ON THE NONLINEAR CONDUCTIVITY TENSOR FOR AN UNMAGNETIZED RELATI--ETC(U)

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On the Nonlinear Conductivity Tensor for an Unmagnetized Relativistic Turbulent Plasma

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The symmetry properties of the second-order nonlinear conductivity tensor for an unmagnetized, relativistic, and weakly turbulent plasma are important in the analysis of the collective bremsstrahlung instability. This tensor has some exact symmetries that, if resonant wave-particle interactions are neglected, become the widely known symmetry related to the Manley-Rowe relations, crossing symmetry, and the nondissipative nature of the
20. ABSTRACT (Cont'd)

Nonlinear current. Using the well-known expression for the conductivity from plasma turbulence theory, a polynomial representation for the tensor is obtained in which all derivatives are removed and the pole structure is clearly exhibited. The exact symmetries are obtained by a lengthy algebraic reduction using this representation.
CONTENTS

1. INTRODUCTION ......................................................... 5

2. POLYNOMIAL REPRESENTATION OF SECOND-ORDER NONLINEAR CONDUCTIVITY TENSOR ......................................................... 6

3. THE EXACT SYMMETRIES .............................................. 10

4. CONCLUSION .............................................................. 19

LITERATURE CITED .......................................................... 20

DISTRIBUTION ............................................................... 23
1. INTRODUCTION

The nature of the collective bremsstrahlung instability and its possible importance to plasma astrophysics and relativistic beam-plasma systems has been explored in considerable detail by Tsytovich and Akopyan.1-5 For an unmagnetized plasma, the photonic growth rate depends on the collective bremsstrahlung probability. Nonlinear bremsstrahlung associated with the three-plasmon dynamic polarization vertex has been shown to make an important contribution to the bremsstrahlung probability.1,2 In vacuum quantum electrodynamics, because of Furry's theorem,6 the analogous diagram is vanishing and is therefore absent in the standard Bethe-Heitler bremsstrahlung cross section.7 In plasma turbulence theory, the collective bremsstrahlung probability depends on the nonlinear bremsstrahlung amplitude through the second-order nonlinear conductivity tensor. The symmetry properties of this tensor are especially important in reducing the complex expression for the collective bremsstrahlung recoil force in a relativistic, weakly turbulent nonequilibrium plasma. The latter is needed to determine the collective bremsstrahlung probability.1,2 The symmetry properties were documented to some extent by Tsytovich and were shown to be related to the approximate nondissipative nature of the nonlinear current and also to crossing symmetry in three-plasmon interactions.1,8,9

The symmetry properties of the nonlinear conductivity tensor have been investigated also in other work. For nonrelativistic, weakly turbulent plasmas, they were established long ago.10 For relativistic plasmas in which the fields and the particle distributions are such that resonant wave-particle interactions can be ignored, the symmetry relating principal parts only has been demonstrated to all orders.11-13 Others have investigated also the relationship between the approximate symmetry and the fact that the total energy dissipated by the nonlinear current is vanishing.6,13,14 The relationship to crossing symmetry was also investigated in analyses of three-wave coupling between Langmuir, sound, and transverse waves.11,15 The relationship to generalized Onsager relations also has been addressed.11 The symmetry properties have been related to those of the Poisson brackets in a perturbation-theoretic Hamiltonian formulation.10,11 Moreover, in coherent three-wave interactions and the weak turbulence equations, it follows from the symmetry properties that wave energy and momentum are approximately conserved, and the Manley-Rowe relations obtain.10,16-21 To a limited extent, symmetry-breaking effects associated with violation of the Manley-Rowe relations have been addressed.16,21,22

*See references in Literature Cited section.
This paper reports two exact symmetry properties of the nonlinear conductivity tensor for an unmagnetized, relativistic, weakly turbulent plasma. The symmetries are not limited to the principal part. Their principal parts reduce to the well-known approximate symmetry. One of the exact symmetries was obtained previously by using the standard expression for the second-order conductivity tensor in plasma turbulence theory, resulting from straightforward iteration of the Vlasov equation. However, this symmetry was defined in an unphysical region of wave vector space, at least in its relationship to the second-order nonlinear current density. Another exact symmetry defined in the physical region is developed here. The iterative approach furnishes a new polynomial representation of the second-order conductivity tensor in which all derivatives are removed and the pole structure is clearly exhibited. It is hoped that the present elaboration of these symmetries will facilitate deeper understanding of the collective bremsstrahlung instability and the dissipative properties of the three-plasmon vertex.

In section 2, the polynomial representation of the second-order nonlinear conductivity tensor is presented. In section 3, the exact symmetries and their derivations by means of the polynomial representation are discussed. It is also shown that, ignoring resonant wave-particle interactions, the exact symmetries are reducible to the well-known symmetry relation for the nonrelativistic longitudinal case. In section 4, the results are briefly summarized.

