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OPTIMAL SENSOR SCHEDULING FOR MULTIPLE HYPOTHESIS TESTING[1]

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ABSTRACT

The generic problem of selecting the sequence of sensors which optimizes the information received about a number of discrete hypotheses is considered. The optimization criterion penalizes the uncertainty present about pairs of hypotheses in a form which has an eigenfunction property with respect to a Bayes update of the conditional probability distribution. Application of the Portryagin minimum principle yields elegant solutions to an interesting class of problems. Applications in surveillance, failure detection, and nondestructive testing are possible.

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OPTIMAL SENSOR SCHEDULING FOR MULTIPLE HYPOTHESIS TESTING

I. Introduction

The problem: Often several competing hypotheses exist about the state of a particular entity, and real time observations must be used to discriminate between them. Once the set of sensors to be used has been specified, the observations can be used to update prior information in a number of ways, although Bayes' theorem underlies some of the most common techniques [1]. In this framework, the net information is captured in the posterior probability distribution over the hypotheses.

In cases where several sensors are available but are mutually exclusive in their use (either due to interference, or because one physical sensor must be pointed in one of a number or directions), an additional problem arises in determining, also in real time, that sequence of sensors which should be activated to provide the above information. The efficacy of a particular sensor sequence must be related to the character of the resulting posterior probabilities; these should clearly discriminate among the hypotheses.

Mathematically, this can be viewed as a problem of selecting, at each point in time, one of M sensors to obtain information about a set of K hypotheses. By defining an interesting cost function on the se of posterior distributions, we can seek an optimal sensor scheduling procedure.

Applications: A number of generic application problems exhibit this structure. In surveillance problems [e.g. 2], the hypotheses would present the presence and type of target that exists at each point in the surveillance

volume, and the sensors would be radars or detectors which must be pointed in azimuth and elevation. In failure detection and identification [3], the hypotheses may represent different types of failure and onset time of the failure. In nondestructive fault localization [4], the hypotheses are types and locations of faults in some medium, and the sensors report the attenuation of energy as it passes through the medium in some direction to be specified. Search problems [5,6] fall into this structure as the hypothetical locations of an object are observed from various vantage points.

Perspective: A number of authors have addressed scheduling problems in this context. Some make special assumptions regarding the observation probability distributions [7,8], others treat the more general problem [9,10]. However, two common threads connect these approaches: they obtain feedback laws mapping posterior destributions into sensor selections, and they use optimization criteria based on weighted probabilities of error (i.e. Bayes' risk criteria). A common result of these two factors is a very ungainly decision rule unless relatively restrictive assumptions are made.

The point of departure of the solution found here is precisely these two characteristics. This solution is content with being open loop (although it is often actually easier to use in an open loop feedback manner), and is based on a criterion which measures the uncertainty in the posterior distribution (and incidentally provides an upper bound on weighted probabilities of error).

Overview: The strategy taken in the sequel is to first pose the problem in discrete time, with emphasis on the new cost function and its

interpretation. Then the stochastic problem is reduced to a deterministic control problem, in continuous time, with a convex control set. The Pontryagin minimum principle can be brought to bear on the problem, and the resulting necessary conditions for the optimal schedule define a two point boundary value problem with sectorwise linear dynamics. The general structure of the solution is then available; certain structural assumptions lead to an iterative solution for the general case, and an elegant solution for a more restricted set of problems. An interesting side result is a geometrical characterization of each sensor by a vector of parameters decribing its capabilities to distinguish between various pairs of hypotheses.

II. Problem Statement.

Hypotheses: The K hypotheses, one of which may be valid, are denoted[3] H_k, $k = 1,\ldots,K$. Prior knowledge provides a probability distribution at time $t = 0$, denoted by $\vec{\pi}(0)$, where

$$\vec{\pi}_k(0) = p(H_k) \tag{1}$$

Observations: Sensor outputs obtained at time t from sensor j are denoted $\vec{y}_j(t)$. The statistics of $\vec{y}_j(t)$ are independent of everything except the sensor j and the underlying hypothesis H_k; in particular

$$p(\vec{y}_j(t)|H_k, \vec{y}_\ell(s)) = p(\vec{y}_j(t)|H_k) \qquad s \neq t \tag{2}$$

(In continuous time, this implies that $\vec{y}_j(t)$ is a process with independent increments [11]; the continuous time case will be addressed more fully in section III). However, no assumption of stationarity of these distributions

[3] Notation is reviewed in Appendix A.

need be made.

If

$$\pi_k(t) = p(H_k | \vec{y}(1), \ldots, \vec{y}(t)) \tag{3}$$

then by Bayes' law

$$\pi_k(t+1) = \frac{p(y(t+1) | H_k)\, \pi_k(t)}{p(\vec{y}(t+1))} \tag{4}$$

where subscripts denoting sensor choices have been omitted. Equations (1) and (4) give the dynamic equations for the evolution of the posterior distribution as observations are obtained.

Cost: The objective of a detection or identification algorithm is to produce correct estimates of the true state of a system. It is also beneficial if these estimates come with high confidence levels. Thus, if one is seeking to drive posterior distributions to some values, the best values are near the extremes, where the true hypothesis is known with almost certainty.

Consider the binary hypothesis case. Figure 1 shows three candidate penalty functions which have reasonable qualitative characteristics - they are all minimum at the extremes and convex downward. Number 1 is the minimum probability of error incurred if a decision between H_1 and H_2 had to be made. Number 2 is a direct measure of the uncertainty in the distribution: it is the entropy (scaled by 1/2)

$$-\frac{1}{2} \sum_{k=1}^{z} \pi_k \log_2 \pi_k \tag{5}$$

The third is similar to the second

$$v(\pi_1, \pi_2) = \sqrt{\pi_1 \pi_2} \tag{6}$$

Note that the last two indicate that an improvement in the probability of

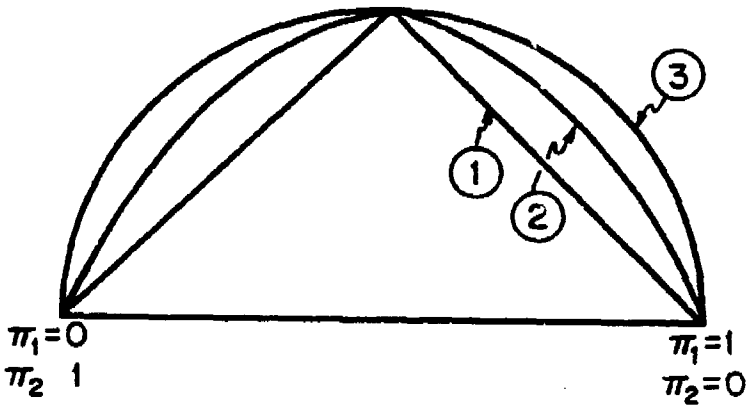


FIGURE 1

Candidate Penalty Functions on $\vec{\pi}$ - Binary Case

error from, say, 10% to 1% is much more rewarding than one from 49% to 40%, and thus greatly encourage extremal distributions.

The third form posesses unique analytical properties, as will be seen in section III. It can be generalized to the form

$$v(\pi_1,\ldots,\pi_k) = \prod_{k=1}^{K} \pi_k^{r_k} \tag{7}$$

$$1 = \sum_{k=1}^{K} r_k \qquad r_k \geq 0$$

and to sums of terms of this form without compromizing these properties.

Definition: An <u>obscurity function</u> $v(\vec{\pi})$ is of the form

$$v(\vec{\pi}) = \sum_{i=1}^{N} b_i \, v_i(\vec{\pi}) \tag{8}$$

with each $v_i(\vec{\pi})$ having the form of (7).

