUMERICAL AND SERIES DERIVATIVE SOLUTION AT ETA=ETAM.
C
BA(K)=UCDS-UCDM
30 CONTINUE
C
COMBINE TWO NUMERICAL SOLUTIONS TO OBTAIN THE TRUE VALUES OF U(J) AND COUT
C
B1=-BA(2)/:BA(1)-BA(2))
\[ B_2 = 1 - B_1 \]
\[ C_{OUT} = B_1 \cdot C_0(1) + B_2 \cdot C_0(2) \]
\[ \text{DO 60} \]
\[ J = 1, N_0 \]
\[ \text{U(J)} = B_1 \cdot U(J, 1) + B_2 \cdot U(J, 2) \]
\[ \text{60 CONTINUE} \]
\[ \text{RETURN} \]
\[ \text{END} \]
SUBROUTINE USERIXKAPBETAC(CONTRER.A.UiDEPSI)
DATA NMAX/100/

SUBROUTINE USER EVALUATES THE VELOCITY DEFECT FUNCTION U1
AND ITS DERIVATIVE UID AT THE INPUT VALUE OF ETA, FROM
A SERIES SOLUTION (OBTAINED BY THE METHOD OF FROEBENIUS)
WHICH IS VALID FOR SMALL ETA AND WHICH SATISFIES THE
ORDINARY DIFFERENTIAL EQUATION FOR THE SIMILARITY DEFECT
PROFILE U1.

INPUT PARAMETERS:
XKAP - VON KARMAN #CONSTANT#
BETAC - CLAUSER PRESSURE GRADIENT PARAMETER
COUT - OUTER REGION LOG-LAW #CONSTANT#
EPS - CONVERGENCE TOLERANCE

U1=UID=0,
X1=ALOG(ETA)
X2=.BETAC
X3=X2+1.
FACA=1./XKAP
FACB=COUT
ASUM=FACA
BSUM=FACB
ASUM=BSUM=X0.
DO 10 I=1,NMAX
X4=FLOAT(I-1)
X5=X4+1.
X6=1./(X5*X5*XKAP)
X7=(X2*X4*X3)*X6
FACB=(-FACB*X7+(X7-X6)*FACA/X5)*ETA
FACA=-FACA*ETA*X7
ASUM=ASUM+FACA
BSUM=BSUM+FACB
TESTA=ASUM
TESTB=BSUM
ASUM=ASUM+X5*FACA
BSUM=BSUM+X5*FACB
IF(I.LE.2)GO TO 10
IF(ASUM.EQ.0.0)GO TO 5
IF(ABS(ASUM/TESTA.GT.EPS))GO TO 20
10 CONTINUE
PRINT 30, NMAX,XKAP,BETAC,COUT,ETA,EPS
30 FORMAT(/,'SERIES CALCULATION IN USER HAS NOT CONVERGED AFTER',
+ '14, 2X,*TERMS#, /,1X,*XKAP= #,E12.5, 3X,#BETAC= #,E12.5,
+ 3X,#COUT= #,E12.5, 3X,#TOLERANCE= #,E12.5,/
STOP
20 U1=BSUM+X1*ASUM
UID=(BSUM+X1*ASUM+ASUM)/ETA
RETURN
END
SUBROUTINE UTAUF(CAPK, XKAP)

COMMON/RDATA/NPAR,NDNP,XNU,UI,DUIDK,YJ(90),UD(90),DLSTR,UA(90),
+ YDPL(1000),UAPL(1000),NPPT

COMMON/OUT/NOETA(400),DETA(400),J(400),ETAM,ETA0,COUT,CIN.

1S, TNOTP, DELIN, DELOUT, USTAR, BETAC

DATA EPS,x4AX/1.E-1,20 /

C C UTAUF COMPUTES THE VALUE OF UTAU ITERATIVELY USING
C THE MATCH CONDITION AND ALSO THE OUTER REGION PROFILE

C EPS - EXIT TOLERANCE FOR TWO SUCCESSIVE ITERATES FOR UTAU
C MAX - MAXIMUM NUMBER OF ITERATIONS IN UTAUF

C CIN=CINF(S,TNOTP,XKAP,0.,1.)