2. POLYNOMIAL REPRESENTATION OF SECOND-ORDER NONLINEAR CONDUCTIVITY TENSOR

The second-order nonlinear conductivity tensor $S_{ijl}(k,k_1,k_2)$ in plasma turbulence theory is defined by

$$S_{ijl}(k,k_1,k_2) = e^2 \int \frac{d^3p}{(2\pi)^3} \frac{v_j}{\omega - \mathbf{k}_1 \cdot \mathbf{v} + i\delta} \left[ \frac{3}{\partial p_j} \right]$$

and is related to the second-order nonlinear current $j^{(2)}_k$ by

$$j^{(2)}_k = -e \int \frac{dk_1 dk_2 \delta(k-k_1-k_2)}{(\omega_1 + i\delta)(\omega_2 + i\delta)} S_{ijl}(k,k_1,k_2) E_{k_1} E_{k_2 l}.$$
It should be mentioned here that the tensor defined by equations (18) and (20) of Akopyan and Tsytovich\(^2\) differs implicitly from equation (1) here in that, instead, the first complex denominator \(\omega - \mathbf{k} \cdot \mathbf{v} + i\delta\) in equation (1) is implicitly \(\omega - \mathbf{k} \cdot \mathbf{v} - i\delta\) there. The notation of equations (1) and (2) corresponds more directly to equations (10) and (12) of Tsytovich.\(^7\) The symmetries discussed here pertain to the two symmetrized forms

\[
\mathbf{s}_{ijl}^{\pm}(k,k_1,k_2) \equiv \mathbf{s}_{ijl}(k,k_1,k_2) \pm \mathbf{s}_{ijl}(k,k_2,k_1) .
\]

The tensor \(\mathbf{s}_{ijl}^{\pm}(k,k_1,k_2)\) is the same as \(\mathbf{s}_{ijl}(k,k_1,k_2)\) of earlier work.\(^{14,24,26}\) It is convenient also to define the antisymmetrized form \(\mathbf{s}_{ijl}(k,k_1,k_2)\) above.

First integrating equation (1) by parts, dropping surface terms, then using the relativistic kinetic relations, performing the differentiations, and combining terms, one obtains

\[
\mathbf{s}_{ijl}(k,k_1,k_2) = \epsilon^2 c^2 \int \frac{d^3p}{(2\pi)^3} \left\{ \mathbf{R}(0) \right\} \frac{1}{\epsilon^2} \left[ \frac{\alpha_1 + \bar{\alpha}_1 \Omega_1}{\Omega + i\delta} + \frac{\alpha_2 + \bar{\alpha}_2 \Omega_1}{(\Omega + i\delta)(\Omega_2 + i\delta)} \right. \\
+ \left. \frac{\alpha_3 + \bar{\alpha}_3 \Omega_1}{(\Omega + i\delta)(\Omega_2 + i\delta)^2} + \frac{\alpha_4 + \bar{\alpha}_4 \Omega_1}{(\Omega + i\delta)^2} \right] \\
+ \frac{\alpha_5 + \bar{\alpha}_5 \Omega_1}{(\Omega + i\delta)^2(\Omega_2 + i\delta)} + \frac{\alpha_6 + \bar{\alpha}_6 \Omega_1}{(\Omega + i\delta)^2(\Omega_2 + i\delta)^2} \\
+ \frac{\alpha_7 + \bar{\alpha}_7 \Omega_1}{(\Omega + i\delta)^3} + \frac{\alpha_8 + \bar{\alpha}_8 \Omega_1}{(\Omega + i\delta)^3(\Omega_2 + i\delta)} \right] ,
\]

where

\[
\epsilon = \left( m^2 c^4 + p^2 c^2 \right)^{1/2} , (5)
\]

\[
\{ \Omega, \Omega_1, \Omega_2 \} = \{ \omega - \mu, \omega_1 - \mu_1, \omega_2 - \mu_2 \} , (6)
\]

\[
\{ \mu, \mu_1, \mu_2 \} = \{ \mathbf{k} \cdot \mathbf{\hat{v}}, \mathbf{k}_1 \cdot \mathbf{\hat{v}}, \mathbf{k}_2 \cdot \mathbf{\hat{v}} \} , (7)
\]
and where \( \{\alpha_n, n = 1, 8\} \) and \( \{\tilde{\alpha}_n, n = 1, 8\} \) are complicated tensor polynomials in the components of \( \vec{v}, \vec{k}, \vec{k}_1, \) and \( \vec{k}_2. \) They are simply a renaming of the coefficients \( C_{n1}^{x}. \) For notational convenience, the tensor indices of the \( \alpha_n \equiv \alpha_{nijkl}(k, k_1, k_2) \) and \( \tilde{\alpha}_n \equiv \tilde{\alpha}_{nijkl}(k, k_1, k_2) \) are suppressed. Explicitly, the \( \alpha_n \) are given by

\[
\alpha_1 = \left( c^2 k_{i1} - u_1 v_1 \right) \delta_{j1} - \left( c^2 k_{i2} - u_1 v_1 \right) \delta_{ij} - u_1 v_j \delta_{i1} \\
- 2k_{1i} v_j v_1 + 3c^{-2} u_1 v_1 v_j v_1 \\
(8)
\]

