STRUCTURE OF THE SET OF STEADY-STATE SOLUTIONS AND ASYMPTOTIC ETC(U)
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BEHAVIOUR OF SEMILINEAR
HEAT EQUATIONS

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We give a precise geometrical description of the set of steady-state solutions for general classes of semilinear heat equations. This enables us to prove global results about the asymptotic behaviour of the solutions of the initial value problem.

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Semilinear heat equations - i.e. heat equations perturbed by a nonlinearity acting only on the lowest order term - arise in many contexts. It is well known that when studying the asymptotic behaviour of the solutions as the time $t$ tends to infinity, a crucial role is played by the steady-state solutions. In this paper we present a global geometrical description of the set of steady-state solutions and this description enables us to give global results on the asymptotic behaviour of the solutions of the initial value problem.
STRUCTURE OF THE SET OF STEADY-STATE SOLUTIONS
AND ASYMPTOTIC BEHAVIOUR OF SEMILINEAR
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Introduction: Let $\Omega$ be a bounded, smooth, connected domain in $\mathbb{R}^N$. We consider both the initial value problem:

\[
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u = f(u) & \text{in } \Omega \times (0,\infty) \\
u(x,t) = 0 & \text{on } \partial \Omega \times (0,\infty), \ u(x,0) = u_0(x) & \text{in } \Omega
\end{cases}
\]

(IVP)

and the stationary problem:

\[
-Au = g(u) \text{ in } \Omega, \ u = 0 \text{ on } \partial \Omega
\]

(SP)

where $u_0$ is some given initial condition and $f$ is a given nonlinearity of class $C^1$.

To explain our results, let us take, to simplify, the case when $f$ is bounded on $\mathbb{R}$. We denote by $S$ the set of solutions of (SP) (in $C^2(\Omega)$) and by $S_+$ (resp. $S_-$, resp. $S_0$) the set of functions $u$ in $S$ such that:

\[
\lambda_1(-\Delta + c(x)) > 0 \text{ (resp. } < 0, \text{ resp. } = 0)
\]

where $\lambda_1(-\Delta + c(x))$ denotes the first eigenvalue of $-\Delta + c(x)$ acting over $H^1_0(\Omega)$.

We prove here that we have:

i) $S_+$ consists of a at most countable number of isolated points,

ii) Every closed connected subset of $S_0$ is a totally ordered $C^1$ curve,

iii) If $C$ is a connected subset of $S_-$ then $\overline{C}$, the closure of $C$, is contained in $S_-$. And this implies that for every $u_0$, the $w$-limit set $\omega(u_0)$ (i.e., the set of functions $u$ such that there exists $t_n \to \infty$ with $u(*)(t_n) \to u(*)$) is contained either in $S_+$, either in $S_0$, or in $S_-$. We next define:

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We next define:

\[ I_+ = \{ u_0 \in C \mid \omega(u_0) \subset S_+ \} \]

\[ I_- = \{ u_0 \in C \mid \omega(u_0) \subset S_- \} \]

\[ I_0 = \{ u_0 \in C \mid \omega(u_0) \subset S_0 \} \]

We then prove that we have:

(i) \( I_+ \cup I_0 \) contains an open dense set;

(ii) On \( I_+ \cup I_0 \), \( \omega(u_0) \) is a singleton;

(iii) If \( u_0 \in I_- \), then there exists \( \varepsilon > 0 \), \( v_+ \in S_+ \cup S_0 \), \( v_- \in S_+ \cup S_0 \) such that, for all \( \tilde{u}_0 \in B(u_0, \varepsilon) \) = \( \{ u - u_0 \| < \varepsilon \} \) (for some norm \( \| \cdot \| \) precised below), we have:

\[
\begin{cases}
\tilde{u}_0 > u_0, \quad \tilde{u}_0 \not\in I_0 & \Rightarrow \omega(\tilde{u}_0) = \{ v_+ \} \\
\tilde{u}_0 < u_0, \quad \tilde{u}_0 \not\in I_0 & \Rightarrow \omega(\tilde{u}_0) = \{ v_- \}.
\end{cases}
\]

In addition we can identify \( v_+ \) and \( v_- \): for example, \( v_+ \) is the minimum element of \( S \) above any function of \( \omega(u_0) \). In particular in view of (i), (ii), generically in \( u_0 \), the solution \( u(x,t) \) of (IVP) converges as time goes to \( +\infty \) to a stable solution (in a linearized sense) of (SP).

This study is, in some sense, the sequel of [13], where we studied the case when \( f \) is convex; and in the study of the (IVP) we will use strongly some results of [13].

The fact that, on \( I_0 \), \( \omega(u_0) \) reduces to only one point is related to a recent work of J. K. Hale and P. Massatt [10] (and actually can be deduced from [10] and the description of \( S_0 \) given above). The only global results describing the instability of steady-state solutions in \( S_- \) are given in H. Fujita [9], P. L. Lions [13], and D. Henry [11], [12] - let us remark that in [9] only a very special case is studied, while in D. Henry [11], [12], it is assumed that not only \( S_0 \) is empty, but that, for all \( u \) in \( S_- \), the linearized operator \(-A - f'(u)\) is one to one. Finally let us mention the related works of H. Matano [14], [15]; N. Chafee [4]; N. Chafee and E. Infante [5]; W. H. Fleming [8]; M. Crandall, P. Fife and L. A. Peletier [6]; M. Bertsch, P. L. Lions and L. A. Peletier [3].
In section I below, we study the sets \( S_+, S_-, S_0 \) and we prove claims i), ii) and iii). In section II, we give the proofs of the assertions concerning the asymptotic behaviour of the solutions of (IVP) in the case when \( f \) is such that the orbits are compact. Finally, in section III, we give various remarks and extensions.

Remark: The results and proofs given below remain trivially valid if the operator \(-\Delta\) is replaced by a general self-adjoint operator \( A \)

\[
A = -\sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij}(x)) + c(x)
\]

(with \((a_{ij})\) uniformly elliptic); if \( f(t) \) is replaced by a nonlinearity of the form \( f(x,t) \) and if the Dirichlet boundary condition is replaced by a general boundary condition (insuring the maximum principle) as for example:

\[
\frac{\partial u}{\partial n} + b(x)u = \varphi(x) \quad \text{on } \partial \Omega
\]

where \( n \) is the unit outward normal, \( b > 0, \varphi \) is given.

Let us also mention that the results of section I remain true for a general second-order elliptic operator (even if not self-adjoint).

I. Structure of the set of steady-state solutions

Let \( f \in C^1(\mathbb{R}) \), we denote by \( S \) the set of solutions \( u \in C^2(\Omega) \) of (SP):

(\text{SP}) \quad -\Delta u = f(u) \quad \text{in } \Omega, \quad u \in C^2(\Omega), \quad u = 0 \quad \text{on } \partial \Omega.

If we denote by \( \lambda_1(-\Delta + c(x)) \) the first eigenvalue of the operator \((-\Delta + c)\) on the space \( H_0^1(\Omega) \) (for any \( c \in L^\infty(\Omega) \)), then we introduce \( S_+ \) (resp. \( S_- \), resp. \( S_0 \)):

\[
S_+ = \{ u \in S, \lambda_1(-\Delta - f'(u)) > 0 \}
\]

\[
S_- = \{ u \in S, \lambda_1(-\Delta - f'(u)) < 0 \}
\]

\[
S_0 = \{ u \in S, \lambda_1(-\Delta - f'(u)) = 0 \}
\]

In the result which follows, the topology is the one of the space \( C_0^0(\overline{\Omega}) = \{ u \in C(\overline{\Omega}), \quad u = 0 \quad \text{on } \partial \Omega \} \).

