ON A CONJECTURE OF C. A. MICHELLEI
CONCERNING CUBIC SPLINE INTERPOLATION
AT A BIINFINITE KNOT SEQUENCE

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ABSTRACT

It is shown that if the knot sequence \( t_i = (t_i)_{i=-\infty}^{\infty} \) satisfies

(i) For some \( m \in \left[ 1, \frac{3+\sqrt{5}}{2} \right] \),

\[
    m^{-1} \leq \liminf_{r \to \infty} \frac{t_{i+r+1} - t_{i+r}}{t_{i+1} - t_i} \leq \limsup_{r \to \infty} \frac{t_{i+r+1} - t_{i+r}}{t_{i+1} - t_i} \leq m
\]

and

(ii) \( \bar{m} = \sup_{|i-j| \leq 1} \frac{t_{i+1} - t_i}{t_{j+1} - t_j} = \infty \),

then for any given bounded sequence \( y \in m(\mathbb{Z}) \) there exists exactly one cubic spline \( s \) with knots \( t_i \) such that

\[ s(t_i) = y_i , \text{ for all } i \in \mathbb{Z} . \]

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Cubic spline interpolation provides a good and handy method to approximate a given function or to fit a given set of points. However, such an interpolation process does not always converge. It is known that the local mesh ratio (that of the lengths of two consecutive intervals) is less than \( \frac{3+\sqrt{5}}{2} \), the interpolation process works for any given bounded data.

This paper continues such investigation. It is shown that the above restriction on the knots may be relaxed. Thus, for a wider class of knot sequences, the cubic spline interpolation can be still applied. Hopefully, this would make such interpolation process more feasible in practice.
ON A CONJECTURE OF C. A. M. MICCHELLI CONCERNING CUBIC SPLINE INTERPOLATION AT A BIINFINITE KNOT SEQUENCE

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1. Introduction. Let \( t: \mathbb{N} \to \mathbb{R} \) be a biinfinite, strictly increasing sequence, set

\[
t_i := \lim_{i \to \pm \infty} t_i,
\]

let \( k \) be an integer, \( k > 2 \), and denote by \( \mathcal{S}_{k, t} \) the collection of spline functions of degree \( < k \) with knot sequence \( t \). Explicitly, \( \mathcal{S}_{k, t} \) consists of exactly those \( k - 2 \) times continuously differentiable functions on

\[
I := (t_i, t_{i+1})
\]

which, on each interval \( (t_i, t_{i+1}) \), coincide with some polynomial of degree \( < k \). Let

\[
m_{k, t} := \mathcal{S}_{k, t} \cap m(I),
\]

i.e., the normed linear space of splines for which

\[
\|s\| := \sup_{t \in I} |s(t)|
\]

is finite. We are interested in the

**Bounded Interpolation Problem (B.I.P.).** To construct, for given \( y \in m(I) \), some

\[
s \in m_{k, t}
\]

for which

\[
s|_{t} = y
\]

We will say that the B.I.P. is correct (for the given knot sequence \( t \)) if it has exactly one solution for each \( y \in m(I) \).

In case \( k = 4 \) (cubic spline interpolation), de Boor [2] showed that if the local mesh ratio

\[
m_k := \sup_{i} \frac{\Delta t_i}{\min |t_{i+1} - t_i|}
\]

is less than \( \frac{1 + \sqrt{5}}{2} \), then the B.I.P. is correct. A similar result was also obtained independently by Zmatrov [7]. The basic idea of [2] was the exponential decay law, which

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could be traced back to [1]. This idea was developed in de Boor [3] and Micchelli [6].

For cubic spline interpolation, Micchelli further raised the following conjecture (see [6], p. 236).

**Conjecture.** If

$$ m^{-1} < \lim \inf \left( \frac{1}{r} \right) \leq \lim \sup \left( \frac{1}{r} \right) < m $$

for some $m \in [1, \frac{3\sqrt{5}}{2})$, then the B.I.P. is correct for $k = 4$.

In hindsight, it is easy to see that this conjecture is faulty. This can be clarified by the following example.