\[
\alpha_2 = \left( u_1 u_2 - c^2 k_1 \cdot k_2 \right) v_1 \delta_{ij} + 4c^{-2} u_1 u_2 v_1 v_j v_1 \\
+ c^2 k_{1i} v_j k_{21} + c^2 k_{1j} k_{21} v_1 - u_1 k_{21} v_j v_1 \\
(9)
\]

\[
\alpha_3 = \left( c^2 k_{2j} - u_2 \right) k_{i1} v_j v_1 + \left( c^2 u_1 v_2 - u_1 k_{2j} \right) v_j v_1 \\
(10)
\]

\[
\alpha_4 = \left( c^2 \cdot k_1 - u u_1 \right) \left( v_i \delta_{j1} + v_j \delta_{i1} \right) - c^2 v_i k_j k_{11} + u_1 v_i k_j v_1 \\
- 2u_1 v_i v_j k_1 - u k_{1i} v_j v_1 + c^2 k_{1i} v_j k_1 \\
+ \left( 5c^{-2} u u_1 - 3k \cdot k_1 \right) v_j v_1 \\
(11)
\]

\[
\alpha_5 = \left( -c^2 k_1 \cdot k_2 + u_1 u_2 \right) v_i k_j v_1 + \left( c^2 k_1 \cdot k_1 - u u_1 \right) v_j k_{21} \\
+ \left( c^2 k_1 \cdot k_1 - u u_1 \right) v_i k_{2j} v_1 + \left( c^2 k_1 \cdot k_1 - u u_1 \right) k_{2j} v_j v_1 \\
+ \left( c^2 k_1 \cdot k_2 - u u_2 \right) k_{1i} v_j v_1 + 2 \left( 3c^{-2} u u_1 u_2 - 2k \cdot k_1 u_2 \right) \\
(12)
\]

\[
- k \cdot k_2 u_1 v_j v_1 \\
\]
\[ \alpha_6 = \left( c^2 k_1^2 k_2^2 - \mu v_1 k_2^2 - k_1^2 \mu_2^2 + c^{-2} \mu v_1^2 \mu_2^2 \right) v_1 v_j v_1, \]  

(13)

\[ \alpha_7 = 2 \left( c^2 k_1 - \mu v_1 \right) v_1 v_j k_1 - 2 \left( k_1^2 \mu - c^{-2} \mu_2^2 v_1 \right) v_1 v_j v_1, \]  

(14)

\[ \alpha_8 = 2 \left( c^2 k_1^2 k_2 - k_1^2 \mu_1^2 - k_1^2 \mu_2 + c^{-2} \mu_1^2 \right) v_1 v_j v_1. \]  

(15)

The \( \tilde{\alpha}_n \) are given by

\[ \tilde{\alpha}_1 = -v_1 \delta_{j1} - v_j \delta_{11} - v_1 \delta_{ij} + 3c^{-2} v_1 v_j v_1, \]  

(16)

\[ \tilde{\alpha}_2 = \left( c^2 k_2 - 2 \mu_2 v_1 \right) \delta_{ij} - k_2 v_j v_1 - v_1 k_2 v_1 - v_1 v_j k_2 + 4c^{-2} \mu_2 v_1 v_j v_1, \]  

(17)

\[ \tilde{\alpha}_3 = \left( c^2 k_2^2 - \mu_2^2 \right) v_1 \delta_{ij} + \left( c^{-2} \mu_2^2 - k_2^2 \right) v_1 v_j v_1, \]  

(18)

\[ \tilde{\alpha}_4 = \left( c^2 k_1 - \mu v_1 \right) \delta_{ij} + \left( c^2 k_2 - \mu v_1 \right) \delta_{ij} - \mu v_1 \delta_{ij}, \]  

(19)

\[ \tilde{\alpha}_5 = \left( c^2 k_1 k_2 - \mu \mu_2 \right) v_1 \delta_{ij} + 2 \left( 3c^{-2} \mu_2 - k_1^2 \mu_2 \right) v_1 v_j v_1 \] 

\[ + c^2 k_2 k_1 v_1 + c^2 v_1 k_2 v_1 + c \] 

\[ - \mu v_1 k_2 v_1 - \mu v_1 k_2 v_1 - \mu k_2 v_1 v_1 - 3 \mu_2 v_1 k_2 v_1, \]  

(20)
\[ \sigma_6 = (c^2 k_2^2 - u_2^2)v_1 k_j v_1 + (c^{-2} mu_2^2 - \mu k_2^2)v_1 v_j v_1, \]  

(21)

\[ \sigma_7 = 2c^2 v_1 k_j k_1 - 2\mu v_1 k_j v_1 - 2\mu v_1 v_j k_1 + 2c^{-2} \mu^2 v_1 v_j v_1, \]  

(22)

\[ \sigma_8 = 2(c^2 k_2^2 - \mu u_2)v_1 k_j v_1 - 2(k^2_2 \mu - c^{-2} \mu^2_2)v_1 v_j v_1. \]  

(23)

The polynomial representation given by equation (4) contains no derivatives, and the explicit pole structure is clearly exhibited. In section 3, equation (4) is used to obtain the exact symmetries.

