A SPECTRAL MAPPING THEOREM FOR THE EXPONENTIAL FUNCTION AND SO—ETC(U)

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A SPECTRAL MAPPING THEOREM
FOR THE EXPONENTIAL FUNCTION,
AND SOME COUNTEREXAMPLES

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Elementary proofs are given for the (known) theorems that (1) \( \sigma(A) = \sigma(e^{tA}) \) if \( A \) is the generator of a \( C_0 \)-semigroup \( \{e^{tA}\} \) of linear operators on a Banach space \( X \), and that (2) \( e^{\sigma(A)} = \sigma(e^{tA}) \setminus \{0\} \) if \( \{e^{tA}\} \) is a holomorphic semigroup. Also a large class of strongly continuous groups \( \{e^{tA}\} \) on a Hilbert space \( H \) is given such that \( \sigma(A) \) is empty. Note that \( \sigma(e^{tA}) \) is not empty, and is away from zero, if \( \{e^{tA}\} \) is a group. Some related remarks are given on the relationship between the spectral bound of \( A \) and the type of \( \{e^{tA}\} \).

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SIGNIFICANCE AND EXPLANATION

If one solves a system of linear differential equation \( \frac{dx}{dt} = Ax \), where \( x = x(t) \) is an \( n \)-vector and \( A \) is an \( n \times n \) matrix, the solution may be written \( x(t) = e^{tA}x(0) \). Here the matrix \( e^{tA} \) may be given by the Taylor series \( \sum_{k=0}^{\infty} \frac{t^kA^k}{k!} \), or it may be more easily computed if the Jordan canonical form of \( A \) is known. In any case if \( \lambda_1, \ldots, \lambda_n \) (some of which may be equal) are the eigenvalues of \( A \), then \( e^{t\lambda_1}, \ldots, e^{t\lambda_n} \) are the eigenvalues of \( e^{tA} \) (this is a special case of the so-called spectral mapping theorem).

It follows that the growth rate of \( x(t) \) in any norm is at most exponential: \( \|x(t)\| \leq M e^{\omega t} \|x(0)\| \) for \( t > 0 \). The infimum of all possible \( \omega \) is called the type of the semigroup \( \{e^{tA}\} \), and will be denoted by type \( A \) in the sequel. The spectral mapping theorem mentioned above implies that \( \omega = \max\{\Re \lambda_i\} \), which is called the spectral bound of \( A \) and will be denoted by \( \text{spb} \ A \). Thus one has the relation \( \text{spb} \ A = \text{type} \ A \).

If the matrix \( A \) is replaced with a linear operator \( A \) in an infinite-dimensional Banach space \( X \), one can still solve \( \frac{dx}{dt} = Ax \) for \( x = x(t) \in X \) under certain conditions on \( A \), to obtain a unique solution \( x(t) = e^{tA}x(0) \) for \( t > 0 \). A is called the generator of the semigroup \( \{e^{tA}\} \). Many problems in linear partial differential equations can be covered by semigroup theory. Here again one can define type \( A \) via the optimal growth rate for \( \|x(t)\|/\|x(0)\| \), and \( \text{spb} \ A \) using the spectrum \( \sigma(A) \) of \( A \) rather than the eigenvalues. It turns out, however, that the spectrum mapping theorem (which would now take the form \( e^{\sigma(A)} = \sigma(e^A) \setminus \{0\} \)) need not hold and, consequently, \( \text{spb} \ A \) and type \( A \) are in general different.

Nevertheless, we show that the previous results are true for a special class of generators \( A \), which are roughly those appearing in parabolic partial differential equations. Also we give a wide class of counterexamples, which are not at all pathological, in which \( -\infty = \text{spb} \ A < \text{type} \ A < +\infty \).

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§1. Introduction

Let \( A \) be a linear operator in a complex Banach space \( X \). Consider the mapping
\[ A + e^A \]
and the validity of the spectral mapping theorem
\[ (0.1) \quad \sigma(e^A) = e^{\sigma(A)}, \]
where \( \sigma \) denotes the spectrum.