Write $h_i := t_{i+1} - t_i$. Let

$$ t_0 = 0, 
\begin{align*}
  h_{2n} &= \frac{1}{2^n}, 
  h_{2n+1} &= \frac{1}{2^{n+1}} \text{ for integer } n \geq 0, \\
  h_{-1-i} &= h_i \text{ for integer } i > 0.
\end{align*}$$

Then

$$ \frac{1}{2} < \lim \inf \left( \frac{h_{i+r}}{h_{i-1}} \right) \leq \lim \sup \left( \frac{h_{i+r}}{h_{i-1}} \right) < 2, $$

but

$$ m = \sup_{t_{i+1} - t_i} \left( \frac{t_{i+1} - t_i}{t_{j+1} - t_j} \right) = \infty. $$

Let $m^+$ and $m^-$ have the same meaning as in [4]. Since the knot sequence $t$ is symmetric with respect to the origin, we must have $m^+ = m^-$. If this B.I.P. is correct, then [4] tells us that $m < \infty$, which is a contradiction.

The above example suggests to us that the condition $m < \infty$ should be added to the assumption. Thus we will prove the following

**Theorem 1.** If a knot sequence $t$ satisfies (2) with $1 < m < \frac{3\sqrt{5}}{2}$ and

$$ m < \infty, $$

then the B.I.P. is correct for $k = 4$.

**Remark.** This theorem covers de Boor's results for cubic splines with bounded global mesh ratio or with local mesh ratio $< \frac{3\sqrt{5}}{2}$ (see [2] and [3]).
2. The basic formulae for cubic spline interpolation. For a given knot sequence \( t \), let

\[ h_i := t_i - t_{i-1}, \quad \lambda_i = h_{i+1}/(h_i + h_{i+1}), \quad \mu_i = h_i/(h_i + h_{i+1}) \quad i \in \mathbb{Z}. \]

If \( s \in \mathcal{M}(S) \) satisfies (1) for some \( y \in \mathcal{M}(S) \), then

\[ \lambda_i s'(t_{i-1}) + 2s'(t_i) + \mu_i s'(t_{i+1}) = 3\lambda_i \frac{y_i - y_{i-1}}{h_i} + 3\mu_i \frac{y_{i+1} - y_i}{h_{i+1}}, \quad i \in \mathbb{Z}. \]

Moreover, it is easy to check that

\[
s(x) = \frac{(x-t_{i-1})(t-x)^2}{h_i^2} s'(t_{i-1}) + \frac{(x-t_i)^2(t-x)}{h_i^2} s'(t_i) + \frac{y_{i-1} + y_i}{2} + \frac{1}{h_i^2} \frac{(t_{i-1} + t_i)(t-x)^2 + 4(t_i-x)(x-t_{i-1}) + (x-t_{i-1})^2}{t_{i-1} - x} \]

for \( x \in [t_{i-1}, t_i] \).

Let

\[
\forall i \in \mathbb{Z}, \quad A(i,j) := \begin{cases} 2 & \text{for } j = i, \\ \lambda_i & \text{for } j = i-1, \\ \mu_i & \text{for } j = i+1, \\ 0 & \text{for } j \in \mathbb{Z}\setminus\{i-1, i, i+1\}. \end{cases}
\]

Then \( A \) is a tridiagonal \( \mathbb{Z} \times \mathbb{Z} \)-matrix. For any \( y \in \mathcal{M}(S) \) we have

\[
\forall i \in \mathbb{Z}, \quad |\lambda_i \beta_{i-1} + 2\beta_i + \mu_i \beta_{i+1}| < \lambda_i |\beta_{i-1}| + 2|\beta_i| + \mu_i |\beta_{i+1}| = 3|\beta_i|,
\]

showing \( \|A\| < 3 \). Here, we view \( A \) as a mapping from \( \mathcal{M}(S) \) to \( \mathcal{M}(S) \). Furthermore,

\[
\forall i \in \mathbb{Z}, \quad |\lambda_i \beta_{i-1} + 2\beta_i + \mu_i \beta_{i+1}| > 2|\beta_i| - \lambda_i |\beta_{i-1}| - \mu_i |\beta_{i+1}| = 2|\beta_i| - 3|\beta_i| = -|\beta_i|.
\]

Hence \( \|A\| > \sup_{i \in \mathbb{Z}} (2|\beta_i| - 3|\beta_i|) = 18\| \). This shows that \( A^{-1} \) exists and \( \|A^{-1}\| < 1 \).