The obscurity function measures the lack of knowledge abou the hypotheses. It is minimum when $\vec{\pi}$ is pure, i.e. when all but one component are zero. The coefficients b_i represent weights attached to varying types of obscurity as indicated by the form of the associated $v_i(\vec{\pi})$.

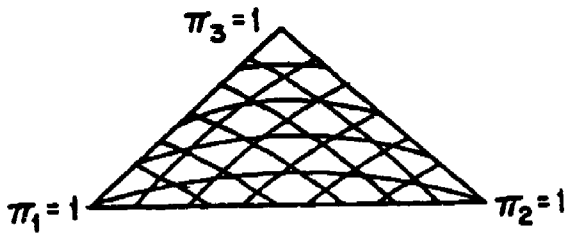
Example: Consider the ternary hypothesis testing problem. Two candidate obscurity functions are

$$v^1(\vec{\pi}) = (\pi_1 \, \pi_2 \, \pi_3)^{1/3} \tag{9}$$

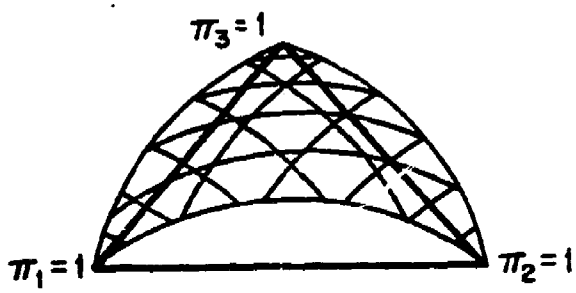
and

$$v^2(\vec{\pi}) = (\pi_1 \, \pi_2)^{1/2} + (\pi_1 \, \pi_3)^{1/2} + (\pi_2 \, \pi_3)^{1/2} \tag{10}$$

Both are zero for pure $\vec{\pi}$; both have their maxima at $\vec{\pi} = [1/3 \; 1/3 \; 1/3]^T$. However, the former includes as minima <u>all</u> distributions with one component zero, the latter has only three minimum points (Figure 2). The former is



$V^1(\vec{\pi})$



$V_2(\vec{\pi})$

FIGURE 2

Example Obscurity Functions

minimized when any hypothesis is eliminated; the latter, when any hypothesis is confirmed.

The above definition is quite general; in the sequel we will assume all terms in the obscurity function are of the form

$$v_i(\vec{\pi}) = (\pi_{k_1} \pi_{k_2})^{1/2} \tag{11}$$

All results will generalize to the earlier case, but this makes clear that each term in $v(\vec{\pi})$ represents the degree to which a pair of hypotheses can be distinguished, and thus a different type of obscurity.

The selection of the obscurity function provides a great deal of flexibility. For instance, if one is only interested in determining whether or not H_1 is true, a function of the form

$$v(\pi_i) = \sum_{k=2}^{K} [\pi_1 \pi_k]^{1/2} \tag{12}$$

is appropriate, as it penalizes ambiguity between H_1 and any other hypothesis without including the obscurity between the others.

Special Cases: There are two special cases of obscurity function of note.

Definition: The uniform obscurity function is

$$v^u(\vec{\pi}) = \sum_{k_1=1}^{K} \sum_{\substack{k_2=1 \\ k_2 \neq k_1}}^{K} \frac{1}{2} (\pi_{k_1} \pi_{k_2})^{1/2} \tag{13}$$

The uniformity stems from the equal penalizing of all pairs of hypotheses; useful interpretation of v^u is given in

Theorem 1: $v^u(\vec{\pi})$ is an upper bound to the minimum probability of error achieved by a decision rule selecting an estimate of H_1 at each point in time, based on the distribution $\vec{\pi}$.

Proof: The minimum probability of error is achieved by selecting the $\hat{H}_k$ which has maximum probability $\hat{\pi}_k$; the resulting error probability is $1 - \hat{\pi}_k$. Now

$$v^u(\vec{\pi}) = \sum_{k_1=1}^{K} \sum_{\substack{k_2=1 \\ k_2 \neq k_1}}^{K} \frac{1}{2} (\pi_{k_1} \pi_{k_2})^{1/2}$$

$$\geq \sum_{\substack{k_2=1 \\ k \neq \hat{k}}}^{K} (\hat{\pi}_k \pi_{k_2})^{1/2} \tag{14}$$

$$\geq \sum_{\substack{k_2=1 \\ k_2 \neq \hat{k}}}^{K} \pi_{k_2}$$

$$= (1 - \hat{\pi}_k)$$

□

The second special case concerns a set of hypotheses which is the Cartesian product of two or more sets of subhypotheses. If the posterior distributions on the composite set always factor into distributions on the component sets, a natural obscurity function arises which additively decomposes. Let $H_{k\ell}$ be the composite hypothesis $(H_k^1 \; H_\ell^2)$, where superscripts denote the component hypothesis sets. Independence of all posterior distributions

$$\pi_{k\ell}(t) = \pi_k^1(t) \; \pi_\ell^2(t) \tag{15}$$

is implied by the conditions

$$\text{(a)} \quad \forall_{k,\ell}: \pi_{k\ell}(0) = \pi_k^1(0)\ \pi_k^2(0) \tag{16}$$

$$\text{(b)} \quad \forall_j : p(y_j(t)\,|\,(H_{k_1}\ H_\ell)) = p(y_j(t)\,|\,H_k) \tag{17}$$

$$\text{or}$$

$$p(y_j(t)\,|\,(H_{k_1}\ H_\ell)) = p(y_j(t)\,|\,H_\ell)$$

That is, the prior distribution has the component hypotheses independent, and subsequent observations change the distribution on either H_k or H_ℓ , but never both.

Definition: A Cartesian obscurity function $v^c(\vec{\pi})$ is of the form

$$v^c(\vec{\pi}) = v^1(\vec{\pi}) + v^2(\vec{\pi}) \tag{18}$$

$$v^1(\vec{\pi}) = \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} b^1_{k_1,k_2} \sum_{\ell=1}^{L} (\pi_{k_1\ell}\ \pi_{k_2\ell})^{1/2} \tag{19}$$

$$v^2(\vec{\pi}) = \sum_{\ell_1=1}^{L} \sum_{\ell_2=1}^{L} b^2_{\ell_1\ \ell_2} \sum_{k=1}^{K} (\pi_{k\ \ell_1}\pi_{k\ \ell_2})^{1/2} \tag{20}$$

Independence (15) then implies

$$v^1(\vec{\pi}) = \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} b^1_{k_1\ k_2} (\pi^1_{k_1}\ \pi^2_{k_2})^{1/2} \tag{21}$$

and similarly for $v^2(\vec{\pi})$. The Cartesian obscurity function thus additively decomposes into separate obscurity functions defined on the two component sets.

This result will be used in an example in section VI.

<u>Conclusion</u>: The above discussion has dwelt on the interpretation and structure of the obscurity function. Assuming it provides a reasonable

measure of the poorness of the state of information defined by the posterior distribution $\vec{\pi}(t)$, the objective of minimizing its sum over some time horizon is appealing. The optimization problem is thus: select the sequence of sensors $j(t)$ from which observations are taken, to minimize the functional

$$E \left\{ \sum_{t=1}^{T} v(\vec{\pi}(t)) \right\} \tag{22}$$

where the expectation is with respect to observations $\vec{y}_j(t)$. The optimization is subject to the initial conditions (1), dynamics (4), and distributions on observations (2).