**Theorem 1.1:**

1) \( S_+ \) consists of a at most countable number of isolated points,
ii) Every closed connected subset of $S_0$ is a totally ordered $C^1$ curve (for the partial order: $u < v$, if $u(x) < v(x)$ $\forall x \in \Omega$).

iii) If $C$ is a connected subset of $S_-$ then $\overline{C} \subset S_-.$

iv) For each $u \in S_+$, if the set $M_+(u) = \{ \tilde{u} \in S, \tilde{u} > u, \tilde{u} \not\in u \}$ is not empty then it has a minimum element $m_+(u)$ and $m_+(u) \in S_+ \cup S_0$. Similarly, if $M_-(u) = \{ \tilde{u} \in S, \tilde{u} < u, \tilde{u} \not\in u \} \neq \emptyset$, then this set has a maximum element $m_-(u)$ and $m_-(u) \in S_- \cup S_0$.

v) If $C$ is a connected component of $S_-$ such that there exists $u \in C$ with $M_+(u) \neq \emptyset$ (resp. $M_-(u) \neq \emptyset$) then $M_+(v) \neq \emptyset$ (resp. $M_+(v) \neq \emptyset$) for all $v \in C$ and $m_+(v)$ (resp. $m_-(v)$) is constant on $C$.

Remark I.1: By a simple use of the strong maximum principle (and of Hopf maximum principle) we have:

1) $m_+(u) > u > m_-(u)$ in $\Omega$, $\frac{3}{\delta_n} (m_+(u)) < \frac{3}{\delta_n} (m_-(u))$ on $\partial \Omega$.

Before giving the proof of Theorem I.1, we mention first an easy application of Theorem I.1 and state a result insuring that $M_+(u)$ or $M_-(u)$ are not empty.

Corollary I.1: Let $C$ be a closed connected subset of $S$ then either $C \subset S_+$, either $C \subset S_-$ or $C \subset S_0$.

Proof: If $C \cap S_+ \neq \emptyset$, then because of i) we have obviously:

$$C = \{ u \} \subset S_+.$$ 

On the other hand if $C \cap S_- \neq \emptyset$, then by the preceding argument we have necessarily:

$$C \cap S_+ = \emptyset.$$ 

Thus:

$$C = (C \cap S_-) \cup (C \cap S_0), \ (C \cap S_-) \cap (C \cap S_0) = \emptyset.$$ 

But in view of ii), $C \cap S_-$ is closed since $\overline{C \cap S_-} \subset S_-$. On the other hand, $C$ and $S_0$ being closed, $C \cap S_0$ is closed. Now, since $C$ is connected this implies: $C \cap S_- \neq \emptyset$ or $C \cap S_0$ is empty; and since we assumed $C \cap S_- \neq \emptyset$, we conclude: $C \cap S_0 = \emptyset$ or $C \subset S_-.$

Proposition I.1: Let $u \in S_-$.

1) Then $M_+(u)$ is not empty if and only if there exists $w \in C^2(\overline{\Omega})$ such that:

2) $-\Delta w \in f(\overline{\Omega})$, $w \in u \in \Omega$, $w \not\in u$. 

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(ii) Similarly \( M_* (u) \) is not empty if and only if there exists \( w \in C^2(\Omega) \) such that
\[
-\Delta w < f(w) \quad \text{in} \quad \Omega, \ w < u \quad \text{in} \quad \Omega, \ w \notin u.
\]

The proof of Proposition 1.1 will be given below, together with the proof of iv).

**Remark 1.2:** If \( f(t) \) satisfies:
\[
\lim_{t \to +\infty} f(t) t^{-1} < \lambda_1 = \lambda_1 (-\Delta)
\]
then \( M_+ (u) \) is not empty for all \( u \in S_+ \). Indeed let \( u \) be such that:
\[
\lim_{t \to +\infty} f(t) t^{-1} < u < \lambda_1
\]
we have for some \( C > 0 \):
\[
f(t) < ut + C, \quad \text{for all} \quad t > 0.
\]

Next, let \( w_{\lambda} \) be the solution of
\[
-\Delta w_{\lambda} - \mu w_{\lambda} = \lambda C \quad \text{in} \quad \Omega, \ w_{\lambda} \in C^2(\Omega), \ w_{\lambda} = 0 \quad \text{on} \quad \partial \Omega
\]
with \( \lambda > 1 \).

For \( \lambda \) large, we have: \( w_{\lambda} > u \quad \text{in} \quad \Omega \); and since we have
\[
-\Delta w_{\lambda} - \mu w_{\lambda} + C \lambda > \mu w_{\lambda} + C > f(w_{\lambda}) \quad \text{in} \quad \Omega,
\]
we conclude by a simple application of Proposition 1.1. Similarly, if \( f(t) \) satisfies:
\[
\lim_{t \to +\infty} f(t) t^{-1} < \lambda_1 = \lambda_1 (-\Delta),
\]
then \( M_- (u) \) is not empty for all \( u \in S_- \).

We now turn to the proof of Theorem 1.1:

**Proof of i.** Since, by definition, for every \( u \in S_+ \) the linearized operator \((-\Delta - f'(u))\) is one to one, \( u \) is an isolated point in \( S \). In addition, in view of the equation \((\#)\) and the Schauder estimates the set \( \{ u \in S, \ sup \ < R \} \) is compact for \( C(\Omega) \) every \( R < +\infty \). These two facts prove i).

**Proof of ii.** We first prove that if \( C \) is a closed connected subset of \( S_0 \) then \( C \) is totally ordered. Of course, we may assume that \( C \) contains more than only one point; in this case each point \( u \) of \( C \) is an accumulation point, that is:
\[
\forall u \in C \quad \exists \{ u_n \} \subset C, \ u_n \neq u, \ u_n \to u.
\]

Then we have
\[ \begin{cases} -\Delta(u_n - u) = \{(f(u_n) - f(u))(u_n - u)^{-1}\}(u_n - u) \text{ in } \Omega \\
 u_n - u \in C^2(\Omega), u_n - u = 0 \text{ on } \partial \Omega, u_n - u \not= 0 ; \end{cases} \]

where \( (f(u_n) - f(u))(u_n - u)^{-1} = c_n(x) = f'(u(x)) \) if \( u_n(x) = u(x) \). Thus \( 0 \) is some eigenvalue of the operators \((-\Delta - c_n)\) and since \( c_n \) converges (uniformly) to \( c(x) = f'(u(x)) \), we deduce:

\[ \lambda_1(-\Delta - c_n) = 0 \text{ for } n \text{ large enough ;} \]

but this implies \( \not(u_n - u) \not= 0 \text{ in } \Omega \). Thus, we have shown that:

\[ \forall u \in C, \exists \varepsilon > 0, \forall v \in C : |v - u| < \varepsilon \not= 0 \Rightarrow \lambda_1\left(-\Delta - \frac{f(v) - f(u)}{v - u}\right) = 0 . \]

Next, we define the map from \( C \times C \) into \( \mathbb{R} \):

\[ A(u, v) = \lambda_1\left(-\Delta - \frac{f(v) - f(u)}{v - u}\right) . \]

It is clear that \( A \) is continuous, therefore the set \( \tilde{C} = A^{-1}(0) \) is closed. But on the other hand one can prove in the same way as above that if \( A(u, v) = 0 \) then:

\[ \forall u, v \in C, \exists \varepsilon > 0 : |v - u| < \varepsilon \Rightarrow A(u, v) = 0 . \]

(Indeed \( 0 \) is an eigenvalue of \((-\Delta - (f(v) - f(u))(v - u)^{-1}\) and as \( \varepsilon \) goes to \( 0 \), \( (f(v) - f(u))(v - u)^{-1} \) converges to \( (f(v) - f(u))(v - u)^{-1} \), since \( A(u, v) = 0 \), this implies \( A(\tilde{u}, \tilde{v}) = 0 \) for \( \varepsilon \) small enough.)