According to the general spectral mapping theorem, (0.1) is true if \( A \in \mathfrak{B}(X) \) (bounded linear operators with domain \( X \)). If \( A \) is unbounded, (0.1) need not be true even when \( A \) is the generator of a strongly continuous group \( \{U(t) = e^{tA}; -\infty < t < \infty\} \). A striking counterexample is given in Hille-Phillips [1, p. 665], in which \( U(t) \) is the Riemann-Liouville fractional integration of the imaginary order on \( X = L^p(0,1) \). Here \( \sigma(A) \) is empty but \( \sigma(e^A) \) is nonempty and is away from zero.

On the other hand, Hille-Phillips [1, p. 460] shows that (0.1) is true (except for zero) if \( A \) is the generator of a semigroup \( \{e^{tA}; t > 0\} \) which is norm-continuous for \( t > \gamma \) for some constant \( \gamma > 0 \). The proof in [1] is difficult, however, based on the Gel'fand theory of normed rings.

The purpose of the present note is twofold. First we give an elementary proof of the Hille-Phillips theorem in the special case when \( A \) generates a holomorphic semigroup. We shall then give a wide class of generators \( A \) of groups for which \( \sigma(A) \) is empty, including the fractional integrals mentioned above as a special case.

§2. A spectral mapping theorem.

we begin with a one-sided inclusion in (0.1) for the generator of a \( C_0 \)-semigroup.

Theorem 1. Let \( A \) be the generator of a semigroup \( \{e^{tA}; t > 0\} \) of class \( C_0 \). Then
\[ e^{\sigma(A)} \subseteq e^{\sigma(e^A)}. \]

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Remark. This is a special case of Corollary 2 to Lemma 16.3.2 of [1, p. 457], in which A may be the generator of any semigroup of class A, but our proof is elementary and short.

Proof. For any complex number \( z_0 \), set

\[
T = e^{z_0} \int_0^t e^{(A-z_0)u} du b(x).
\]

Since \( (A-z_0)u = (d/dt)e^{A} \) for \( u \in D(A) \), it follows on integration that

\[
T(A-z_0) = (A-z_0)T = e^A - e^{z_0}.
\]

If \( e^{0} \notin \rho(e^A) \) (\( \rho \) denotes the resolvent set), (1.2) gives

\[
(A-z_0)^{-1} = (e^A - e^{z_0})^{-1} T \in B(x),
\]

so that \( z_0 \notin \rho(A) \). In other words, \( z_0 \in \sigma(A) \) implies \( e^{z_0} \in \sigma(e^A) \), q.e.d.

Theorem 2. Let A be the generator of a holomorphic \( C_0 \)-semigroup. Then \( \sigma(A) = \sigma(e^A) \). \( \sigma(A) \notin \{0\} \).

Remarks. (a) By a holomorphic \( C_0 \)-semigroup we mean a semigroup of class \( C_0 \) which has an analytic continuation to a sector containing the positive t-axis. It is known (see [1, Theorem 12.8.1]) that A generates such a semigroup if and only if \( \rho(A) \) contains a sector

\[
\Sigma = \{z ; \arg(z-y) < \omega\}.
\]

where \( y \) is a complex number and \( \omega > \pi/2 \), and

\[
|z-A|^{-1} < M_{e^A}|z-y|^{-1} \text{ for } |\arg(z-y)| < \omega - \epsilon
\]

for each \( \epsilon \in (0, \omega) \).

(b) Removing 0 from \( \sigma(e^A) \) in Theorem 2 is natural since \( \sigma(e^A) \) never contains 0 but \( \sigma(e^A) \) may well do.

(c) Theorem 2 is a special case of Theorem 16.4.1 of [1].