III. Reduction to a Continuous Deterministic Optimal Control Problem

The problem stated above is a stochastic optimization problem, where the original, imperfectly observable state H has been replaced with the conditional probability $\vec{\pi}$, which can be determined exactly. State transitions are still stochastic, due to the appearance of $\vec{y}(t)$ in (4), but possess a Markov property. This is a standard approach [12] to dealing with this type of problem; the next step might be to use dynamic programming to obtain a feedback solution, where $j(t)$ would be selected on the basis of $\vec{\pi}(t)$.

Due to the lack of success of this approach in producing implementable solutions for the general, multiple hypothesis problem consider a less ambitious goal: finding the optimal <u>open loop</u> schedule (i.e. select the best sequence of sensors based only on the prior distribution). Not only are solutions of this form applicable in some cases where feedback solutions cannot be implemented, but they can be used as open loop feedback solutions where the entire schedule is effectively recomputed at each time, using $\vec{\pi}(t)$

as the prior distribution, and the first selection given by that schedule is implemented.

Reduction to a deterministic problem in this case involves performing the expectation in (22). This results in a discrete time problem and a characterization of each sensor by a set of coefficients. After some discussion of the interpretation of these coefficients, a continuous time approximation will be constructed for further study (or, the continuous time problem can be found directly in the case of continuous time observations). After a final detail, where the control set is extended from discrete points to a connected set, the equivalent deterministic optimal control problem is presented.

Reduction to Deterministic Dynamics: The form of the obscurity function was selected for its qualitative properties and because of:

Theorem 2: Functions of the form (7) are eigenfunctions of the expectation/Bayes update operation and the associated eigenvalue is completely determined by sensor characteristics.

Proof: Let[4]

$$v_i(\vec{\pi}(t+1),\ t+1) = x_i(t+1)\ (\pi_1\ \pi_2)^{1/2}$$

Then

$$v_i(\vec{\pi}(t),\ t) = \underset{\vec{y}(t+1)}{E}\{v_i(\vec{\pi}(t+1),\ t+1)\} \tag{23}$$

with $\vec{\pi}(t+1)$ given by (4). Substituting,

$$v_i(\vec{\pi}(t),\ t) = \int_{-\vec{\infty}}^{\vec{\infty}} p(\vec{y}_j(t+1)\ x_i(t+1)\left(\frac{p(\vec{y}_j(t+1)|H_1)\ \pi_1(t)\ p(\vec{y}_j(t+1)|\ H_2)\pi_2(t)}{(p(\vec{y}_j(t+1)))^2}\right)^{1/2}$$

$$= \alpha_{ij}(t+1)\ x_i(t+1)\ (\pi_1(t)\ \pi_2(t))^{1/2} \tag{24}$$

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[4] For notational simplicity here, assume the ith term of the obscurity function involves H_1 and H_2.

where j is the sensor selected for t+1 and

$$\alpha_{ij}(t+1) = \int_{-\vec{\infty}}^{\vec{\infty}} (p(\vec{y}_j(t+1)|H_1)\ p(\vec{y}_j(t+1)|H_2))^{1/2}\ d\vec{y}_j \tag{25}$$

By (24), $v_i(\vec{\pi}(t+1),t+1)$ is an eigenfunction of the update (23), with $\alpha_{ij}(t+1)$ is eigenvalue.

□

Now, for a fixed sequence of sensors $\tilde{j} = \{j(t),\ t = 1,\ldots,T\}$, define the expected cost to-go at time t with conditional distribution $\overline{\pi}(t)$ as

$$V_{\tilde{j}}(\vec{\pi}(t),\ t) = E\{\sum_{s=t+1}^{T} v(\vec{\pi}(s))\} \tag{26}$$

where $V_{\tilde{j}}(\vec{\pi}(T),T) = 0$ at the terminal time. The key result is then

Theorem 3: At each time t, the cost-to-go takes the form

$$V_{\tilde{j}}(\vec{\pi}(t),t) = \sum_{i=1}^{N} x_i(t)v_i(\vec{\pi}) \tag{27}$$

where

$$x_i(t) = \alpha_{ij}(t+1)\ x_i(t+1) + b_i \tag{28}$$

$$x_i(T) = 0 \tag{29}$$

Proof: By reverse induction. At t = T, (29) implies the cost-to-go is uniformly zero. Assume (27) holds at time t+1,so

$$V_{\tilde{j}}(\vec{\pi}(t),t) = E\{V_{\tilde{j}}(\tilde{\pi}(t+1),\ t+1)\} + b_i\ v_i(\vec{\pi}(t)) \qquad \text{by (8,22)}$$

$$= E\{\sum_{i=1}^{N} x_i(t+1)\ v_i(\vec{\pi}(t+1))\} + b_i\ v_i(\vec{\pi}(t)) \qquad \text{by (27)}$$

$$= \sum_{i=1}^{N} \alpha_{ij}(t+1)\ x_i(t+1)\ v_i(\vec{\pi}(t)) + b_i\ v_i(\vec{\pi}(t)) \qquad \text{by Thm 2}$$

$$= \sum_{i=1}^{N} (\alpha_{ij}(t+1)\ x_i(t+1) + b_i)\ v_i(\vec{\pi}(t))$$

and $v_i(\vec{\pi})$ is of the fundamental form (7). □

This gives a deterministic <u>linear</u> dynamical problem (28,29) with states $x_i(t)$ representing the amplitudes of a finite number of modes of the cost-to-go function excited by the terms of the obscurity function. The coefficients $\alpha_{ij}(t)$ represent the decay of $x_i(t)$ when sensor j is selected at time t, and the driving terms b_i representing the relative importance of each term. Moreover, the $x_i(t)$ are truely states as their evolution depends only on selections j made betweeen t and T, although this property holds in <u>reverse time</u>.

Corollary 3a: The total cost is

$$V_j(\vec{\pi}(0),0)) = \sum_{i=1}^{N} x_i(0)\ v_i(\vec{\pi}(0)) \tag{30}$$

Proof: Immediate from Theorem 3 when $t = 0$. □

If (known) parameters c_i are defined as

$$c_i = V_i(\vec{\pi}(0)) \tag{31}$$

then the equivalent deterministic optimal control problem is to select $\tilde{j}$

to minimize

$$\sum_{i=1}^{N} x_i(0)\ c_i \tag{32}$$

subject to

$$x_i(t-1) = \alpha_{ij}(t)\ x_i(t) + b_i \tag{33}$$
$$x_i(T) = 0$$

Interpretation of the α_{ij}: These parameters measure the ability of sensor j to contribute to the reduction of each term of the obscurity function. The set $\{\alpha_{ij} \mid i = 1,\ldots,N\}$ describe the information gathering ability of j in all directions which are contained in $v(\vec{\pi})$. For example, sensor 1 may be able to distinguish H_1 from H_2 and H_3, but not between the latter, while sensor 2 only separates H_2 from H_3. The information from each sensor alone is incomplete; the set above paves the way towards a geometric interpretation of information.

Gross properties of the α_{ij} are

Theorem 4: For all i,j,t,

$$0 \leq \alpha_{ij}(t) \leq 1 \tag{34}$$

with the lower limit obtained iff it is possible to completely eliminate one of the hypotheses in v_i with any single observation $\vec{y}_j(t)$, and the upper iff $\vec{y}_j(t)$ is independent of the hypotheses in $v_i(\vec{\pi})$.