C REX=UI/XNU

USTRA=USTAR

DO 10 I=1,MAX

DELOUT=DLSTR/USTRA

DELIN=1./(REX*USTRA)

BETAC=-(DUIDK*DELOUT/(UI*USTRA))

IF(BETAC.LT.-.50) BETAC=-.50

CALL UOUTER(XKAP, CAPK)

XA=1./(ALOG(DELOUT/DELIN)/XKAP+CIN-COUT)

IF(I.NE.1) GO TO 20

FB=USTRA-XA

USTRA=USTRA

USTRA=XA

GO TO 10

20 FA=USTRA-XA

XI=USTRA-FA*(USTRA-XA)/(FB-FA)

TEST=ABS(1.-USTRA/XI)

IF(TEST.LT.EPS) GO TO 30

IF(ABS(FB).LT.ABS(FA)) GO TO 15

USTRA=USTRA

FB=FA

15 USTRA=XI

10 CONTINUE

WRITE(6,40) MAX, USTRA, XI

40 FORMAT(/,1X,#ITERATION HAS FAILED TO CONVERGE IN UTAUF,

AFTER#,14,# ITERATIONS#,1X,#LAST TWO ITERATES FOR UTAU ARE#,22E16.8,/) 

30 USTAR=USTRA

RETURN

END
FUNCTION CINF(S, TNOTP, XKAP, PPLUS, ROOTPR)
C
FUNCTION CINF USES THE WALL COMPATIBILITY CONDITIONS TO
C CALCULATE THE INNER REGION CONSTANTS CINEP AND TNOTP FOR A
C GIVEN VALUE OF S FOR BOTH THE TEMPERATURE AND THE VELOCITY
C PROFILE IN THE INNER LAYER OF A TURBULENT BOUNDARY LAYER.
C* ROOTPR = 1. FOR THE VELOCITY PROFILE CASE.
C* PPLUS = 0. FOR THE TEMPERATURE PROFILE CASE.
C
DATA GM, ROOTPI, EPSI, ITMX = 0.604539368109179, 1.77245385890552, 1.E-1 + 0.20/
JC = 0
X1 = 0.5*S*ROOTPR*ROOTPI*XKAP
S2 = S*S
X2 = X1*XKAP*S2*PPLUS/3.
AL = EXP(-1.-X1-X2)
DO 10 K = 1, ITMX
X3 = ALOG(AL)
X4 = AL + 1.
AL2 = AL*AL
X5 = SQRT(1. - AL2)
C = AL
AL = AL - (X1*X5 + X4*X3 + 1. - AL + X2*(AL2 + 2.*AL + 1.) / X4) / (-AL*X1/X5 +
+ X3 + 1.) / AL*X2*(AL2 + 2.*AL + 3.) / (X4*X4))
IF(ABS(AL/C - 1.) .LE. EPSI) GO TO 20
CONTINUE
JC = 1
20 AL2 = AL*AL
X4 = 1. - AL2
TNOTP = AL2*PPLUS/X4
X5 = SQRT(X4)
CINF = X1*X5 + GM - 0.5*AL2 + TNOTP + 1. / (AL + 1.)
+ (0.25*X2*(1. - AL2) - PPLUS*TNOTP*XKAP) / (3. * AL2 - 4. * AL + 1.) / 3. / X4
CINF = CINF / XKAP
IF(JC .EQ. 0) GO TO 30
WRITE(6, 940) JK*TNOTP, CINF, AL, C
STOP
C
40 FORMAT(1X, 30HNO CONVERGENCE IN CINF AFTER 14, 11H ITERATIONS, /,
+ 1X, 7HTNOTP =, E15.5, 10X, 6HCINF =, E15.5, 10X, 4HAL =, E15.5, 10X, 5HALP
+ =, E15.5, /)
C
30 RETURN
END
FUNCTION UP(YP,S,TNOTE,CIN,XKAP,PPLUS)
C** FUNCTION UP CALCULATES THE TIME-MEAN TEMPERATURE OR VELOCITY
C** PROFILE IN THE WALL LAYER OF A TURBULENT BOUNDARY LAYER
C** FOR SPECIFIED VALUES OF THE PARAMETERS LISTED BELOW.