3. THE EXACT SYMMETRIES

In this section, the polynomial representation, equation (4), of the nonlinear conductivity tensor is used to explicitly establish the following exact symmetries in wave vector space:

\[ \sigma_{ijl}(\pm k_1 \pm k_2, k_1, k_2) = \sigma_{ijl}(k_1, \pm k_1 \pm k_2, k_2). \]  

(24)

The one involving \( \sigma_{ijl}(-k_1 - k_2, k_1, k_2) \) was obtained previously. The one involving \( \sigma_{ijl}(k_1 + k_2, k_1, k_2) \) occurs in the physical region of wave vector space of equation (2) and is therefore of greater interest.

First replacing \( k \) by \( \pm k_1 \pm k_2 \) in equation (4) and then combining terms, one obtains

\[ \sigma_{ijl}(\pm k_1 \pm k_2, k_1, k_2) \]

\[ = e^2 c^2 \int \frac{d^3 p}{(2\pi)^3} \frac{\xi^{R(0)}_p}{e^2} \sum_{n=1}^{24} \beta_n \frac{\xi^{R(0)}_n(\Omega_1 + i\delta)^3(\Omega_2 + i\delta)^3}{\Omega_1 + i\delta + \Omega_2 + i\delta}, \]  

(25)

where
\{ n, n = 1, 24 \} = \{ \Omega_1 \Omega_2, \Omega_1 \Omega_5, \Omega_1 \Omega_6, \Omega_1 \Omega_2^3, \Omega_1 \Omega_2^5, \Omega_1 \Omega_2^6, \Omega_1 \Omega_2^5, \Omega_1 \Omega_2^6 \}

\Omega_1 \Omega_2^3, \Omega_1 \Omega_2^5, \Omega_1 \Omega_2^6, \Omega_1 \Omega_2^5, \Omega_1 \Omega_2^6, \Omega_1 \Omega_2^5, \Omega_1 \Omega_2^6, (26)

\Omega_1 \Omega_2^5, \Omega_1 \Omega_2^5, \Omega_1 \Omega_2^5, \Omega_1 \Omega_2^5, \Omega_1 \Omega_2^5, \Omega_1 \Omega_2^5, \Omega_1 \Omega_2^5, \Omega_1 \Omega_2^5, \Omega_1 \Omega_2^5 \}

and where \( \{ \beta_1, n = 1, 24 \} \) are complicated tensor polynomials in the components of \( \nabla^r, \kappa_1, \) and \( \kappa_2 \). The latter are given by

\[
\beta_1 = (c^2 k_1^2 k_2^2 + c^2 k_1 \cdot k_2 k_1^2 - \mu_1^2 k_1^2 - \mu_1 \mu_2 k_1^2 - \mu_1^2 k_1 \cdot k_2
- \mu_1^2 k_2^2 + c^{-2} \mu_1^2 \mu_2^2 + c^{-2} \mu_1^3 \mu_2) v_1 v_3 v_1 ,
\]

\[
\beta_2 = (c^2 k_1^2 - \mu_1^2) k_1 v_1 v_3 + (c^2 k_1^2 - \mu_1^2) k_2 v_1 v_3 + (c^2 k_1^2 - \mu_1^2) k_2 v_1 v_3
+ (c^2 k_1^2 - \mu_1^2) k_2 v_1 v_3
+ c^{-2} \mu_1^2 v_1 v_3 v_1 ,
\]

\[
\beta_3 = (c^2 k_1^2 - \mu_1^2) v_3 \delta_{11} - (c^2 k_1^2 - \mu_1^2) v_3 \delta_{11} ,
\]

\[
\beta_4 = [3 c^2 k_1^2 \mu_2 + 2 c^2 k_2^2 k_1 \cdot k_2 + 2 c^2 (k_1 \cdot k_2)^2 + 3 c^2 k_1^2 k_1 \cdot k_2
- 3 c_1^2 \mu_2 - 3 c_1 \mu_2 \mu_2 - 2 k_1 \cdot k_2 \mu_2^2 + 4 k_1 \cdot k_2 \mu_1 \mu_2
- 2 k_2^2 \mu_2 - 3 k_2^2 \mu_2 - 3 k_1 \cdot k_2 \mu_2^2 + 2 c^{-2} \mu_1 \mu_2
+ 3 c^{-2} \mu_1 \mu_2 + 3 c^{-2} \mu_1 \mu_2] v_1 v_3 v_1 ,
\]
\[ \beta_5 = (2c^2 k_1 \cdot k_2 k_{11} + 3c^2 k_1^2 k_{11} - 2u_{11} u_{21} k_{11} - 3u_{11}^2 k_{11} \\
+ c^2 k_1 \cdot k_2 k_{21} - u_{21}^2 k_{11} - u_{11} u_{21} k_{21} + c^2 k_2^2 k_{11} \\
+ 3c^2 k_1^2 k_{21} - 3u_{11}^2 k_{21}) v_1 v_j \\
+ \left( c^2 k_1 \cdot k_2 k_{11} - u_{11}^2 k_{11} + c^2 k_2^2 k_{21} - 3u_{11}^2 k_{21} \right) v_j v_1 \\
+ \left( c^2 k_1 \cdot k_2 k_{11} - u_{11}^2 k_{11} + c^2 k_2^2 k_{21} - 3u_{11}^2 k_{21} \right) v_1 v_1 \] (31)