Proof: Since $p(\vec{y}_j(t)|H_k) \geq 0$ for all $\vec{y}_j$, for all $\vec{y}_j$

$$\int_{-\vec{\infty}}^{\vec{\infty}} (p(\vec{y}_j(t)|H_{k_1})\ p(\vec{y}_j(t)|H_{k_2}))^{1/2}\ d\vec{y} \geq 0 \tag{35}$$

with equality iff

$$p(\vec{y}_j(t)|H_{k_1})\ p(y_j(t)|H_{k_2}) = 0 \tag{36}$$

for all $y_j(t)$, i.e. iff the set of $\vec{y}_j(t)$ which may result when H_2 is true is disjoint from that possible when H_{k_2} is true, and hence $\vec{y}_j(t)$ provides perfect information to distinguish between them. Since also

$$\int_{-\vec{\infty}}^{\vec{\infty}} p(\vec{y}_j|H_k)d\vec{y} = 1 \tag{37}$$

the integral

$$\int_{-\vec{\infty}}^{\vec{\infty}} p(\vec{y}_j(t)|H_{k_1})\ p(\vec{y}_j(t)|H_{k_2})d\vec{y} \leq 1 \tag{38}$$

with equality iff

$$p(\vec{y}_j(t)|H_{k_1}) = p(\vec{y}_j(t)|H_{k_2}) \tag{39}$$

for all $\bar{y}_j(t)$.

□

Thus, qualitatively speaking, good schedules use sensors where the α_{ij} are small for terms where c_i or b_i are large.

In preparation for the transition to continuous time, introduce

Definition: The <u>clarification coefficient</u> of sensor j with respect to $v_i(\vec{\pi})$ is $a_{ij}(t)$

$$a_{ij}(t) = -\ln \alpha_{ij}(t) \tag{40}$$

Corollary 4a: Clarification coefficients are nonnegative and unbounded with equality to zero holding iff the sensor produces outputs which are independent of the hypotheses of the associated term.

Proof: Properties of ln. □

Appendix B contains formulae for the clarification coefficients for two common observation processess: Poisson and Gaussian.

Reformulation in Continuous Time: The remainder of this section deals with improving the analytic properties of the problem by replacing the discrete time and control sets with continuous equivalents. The problem as posed (32,33) can be solved using the discrete time minimum principle [13], but the solution has implicit properties which are less cumbersome in a continuous time framework.

Consider the formal continuous time analog of (32,33): minimize

$$\sum_{i=1}^{N} c_i \, x_i(0) \tag{41}$$

with

$$\frac{dx_i}{d(-t)} = -a_{ij}(t)\, x(t) + b_i(t) \qquad x_i(T) = 0 \tag{42}$$

where again the dynamics appear in reverse time. Integrating (42) from time t to time $t - \delta$ gives approximately

$$x_i(t-\delta) - x_i(t) = -a_{ij}(t)\, x_i(t)\delta + b_i(t) \tag{43}$$

$$\begin{aligned} x_i(t-\delta) &= (1 - a_{ij}(t)\delta)\, x_i(t) + b_i(t)\delta \\ &\simeq e^{-a_{ij}(t)\delta} k_i(t) + b_i(t)\delta \end{aligned} \tag{44}$$

provided δ is sufficiently small that second order terms can be neglected, i.e.

$$a_{ij}(t)\delta \ll 1 \tag{45}$$

Setting $\delta = 1$

$$x(t-1) \approx \alpha_{ij}(t)\, x_i(t) + b_i(t) \tag{46}$$

Provided the $a_{ij}(t)$ are the clarification coefficients (4) and $\alpha_{ij}(t)$ the eigenvalues (25), (41,42) may be valid approximations.

The approximations require (45) to hold when δ is the unit discrete time interval. If it is invalid, decreasing δ is suggested - i.e. increase the discrete time sample rate. The principal effect of this is to create more opportunities for changing the sensor selection - i.e. making switch times into more continuous variables. As $\delta \to 0$, the approximation becomes exact;[5] this can be shown specifically for continuous time Gaussian processes either using [14] or more direct techniques (Appendix C).

Convexification of Control Variable: In the problem thus far there have been a discrete set of sensors from which to select. It will be convenient to convexify this set by introducing the M control variables $u_j(t)$, which specify what fraction of an infinitesimal cycle is devoted to each sensor j. Thus

$$\sum_{j=1}^{M} u_j(t) = 1 \qquad\qquad u_j(t) \geq 0 \tag{47}$$

are the constraints which admissible controls must satisfy.

With this interpretation, (44) becomes

$$x_i(t-\delta) \approx e^{-\sum_j a_{ij}(t)u_j(t)\delta}\, x_i(t) + b_i\delta \tag{48}$$

or, as $\delta \to 0$

[5] As the discrete sample rate goes to zero, this expression is exact for all stationary processes, as well as for nonstationary Gaussian and Poisson processes. It provides a piecewise linear approximation for other sampled, nonstationary independent increments processes; however, the sequel will assume $\alpha_{ij}(t)$ to be twice differentiable and thus a more advanced approximation, such as splines, may be necessary.

$$\frac{dx_i}{d(-t)} = -a_i(\vec{u})\, x_i(t) + b_i \tag{49}$$

where

$$a_i(\vec{u}) = \sum_{j=1}^{M} a_{ij}(t)\, u_j(t) \tag{50}$$

This convexification of the control set allows the above interpretation of polling sensors with u_j being the fraction of time devoted to sensor j. Mixed controls (some $u_j \notin \{0,1\}$) do arise in the optimal solution. Were a solution attempted without convexification, the optimal solution would still be forced to achieve this mixture by infinitesimal "time sharing". In practice, either this polling can be approximated or, in open loop feedback uses, it will almost never occur as the set of $\vec{\pi}(0)$ for which it is initially required is of measure zero.

Reversal of the Time Index: Finally, the reverse dynamics that naturally arose in (32,33) and (40,41) are a notational nuisance; replacing the time variable t with another t´

$$t' = T - t \tag{51}$$

yields an identical problem more in line with standard optimal control problems. The only caveat is that the solution to the resulting problem, $\bar{u}^*(t')$, is the reverse of the optimal schedule.

Conclusion: This section has reduced the original problem (1,4,8,22) to an equivalent problem: minimize

$$\sum_{i=1}^{N} c_i\, x_i(T) \qquad c_i = v_i(\vec{\pi}(0)) \tag{52}$$

with

$$\dot{x}_i(t') = -a_i(\vec{u}(t'))\ x_i(t') + b_i \qquad x_i(0) = 0 \qquad (53)$$

The system dynamics $a_i(\vec{u}(t'))$ will always be nonnegative, and larger values correspond to greater clarification by the selected sensor.

IV. Optimal Solution of Reduced Problem

Here the above problem is interpreted using the Pontryagin minimum principle and the geometric structure of the solution emerges. The first section introduces the type of results obtained by examining the binary hypothesis problem where only one dimension of obscurity exists. After stating necessary conditions which the optimal schedule must satisfy and deriving some of its properties, the interpretation of these conditions in terms of sensor clarification coefficients provides some preliminary tests for eliminating sensors from consideration. Further examination of singular (mixed) control arcs yields more basic structure of the schedule as well as a classification of problems in terms of the sensor sets. These will be the general results; section IV will exploit the necessary conditions to compute the optimal schedule.

<u>Preview - Binary Hypothesis Testing:</u> This special case illustrates some of the conclusions that can be drawn about optimal schedules. The obscurity function between the two hypotheses H_1 and H_2 has one term

$$v_1 = (\pi_1\ \pi_2)^{1/2}$$

so the continuous problem to be solved is to minimize

$$c_1\ x_1(T)$$

with

$$\dot{x}_1(t) = a_1(\vec{u}(t))\ x_1(t) + 1 \qquad x_1(0) = 0$$

Here the solution is obvious: choose $\vec{u}(t)$ to <u>maximize</u> the coefficient $a_1(\vec{u}(t))$ at each time t. This corresponds to selecting the sensor with maximal instantaneous clarification coefficient $a_{ij}(t)$ at each t.