C** YP = Y+ SCALING WALL LAYER COORDINATE
C** S = S*CYCLE TIME PARAMETER
C** TNOTE = TO*SIMILARITY PARAMETER
C** CIN = INNER REGION LOG-LAW CONSTANT CI OR BI
C** XKAP = KAPPA (VON-KARMAN CONSTANT) OR KAPPA-THETA
C** PPLUS = WALL LAYER PRESSURE GRADIENT PARAMETER. (PPLUS
C** IS ZERO FOR THE TEMPERATURE PROFILE CASE.)
C** NOTE A CALL TO UP SHOULD NORMALY BE PRECEDED BY A CALL TO
C** CINF WHICH COMPUTES CIN AND TNOTE, GIVEN S.
C
DATA XL,SRPT=-0.40453934,1.189179,1.77245395090516/ 
P(X)=2.*PPLUS*(X+TNOTE)/3.
R(X)=CIN*(0.5*LOG((X+TNOTE)-X1)/X1AP+0.5*PPLUS*(S2+2.*TNOTE) 
Q(X,Y,ZA)=(2.*X*X+1.)*Y+2.*X*ZA/3P1
W(X,Y,ZA)=(-0.75*X*ZA)/(3P1*X*ZAP) 
HA=W/(X+ZAP)
C** PRECIS IS THE VALUE OF X SUCH THAT EXP(-X*X) MAY BE COMPUTED
C** WITHOUT INCURRING AN UNDEFLOW.
C
PRECIS=25.9361455
C
S2=S*S
TPS2=S2+TNOTE
H0=0.5*YP/SQRT(S2+TNOTE)
ERFH=ERF(H0)
EXPH=EXPH0=0.
IF(H-LT.PRECIS)EXPH=EXP(-H*M)
IF(H-LT.PRECIS)EXPH0=EXP(-H0*M0)
UP=TPS2*(R(S2)*Q(H,ERFH,EXPH)+Z(H,ERFH,EXPH)) 
+ -TNOTE*(R(8.*)Q(H,ERFH,EXPH)+Z(H,ERFH,EXPH)) 
IF(PPLUS.EQ.0.)GO TO 10 
IF(H-LT.PRECIS)GO TO 20 
UP=UP+0.5*(S2*S2+2.*S2*TNOTE)*PPLUS 
GO TO 10
20 IF(MH-LT.PRECIS)GO TO 30 
MH=MH(ERFH,EXPH)
M0=H0**4.75 
GO TO 40
30 IF(MH(ERFH,EXPH)) 
M0=M0(H0.ERFH,EXPH) 
GO TO 40 
40 UP=UP+TPS2*(S2)*MH-TNOTE*P1*H0 
10 UP=UP/S2 
RETURN
END
-137-
FUNCTION XI(X)
DATA SRP1,GAMO,EP51/1.772453850985516,0.57721566490153286,1.E-10/
C     * FUNCTION XI EVALUATES THE TRIPLE INTEGRAL IN THE UNSTEADY
C     * WALL LAYER MODEL. THE TOLERANCE EPSI IS THE NUMBER OF
C     * SIGNIFICANT FIGURES DESIRED FOR XI AND ITS DERIVATIVE
C     * XI AND IS MACHINE DEPENDENT.
C
XI=0.SIF=1
IF(X.LE.0.) RETURN
GO TO 1
ENTRY XI
C     X=0.SIF=2
IF(X.LE.0.) RETURN
1 X2=X*X
FAC=2.*X2
M=100
SUN=0.
SUM=0.
TERM=1.
IF(X.GE.5.30) GO TO 110
IF:IF.EQ.2) GO TO 100
TERM=X
ALPHA=1.
DO 2 I=1,M
TERM=TERM*FAC/FLOAT(2*I+1)
SUN=SUN+TERM*ALPHA
IFABS((SUM-SUMT)/SUM).LT.EPSI) GO TO 3
ALPHA=ALPHA*1./FLOAT(I+1)
2
SUN=SUN
3 XI=0.25*EXP(-X2)*SUM
RETURN
100 DO 4 I=1,M
TERM=TERM*FAC/FLOAT(2*I-1)
SUN=SUN+TERM/FLOAT(I)
IFABS((SUM-SUMT)/SUM).LT.EPSI) GO TO 5
4
SUN=SUN
5 XI=0.25*EXP(-X2)*SUM
RETURN
110 IF(IF.EQ.2) GO TO 120
DO 6 I=1,M