This shows that \( A^{-1}(0) = \tilde{C} \) is also open (for the relative topology on \( C \times C \)) but since \( C \times C \) is connected and since \( A(u, u) = \lambda_1\left(-\Delta - f'(u)\right) = 0 \) (\( C \subset S_0 \)) we deduce:

\[ \tilde{C} = C \times C, \text{ or in other words:} \]

\[ \lambda_1\left(-\Delta - (f(u) - f(v))(u - v)^{-1}\right) = 0, \forall u, v \in C . \]

Since we have for all \( u, v \in C \):

\[ \begin{cases} -\Delta(u-v) = \{(f(u) - f(v))(u-v)^{-1}\}(u-v) \text{ in } \Omega \\
 u-v \in C^2(\Omega), u-v = 0 \text{ on } \partial \Omega , \end{cases} \]

we proved that if \( u, v \in C, u \not= v \) then necessarily we have

either:

\[ u > v \text{ in } \Omega, \frac{u}{\delta u} < \frac{v}{\delta v} \text{ on } \partial \Omega \]

or:

\[ u < v \text{ in } \Omega, \frac{u}{\delta u} > \frac{v}{\delta v} \text{ on } \partial \Omega \]

that is \( C \) is totally ordered.
We now prove that C is a $C^1$ curve: without loss of generality we may assume that C contains more than one point. Then we introduce for all $v \in C$:

$$t(v) = \int_{\Omega} v(x)dx.$$ 

This defines a continuous map from C into $\mathbb{R}$: its range is some interval $(t_0, t_1)$ (if C has a maximum element, we take $(t_0, t_1]$ and if C has a minimum element, we take $[t_0, t_1)$). Since C is totally ordered, it is clear that:

$$t(v) = t(v') \implies v = v'.$$

Thus the map from C into I $(v \mapsto t(v))$ is continuous and one to one. In addition since C is totally ordered, we have: $t(v) > t(v') \implies v > v'$ in $\Omega$. We may now define a parametrization of C: $I \ni t \mapsto v_t$ where $v_t$ is given by the solution of $t(v_t) = t$.

It is very easy to check that $v_t$ is continuous for $t \in I$.

We now prove that the map $(t \mapsto v_t)$ is $C^1$ on I and that $v'(t) = w_t$ where $w_t$ is the normalized first eigenfunction of:

$$\begin{cases}
-\Delta v_t = f'(v_t) w_t & \text{in } \Omega, \quad w_t \in C^2(\Omega) \\
 w_t > 0 & \text{in } \Omega, \quad w_t = 0 & \text{on } \partial \Omega, \quad \int_{\Omega} w_t(x)dx = 1.
\end{cases}$$

since the continuity of $w_t$ with respect to $t$ is a standard consequence of the continuity of $f'(v_t)$, we will only prove that:

$$\frac{1}{h}(V(t+h) - V(t)) \xrightarrow{h \to 0^+} w_t.$$ 

But we have:

$$\begin{align*}
-\Delta \left( \frac{1}{h}(v_{t+h} - v_t) \right) &= \{(f(v_{t+h}) - f(v_t))(v_{t+h} - v_t)\}^{-1}\left( \frac{1}{h} v_{t+h} - v_t \right) \\
\frac{1}{h}(v_{t+h} - v_t) &\in C^2(\Omega), \quad \frac{1}{h}(v_{t+h} - v_t) = 0 & \text{on } \partial \Omega, \quad \frac{1}{h}(v_{t+h} - v_t) > 0 & \text{in } \Omega;
\end{align*}$$

and

$$\int_{\Omega} \frac{1}{h}(v_{t+h} - v_t)(x)dx = 1.$$ 

If we denote by: $w_t^h = \frac{1}{h}(v_{t+h} - v_t)$. In view of (7) and (8), we have:

$$\|w_t^h\|_{L^2(\Omega)} \leq C, \quad \|\Delta w_t^h\|_{L^1(\Omega)} \leq C \quad \text{(for some } C \text{ ind. of } h).$$

From well-known regularity results, this implies:

$$\|w_t^h\|_{L^p(\Omega)} \leq C, \quad \text{for all } 1 \leq p < \frac{N}{1-2} \quad (\leq \text{ if } N = 1, 2).$$
but this implies, using (7): \( Iw_{C}^{h} \in \mathcal{L}^p(\Omega) \) for all \( 1 < p < \frac{N}{n-2} \) and by a straightforward bootstrap argument and by Schauder estimates we obtain:
\[
Iw_{C}^{h} \in C^{2,\alpha}(\Omega)
\]
Thus taking if necessary a subsequence, \( w \in C^{2,\alpha}(\Omega) \) solution of (6).

**Remark 1.3:** It is easy to deduce from the above proof that the curve \( C \) has the same regularity than \( f \) (if \( f \in C^{k} \), then \( C \) is of class \( C^{k} \), for all \( 1 < k < \infty \)).

**Proof of iii):** Let \( C \) be a connected subset of \( S_\Omega \), we may assume without loss of generality that \( C \) contains more than one point. Let \( (u_n) \) be a converging sequence in \( C, u_n \rightarrow u \). It is clear that \( u \in S_\Omega \cup S_0 \). Suppose that \( u \in S_0 \) and let us try to obtain a contradiction.

Each \( u_n \) is an accumulation point in \( C \), thus there exists \( u_n^m \in C \) such that:
\[
\frac{u_n^m - u_n}{u_n} \rightarrow u, u_n^m \neq u_n
\]
\[
\begin{cases}
\Delta(u_n^m - u_n) = (f(u_n) - f(u_n^m))(u_n^m - u_n)(u_n^m - u_n) & \text{in } \Omega \\
u_n^m - u_n \in C^2(\Omega), u_n^m - u_n \neq 0, u_n^m - u_n = 0 & \text{on } \partial \Omega
\end{cases}
\]
Therefore \( 0 \) is some eigenvalue of the operator \((-\Delta - c_n)^m\) where
\[
c_n^m = (f(u_n^m) - f(u_n))(u_n^m - u_n).\]
Since \( c_n^m \) converges, as \( m \) goes to infinity, to \( c_n = f'(u_n) \), we deduce that \( 0 \) is an eigenvalue of \((-\Delta - f'(u_n))\). Now, since \( u_n \rightarrow u \) and thus \( f'(u_n) \rightarrow f'(u) \) and since \( \lambda_1(-\Delta - f'(u)) = 0 \), this would imply that for \( n \) large:
\[
\lambda_1(-\Delta - f'(u)) = 0
\]
and this contradicts the assumption: \( u_n \in C \subset S_\Omega \). The contradiction proves iii).