Thus $\vec{u}$ is chosen to maximize (a function of) $a_1(t)$ at each time, and the selected sensor may vary as $a_{ij}(t)$ changes with time. No mixed controls are required here, but multiple hypotheses with multiple terms in $v(\vec{\pi})$ will induce a directionality which requires mixing.

This can be extended to the case where $\vec{u}$ adjusts continuous parameters internal to a sensor; see Appendix D for an example application in data compression.

<u>General Necessary Conditions:</u> Necessary conditions for the dynamic optimization problem (52,53) can be obtained from the Pontryagin minimum principle [15,16]. They are summarized in

Theorem 5: The optimal solution $\vec{u}(t)$ to (52,53) satisfies:

$$\sum_{i=1}^{N} z_i(t)\, a_i(\vec{u}^*(t)) \geq \sum_{i=1}^{N} z_i(t)\, a_i(\vec{u}(t)) \quad \text{for all } \vec{u}(t) \tag{54}$$

$$\dot{z}_i(t) = q_i(t) \qquad z_i(0) = 0 \tag{55}$$

$$\dot{q}_i(t) = a_i(\vec{u}^*(t))\, q_i(t) \qquad q_i(T) = b_i\, c_i \tag{56}$$

Proof: see Appendix E. □

Because of their associations with variables in the proof, (54) will be referred to as the Hamiltonian condition, $z_i(t)$ as states, and $q_i(t)$ as costates. In addition, introduce the M vectors

$$\vec{a}_j(t) = [a_{ij}(t) \quad a_{2j}(t) \;\ldots\ldots\; a_{Nj}(t)]^T \tag{56}$$

of clarification coefficients for each sensor and the NxM composite matrix

$$\underline{A}(t) = \begin{bmatrix} \vec{a}_1 & \vdots & \vec{a}_2 & \vdots & & \vdots & \vec{a}_M \\ & \vdots & & \vdots & \cdots & \vdots & \\ & \vdots & & \vdots & & \vdots & \end{bmatrix} \tag{57}$$

so that

$$\vec{a}(\vec{u}(t)) = \underline{A}(t)\ \bar{u}(t) \tag{58}$$

is the vector of system coefficients $a_i(\bar{u}(t))$.

The necessary conditions can then be rewritten as

$$\langle \vec{z}(t), \vec{a}(\vec{u}(t)) \rangle \geq \langle \vec{z}(t), \vec{a}(\vec{u}(t)) \rangle \qquad \text{for all } \vec{u}(t) \tag{59}$$

$$\dot{\vec{z}}(t) = \vec{q}(t) \qquad \vec{z}(0) = \vec{0} \tag{60}$$

$$\dot{\vec{q}}(t) = \vec{a}(\vec{u}(t)) \cdot \bar{q}(t) \qquad \bar{q}(T) = \bar{b} \bullet \bar{c} \tag{61}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^N$ and $\cdot$ is componentwise multiplication.

<u>Properties of $\vec{z}(t)$ and $\vec{q}(t)$</u>: Some properties of the solution to (59-61) are immediate. Note that these describe a 2N dimensional differential equation with boundary conditions at both ends of the time horizon. The solution approach here will be to focus on the initial conditions for $\vec{z}$ and on the interrelationship between the dynamics of $\vec{z}$ and $\vec{q}$.

Theorem 6: $\vec{z}(t)$ and $\vec{q}(t)$ have the following properties:

a) $\vec{q}(t) > \vec{0}$
b) $\vec{q}(t)$ is monotonically nondecreasing
c) $\vec{z}(T)$ is monotonically increasing
d) if $\vec{q}(0) = \bar{q}_0$ leads to a solution of all conditions except $\bar{q}(T) = \bar{b} \cdot \bar{c}$, then so does $\bar{q}(0) = \lambda q_0$ for all positive λ .

Proof: a) Assume $\vec{q}(t_o) \leq 0$. $\vec{a}(\vec{u}(T)) \geq 0$ for all $\vec{u}(t)$ (from theorem 4), so $\vec{q}(t) \leq 0$ for $t \geq t_o$, and hence $\vec{q}(t) \leq 0$ for $t \geq t_o$. But $\vec{q}(T) = \vec{b} \cdot \vec{c} \geq 0$ in contradiction.

b and c) immediate from a) and theorem 4.

d) If $(\vec{z}(t), \vec{q}(t))$ satisfy all conditions except $\vec{q}(T) = \vec{b} \cdot \vec{c}$, then $(\gamma\vec{z}(t), \gamma\vec{q}(t))$ also do provided $\gamma > 0$. □

Thus $\vec{z}(t)$ moves outward into the positive sector of $\mathbb{R}^N$, from the origin, and $\vec{q}(t)$ moves outward to approach $\vec{q}(T)$.

<u>Properties from the Hamiltonian Condition:</u> The Hamiltonian condition (59) provides the key to the geometrical structure of the problem. The vectors of clarification coefficients $\vec{a}_j(t)$ describe M points which are vertices of a polytope in $\mathbb{R}^N$. The control $\vec{u}(t)$, by taking a convex combination of the $\vec{a}_j(t)$, allow $\vec{a}(\vec{u}(t))$ to be selected anywhere in the interior or on the boundary of this polytope. These candidates for $\vec{a}(\vec{u}(t))$ will be called the <u>control polytope</u>.

The condition (59) implies that $\vec{u}^*(t)$ must select a point in this control polytope which has a projection on $\vec{z}(t)$ at least as great as any other point in the polytope. It follows immediately that $\vec{a}(\vec{u}^*(t))$ must lie on the boundary of the control polytope, and in fact an even stronger condition holds.

Theorem 7: Sensor j_o will not be used at time t for <u>any</u> choice of $\vec{b}$, $\vec{c}$ or t if there exists a nonnegative vector $\vec{r}$ such that

$$\vec{a}_{j_o}(t) \leq \underline{A}(t)\,\vec{r}(t) \tag{62}$$

$$\sum_{j=1}^{n} r_j(t) = 1 \qquad r_{j_o}(t) = 0 \tag{63}$$

Proof: Assume (62,63) hold. Then for any nonegative $\bar{z}$,

$$\langle \vec{z}, \vec{a}_{j_o}(t)\rangle \leq \langle \vec{z}, \underline{a}(t)\, \vec{r}(t)\rangle = \langle \vec{z}, \vec{a}(\vec{r}(t))\rangle \tag{64}$$

so $\vec{u}(t) = \vec{r}(t)$ provides at least as large a value of the Hamiltonian as does $\vec{a}_{j_o}$. Since indeed $\vec{z}(t) \geq 0$ for all time, j_o will never be selected at time t. □

The converse of this theorem is also true, but will not be proven here. The import of this theorem is that it provides a convenient test for determining whether each sensor falls into the following class.

Definition: A sensor is <u>superfluous</u> at time t if it will not appear in the optimal schedule, for any problem, at time t. It is <u>completely superfluous</u> if it will not appear in such a schedule at any time.

Henceforth, consider only the set of nonsuperfluous sensors at each time t.

Returning to the basic geometry of the Hamiltonian condition, the following concepts are helpful:

Definition: Two sensors are <u>adjacent</u> if the line segment connecting their vectors of clarity coefficients lies completely in the convex hull of the control polytype.