TERM=TERM*FAC/(2*I-1)/FAC
TERM=TERM/FLOAT(I)
IF(TERM.LT.EPSI) GO TO 7
6
SUN=SUN+TERM
7 XI=SRPI*(ALOG(X2)+GAMO-SUM)/8.
RETURN
120 DO 8 I=1,M
TERM=TERM*FAC/(2*I-1)/FAC
IF(TERM.LT.EPSI) GO TO 9
8
SUN=SUN+TERM
9 XI=SRPI*(1.+SUM)/(4.*X)
RETURN
END
SUBROUTINE LOGAX(NCYC, XLC, Hght)
C SUBROUTINE LOGAX PLOTS THE X-AXIS LOGARITHMICALLY
C
C INPUT PARAMETERS
C NCYC = NUMBER OF CYCLES
C XLC = LENGTH OF ONE CYCLE
C Hght = HEIGHT OF AXIS LABEL
C = 0 FOR NO LABEL
DIMENSION XLOG(8)
TCK1=Hght
IF(Hght.LE.0.) TCK1=XLC/20.
TCK2=TCK1/2.
C SET UP EIGHT TIC MARKS PER CYCLE
DO 1 I=1,8
1 XLOG(I)=XLOG10(FLOAT(I+1))*XLC
DO 3 N=1,NCYC
X=I
X=I
XLOG=NCYC
IF(IN.EQ.1) CALL PLOT(XLOG11,TCK1,3)
CALL PLOT(XLOGM1,0.,2)
DO 2 I=1,8
X=XLOG11+XLOG1
CALL PLOT(X,0.,2)
CALL PLOT(X,TCK2,2)
2 CALL PLOT(X,0.,2)
XLOG=NCYC
CALL PLOT(XLOG,0.,2)
3 CALL PLOT(XLOG,0.,2)
IF(Hght.LE.0.) GO TO 5
NUM=10**NCYC
FAC=2.*Hght/7.
NUM=FLOAT(9-NCYC)*FAC
NCYC=NCYC+1
DO 4 N=1,NCYC
NUM=NCYC-N
XNUM=FLOAT(INUM)*XLC-TNUM
5 NUM=NUM+FAC
YNUM=-1.5*Hght
CALL NUMBER(XNUM,YNUM,Hght,NUM,0..2HIS)
4 NUM=NUM/10
XSYM=XLC*FLOAT(NCYC)/2.-2.*FAC
YSYM=1.*Hght
CALL SYMBOL(XSYM,YSYM,Hght,2HYS,.8..2)
5 CALL PLOT(0.,0.,3)
RETURN
END
SUBROUTINE SHFTYAX(NLNPLTPPXLIICPLHGNT)
C SUBROUTINE SHFTYAX PLOTS THE Y-AXIS FOR A MULTIPLE NUMBER
C OF CURVES ON THE SAME PLOT WITH SHIFTED Y-AXIS ORIGINS
C
C INPUT PARAMETERS
C NL = NUMBER OF POINTS LABELED NON-ZERO
C NPLTPP = NUMBER OF POINTS LABELED ZERO
C (EQUAL TO NUMBER OF AXIS SHIFTS)
C XL = LENGTH BETWEEN AXIS TICS (SPACING)
C INCPL = 0
C HGHT = HEIGHT OF AXIS LABEL
TCK=HGHT
IF(HGHT.LE.0.) TCK=XL/10.
NL=NPLTPP+NL-1
CALL PLOT(TCK,0.,3)
CALL PLOT(0.,0.,2)
Y=0.
DO 1 I=1,NL
Y=Y+XL
CALL PLOT(0.,Y,2)
CALL PLOT(TCK,Y,2)
1 CALL PLOT(0.,Y,2)
IF(HGHT.LE.0.) GO TO 4
NUM=NL*INCPL
FAC=HGHT/7.
XNUM=-2.5*FAC
YNUM=Y-HGHT/2.
DO 2 I=1,NL
IF(NUM.LE.99) CALL NUMBER(XNUM,YNUM,HGHT,NUM,0.,2H12)
IF(NUM.GT.99) CALL NUMBER(XNUM+FAC,YNUM,HGHT,NUM,0.,2H1)
YNUM=YNUM-XL
2 NUM=NUM+INCPL
XNUM=-1.5*FAC
DO 3 I=1,NPLTPP
CALL NUMBER(XNUM,YNUM,HGHT,NUM,0.,2H11)
3 YNUM=YNUM-XL
XSYM=-3.*HGHT
YSYM=XL*FLOAT(N)/2.-FAC
CALL SYMBOL(XSYM,YSYM,HGHT,2*H+,90.,2)
CALL PLOT(0.,0.,3)
RETURN
END
APPENDIX D