\[ \beta_6 = \left( -3u_{11}^2 + 3c^2 k_1^2 \right) v_j \delta_{11} + \left( c^2 k_1 \cdot k_{11} + c^2 k_1 \cdot k_{11} \right) v_1 \\
+ \left( c^2 k_1 \cdot k_{11} + c^2 k_1 \cdot k_{11} + c^2 k_1 \cdot k_{21} \right) v_j + c^2 k_1 \cdot k_{11} v_1 \\
+ \left( -2u_{21}^2 k_{11} - 4u_{11} k_{11} - 3u_{11}^2 k_{21} \right) v_i v_j \\
+ \left( -2u_{21} k_{11} - u_{11} k_{11} \right) v_i v_1 + \left( -2u_{21}^2 k_{11} - u_{11} k_{11} - 3u_{11} k_{21} \right) v_j v_1 \\
+ \left( -4k_1^2 + 10c^2 u_{21} + 8c^2 u_{11}^2 - 2k_1 \cdot k_2 \right) v_i v_j v_1 \] (32)

\[ \beta_7 = \left( -2u_{11} v_j + c^2 k_1 \cdot k_{11} \right) \delta_{11} - k_{11} \bar{v}_j v_1 - k_{11} \bar{v}_i v_j \\
- k_{11} \bar{v}_j v_1 + 4c^2 u_{11} v_i v_j v_1 \] (33)
\[ \beta_8 = \left[ 3c^2k_1^2k_2^2 + 2c^2k_1^2\mathbf{k}_1 \cdot \mathbf{k}_2 + 3c^2k_2^2\mathbf{k}_1 \cdot \mathbf{k}_2 + 2c^2(\mathbf{k}_1 \cdot \mathbf{k}_2)^2 ight. \\
- 3k_1^2\mu_1^2 - 3k_2^2\mu_1^2\mu_2 - 2\mathbf{k}_1 \cdot \mathbf{k}_2\mu_1^2 - 4\mathbf{k}_1 \cdot \mathbf{k}_2\mu_1\mu_2 \\
- 2k_1^2\mu_1\mu_2 - 3k_2^2\mu_1^2 - 3\mathbf{k}_1 \cdot \mathbf{k}_2\mu_2^2 + 2c^{-2}\mu_1^3\mu_2 \\
+ 5c^{-2}\mu_1^2\mu_2^2 + 3c^{-2}\mu_1^3\mu_2 \right] v_i v_j v_1 . \\
\]

\[ \beta_9 = \left( 3c^2k_1^2k_2^2 + 2c^2\mathbf{k}_1 \cdot \mathbf{k}_2k_1^2 + 3\mathbf{k}_1^2k_2^2 - 2\mu_1^2k_2^2 - 2\mu_1\mu_2k_2^2 ight. \\
+ 2c^2k_1^2\mu_1^2 + 2c^2\mathbf{k}_1 \cdot \mathbf{k}_2\mu_1^2 - 2\mu_1^2k_1^2 + c^2k_2^2k_1^2 \\
- 2\mu_1\mu_2k_1^2 - \left( \mathbf{k}_2 \cdot \mathbf{k}_2 \right) v_i v_j + \left( \mathbf{k}_1 \cdot \mathbf{k}_2 \right) v_i v_j \\
+ 2c^2k_1^2k_2^2 - \mu_1^2k_2^2 - 2\mu_1\mu_2k_2^2 - 2\mu_2^2k_2^2 + 2c^2\mathbf{k}_1 \cdot \mathbf{k}_2k_1^2 \\
- 2\mu_1\mu_2k_1^2 + 3c^2k_2^2k_1^2j - \left( 3\mu_1^2k_1^2 \right) v_i v_1 \\
+ \left( 2c^2\mathbf{k}_1 \cdot \mathbf{k}_2k_1^2 - 2\mu_1\mu_2k_1^2 - 3c^2k_2^2k_1^2 - 3\mu_2^2k_1^2 \\
+ 3c^2k_1^2k_2^2 + 2c^2\mathbf{k}_1 \cdot \mathbf{k}_2k_2^2 - 3\mu_1^2k_2^2 - 2\mu_1\mu_2k_2^2 \right) v_j v_1 \\
+ \left( -10\mathbf{k}_1 \cdot \mathbf{k}_2\nu_1 - 9k_1^2\nu_2 - 10\mathbf{k}_1 \cdot \mathbf{k}_2\nu_2 + 19c^{-2}\mu_1^2 \mu_2^2 \\
+ 19c^{-2}\mu_1^2\mu_2^2 - 2\mu_2^2k_2^2 + 2c^{-2}\mu_2^3 - 9\mu_1k_2^2 - 2\mu_1k_2^2 \\
+ 2c^{-2}\mu_1^3 \right] v_i v_j v_1 . \\
\]
$$\beta_{10} = \left( c^2 k_1^2 + 2 c_2 k_1 \cdot k_2 - \mu_1^2 - 2 \mu_1 \mu_2 + c^2 k_2^2 - \mu_2^2 \right) v_i \delta_{ij}$$