Definition: A <u>face</u> of the control polytype is a collection of mutually adjacent sensors. Faces can be of any dimension, from 1 (a single sensor) to N - 1.

Now we find

Theorem 8: The solution to the Hamiltonian condition (59) is unique unless $\vec{z}$ is normal to the line connecting two adjacent sensors.

Proof: (59) is the maximization of a piecewise linear function of $\vec{u}$ subject to a linear constraint and nonnegativity requirements. It is well known that this maximum occurs at a vertex of the function. It is unique unless

$$\langle \vec{z}, a_{j_1}(t)\rangle = \langle \vec{z}, a_{j_2}(t)\rangle \tag{65}$$

for two vertices j_1 and j_2, i.e.

$$\langle \vec{z}, \vec{a}_{j_1}(t) \quad -\vec{a}_{j_2}(t)\rangle = 0 \tag{66}$$

Also, the maximum occurs at any convex combination of $\vec{a}_{j_1}$ and $\vec{a}_{j_2}$. (66) implies $\vec{z}$ is normal to the line connecting the two vertices; the convexity condition requires that the segment connecting the vertices also be on the boundary of the control polytope, i.e. that j_1 and j_2 be adjacent. □

This defines <u>switch curves</u> from one sensor to an adjacent one; $\vec{u}^*(t)$ takes on different values (and is pure, selecting only one sensor) on opposite sides of the hyperplane defined by (66). All switch curves are linear manifolds including the origin, hence the regions between these curves are sectors of $\mathbb{R}^{2N}$ (Figure 3). Since $\vec{u}^*$ is constant within each sector, the overall dynamics of $(\vec{z}(t), \vec{q}(t))$ are piecewise linear, with the pieces being sectors. As $\vec{z}(t)$ moves from sector to sector, it selects a sequence of sensors, where each sensor follows another which is adjacent to it. This property helps reduce the number of possible sensor sequences - any schedule with nonadjacent sensors in succession is not optimal.

What happens, though, if $\vec{z}(t)$ does not move from sector to sector, but stays on the boundary separating two or more sectors?

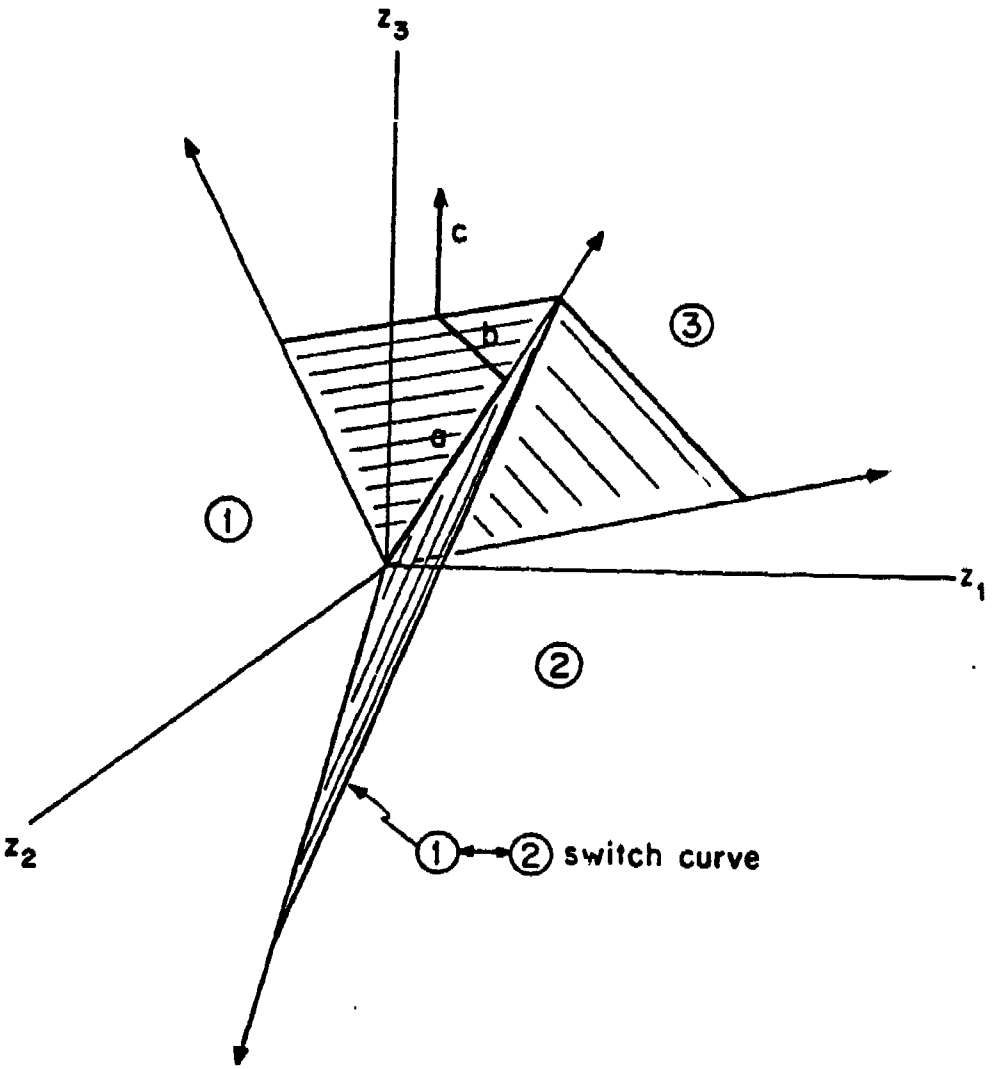


FIGURE 3

Control Sectors and Switch Curves with Singular Arcs

<u>Singular Control Arcs</u>: When $\vec{z}(t)$ stays on a switch curve for a nonzero interval of time, the Hamiltonian does not provide a unique choice for $\vec{u}^*(t)$ for that interval. However, the requirement that $\vec{z}(t)$ stay on the curve for the continuous interval implies additional conditions [17] namely

$$\frac{d^n}{dt^n} \langle \vec{z}(t), \vec{a}_{j_1}(t) - \vec{a}_{j_2}(t) \rangle = 0 \tag{67}$$

This condition can be exploited to find partial optimal paths at the start of an interval.

Specifically, consider singular control between adjacent sensors 1 and 2. Define

$$\vec{n}_{12}(t) = \vec{a}_1(t) - \vec{a}_2(t) \tag{68}$$

as the normal to the switch hyperplane between the two sensors. The selection between them is then made as

$$\begin{array}{lll} \langle \vec{z}(t), \vec{n}_{12}(t) \rangle > 0 & & \text{choose 1} \\ \langle \vec{z}(t), \vec{n}_{12}(t) \rangle = 0 & & \text{ambiguous} \\ \langle \vec{z}(t), \vec{n}_{12}(t) \rangle < 0 & & \text{choose 2} \end{array} \tag{69}$$

In order to maintain the ambiguous case,

$$\langle \vec{q}(t), \vec{n}_{12}(t) \rangle + \langle \vec{z}(t), \dot{\vec{n}}_{12}(t) \rangle = 0 \tag{70}$$

$$\langle \vec{a}(t) \cdot \vec{q}(t), \vec{n}_{12}(t) \rangle + 2\langle \vec{q}(t), \dot{\vec{n}}_{12}(t) \rangle + \langle \vec{z}(t), \ddot{\vec{n}}_{12}(t) \rangle = 0 \tag{71}$$

are necessary where $\vec{a}(t)$ is the vector of coefficients resulting from some convex combination of $\vec{a}_1(t)$ and $\vec{a}_2(t)$