**Remark 1.4:** We proved in fact that if \( u_n \in S_\Omega \) and \( u_n \) is an accumulation point in \( S_\Omega \), then all limit points of the sequence \((u_n^m)\) lie in \( S_\Omega \).

**Proof of iv) and of Proposition 1.1:** We will only prove the assertions concerning \( m_+ \) and \( M_+ \). We first remark that if \( M_+(u) \) is not empty then any \( \tilde{u} \) in \( M_+(u) \) satisfies (2).

Thus it remains to prove that if there exists \( w \) satisfying (2), then there exists a minimum element in \( S \) above \( u \). We will prove that this minimum element \( m_+(u) \) belongs to \( S_\Omega \cup S_0 \).

---
To prove the existence of a minimum solution in $S$ above $u$, we will adapt some general results of H. Amann [1], [2]. Since $u \in S_+$, there exists $\lambda_1 < 0$ and $v_1 \in C^2(\Omega)$ such that:
\[
\begin{cases}
-\Delta v_1 = f'(u)v_1 + \lambda_1 v_1 & \text{in } \Omega, \quad v_1 \in C^2(\Omega), \\
v_1 > 0 & \text{in } \Omega, \quad v_1 = 0 \text{ on } \partial \Omega.
\end{cases}
\]
Thus for $\epsilon$ small enough ($0 < \epsilon < \epsilon_1$):
\[
-\Delta (u + \epsilon v_1) = f(u) + f'(u)\epsilon v_1 + \epsilon \lambda_1 v_1 \in \Omega \\
< f(u + \epsilon v_1) \in \Omega.
\]
On the other hand, if $w$ satisfies (2), from the strong maximum principle we deduce:
\[
w(x) > u(x) \text{ in } \Omega, \quad \frac{3w}{\partial} \frac{\partial w}{\partial n} < \frac{\partial}{\partial n}(u + \epsilon v_1) \text{ on } \partial \Omega.
\]
Therefore for $\epsilon$ small enough ($0 < \epsilon < \epsilon_2$):
\[
u + \epsilon v_1 < w \text{ in } \Omega, \quad \frac{3w}{\partial} \frac{\partial w}{\partial n} < \frac{\partial}{\partial n}(u + \epsilon v_1) \text{ on } \partial \Omega.
\]
Hence we obtained, for $0 < \epsilon < \epsilon_0 = \min(\epsilon_1, \epsilon_2)$,
\[
-\Delta (u + \epsilon v_1) < f(u + \epsilon v_1) \text{ in } \Omega, \quad u + \epsilon v_1 < w \text{ in } \Omega.
\]
We will denote by $K$ a positive constant such that:
\[
f'(t) + Kt > 0, \quad \text{for } t \in [-1, 1], \quad + [1, \infty]
\]
And we introduce the standard iterative method: ($0 < \epsilon < \epsilon_0$)
\[
\begin{cases}
\begin{align*}
u_0 & = u + \epsilon v_1, \\
u_{n+1} & = f(u_n) + K u_n \text{ in } \Omega, \\
u_{n+1} & = C^2(\Omega), \quad u_{n+1} = 0 \text{ on } \partial \Omega.
\end{align*}
\end{cases}
\]
It is then obvious to show that:
\[
\begin{cases}
\begin{align*}u < u_0 < u_1 < \cdots < u_n < u_{n+1} < \cdots < w \text{ in } \Omega, \\
u_{n+1} & \in C^2(\Omega), \quad u_{n+1} = 0 \text{ on } \partial \Omega.
\end{align*}
\end{cases}
\]
and in addition: $u_n < u_n^e < w$ for all $n > 1$, $0 < \epsilon < \epsilon'$, and thus: $u^e < u^e'$. Hence, as $\epsilon$ goes to $0$, $u^e$ converges in $C^2(\Omega)$ to $u \in S$. 

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We now prove that for all \( \epsilon > 0 \) \( u^\epsilon \in S_+ \cup S_0 \), thus \( \tilde{u} \in S_+ \cup S_0 \), and we will deduce that \( u^\epsilon \leq \tilde{u} \) for \( \epsilon \) small enough.

Indeed we have:

\[
\Delta (u^\epsilon - u_{n+1}^\epsilon) + K(u^\epsilon - u_{n+1}^\epsilon) = f(u^\epsilon) + Ku^\epsilon - f(u_n^\epsilon) - Ku_n^\epsilon
\]

\[
> f(u^\epsilon) + Ku^\epsilon - f(u_{n+1}^\epsilon) - Ku_n^\epsilon
\]

\[
> (f(u^\epsilon) - f(u_{n+1}^\epsilon))(u^\epsilon - u_{n+1}^\epsilon) + K(u^\epsilon - u_{n+1}^\epsilon)^{-1} \text{ in } \Omega
\]

and \( u^\epsilon - u_{n+1}^\epsilon > 0 \) in \( \Omega \), \( u^\epsilon - u_{n+1}^\epsilon \in C^2(\Omega) \), \( u^\epsilon - u_{n+1}^\epsilon = 0 \) on \( \partial \Omega \).

This implies:

\[
\lambda_1 (\Delta - c_n^\epsilon) > 0,
\]

where

\[
c_n^\epsilon = \{(f(u^\epsilon) - f(u_{n+1}^\epsilon))(u^\epsilon - u_{n+1}^\epsilon)^{-1}\}.
\]

Next, as \( n \) goes to \( \infty \), \( c_n^\epsilon \to f'(u^\epsilon) \) and thus \( u^\epsilon \in S_+ \cup S_0 \). Hence \( \tilde{u} \in S_+ \cup S_0 \), \( \tilde{u} \neq u \), \( u < \tilde{u} < u^\epsilon \) in \( \Omega \).

But the maximum principle shows that:

\[
u(x) < \tilde{u}(x) \text{ in } \Omega, \quad \frac{\partial u^\epsilon}{\partial n} < \frac{\partial \tilde{u}}{\partial n} \text{ on } \partial \Omega,
\]

and therefore we have for \( \epsilon \) small enough: \( u + \epsilon v_1 < \tilde{u} \) in \( \Omega \), this yields: \( u^\epsilon < \tilde{u} \), \( u^\epsilon < \tilde{u} \) and \( u^\epsilon \leq \tilde{u} \) for \( \epsilon \) small enough.

Now for any \( \tilde{u} \in S \) such that \( \tilde{u} > u \), \( \tilde{u} \neq u \); we obtain from the strong and Hopf maximum principle:

\[
\tilde{u} > u \text{ in } \Omega, \quad \frac{\partial \tilde{u}}{\partial n} < \frac{\partial u}{\partial n} \text{ on } \partial \Omega
\]

and therefore for \( \epsilon \) small enough: \( u + \epsilon v_1 < \tilde{u} \) in \( \Omega \). This implies: \( u^\epsilon < \tilde{u} \), \( v \) and since \( u^\epsilon < \tilde{u} \) we deduce \( \tilde{u} < \bar{u} \). This shows that \( \tilde{u} \) is the minimum element of \( S \) above \( u \). We will denote it by \( m_+(u) \); and we already showed that \( m_+(u) \in S_+ \cup S_0 \).

Remark 1.5: Other arguments for the existence of \( m_+(u) \) can be given but we prefer the above one since it yields a constructive existence proof.