TEST CASE

The turbulent boundary layer prediction code presented in Appendix C is set up to guide the user in understanding the code's operation. Comments are used to explain input variables and to denote where major operations are taking place.

The user supplied code input begins with "PROGRAM MAIN"; an integer value for each of the model parameters $K$, $S$, and $\kappa$ is read in a 312 format. This integer value determines if the parameter is to be optimized or not; an integer value of 0 indicates no optimization for the corresponding variable and 1 indicates parameter optimization. The next variables to be read are "ITERM", "IPRINT", "EPS", and "NA" in 2I4, E6.1,14 format. The "ITERM" variable denotes the maximum number of cycles to be performed in direct search. "IPRINT" is a print control parameter which determines if printing of intermediate optimization results is included in the output. The "EPS" parameter is the exit tolerance for convergence of the optimized parameters and "NA" denotes the number of mesh points for the outer region numerical solution to the similarity equation. A typical value of "NA" is 350 to ensure good accuracy and should be increased in applications where there is an intense velocity variation.
in the outer layer. After reading in all of these variables, the program then calls "SUBROUTINE PROFIT" which reads in experimental data pertinent to the run.

"SUBROUTINE PROFIT" begins by reading in a four digit identification number for the data run "ID". The choice of an identifier is up to the user's discretion and will be successively incremented by one for each subsequent data station that is examined in the data set under consideration. The "ID" identifier is followed by "IUINT" which denotes whether the input experimental data is in English (ID=0) or metric units (ID=1). The selection of metric or English units is important to the input since this determines a conversion factor which is used in the code. A user supplied title for the data set is then read which is printed in the output. The number of data stations "NSTA" is read followed by input parameters for plotting. These plot parameters are read as "NCYC, NPLTPP, XLC, XL, NL, INCPL" and are accompanied by self-explanatory comments describing their function in the code. Their purpose is to allow maximum flexibility in obtaining plotted profiles to meet the requirements of the user. The only plotting comment worth mentioning here is that the "NPLTPP" parameter denotes the number of curves to be plotted on one plot using the shifted Y-axis method. If there are more data stations than the NPLTPP parameter (NSTA>NPLTPP), then another plot will be drawn to plot the remaining curves.
The data that follows is read for each data station and the sequence described in this paragraph is repetitive for subsequent data stations. First, the kinematic viscosity "XNU" is read in E10.3 format followed by "X, UE, UTAUE, DELTA, DELSTR, THETA, DUEDX, BETE" in 7F10.0, F10.2 format; these variables represent the x-location on the wall, local mainstream velocity, experimental value of $u_e$, experimental value of the boundary layer thickness $\delta$, experimental value of displacement thickness $\delta^*$, experimental value of momentum thickness $\delta$, local value of the velocity gradient $du_e/dx$, and the value of the Clauser pressure gradient parameter $\beta_c$ respectively. An optional read statement follows which reads the value of mainstream turbulence level "TUT" for examination of mainstream turbulence effects. The number of experimental velocity data points "NDP" is read next followed by the starting values of the model parameters $K$, $S$, and $\kappa$ denoted by $x_\phi(1)$, $x_\phi(2)$, and $x_\phi(3)$ respectively. These starting values are used to initiate optimization in the direct search subroutine. The experimental velocity profile data points are then read in F10.0 format starting with the x-coordinate values which signify the distance from the wall "YD". These are followed by the corresponding y-coordinates values which denote the nondimensionalized velocities "UD" represented by the ratio of velocity over the local mainstream
velocity \( u/U_e \). "SUBROUTINE PROFIT" then calls "SUBROUTINE DRSRCH" to initiate the optimization of the profile parameters.

The direct search subroutine calls all other subroutines to evaluate the optimization function \( F(x) \). After the optimization has been completed, information pertinent to the model is printed out and a plot of the analytical profile and experimental data points is made. A listing of the output for one data station is presented on the subsequent pages. An output summary similar to that in Table 5.1 for all data stations is printed at the end of the output listings for all data stations.
### UT MAINSTREAM TURBULENCE TEST DATA

- **X**: 40,300 FT  
- **UE**: 98.76 FT/SEC

#### EXPERIMENTAL VALUES
- $\Delta u_e = 0.000$ 1/SEC
- $U_{10} = 4.029$ FT/SEC
- $\Delta T = -0.900$ IN
- $\Delta T_e = 0.004$ IN
- $\theta = 0.0573$ IN

#### EXPERIMENTAL VELOCITY PROFILE

**61 POINTS**

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After 7 iterations in ORSCHM, F(X) = 0.100056E-01 with 1 parameter fit.

CAPK = 0.184185E-01, & = 0.110254E+02, KAPPA = 0.247900E+00, UTAU = 0.415889E+00.

CIN = 0.573116E+01, TPLUS = 0.276168E-02, SETAC = 0.

MATCH CONDITION CRITERIA FOR VELOCITY PROFILE:

ETAN = 0.0206, NE = 399, ETAO = 0.9228.

LENGTH SCALES:

DELIN = 0.05004, DELOUT = 0.16306.
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