Proof. In view of Theorem 1, it suffices to show that

\[
\sigma(e^A) \notin \{0\} \subset \sigma(A)
\]

in the proof we may assume \( y = 0 \) in (1.4), (1.5) without loss of generality.
To prove (1.6), it suffices in turn to show that

\[ 0 + \zeta_0 \not\ni e^{\sigma(A)} \text{ implies } \zeta_0 \ni p(e^A) . \]

To this end we first note that

\[ e^A = \frac{1}{2\pi i} \int_C e^z (z-A)^{-1} dz , \]

where \( C_0 \) is a curve in \( \mathbb{C} \) running from \( e^{-i\theta} \) to \( e^{i\theta} \), where \( \pi/2 < \theta < \pi \) (see e.g. Kato [2, p. 489]).

Given a \( \zeta_0 \) as in (1.7), consider all the complex numbers \( z_j \) (\( j = 1, \ldots, m \)) lying to the left of \( C_0 \) and satisfying \( e^{z_j} = \zeta_0 \). Obviously there are at most finitely many such \( z_j \); they are on a vertical line and equally spaced. We may assume that there is no \( z \in C_0 \) with \( e^z = \zeta_0 \), by deforming \( C_0 \) if necessary.

The assumption in (1.7) implies that \( z \in p(A) \) (\( j = 1, \ldots, m \)). Hence we can find a small circle \( C_j \) about \( z_j \) such that \( C_j \) and its interior are in \( p(A) \). We may assume that \( C_j \), including \( C_0 \), are separated from one another.

We now construct a Dunford-type integral

\[ s = \frac{1}{2\pi i} \int_C e^z (z-A)^{-1} dz \in \mathbb{R}(X) \]

where

\[ C = C_0 + C_1 + \ldots + C_m , \]

the \( C_j \) being assumed to be coherently oriented (so that the \( C_j \) with \( j > 1 \) are negatively oriented). Note that the integral (1.9) exists because the integrand is analytic for \( z \in C \) and decays exponentially at infinity on \( C_0 \).

We note that \( C_0 \) in (1.8) may be replaced by \( C \), since there is no contribution to the integral from the \( C_j \), the integrand being analytic on each \( C_j \) and its interior.

Then we can apply the Dunford integral calculus, to obtain (see e.g. [2, p. 44])

\[ e^A s = \frac{1}{2\pi i} \int_C e^{2z} (z-A)^{-1} dz . \]
Here it should be noted that both $e^z$ and $e^z(e^z - c_0)^{-1}$ are analytic in the closed domain bounded by $C$. The fact that this domain is unbounded causes no difficulty, since these functions decay rapidly at infinity.

Since

$$\frac{e^{2z}}{e^z - c_0} = \left(1 + \frac{c_0}{e^z - c_0}\right)e^z,$$

it follows from (1.8) to (1.10) that

$$e^z = e^z + c_0,$$

Hence

$$(c_0 - e^z)(1 - s) = c_0.$$

It follows that $c_0 \cup (e^z)$, with

$$(c_0 - e^z - 1 = c_0^{-1}(1 - s) \cup B(x).$$

This proves (1.7), q.e.d.

§2. Counterexamples --- fractional powers of accretive operators.

In this section we show that the counterexample of fractional integrals mentioned in §0 is not an isolated phenomenon.

**Theorem 3.** Let $H$ be a Hilbert space. Let $b \in B(H)$ be accretive. Then the fractional powers $b^\alpha$ are well defined and form a holomorphic semigroup for $Re \alpha > 0$, with

$$\|b^\alpha\| < \frac{\sin \pi \xi'}{\pi \xi'(1 - \xi')^{1/2}} e^{\pi |\eta|/2} (\xi = Re \alpha > 0)$$

where $\eta = Im \alpha$ and $\xi' = \xi - [\xi]$. If in particular $0$ is not an eigenvalue of $b$, then $b^\alpha$ is strongly continuous for $Re \alpha > 0$, and $(b^{\alpha}; -\infty < \eta < \infty)$ is a strongly continuous group with

$$\|b^{\alpha \eta}\| < e^{\pi |\eta|/2}.$$

If, in addition, $b$ is quasi-nilpotent, then the generator $iA$ of the group $(b^{\alpha \eta})$ has empty spectrum, while $e^{tA} = e^{t\overline{A}}$ have nonempty spectra away from $0$.