**Proof of v):** Again we will only prove the assertions concerning \( m_+ \), \( M_+ \). Let \( 
C \) be a connected component of \( S_- \) and suppose there exists \( u_0 \) with \( M_+(u_0) \neq \emptyset \). Let \( C' \) be the connected component of the set \( (\tilde{u} \in S_-, M_+(\tilde{u}) \neq \emptyset) \) containing \( u_0 \). We first show that \( C' \) is open (for the relative topology); indeed if \( u \in C' \), \( m_+(u) \) satisfies:

\[
\text{-10-}
\]
Therefore for \( v \in S, v \) near \( u \), we still have: \( v < m_+(u) \) in \( \mathbb{R} \), thus \( m_+(v) \neq \emptyset \) and this shows that \( C' \) is open.

We next show that \( m_+ \) is continuous on \( C' \): let \( u_n \in C', u_n \rightarrow u \) and \( u \in C' \); let us prove that \( m_+(u_n) \rightarrow m_+(u) \). We first remark that for \( n \) large enough, as we proved above, we have: \( u_n < m_+(u) \) in \( \mathbb{R} \). Thus: \( u_n < m_+(u_n) \) in \( \mathbb{R} \), for \( n \) large enough. This proves in particular that \( m_+(u_n) \) is bounded in \( \mathbb{R} \), and using the equation (SP) and regularity estimates we deduce that \( m_+(u_n) \) is bounded in \( C^2, \alpha(\mathbb{R}) \) 

\((0 < \alpha < 1) \). Now (taking if necessary a subsequence) \( m_+(u_n) \) converges in \( C^2(\mathbb{R}) \) to \( \tilde{u} \in S_+ \cup S_0 \) and \( u \in \tilde{u} \leq m_+(u) \).

Since \( \tilde{u} = S_+ \cup S_0 \), \( \tilde{u} \neq u \) and thus from the definition of \( m_+(u) \) this shows that \( \tilde{u} = m_+(u) \). This proves the continuity of \( m_+ \) on \( C' \).

We now prove that \( m_+ \) is constant on \( C' \) and thus

\[ m_+(u) = m_+(u_0), \quad \forall u \in C'. \]

Indeed, \( m_+(C') \) is a connected set \( \subseteq S_+ \cup S_0 \). But because of i), if \( m_+(C') \cap S_+ \neq \emptyset \) then \( m_+(C') = \{ m_+(u_0) \} \subseteq S_+ \). And if \( m_+(C') \cap S_+ = \emptyset \), we then have \( m_+(C') \subseteq S_0 \); and from ii), we deduce that \( m_+(C') \) is totally ordered. This will enable us to show that \( m_+ \) is locally constant and this concludes the proof since \( m_+ \) is continuous and \( C' \) is connected. Let \( v \in C' \); if \( m_+ \) is not constant in a relative neighborhood of \( v \), then there exists \( v_n \in C', v \neq v_n \) and \( m_+(v) \neq m_+(v_n) \). Since \( m_+(v_n) \) and \( m_+(v) \) can be compared we have:

either

\[ m_+(v) < m_+(v_n) \in \mathbb{R} \]

or

\[ m_+(v) > m_+(v_n) \in \mathbb{R} \]

If the first case happens for \( n \) large enough, recalling that \( m_+(v_n) \neq m_+(v) \), we should have: \( v < m_+(v_n) \) in \( \mathbb{R} \). And this contradicts the definition of \( m_+(v) \).

Now, if the second case happens for \( n \) large enough, recalling that \( v \neq v_n \)

\[ \mathbb{R} \) and \( m_+(v) \neq m_+(v_n) \), \( v \neq m_+(v_n) \) in \( \mathbb{R} \), this would imply: \( v < m_+(v) \neq m_+(v_n) \) in \( \mathbb{R} \). And this contradicts the definition of \( m_+(v_n) \). Thus \( m_+ \) is locally constant in \( C' \) and this shows that \( m_+ \) is constant on \( C' \): \( m_+(u) = m_+(u_0), \forall u \in C' \).
We may now conclude by proving that \( C' \) is closed (for the relative topology) and since \( C \) is connected, this will show that \( C = C' \). Therefore let \( u_n \in C', \ u_n \xrightarrow{n} u \in C \). We have just proved

\[
\exists \, n \in \mathbb{N} : u_n = u_{n+1} = u_{n+2} = \ldots \quad \text{and} \quad u \notin \text{int}(C).
\]

Thus: \( u \notin \text{int}(C) \) and \( u \notin \text{int}(C') \). This shows that \( N_u \) is not empty and that \( u \notin C' \).

\textbf{Remark 1.6:} We would like to point out that because of the local compactness of \( S \), Theorem 1.1 is still valid (with the same proof) for the topology of any space like \( W^{1,\infty}(\Omega), C^0(\overline{\Omega}), C^2(\overline{\Omega}) \) - where we denote by \( X_0 \) the subspace of any functional space \( X \) of functions vanishing on \( \partial \Omega \).

\textbf{Remark 1.7:} As it was mentioned in the Introduction, Theorem 1.1 and its proof are still valid for any uniformly elliptic second-order operator instead of \(-A\), for general nonlinearities \( f(x,t) \) instead of \( f(x) \) and for general boundary conditions (satisfying the maximum principle).

\textbf{II. Asymptotic behaviour for quasi-bounded nonlinearities.}

We consider now the (IVP)

\[
\begin{aligned}
\begin{cases}
\frac{\partial u}{\partial t} - Au = f(u) & \text{in } \Omega \times (0,\infty), \ u \in C^2(\Omega) \cap C(\overline{\Omega}) \\
u(x,t) = 0 & \text{on } \partial \Omega \times (0,\infty), \ u(x,0) = u_0(x) \text{ in } \overline{\Omega}
\end{cases}
\end{aligned}
\]

(IVP)

where \( \Omega = \Omega \times (0,\infty), u_0 \) is some given initial condition in the space \( X = W^{1,\infty}(\Omega) \) (for example).

We will assume that \( f(\mathcal{C})(\mathbb{R}) \) satisfies:

\[
\lim_{|t| \to \infty} f(t)/t < 0 \quad \text{and} \quad \lim_{|t| \to 0} f(t)/t = 0.
\]

This insures for example that, for any \( u_0 \) in \( X \), there exists a unique solution \( u(x,t) \) of (IVP) and that:

\[
|u(x,t)|_C(\overline{\Omega}) < C \quad \text{(indep. of } t > 0) .
\]

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This implies that $u(\cdot,t)$ is bounded in $C^2_{\alpha}(\overline{\Omega})$ (for $t > \delta > 0$) for all $0 < \alpha < 1$.

In particular the orbit $(u(\cdot,t))_{t>\delta}$ is compact in $X$ (for all $\delta > 0$).

Of course $u(x,t)$ defines a (nonlinear) semigroup: $u(\cdot,t) = S(t)u_0$. Finally, since the orbit is compact in $X$, by well-known results (see for example C. M. Dafermos [7]) we have, denoting by $w(u_0)$ the $\omega$-limit set of $(S(t)u_0)_{t>0}$ i.e.,

$$w(u_0) = \{ u \in X, \exists t_n \uparrow 0 \ u(\cdot,t_n) \to u \}.$$  

(10) $w(u_0)$ is a connected compact subset of $X$.

In addition, since we have a Lyapunov function namely:

$$S(v) = \int_0^1 \frac{1}{2} |\dot{v}|^2 - F(v) \, dx,$$

where $F(t) = \int_0^t f(s) \, ds$; we have:

(11) $w(u_0) \subset S_0$ in $C^1,\alpha(\overline{\Omega})$ (0 < $\alpha$ < 1).