For states and costates which satisfy (69-71), the behavior of their trajectories and the conditions for singular control are summarized in

Theorem 9: If $\vec{z}_1(\tau)$, $\vec{q}(\tau)$ are the trajectories resulting from application of sensor 1 for a small period of time following t, ($\tau \in [t,t+\delta]$) and $\vec{z}_2(\tau)$, $\vec{q}_2(\tau)$ likewise for sensor 2, both with initial conditions at t satisfying. (69,70), then

a) $$<\vec{z}_1(\tau), \vec{n}_{12}(\tau)> \geq <\vec{z}_2(\tau), \vec{n}_{12}(\tau)> \tag{72}$$

b) $<\vec{z}_1(\tau), \vec{n}_{12}(\tau)> \geq 0$ iff

$$< \vec{a}_1(t)\ \vec{q}(t), \vec{n}_{12}(t)> + 2 <\vec{q}(t), \dot{\vec{n}}_{12}(t)> + <\vec{z}(t), \ddot{\vec{n}}_{12}(t)> \geq 0 \tag{73}$$

c) $< \vec{z}_2(\tau), \vec{n}_{12}(\tau)> \leq 0$ iff

$$<\vec{a}_2(t)\ \vec{q}(t), \vec{n}_{12}(t)> + 2 <\vec{q}(t), \dot{\vec{n}}_{12}(t)> + <\vec{z}(t), \ddot{\vec{n}}_{12}(t)> \leq 0 \tag{74}$$

d) Singular control arise if and only if both (b) and (c) hold.

Proof: The proof is primarilly algebraic and deals with the evolution of the projection of $\vec{z}$ onto the normal to the the switch hyperplane. Details are contained in Appendix F. □

Theorem 9 provides a way to determine whether singular controls may exist at a specific point; as such it is rather tedious in general. A broader condition is:

Corollary 9A: A sufficient condition [5] for (73) to hold <u>at every point on</u> a singular are along the switch curve, at time t, is that there exists a $\gamma > 0$ such that either

$$\gamma\ \ddot{\vec{n}}_{12}(t) \geq \ddot{\vec{n}}_{12}(t) \qquad \text{and} \tag{75}$$

$$\gamma\vec{a}_1(t)\cdot\vec{n}_{12}(t) + 2\ \dot{\vec{n}}_{12}(t) \geq \vec{n}_{12}(t) \tag{76}$$

[5] This is also necessary if the statement is to hold for <u>all</u> such points; this will not be proven here.

or

b) $\gamma \dddot{\vec{n}}_{12}(t) \geq -\dot{\vec{n}}_{12}(t)$ and (77)

$$\gamma \vec{a}_1(t) \cdot \vec{n}_{12}(t) + 2\,\dot{\vec{n}}_{12}(t) \geq -\vec{n}_{12}(\tau) \tag{78}$$

A similar condition, with $\vec{a}_2(t)$ replacing $\vec{a}_1(t)$ and inequalities reversed, guarantees (74) at every point.

Proof : Assume (a). Since $\vec{z}(t) \geq 0$, (75) implies

$$\gamma < \vec{z}(t), \dddot{\vec{n}}_{12}(t) > \geq < \vec{z}(t), \dot{\vec{n}}_{12}(t) > \tag{79}$$

Since $\bar{q}(t) \geq 0$ also, (76) yields a similar result which, when combined with (79), gives

$$\gamma \frac{d^2}{dt^2} < \vec{z}(t), \vec{n}_{12}(t) > \geq \frac{d}{dt} < \vec{z}(t), \vec{n}_{12}(t) > \tag{80}$$

By (70) both of these are nonnegative at each point on the curve; since $\gamma > 0$, (73) must hold.

Assume (b). Identical arguments give

$$\gamma \frac{d^2}{dt^2} < \vec{z}(t), \vec{n}_{12}(t) > \geq - \frac{d}{dt} < \vec{z}(t), \vec{n}_{12}(t) > \tag{81}$$

and (70) again gives the latter as equal to zero along a singular curve, so (73) still holds.

The condition for (74) is proven in the same way. □

This provides a quick test to see if singular controls can be maintained for arbitrary periods along a switch curve. Failure of the conditions of corollary 9A to hold require more detailed examination for singularity using theorem 9.

A final point on the topic of singular controls is to note that it is possible to maintain singular arcs using more than two sensors. Since this requires $\vec{z}(t)$ to stay on the switch curve between each pairs of sensors, it is necessary that all sensors involved be adjacent. The conditions for the existence of singular controls are the union of the pairwise conditions above; for computation of the actual mixture of sensors, see appendix F.

Thus singular controls can exist; they will form the backbone of the solution to a well structured class of problems discussed in section V.

Crossing of Switch Curves: While theorem 9 gives conditions for singular controls, it also provides a great deal of information concerning how and when various switch curves may be crossed. Successive sensors in an optimal schedule must be adjacent to one another and this limits the set of candidate orderings; exploitation of theorem 9 allows further limitations to be considered.

The focus of this development is on the linear manifold in $\mathbb{R}^{2N}$

$$\langle \vec{z}(t), \vec{n}_{12}(t) \rangle = 0 \tag{82}$$

$$\langle \vec{q}(t), \vec{n}_{12}(t) \rangle + \langle \vec{z}(t), \dot{\vec{n}}_{12}(t) \rangle = 0 \tag{83}$$

Being linear, this manifold separates $\mathbb{R}^{2N}$ into four subsets corresponding to the possible inequalities replacing = in (82, 83). The partition (82) in $\vec{z}$ - space is the switch curve; the companion partition in $\vec{q}$ - space (83) determines which direction $\vec{z}$ will be driven from the switch curve by the current value of $\vec{q}$. For simplicity, consider only the half space

$$\langle \bar{z}(t), \bar{n}_{12}(t) \rangle > 0 \tag{84}$$

Identical results follow for the other halfspace using $\vec{n}_{21}(t)$, and theorem 9

dealt with the case of equality.

Suppose $\vec{z}(t)$ is in the sector where $\vec{a}_1(t)$ is optimal, and $\vec{q}(t)$ in the half space

$$< \vec{q}(t), \vec{n}_{12}(t)> + < \vec{z}(t), \dot{\vec{n}}_{12}(t)> \geq 0 \tag{85}$$

Since this implies

$$\frac{d}{dt} < \vec{z}(t), \vec{n}_{12}(t)> \geq 0 \tag{86}$$

$\vec{z}(t)$ must be moving further into the interior of the sector and <u>away</u> from the switch curve.

If $\vec{q}(t)$ is in the open half space complementary to (85), then

$$\frac{d}{dt} < \vec{z}(t), \vec{n}_{12}(t)> < 0 \tag{87}$$

and $\vec{z}(t)$ is <u>approaching</u> the switch curve. If conditions can be found to quarantee that $\bar{q}(t)$ lies in one or the other of these half spaces for all time, then an important characterization of possible switches is obtained.

This involves conditions under which $\vec{q}(t)$ crosses the boundary (83) when $\vec{a}_1(t)$ is applied. Suppose (85) holds but $\vec{q}(t)$ reaches the surface (83). If

$$\frac{d}{dt} < \vec{q}(t), \vec{n}_{12}(t)> + \frac{d}{dt} < \vec{z}(t), \dot{\vec{n}}_{12}(t)> \geq 0 \tag{88}$$

$\vec{q}$ will not move off of the boundary and enter the interior of the complement (85). Hence (86) holds, and hence $\vec{a}_2(t)$ will not become optimal. Fortunately, (88) is the same as (73) for which a sufficient condition was given in corollary 9A.