$$+ \left( -3 \mu_1^2 + 3 c^2 k_1^2 \right) v_j \delta_{ij} + \left( c^2 k_2^2 - \mu_2^2 \right) v_i \delta_{ij}$$

$$+ \left( 2 c^2 k_1 k_1 k_{1j} + c^2 k_2 k_{2j} + 3 c^2 k_{1j} k_{2j} \right) v_i$$

$$+ \left( 2 c^2 k_1 k_{1i} + 2 c^2 k_2 k_{2j} + 3 c^2 k_{2j} k_{1i} \right) v_j$$

$$+ \left( c^2 k_1 k_{1j} + c^2 k_2 k_{2i} + 3 c^2 k_{1j} k_{2i} \right) v_i$$

$$+ \left( -3 \mu_1 k_{1j} - 8 \mu_1 k_{2i} - 8 \mu_2 k_{2j} - 4 \mu_2 k_{1i} \right) v_i v_j$$

$$+ \left( -2 \mu_1 k_{1j} - 7 \mu_2 k_{1i} - 8 \mu_1 k_{2i} - 2 \mu_2 k_{2j} \right) v_j v_i$$

$$+ \left( -3 \mu_1 k_{2j} - 2 \mu_1 k_{1j} - 7 \mu_2 k_{1j} - 4 \mu_2 k_{2j} \right) v_i v_1$$

$$+ \left( 14 c^{-2} \mu_1^2 + 32 c^{-2} \mu_1 \mu_2 - 8 k_1 \cdot k_2 - 6 k_1^2 \right)$$

$$- 3 k_2^2 + 7 c^{-2} \mu_2^2 \right) v_i v_1 v_1 \right) ,$$

$$\beta_{11} = \left( c^2 k_1 + c^2 k_2 - 2 \mu_1 v_i - 2 \mu_2 v_j \right) \delta_{ij}$$

$$+ \left( c^2 k_2 - 2 \mu_2 v_j \right) \delta_{ij} + \left( c^2 k_1 - \mu_1 v_j \right) \delta_{ij}$$

$$- 2 \left( k_{2j} + 2 k_{1j} \right) v_i v_j - 2 \left( k_{1j} + 2 k_{2j} \right) v_j v_i$$

$$- 2 \left( k_{2j} + 2 k_{1j} \right) v_1 v_i + 8 \left( k_2 + 2 \mu_1 \right) c^{-2} v_i v_j v_i$$

$$\beta_{12} = \left( -v_i \delta_{ij} - v_1 \delta_{ij} - v_j \delta_{ij} + 3 c^{-2} v_i v_j v_1 \right) ,$$
\[ e_{13} = \left( c^2 k_1^2 k_2^2 + c^2 k_1^* k_2^2 - u_1^2 k_1^2 - u_1 u_2 k_2^2 - u_2^2 k_1^* k_2 \right. \\
\left. - u_2^2 k_1 + c^{-2} u_2^2 u_1^2 + c^{-2} u_1 u_3^2 \right) v_1 v_3 v_1, \]

\[ e_{14} = \left( c^2 k_1^* k_2^* k_1 - u_1 u_2 k_1^2 + 3c^2 k_2^2 k_1^2 - 3u_2^2 k_1^4 - c^2 k_1^2 k_2 \right. \\
\left. + c^2 k_1^* k_2^* k_2^* - u_1^2 k_2^2 - u_1 u_2 k_2^2 \right) v_1 v_1 \\
\left. + \left( c^2 k_1^2 k_2^2 + u_1 u_2 k_2^2 + c^2 k_1^* k_2^* k_1 - u_1^2 k_2 \right) v_1 v_j \\
\left. + \left( c^2 k_2^2 k_2^2 + 2c^2 k_1^* k_2^* k_2^2 - u_1^2 k_2^2 - 2u_1 u_2 k_2^2 + c^2 k_1^* k_2^* k_1 \right) v_1 v_j \\
\left. - u_1 u_2 k_1^2 + 3c^2 k_1^2 k_2^2 + 3c^2 k_2^2 k_2^2 - 3u_2^2 k_1^2 - 3u_2^2 k_2^2 \right) v_1 v_1 \\
\left. + \left( -6k_1^* k_2^* u_2^2 - 4k_1^2 u_2^2 + 8c^{-2} u_1^2 u_2^2 - 7k_1^2 u_1^2 - 3k_2^2 u_2 \\
\left. + 13c^{-2} u_1 u_2^2 + 3c^{-2} u_2^2 - 4k_1^* k_2^* u_1 \right) v_1 v_3 v_1, \]