3. The exponential decay. The following lemma plays an essential role in this paper.

Lemma 1. For any knot sequence \( t \),

\[
|A^{-1}(j,i)| < 2^{-|j-i|}, \quad \forall i, j \in \mathbb{Z}.
\]

and

\[
A^{-1}(j,1)A^{-1}(j+1,1) < 0, \quad \forall i, j \in \mathbb{Z}.
\]

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Moreover, it satisfies the following condition: for some integer \( r > 0 \)
\[
(i) \quad \frac{h_j}{h_j} < \frac{3}{4} m_0 |i-j| \quad \text{whenever} \quad |i-j| > r ,
\]
then
\[
(ii) \quad |A^{-1}(j,i)| < \left(1 + m_0^{-1} + \sqrt{1 + m_0^{-1} + m_0^{-2}}\right)^{-|i-j|} \quad \text{whenever} \quad |i-j| > r .
\]

Proof. For simplicity we fix \( i \) and write \( b_j := A^{-1}(j,i) \). Since \( AA^{-1} = 1 \), \( b(\ast) \notin m(\mathbb{H}) \). By \( AA^{-1} = 1 \) we have
\[
b(j,i) = \left\{ \begin{array}{ll} 1 & \text{for} \quad j = i , \\ 0 & \text{for} \quad j \neq i . \end{array} \right.
\]

Then by induction on \( j \), we can easily show that \( |b_{j-1}| > 2 |b_j| \) for all \( j < j_0 \). This contradicts the fact \( b \notin m(\mathbb{H}) \). Similar to (13), the following also holds:
\[
|b_j| < |b_{j-1}| \quad \text{for all} \quad j > 1 .
\]

Now (12) and (13) yield that
\[
|b_{j+1}| = |2b_j + \lambda_j b_{j-1}| / \mu_j > (2 |b_j| - \lambda_j |b_{j-1}|) / \mu_j
\]
\[
(14) \quad > (2 - \lambda_j) |b_j| / \mu_j = |b_j| \ast (1 + \mu_j / \mu_j > 2 |b_j|) \quad \text{for} \quad j < 1 .
\]

Similarly
\[
|b_{j-1}| > 2 |b_j| \quad \text{for} \quad j > 1 .
\]

In particular, \( |b_{k-1}| < |b_k| \) and \( |b_{k+1}| < |b_k| \). In connection with (12) we obtain
\[
1 = \lambda_k b_{k-1} + 2b_k + \mu_k b_{k+1} = |\lambda_k b_{k-1} + 2b_k + \mu_k b_{k+1}| \quad >
\]
\[
2 |b_k| - \lambda_k |b_{k-1}| - \mu_k |b_{k+1}| > (2 - \lambda_k - \mu_k) |b_k| = |b_k| . \]
This proves (8) for \( j = i \). For \( j \neq i \), (8) comes from \((14), (14')\) and \((15)\). For the rest of the proof we may assume \( j < i \) without any loss. To prove (9) we argue indirectly. If \( b_{j+1} b_{j} > 0 \) for some \( j < i \), then

\[
|b_{j+1}| > |\lambda_j b_{j+1}| = |2b_j + \mu_j b_{j+1}| > 2|b_j|.
\]

Comparing the above inequality with \((13)\), we must have \( b_j = 0 \) for all \( j < j_0 \). It would cause all \( b_j = 0 \), which is absurd. Now we can write down

\[
-b_{j+1}/b_j = 2 + 2q_j \quad \text{for} \quad j < i
\]

with \( q_j > 0 \). Let \( q_j := h_{j+1}/h_j \). We deduce from \((12)\) that, for \( j < i-1 \),

\[
-b_{j+1}/b_j = 2 + 2q_j \left(1 - \frac{1}{2(2+2q_j)}\right) = 2 + 2q_j + \frac{4q'_j + 3}{4q'_j + 4}.
\]

This shows that

\[
q_j + 1 = q_j + \frac{4q'_j + 3}{4q'_j + 4} \quad \text{for} \quad j < i-1.
\]

Let

\[
p_j := \frac{4q'_j + 4}{4q'_j + 3}.
\]

It is easy to verify that

\[
2 + 2q_j = 1 + p_j + \sqrt{1 + p_j + p_j^2}.
\]

Now \((16)\) and \((19)\) give us

\[
|b_j/b_j| = \prod_{k=j}^{i-1} |b_{k+1}/b_k| = \prod_{k=j}^{i-1} (2 + 2q_k) = \prod_{k=j}^{i-1} (1 + p_k + \sqrt{1 + p_k + p_k^2}).
\]

It follows from \((17)\) and \((18)\) that

\[
\prod_{k=j}^{i-1} \frac{4q'_k + 4}{4q'_k + 3} = \prod_{k=j}^{i-1} \left(\frac{4q'_k + 4}{4q'_k + 3} \cdot \frac{4q'_{k-1} + 3}{4q'_{k-1} + 4} \cdot \frac{4q'_k + 4}{4q'_k + 3} \cdot \frac{4q'_{k-1} + 3}{4q'_{k-1} + 4} \right) = \frac{4q'_1 + 4 + 3}{4q'_1 + 4 + 3} \cdot \frac{4q'_1 + 4 + 3}{4q'_1 + 4 + 3} \cdot \frac{4q'_1 + 4 + 3}{4q'_1 + 4 + 3} \cdot \frac{4q'_1 + 4 + 3}{4q'_1 + 4 + 3} = 1.
\]