Now applying Corollary I.1, we see that only three possibilities may happen:

1) $w(u_0) \subset S_+$ or ii) $w(u_0) \subset S_0$ or iii) $w(u_0) \subset S_-$. It is then natural to introduce the three sets (which are disjoint):

$$I_+ = \{ u_0 \in X, w(u_0) \subset S_+ \}$$

$$I_0 = \{ u_0 \in X, w(u_0) \subset S_0 \}$$

$$I_- = \{ u_0 \in X, w(u_0) \subset S_- \};$$

we just explained why we have: $X = I_+ \cup I_0 \cup I_-$.

We will denote by $B(u_0,\varepsilon) = \{ u \in X, \| u - u_0 \| < \varepsilon \}$. Our main result concerning the asymptotic behaviour of (IVP) is the following:

Theorem II.1: Under assumption (9) and if $f \in C^2(\mathbb{R})$ we have:

i) $I_+ \cup I_0$ contains an open dense set;

ii) For all $u_0 \in I_+ \cup I_0$, $w(u_0)$ is a singleton;

iii) If $u_0 \in I_-$, there exists $\varepsilon > 0$ such that:

(13) $\forall \ v \in B(u_0,\varepsilon), \ v > u_0, \ v \notin \mathbb{R} \Rightarrow w(v) = [u^+] \subset S_+ \cup S_0$

where $u^+ = \max(v), \ v \in B(u_0)$;

(14) $\forall \ v \in B(u_0,\varepsilon), \ v < u_0, \ v \notin \mathbb{R} \Rightarrow w(v) = [u^-] \subset S_- \cup S_0$

where $u^- = \min(v), \ v \in B(u_0)$.

In particular if $v \in B(u_0,\varepsilon), \ v > u_0$ on $v < u_0$ then $v \in I_+ \cup I_0$. 

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Remark II.1: Remark first that in view of Remark 1.2 and assumption (9), $m^+$ and $m^-$ are defined on $S_-$ and are constant on each connected component of $S_-$, and thus $m^+$, $m^-$ are constant on $w(u_0)$ since $w(u_0)$ is connected (for $u \in I_-$).

Remark II.2: The fact that $w(u_0)$ is a singleton, for $u_0$ in $I_0$ can be deduced from the fact that by part ii) of Theorem 1.1 $w(u_0)$ is a $C^1$-curve and from general results on dynamical systems of J. Hale and P. Massart [10]. However we give a different proof which is a trivial consequence of the fact that $w(u_0)$ is totally ordered.

Remark II.3: The above result shows in particular that, generically in $X$, $S(t)u_0 + u \in S_+ \cup S_-$ (that is a solution of (SP) such that: $\lambda_1(A - f'(u)) > 0$).

Remark II.4: The above result is still valid if we replace $W^1_0(\Omega)$ by $C_0^1(\Omega)$ or $C_0^2(\Omega)$ or the subspace $E$ of $C_0^1(\Omega)$ defined by: $E = \{ u \mid \lambda \in (-\delta, \delta) \}$ where $\delta(x) = \text{dist}(x, \partial \Omega)$ and $\lambda \in E$ where $[u,v] = \{ w \in C_0^1(\Omega), u < w < v \}$, $E$ is equipped with the order unit norm:

$$\text{inf}(\lambda > 0 / -\lambda \delta < u < \lambda \delta)$$

$E$ with this norm is a Banach space - for more details, see H. Amann [21].

Remark II.5: Similar results hold for general self adjoint uniformly elliptic second-order operators, for general nonlinearities of the form $f(x,t)$ and for general boundary conditions preserving the maximum principle.

Before giving the proof of Theorem II.1, we mention the following standard comparison principle: let $u_0 < v_0$, $u_0 \not\in v_0$ then for all $t > 0$ we have:

$$S(t)u_0(x) < S(t)v_0(x) \text{ in } \Omega, \frac{3}{3_n}(S(t)u_0) > \frac{3}{3_n}(S(t)v_0) \text{ on } \partial \Omega.$$  

Proof of ii): Because of (10) and of part i) of Theorem 1.1, then it is trivial that $w(u_0)$ is a singleton if $u_0 \in I_0$. Next, let $u_0 \in I_0$, $w(u_0)$ is a compact connected set $\subset S_0$, thus $w(u_0)$ is totally ordered. If $w(u_0)$ is not a singleton, there exist $u_1, u_2, u_3$ in $w(u_0)$ such that: $u_1 < u_2 < u_3$ in $\Omega$, $\frac{3}{3_n}(u_1) > \frac{3}{3_n}(u_2) > \frac{3}{3_n}(u_3)$ on $\partial \Omega$. Since $u_1 \in w(u_0)$, there exists $(t_n)_{n}$ such that $u(x,t_n) \in w(u_0)$ in $\Omega$. Therefore for $n$ large enough, we have: $u(x,t_n) < u_2(x)$ in $\Omega$. And this gives:

$$u(x,t) < S(t-t_n)u_2(x) = u_2(x), \text{ for } t > t_n.$$  

And this contradicts the fact that $u_3 \in w(u_0)$.

Proof of iii): We will only prove the part concerning (13). Let $u_0 \in I_0$, we recall a
few results proved in P. L. Lions [13]: for $t$ large enough we have:

$$\exists \alpha > 0, \lambda_1(-\Delta - f'(u(x,t))) \leq -\alpha < 0$$  

(this is true for $t = +\infty$ since $w(u_0) \subset S_\alpha$, and by continuity this remains true for $t$ large).

Let $v_1(x,t)$ be a normalized eigenfunction of

$$
\begin{cases}
-\Delta v_1 = f'(u(x,t))v_1 + \lambda_1(t)v_1 & \text{in } \Omega \\
v_1 \in C^2(\Omega), v_1 > 0 \text{ in } \Omega, v_1 = 0 \text{ on } \partial \Omega, |v_1|_{L^2(\Omega)} = 1
\end{cases}
$$

(and thus by (15), $\lambda_1(t) \leq -\alpha < 0$).

In [13], it is proved that: $\frac{3}{3\varepsilon} (\lambda_1(t)) \to \infty, \exists v_1(x,t) \in C^1(\Omega)$. Then we have for $\varepsilon > 0$:

$$\frac{3}{3\varepsilon} (u+\varepsilon v_1) - \Delta (u+\varepsilon v_1) = f(u) + f'(u)\varepsilon v_1 + \lambda_1(t)v_1 + \varepsilon (\frac{3}{3\varepsilon} v_1)$$

and for $\varepsilon$ small enough, we deduce:

$$\frac{3}{3\varepsilon} (u+\varepsilon v_1) - \Delta (u+\varepsilon v_1) < f(u+\varepsilon v_1) - \frac{\alpha}{2} \leq v_1 + \varepsilon (\frac{3}{3\varepsilon} v_1)$$

Since for $t$ large enough: $\frac{3}{3\varepsilon} (v_1) \leq \frac{\alpha}{2} v_1$ in $\Omega$, we obtain finally:

$$\exists T_0 > 0, T > 0 \text{ such that for } \varepsilon \in (0,T_0), \text{ for } t > T_0 \text{ we have:}$$

$$\exists \varepsilon_0 > 0, \exists T_0 > 0 \text{ such that for } \varepsilon \in (0,T_0), \text{ for } t > T_0 \text{ we have:}$$

$$\exists \varepsilon_0 > 0, \exists T_0 > 0 \text{ such that for } \varepsilon \in (0,T_0), \text{ for } t > T_0 \text{ we have:}$$

Remark that we also have:

$$\exists \varepsilon_0 > 0, \exists T_0 > 0 \text{ such that:}$$

(17) \quad $\forall \delta(x) \leq v(x,t) \leq C\delta(x)$ in $\Omega$

(recall that $\delta(x) = \text{dist}(x,\partial \Omega)$).