\[ e_{15} = \left( -u_1^2 + c^2 k_1^2 \right) v_3 v_1 v_1 + \left( -3u_2^2 + 3c^2 k_2^2 \right) v_1 v_3 v_1 \\
\left. + \left( -u_1^2 - 2u_1 u_2 + c^2 k_1^2 - 2c^2 k_1^* k_2 - u_2^2 + c^2 k_2^2 \right) v_1 v_3 v_1 \\
\left. + c^2 \left( k_1^2 k_1^* + 3k_1^* k_2^2 + 2k_2^2 k_2 \right) v_1 \\
\left. + c^2 \left( k_1^2 k_1^* + 3k_1^* k_2^2 + k_2^2 k_2 \right) v_1 \\
\left. + c^2 \left( 2k_2^2 k_2^2 + 3k_2^2 k_2^2 + 2k_2^2 k_2 \right) v_1 \\
\left. + \left( -7u_1 k_2^2 - 2u_2 k_2^2 - 4u_1 k_1^2 - 2u_2 k_1^2 \right) v_1 v_3 \\
\left. + \left( -u_1 k_1^2 - 8u_2 k_1^2 - 8u_2 k_2^2 - 4u_1 k_2^2 \right) v_1 v_1 \\
\left. + \left( -u_1 k_1^2 - 8u_2 k_1^2 - 2u_2 k_2^2 - 7u_1 k_2^2 \right) v_3 v_1 \\
\left. + \left( 7c^{-2} u_1^2 + 32c^{-2} u_1 u_2 + 14c^{-2} u_2^2 - 3k_1^2 - 8k_1^* k_2 \\
\left. - 6k_2^2 \right) v_1 v_3 v_1, \right. \]
\[ \beta_{16} = 2 \left( c^2 k_{11} + c^2 k_{21} \right) \delta_{11} + 3c^2 k_{21} \delta_{1j} + 3c^2 k_{1j} \delta_{i1} \\
- 6u_1 v_j \delta_{11} - 6u_2 v_1 \delta_{i1} - 4(u_1 + u_2) v_1 \delta_{j1} - 5(k_{11} + k_{21}) v_1 v_j \\
- 5(k_{1i} + k_{2i}) v_j v_1 - 5(k_{1i} + k_{2i}) v_i v_1 \\
+ 20 \left( c^{-2} u_1 + c^{-2} u_2 \right) v_i v_j v_1 , \]  
(42)

\[ \beta_{17} = -3v_1 \delta_{j1} - 3v_1 \delta_{ij} - 3v_1 \delta_{i1} + 9c^{-2} v_1 v_j v_1 , \]  
(43)

\[ \beta_{18} = \left( c^2 k_{21} k_{1j} + c^2 k_{2j} k_{21} - u_2 k_{1j} - u_2 k_{2j} \right) v_1 v_1 \\
+ \left( 2c^{-2} u_1 u_2^2 + c^{-2} u_2^3 - 2u_1 k_2^2 - u_2 k_2^2 \right) v_1 v_j v_1 \\
+ \left( c^{-2} k_2^2 - u_2^2 \right) k_{1j} v_j v_1 , \]  
(44)

\[ \beta_{19} = c^2 k_{11} k_{21} v_j + \left( c^2 k_{11} k_{2j} + c^2 k_{1j} k_{2i} + c^2 k_{2j} k_{2i} \right) v_1 v_1 \\
+ \left( c^2 k_{1j} k_{21} + c^2 k_{2j} k_{21} \right) v_1 v_1 + \left( 3c^2 k_2^2 - 3u_1^2 \right) v_1 \delta_{i1} \\
- \left( u_2 k_{21} + 2u_1 k_{2j} \right) v_1 v_j - \left( u_2 k_{2j} + 2u_1 k_{2j} + 3v_1 k_{1j} \right) v_j v_1 \\
- \left( 2u_1 k_{2j} + 3v_1 k_{1j} + 4u_2 k_{2j} \right) v_1 v_1 \\
+ 2 \left( 5c^{-2} u_1 u_2 - 2k_2^2 + 4c^{-2} u_2^2 - k_1 + k_2 \right) v_1 v_j v_1 , \]  
(45)

\[ \beta_{20} = \left( 3c^2 k_2^2 - 6u_2 v_1 \right) \delta_{1j} + \left( c^2 k_{1j} - 2u_1 v_j \right) \delta_{11} \\
+ \left( c^2 k_{1i} + c^2 k_{2i} - 2u_1 v_i - 2u_2 v_i \right) \delta_{j1} \\
+ 8 \left( c^{-2} u_1 + 2c^{-2} u_2 \right) v_1 v_j v_1 - 2 \left( k_{1j} + 2k_{2j} \right) v_1 v_1 \\
- 2 \left( k_{1i} + 2k_{2i} \right) v_1 v_j - \left( 2 k_{1i} + 2k_{2i} \right) v_j v_1 , \]  
(46)
\[
\beta_{21} = -3v_1^2 \delta_{11} - 3v_j^2 \delta_{11} - 3v_i^2 \delta_{ij} + 9c^{-2} v_1 v_j v_1, \\
\beta_{22} = (c^2 k_2^2 - \mu_2^2) v_1 \delta_{ij} + (c^{-2} \mu_2^2 - k_2^2) v_1 v_j v_1, \\
\beta_{23} = (-2u_2 v_1 + c^2 k_2) \delta_{ij} - k_1 v_j v_1 - k_2 v_i v_1 \\
- k_2 v_1 v_j + 4c^{-2} u_2 v_1 v_j v_1, \\
\beta_{24} = -v_1^3 \delta_{11} - v_j^3 \delta_{11} - v_i^3 \delta_{ij} + 3c^{-2} v_1 v_j v_1.
\]

Similarly, by reducing \( \sigma_{ii}((-k_1, +k_1, +k_2, -k_2)) \), comparing the result with equation (25) term by term, and then using the reality property of the current and the electric field, the exact symmetry relations given by equations (24) follow.