**Remark.** The estimate (2.2) is sharp; equality holds for $k(x,y) = 1$ (see [1, p. 165]).
Proof. Theorem 3 was proved in Kato [3] except for the last assertion regarding the case when \( H \) is quasi-nilpotent. In this case the semigroup \( \{s^\xi; \xi \geq 0\} \) is of type \(-\infty\) (i.e., \( \lim_{\xi \to \infty} \xi^{-1} \log s^\xi = -\infty \)), so that its generator \( A \) has empty spectrum. (Note that \( s^\xi \) has generator \( A \) if \( s^{i\eta} \) has generator \( iA \), since \( e^{i\eta} \) is a group, on the other hand, it is obvious that \( e^{i\eta} = e^{\eta} \in \mathcal{B}(X) \) have nonempty spectra away from 0.

Example. There are abundant examples of operators \( B \) satisfying the conditions of Theorem 3. Let \( k(x,y) \) be a continuous, hermitian symmetric, nonnegative-definite kernel on \([0,1] \times [0,1]\). \( k(x,y) \) defines an integral operator \( K \in \mathcal{B}(H) \), where \( H = L^2(0,1) \), such that \( K^* = K > 0 \). Let \( B \) be the associated Volterra operator:
\[
Bu(x) = \int_0^x k(x,y)u(y)dy \quad (u \in H).
\]
Then \( B \) is quasi-nilpotent and accretive, since \( 2 \text{Re}(Bu,u) = (Ku,u) > 0 \). \( B \) has no eigenvalue 0 if \( K \) is strictly positive, since \( Bu = 0 \) implies \( (Ku,u) = 0 \) by the remark above. But \( B \) may have no eigenvalue 0 even when \( K \) is only semi-definite. The simplest example of \( B \) is given by \( k(x,y) = 1 \). Then \( B \) is a simple integration, and \( \{B^\alpha\} \) is exactly the fractional integrals considered in §0.

§3. The spectral bound and the type.

If \( A \) is a closed linear operator in \( X \), we define the spectral bound of \( A \) by
\[
\text{spb} A = \sup \text{Re} \sigma(A) = \sup \{\text{Re} \lambda; \lambda \in \sigma(A)\}.
\]
We set \( \text{spb} A = -\infty \) if \( \sigma(A) \) is empty.

If \( A \) generates a strongly continuous semigroup \( \{e^{tA}; t > 0\} \), the type of \( A \) is defined by
\[
\text{type} A = \lim_{t \to +\infty} \frac{\log \|e^{tA}\|}{t} = \log \text{spr} e^A = +\infty,
\]
where \( \text{spr} \) denotes the spectral radius. (Type \( A \) is usually referred to the semigroup \( \{e^{tA}\} \) rather than to the generator \( A \), but we use the notation type \( A \) as a convenient abuse.)
Theorem 1 shows that

\[(3.3)\quad \text{spb}\ A \leq \text{type}\ A \quad (\leq \infty)\]

if \(A\) generates a \(C_0\)-semigroup. (Actually it is true for more general semigroups.)

Theorem 2 shows that equality holds in (3.3) if \(A\) generates a holomorphic semigroup. (Again it is true for more general semigroups.)

Theorem 3 shows that equality in (3.3) need not hold even for the generator \(A\) of a strongly continuous group; indeed one has \(\text{spb}\ A = \infty\) while \(|\text{type}\ A| < \pi/2\) in Theorem 3.

It may be noted that Greiner-Voigt-Wolff [4] gives examples of positivity-preserving \(C_0\)-semigroups \(\{e^{tA}\}\) on certain function spaces for which \(\text{spb}\ A = \infty\) and type \(A = 0\).

In fact little is known, beyond holomorphic semigroups (or, more generally, norm-continuous semigroups), about the question of when equality holds in (3.3).
REFERENCES


A Spectral Mapping Theorem for the Exponential Function, and some Counterexamples

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C₀-semigroups, holomorphic semigroups, spectral mapping theorem, spectral radius, spectral bound, type