Next, let $u^+ = u^+(w) (v \in \partial w(u_0))$, we already showed that $u_+^+$ is constant on $w(u_0)$ - see Remark II.1 above). A simple continuity argument shows that there exists

$$T > 0, \exists \delta(x) \leq \frac{\alpha}{2} v_1$$

Next let $u$ be such that, for $v \in \partial w(u_0), S(T)v \leq u^+$ in $\Omega$ (use the continuity of the map $S(T)$ from $X$ into $C^1(\Omega)$).
We now take \( v \in \mathcal{W}(u_0, \varepsilon) \), \( v > u_0 \), \( v \neq u_0 \) and we are going to prove that
\[
b(t)v \to u^*. \]
From the choice of \( \varepsilon \) above we surely \( w \notin u^*, \) \( w \in \mathcal{S} \) if \( w \in w(v) \).
From the definition of \( u^* \), it just remains to prove if \( w \in w(v) \) then there exists
\( \tilde{u} \in \mathcal{W}(u_0) \) such that: \( w \geq \tilde{u}, \ w \neq \tilde{u} \). And this will be achieved with the help of (16) - (17). Indeed, we have:
\[
S(T_0)v(x) > S(T_0)u_0(x) \text{ in } \Omega, \ \frac{3}{\delta_n} (S(T_0)v) < \frac{3}{\delta_n} (S(T_0)u_0) \text{ on } \partial \Omega,
\]
and thus there exists \( \varepsilon \) small enough such that:
\[
S(T_0)v > S(T_0)u_0 + \varepsilon v_1(x, T_0) \text{ in } \Omega.
\]
Next because of (16), this yields:
\[
S(t)v(x) > u(x, t) + \varepsilon v_1(x, t) \text{ in } \Omega, \text{ for } t > T_0 ;
\]
or in view of (17):
\[
S(t)v(x) > u(x, t) + \varepsilon \delta(x) \text{ in } \Omega, \text{ for } t > T_0 .
\]
Now it is easy to conclude, since if \( S(t)v + w \in w(v) \), there exists a subsequence \( t_n \) such that \( u(t_n, t_n) \to \tilde{u} \in \mathcal{W}(u_0) \) and we have:
\[
w > \tilde{u} + \varepsilon \delta \text{ in } \Omega ;
\]
thus \( w \geq \tilde{u} \) and since we already know \( w < u^* = m^*(u) \), we conclude from the definition of \( m^* \): \( w = m^*(u) = u^* \).

Proof of i): First, let us remark that it is well-known that \( I_+ \) is an open set (see for example D. Henry [11]). To prove i), we are going to exhibit, for each \( u_0 \) in \( X \), an open set \( U_{u_0} \) such that:
\begin{itemize}
  \item[1)] \( U_{u_0} \subset I_+ \cup I_0 \),
  \item[2)] \( u_0 \notin \overline{U}_{u_0} \).
\end{itemize}

Indeed if \( u_0 \in I_+ \), we take \( U_{u_0} = I_+ \). Next, if \( u_0 \in I_0 \), we have just seen that there exists \( \varepsilon > 0 \) such that:
\[
\forall v \in X, u_0 < v < u_0 + \varepsilon \Rightarrow w(v) = \{u^*\} = \{m^*(v)\}, \ \forall v \in w(u_0) .
\]
We then define:
\[
U_{u_0} = \{v \in X, \ \exists 0 < \eta_1 < \eta_2 < \varepsilon, u_0 + \eta_1 \delta < v < u_0 + \eta_2 \delta \text{ in } \Omega \} .
\]
Obviously \( u_0 \notin \overline{U}_{u_0} \) and for all \( v \) in \( U_{u_0} \), \( w(v) = \{u^*\} \subset I_+ \cup I_0 \) and thus \( v \in I_+ \cup I_0 \). Finally if \( u_0 \in I_0 \), two cases are possible:

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1st case: \( 3v_n \in u_n, v_n \in I^*, \) In this case we define:
\[
O_n = \bigcup_{n=0}^{v_n} (O_n^* \text{ has been defined above}).
\]
Thus \( O_0 \subset I^* \cup I_0 \) and \( u_0 = \lim_{n \to \infty} v_n \in O_0 \).

2nd case: \( \exists \epsilon_0 > 0, \forall v, v \in B(u_0, \epsilon_0), v > u_0, v \notin \mathbb{R} \Rightarrow v \in I^* \cup I_0. \)

We then define
\[
O_0 = \{v \in \mathbb{R} \mid 0 < \epsilon_1 < \epsilon_2 < \epsilon_0, u_0 + \epsilon_1 < v < u_0 + \epsilon_2 \text{ in } \mathbb{R} \}.
\]
and we conclude.

Remark II.6: i) Let \( S_+ = \{u_j, 1 \leq j \leq N (N \in \mathbb{N}) \} \), we may define:
\[
I_+^j = \{u_0 \in \mathbb{R} \mid \mathbb{R} = (u_j) \}. \quad \text{Then it is quite obvious to show that } I_+^j \text{ are the connected components of } I_+ \text{ (and thus are open)}
\]

ii) Let \( S_- = \bigcup_{j \in \mathbb{N}} C_j (N \in \mathbb{N}) \) where \( C_j \) are the connected components of \( S_- \), we may define:
\[
I_-^j = \{u_0 \in \mathbb{R} \mid \mathbb{R} = (u_j) \}.
\]
It is also easy to show that \( I_-^j \) are the connected components of \( I_- \) and that for all \( u_0 \in I_-^j \), there exists \( \epsilon > 0 \) such that:
\[
\begin{cases}
\forall v \in B(u_0, \epsilon), v > u_0, v \notin \mathbb{R} \Rightarrow \mathbb{R} = (u_j) \\
\forall v \in B(u_0, \epsilon), v < u_0, v \notin \mathbb{R} \Rightarrow \mathbb{R} = (u_j)
\end{cases}
\]

where \( u_j^+ \) is the minimum solution of (SP) above any element of \( C_j \) and \( u_j^- \) is the maximum solution of (SP) below any element of \( C_j \) (and \( u_j^+, u_j^- \in S_+ \cup S_0 \)).