Now we can write down

\[
|b_j/b_j| = \prod_{k=j}^{i-1} |b_{k+1}/b_k| = \prod_{k=j}^{i-1} (2 + 2q_k) = \prod_{k=j}^{i-1} (1 + p_k + \sqrt{1 + p_k + p_k^2}).
\]

It follows from \((17)\) and \((18)\) that

\[
\prod_{k=j}^{i-1} \frac{4q'_k + 4}{4q'_k + 3} = \prod_{k=j}^{i-1} \left(\frac{4q'_k + 4}{4q'_k + 3} \cdot \frac{4q'_{k-1} + 3}{4q'_{k-1} + 4} \cdot \frac{4q'_k + 4}{4q'_k + 3} \cdot \frac{4q'_{k-1} + 3}{4q'_{k-1} + 4} \right) = \frac{4q'_1 + 4 + 3}{4q'_1 + 4 + 3} \cdot \frac{4q'_1 + 4 + 3}{4q'_1 + 4 + 3} \cdot \frac{4q'_1 + 4 + 3}{4q'_1 + 4 + 3} \cdot \frac{4q'_1 + 4 + 3}{4q'_1 + 4 + 3} = 1.
\]

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If $\xi$ satisfies (8) and $|i-j| > r$, then

\[ (22) \quad \prod_{k=j}^{i-1} \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{4} + \frac{4}{3} \frac{|i-j|}{\delta_0} = \frac{|i-j|}{\delta_0}. \]

Therefore lemma 1 will be proved, once the following lemma is established:

Lemma 2. Suppose $p_1, \ldots, p_n$ and $p$ are nonnegative real numbers with $p^n = P_1 P_2 \ldots P_n$. Then

\[ \prod_{i=1}^{n} \left( 1 + p_i + \sqrt{1 + p_i + p_i^2} \right) > (1 + p + \sqrt{1 + p + p^2})^n. \]

Proof. Let

\[ F(p_1, \ldots, P_n-1, P_n) := \prod_{i=1}^{n} \left( 1 + p_i + \sqrt{1 + p_i + p_i^2} \right). \]

We want to determine the minima of the function $F$ under the constraint $\prod_{i=1}^{n} p_i = c$, where $c$ is a constant, $c = p^n$. If some $p_i > (2+2p)^n$, then

\[ F(p_1, \ldots, p_n) > (2 + 2p)^n \prod_{i=1}^{n} \left( 1 + p + \sqrt{1 + p + p^2} \right) > \inf \{ F(p_1, \ldots, p_n) \}. \]

Hence

\[ \inf \{ F(p_1, \ldots, p_n) \} = \inf \{ F(p_1, \ldots, p_n) \} \quad \forall \prod_{i=1}^{n} p_i = c. \]

Thus there exists a point $(p_1, \ldots, p_n)$ with $\prod_{i=1}^{n} p_i^0 = c$ such that $F(p_1, \ldots, p_n) = \inf \{ F(p_1, \ldots, p_n) \}$. To find $(p_1, \ldots, p_n)$ we shall use the method of Lagrange multipliers and set

\[ \theta(p_1, \ldots, p_n) := F(p_1, \ldots, p_n) - \lambda p_1 \ldots p_n. \]

Then

\[ \frac{\partial \theta}{\partial p_i}(p_1, \ldots, p_n) = 0, \quad i = 1, \ldots, n; \quad \text{that is} \]

\[ \prod_{j \neq i}^{n} \left( 1 + p_i^0 + \sqrt{1 + p_i^0 + (p_j^0)^2} \right) \left( 1 + \frac{2p_i^0 + 1}{2\sqrt{1 + p_i^0 + (p_j^0)^2}} \right) - \lambda \prod_{j \neq i}^{n} p_j^0 = 0. \]

It follows that

\[ p_i^0 \frac{\left( 1 + \frac{2p_i^0 + 1}{\sqrt{1 + p_i^0 + (p_j^0)^2}} \right)}{2\sqrt{1 + p_i^0 + (p_j^0)^2}} = \lambda \prod_{j \neq i}^{n} p_j^0 \frac{\sqrt{1 + p_i^0 + (p_j^0)^2}}{2\sqrt{1 + p_i^0 + (p_j^0)^2}}. \]
Therefore

\[ f(p_k^0) = r(p_k^0) \text{ for all } i, k \in \{1, \ldots, n\}, \]

where

\[ f(x) := \frac{x(1 + \frac{2x + 1}{x + x^2})}{1 + x + 1 + x + x^2} = \frac{1 + \frac{x - 1}{2}}{2 + x + x^2}. \]

An easy calculation yields

\[ f'(x) = 3(x + 1), \]

which is strictly increasing on \([0, \infty)\). Thus (23) and (24) give

\[ p_0^0 = \cdots = p_n^0 = p. \]

This ends the proof of lemma 2. Also the proof of lemma 1 is complete.