The principal parts of equations (24) correspond to the well-known symmetry of the second-order nonlinear conductivity tensor. This correspondence results from ignoring resonant wave-particle interactions. In that case, the well-known symmetry relation for the nonrelativistic longitudinal case, equation (2.83) of Tsytovich, also follows. To see this, one first notes that the pure longitudinal nonrelativistic second-order conductivity \( S_{k_1 k_2} \) is given by

\[
S_{k_1 k_2} = \frac{e^3}{|k_1| |k_2|} \int \frac{k_1 \cdot \mathbb{v}}{\omega - k_1 \cdot \mathbb{v} + i\delta} \left( \sigma_{k_2 \mathbb{v}} \right) \frac{1}{\omega_2 - k_2 \cdot \mathbb{v} + i\delta} \\
\times \left( k_2 \cdot \mathbb{v} \right) \frac{d^3 \mathbb{p}}{(2\pi)^3}.
\]

Comparing equation (51) with equation (1) then yields

\[
S_{k_1 k_2} = -e \frac{k_1 \cdot k_1}{|k_1|} \frac{k_2 \cdot k_1}{|k_2|} \frac{1}{\omega_1 \omega_2} S_{ij1}(k_1, k_1, k_2).
\]

By using equation (3) and equation (52), it follows that
\[
\frac{1}{\omega_2} \left( S_{k_2, k_1+k_2, -k_1} + S_{k_2, -k_1, k_1+k_2} \right) = -e \frac{(k_{1j} + k_{2j})_{j=1}^2 \alpha_{1jl}^+ (k_{2j}, k_1+k_2, -k_1)}{|k_1 + k_2| |k_1| |k_2| (\omega_1 + \omega_2) \omega_1 \omega_2} \tag{53}
\]

\[
and \quad - \frac{1}{\omega_1 + \omega_2} \left( S_{-k_1-k_2, k_1, -k_2} + S_{-k_1-k_2, -k_2, -k_1} \right) = -e \frac{(k_{1j} + k_{2j})_{j=1}^2 \alpha_{1jl}^+ (-k_{2j}, k_1-k_2, -k_1)}{|k_1 + k_2| |k_1| |k_2| (\omega_1 + \omega_2) \omega_1 \omega_2} \tag{54}
\]

If one ignores the imaginary part \( \delta \) in equation (1), then using equation (3) one immediately obtains the following approximate relation:

\[
\alpha_{1jl}^+ (-k, k_1, k_2) = -\alpha_{1jl}^+ (k, k_1, k_2) \tag{55}
\]

By using equation (55) together with one of the exact symmetry relations equations (24), it follows that

\[
\alpha_{jl}^+ (-k_1-k_2, -k_2, -k_1) = \alpha_{jl}^+ (k_2, k_1+k_2, -k_1) \tag{56}
\]

Substituting equation (56) into equation (54) and comparing the result with equation (53), one obtains

\[
\frac{1}{\omega_2} \left( S_{k_2, k_1+k_2, -k_1} + S_{k_2, -k_1, k_1+k_2} \right) = - \frac{1}{\omega_1 + \omega_2} \left( S_{-k_1-k_2, k_1, -k_2} + S_{-k_1-k_2, -k_2, -k_1} \right) \tag{57}
\]

Equation (57) is the well-known symmetry relation for the nonrelativistic longitudinal case (eq (2.83) of Tsytovich\(^9\)). Ignoring the imaginary part \( \delta \) in equation (55) is equivalent to ignoring resonant wave-particle interactions and including only the principal part. The approximate symmetry can also be shown to follow from the other exact symmetry of equations (24).
4. CONCLUSION

In conclusion then, the following exact symmetry relations hold for the second-order nonlinear conductivity tensor of an unmagnetized relativistic weakly turbulent plasma:

$$\sigma^\pm_{ijl}(\pm k_1 \pm k_2, k_1, k_2) = \sigma^\mp_{jil}(\pm k_1 \pm k_2, k_2),$$

(58)

where $\sigma^\pm_{ijl}(k,k_1,k_2)$ are defined by equations (1) and (3). Also, a polynomial representation, equation (4), for the tensor has been obtained in which all derivatives are removed and the pole structure is clearly exhibited. The principal part of the exact symmetries, equation (58), is the well-known approximate symmetry that applies when resonant wave-particle interactions are negligible, the Manley-Rowe relations obtain, and the nonlinear current is nondissipative.

The symmetry properties are especially useful in the calculation of the bremsstrahlung recoil force in a relativistic nonequilibrium plasma. The latter is necessary to determine the collective bremsstrahlung probability and to investigate the conditions for the occurrence of a bremsstrahlung instability.28
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