III. Variants and extensions:

III.1. Unbounded nonlinearities:

If we no more assume \( (9) \), the solution may not exist for all time. One way to get rid of this difficulty is to restrict our attention to an invariant domain \( K \) such that for any \( u_0 \in K \), the solution of (IVP) exists for all \( t > 0 \) and remains bounded as \( t \to \infty \). Then Theorem II.1 remains valid for \( u_0 \) in \( K \) (and for the relative topology of \( K \) in \( X \)).
Let us give two natural examples:

(i) \( K = \left\{ u_0 \in X/u(.,t) \text{ exists for all } t > 0, \int u(.,t) = C \in C(u_0) \text{ for } t > 0 \right\} \)
(remark that \( K \neq \emptyset \) if and only if \( S \neq \emptyset \));

(ii) \( K = \left\{ u_0 \in X/u < u_0 < u \text{ in } \Omega \right\} \)
where \( u, u \) are sub and supersolutions of \( (SP) \) that is satisfying:

\[
\begin{aligned}
-\Delta u &< f(u) \text{ in } \Omega, \quad -\Delta u > f(u) \text{ in } \Omega \\
\frac{\partial u}{\partial v} &\leq C(\Omega), \quad u < u \text{ in } \Omega, \quad u < 0 \text{ on } \partial \Omega.
\end{aligned}
\]

In these two cases we define:

\[
I_+ = \left\{ u_0 \in X/u(0) \subset S_+ \right\}
\]

\[
I_- = \left\{ u_0 \in X/u(0) \subset S_- \right\}
\]

\[
I_0 = \left\{ u_0 \in X/u(0) \subset S_0 \right\}
\]

We then have:

**Theorem III.1:** Let \( f \in C^2(\mathbb{R}) \) and let \( K \) be defined by (i) or (ii), we have:

1) There exists an open set \( \Omega \) such that \( \Omega \subset I_+ \cup I_0, \emptyset \cap K = K; \)

2) For all \( u_0 \) in \( I_+ \cup I_0, \omega(u_0) \) is a singleton;

3) If \( u_0 \in I_+, \) there exists \( \varepsilon > 0 \) such that:

\[
\begin{aligned}
\forall (u_0, \varepsilon) \in K, \quad \forall < u_0, \quad \forall \neq u_0 \Rightarrow u(\varepsilon) = (u^+) \subset S_+ \cup S_0
\end{aligned}
\]

where \( u^- = m^-(w), \quad \forall \varepsilon \in \omega(u_0); \)

\[
\begin{aligned}
\forall (u_0, \varepsilon) \in K, \quad \forall < u_0, \quad \forall \neq u_0 \Rightarrow u(\varepsilon) = (u^-) \subset S_+ \cup S_0
\end{aligned}
\]

where \( u^+ = m^+(w), \quad \forall \varepsilon \in \omega(u_0). \)

In particular if \( \forall \varepsilon \in (u_0, \varepsilon) \in K, \quad \forall \neq u_0 \) or \( \forall < u_0, \quad \forall \neq u_0 \) then \( \forall \in I_+ \cup I_0. \)

**Remark III.1:** The proof of this result is totally identical to the one of Theorem III.1.

We just need to remark that if \( K \) is given by (i) and if there exist \( u_0 \in I_-, \forall \in K \) such that \( \forall > u_0, \forall \neq u_0 \) then \( m^+(w) \) is not empty on the connected component of \( S_- \) containing \( \omega(u_0). \)
Remark III.2: In some sense, the above result contains the results of P. L. Lions [13] (except some geometrical descriptions heavily dependent on the convexity of the nonlinearity $f$ assumed in [13]).

Remark III.3: Most of the remarks made in the preceding section are still valid here (with some obvious adaptations).

### III.2 Iterative schemes:

To simplify, we will consider the case of a nonlinearity satisfying (9) and we will assume in addition that we have:

$$ SK_0 > 0, \ f(t) + K_0 t \ \text{is nondecreasing for} \ t > 0. \ (19)$$

We consider the asymptotic behaviour of iterative schemes like: $u_0 \in X$ and $(u_n)_{n \geq 1}$ is defined by

$$ -\Delta u_n + Ku_n = f(u_{n-1}) + Ku_{n-1} \ \text{in} \Omega, \ u_n \in C^2(\Omega), \ u_n = 0 \ \text{on} \ \partial \Omega. \ (20)$$

From (9), it is easy to deduce that $u_n$ is bounded in $L^\infty(\Omega)$ and thus in $C^2(\Omega) (0 < \alpha < 1)$.

Now, if $K > K_0$, we have, multiplying (20) by $u_n - u_{n-1}$ and integrating by parts over $\Omega$:

$$ \int |Du_n|^2 + Ku_n^2 dx - \int Du_n \cdot Du_{n-1} + Ku_{n-1} u_{n-1} dx = \int (f(u_{n-1}) + Ku_{n-1})(u_n - u_{n-1}) $$

and because of (19), this is less than:

$$ < \int |Du_n|^2 + Ku_n^2 dx - \int F(u_{n-1}) + Ku_{n-1}^2 u_{n-1} dx $$(recall that $F(t) = \int_0^t f(s) ds$). On the other hand, since we have:

$$ \int Du_n \cdot Du_{n-1} + Ku_{n-1} u_{n-1} dx = \frac{1}{2} \int |Du_n|^2 + Ku_n^2 dx + \frac{1}{2} \int |Du_{n-1}|^2 + Ku_{n-1}^2 dx - \frac{1}{2} \int |Du_n - u_{n-1}|^2 + Ku_{n-1} (u_n - u_{n-1})^2 dx,$$

we finally obtain, if $K > K_0$: 

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\[ f_\Omega \left( \frac{1}{2} |D u_n|^2 - F(u_n) \right) dx - f_\Omega \left( \frac{1}{2} |D u_{n-1}|^2 - F(u_{n-1}) \right) dx < \\
\leq - \frac{1}{2} f_\Omega |D(u_n - u_{n-1})|^2 + K (u_n - u_{n-1})^2 dx \]

and this enables us to show that: \( u_n - u_{n-1} \to 0 \) in \( C^{2,\alpha}(\Omega) \) \((0 < \alpha < 1)\). Therefore, if we denote by \( \omega(u_0) \) the set of limit points of the sequence \( (u_n)_{n \to 1} \), we have:

(21) \( \omega(u_0) \) is a compact, connected set, contained in \( S \).

Therefore we may define again:

\[ I_+ = \{ u_0/\omega(u_0) \subset S \}, I_- = \{ u_0/\omega(u_0) \subset S \}, I_0 = \{ u_0/\omega(u_0) \subset S_0 \} \]

Of course \( I_+ \cup I_- \cup I_0 = X \).

Then, we have:

**Theorem III.2:** Under assumptions (9), (19) and if \( K > K_0 \), then we have:

1) \( I_+ \cup I_0 \) contains an open dense set,

ii) For all \( u_0 \) in \( I_+ \cup I_0 \), \( \omega(u_0) \) is a singleton,

iii) If \( u_0 \in I_- \), there exists \( \epsilon > 0 \) such that:

(13) \( \forall \forall \in S(u_0, \epsilon), \forall u_0', \forall \neq u_0 \Longrightarrow \omega(v) = \{ u^+ \} \subset S_+ \cup S_0 \)

where \( u^+ = \omega^+(w), \forall \in \omega(u_0) \),

(14) \( \forall \forall \in S(u_0, \epsilon), \forall u_0', \forall \neq u_0 \Longrightarrow \omega(v) = \{ u^- \} \subset S_+ \cup S_0 \)

where \( u^- = \omega^-(w), \forall \in \omega(u_0) \).

In particular if \( \forall \in S(u_0, \epsilon), \forall \neq u_0 \) or \( \forall \neq u_0', \forall \neq u_0 \) then \( \forall \in I_+ \cup I_0 \).

The proof of this result is identical to the one of Theorem II.1 and we will skip it. In addition, all remarks made in the preceding sections are still valid with some obvious modifications.
REFERENCES


Structure of the Set of Steady-State Solutions and Asymptotic Behaviour of Semilinear Heat Equations

We give a precise geometrical description of the set of steady-state solutions for general classes of semilinear heat equations. This enables us to prove global results about the asymptotic behaviour of the solutions of the initial value problem.