4. The proof of theorem 1. By the hypothesis (2) there exist a positive integer \( r \) and a real number \( m_0 \) with \( m < m_0 < \frac{3 + \sqrt{2}}{2} \) such that

\[ h_j / h_j < \frac{3}{4} m_0 |i-j| \text{ whenever } |i-j| > r. \]

Then by lemma 1,

\[ \forall i, j \in \mathbb{Z}, |b^{i-j}(j,i,j)| \leq \begin{cases} 2^{-|i-j|} & \text{if } |i-j| < r, \\ (1 + m^{-1} + \sqrt{1 + m^{-1} + m_0^{-2}})^{-|i-j|} & \text{if } |i-j| > r. \end{cases} \]

Let

\[ M := (m_0)^r < = \]

and

\[ c_k = \frac{3 \lambda_i - \frac{y_i - y_{i+1}}{h_i}}{h_i} + 3 \mu_i \cdot \frac{y_i + y_{i+1}}{h_i+1}, i \in \mathbb{Z}. \]

Then it follows that

\[ s^*(t^j_i) = \sum_{i-j} |b^{i-j}(j,i)\lambda_i c_k| + \sum_{|i-j|<r} \lambda^{i-j}(j,i) c_k + \sum_{|i-j|>r} \lambda^{i-j}(j,i) c_k. \]

By the hypotheses of theorem 1 we have the following estimates for \( c_k \):
\[
|c_l| < b \cdot \|y\| \cdot (1 + \alpha) \quad \text{if } |j - l| < \tau \;
\]
\[
|c_l| < 6 \cdot \|y\| \cdot (1 + \alpha) \quad \text{if } |j - l| > \tau.
\]

Write
\[
\theta := m_0 \left( 1 + m_0^{-1} + \sqrt{1 + m_0^{-1} + m_0^{-2}} \right)^{-1}.
\]

Then \( \theta < 1 \) as long as \( m_0 < \frac{3 + \sqrt{5}}{2} \). Applying (25) and (27) to (26), we obtain
\[
|s'(c_j)| < 6M(1 + \alpha) \cdot \|y\| \quad \left| \frac{1}{h_j} - \sum_{|j - l| < \tau} \frac{1}{h_j} \right| 2^{-|l - j|} + 6(1 + \alpha) \|y\| \quad \left| \frac{1}{h_j} - \sum_{|j - l| > \tau} \theta^{1/2} \right| \quad (28)
\]
\[
< \text{const} \cdot \|y\| \cdot \frac{1}{h_j}.
\]

Furthermore (6) tells us
\[
\max_{t_j-1 \leq t \leq t_j} |s(x)| < \text{const}(h_{j-1} |s'(t_{j-1})| + h_j \cdot |s'(t_j)| + \|y\|).\]

which in connection with (28) yields the desired result
\[
|s\| \leq \text{const} \cdot \|y\|.
\]

Our proof is complete.
REFERENCES

On a Conjecture of C. A. Micchelli Concerning Cubic Spline Interpolation at a Biinfinite Knot Sequence

It is shown that if the knot sequence \( t := (t_i)_{i=-\infty}^{\infty} \) satisfies

(i) For some \( m \in \left[ 1, \frac{3+\sqrt{5}}{2} \right] \),

\[
-1 < \liminf_{r \to 0} \frac{t_{i+r+1} - t_{i+r}}{t_{i+1} - t_i} \leq \limsup_{r \to 0} \frac{t_{i+r+1} - t_{i+r}}{t_{i+1} - t_i} < m
\]

and

(continued)
(ii) \[ m_i := \sup_{t \in Z} \frac{t_{i+1} - t_i}{t_{j+1} - t_j} < \infty, \]

then for any given bounded sequence \( y \in m(Z) \) there exists exactly one cubic spline \( s \) with knots \( t_i \) such that

\[ s(t_i) = y_i, \quad \text{for all } i \in Z. \]