Definition: The switch surface between two adjacent sensors, say 1 and 2,

is <u>closed</u> under sensor 1 if $\vec{q}(t)$ cannot leave the sector (85) when sensor 1 is being applied.

Corollary 9B: A sufficient condition for the surface between 1 and 2 to be closed is (75-78).

Proof: Above (note 75-78) do not involve either either $\vec{z}(t)$ or $\vec{q}(t)$ explicitly). □

The principal implication of this development is that if sensor 1 is activated by $\bar{z}(t)$ entering the interior of its sector, $\vec{q}(t)$ satisfies (85) at that point, and the surface between 1 and 2 is closed, then sensor 2 cannot follow sensor 1 in an optimal sequence.

Corollary 9C: A surface which is closed in both directions supports singular arcs.

Proof: Closure implies that the application of each control represented on either side of the surface will drive a $\vec{q}(t)$ on the surface into the respective interiors of the halfspaces. Hence the conditions (73,74) of theorem 9 are met and singular controls are possible. □

Thus there is an interesting relationship between closed surfaces and singular controls: while a surface closed in both directions precludes scheduling 1 before 2, or vice versa, and hence cannot be crossed, it does support singular arcs which follow the surface and branch at some point into one or the other.

Example: Consider the stationary clarification vectors

$$\vec{a}_1^T = [1 \quad 2 \quad 3]$$
$$\vec{a}_2^T = [2 \quad 3/2 \quad 1]$$
$$\vec{a}_3^T = [5/4 \quad 7/4 \quad 13/4]$$

The surface between $\vec{a}_1$ and $\vec{a}_2$ is closed, since

$$\vec{n}_{12}^{\,T} = [-1 \quad .1/2 \quad 2]$$

and (75,76) satisfied with $\gamma = 2/3$. (76) becomes

$$2/3 \begin{bmatrix} 1 & . & -1 \\ 2 & . & 1/2 \\ 3 & . & 2 \end{bmatrix} \geq \begin{bmatrix} -1 \\ 1/2 \\ 2 \end{bmatrix}$$

It is closed in the opposite direction also, since $\bar{n}_{12} = -\bar{n}_{12}$ and $\gamma = 1$ satisfies

$$\gamma\, \vec{a}_2 \cdot \vec{n}_{21} \geq \vec{n}_{21} \tag{89}$$

Routine verification shows that

$$\vec{a} = 1/3\, \vec{a}_1 + 2/3\, \vec{a}_2 \tag{90}$$

does indeed maintain a singular arc as predicted by corollary 9C. □

However, the surface between $\vec{a}_1$ and $\vec{a}_3$ is <u>not</u> closed in either direction. Note that in the stationary case, the partition of $\vec{q}$ space is independent of $\vec{z}(t)$ as the second term in (85) drops out. (85) then requires

$$q_2 = q_1 + q_3 \tag{91}$$

Applying theorem 9 at each point of this surface gives

$$< \vec{a}_1 \cdot \vec{q}\,(t),\ \vec{n}_{13} > \geq 0 \qquad \text{iff} \quad q_1 \geq q_3 \tag{92}$$

$$< \vec{a}_3 \cdot \vec{q}(t),\ \vec{n}_{13} > \leq 0 \qquad \text{iff} \quad q_1 \leq 3q_3 \tag{93}$$

Figure 4 shows a view along this switch surface; the upper arrows indicate motion towards or away from the surface when $\vec{z}$ calls for $\vec{a}_1$; while the lower do so for $\vec{a}_2$. Three regions exist: $\vec{q}$ pierces the surface regardless

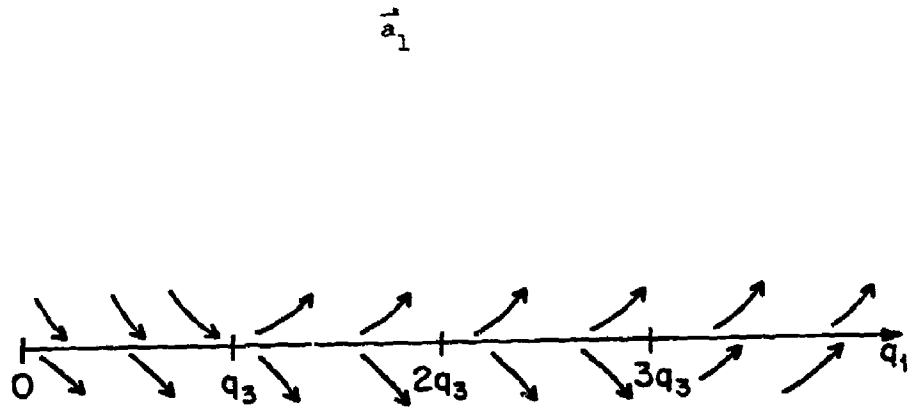
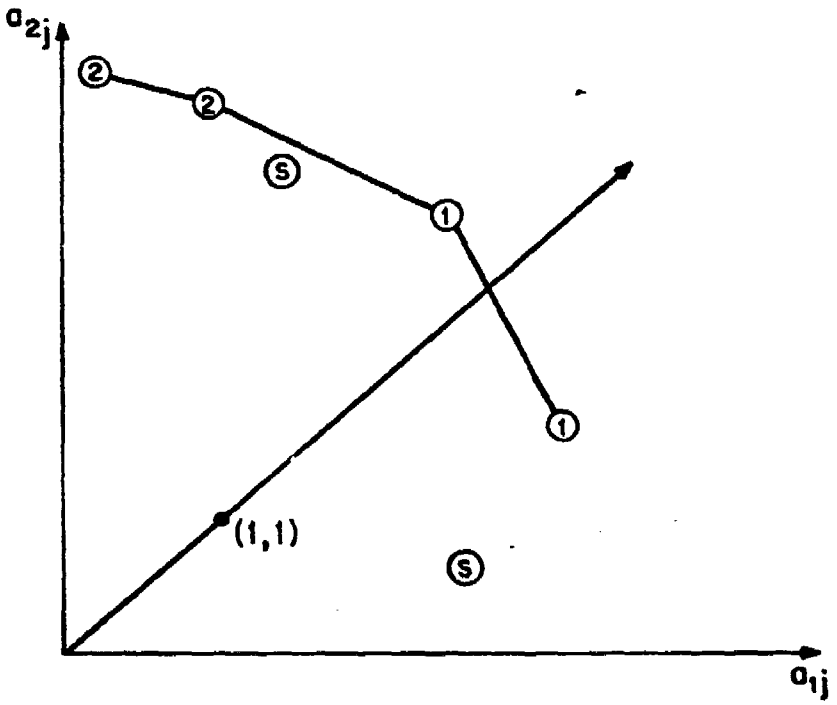


FIGURE 4

Example of Switch Surface Structure



convex hull of control polytype

1 primary sensor

2 secondary sensor

S superflous sensor

FIGURE 5

Example of Sensor Types

of $\vec{z}$ if $q_1 < q_3$; singular arcs may exist for $q_2 \leq q_1 \leq 3q_3$; and $\vec{q}$ crosses the boundary towards the (85) halfspace if $q_1 > q_3$. Any subset of these three types of behavior may occur in a boundary which fails to satisfy corollary 9A.

These results give a final classification for sensors.

Definition: A <u>primary sector</u> j of the optimal dynamics is one in which all switch surfaces with adjacent sensors are closed, for all t, from each of them into j. A <u>primary sensor</u> is one activated when $\vec{z}(t)$ lies in the corresponding primary sector. All sectors and sensors which are not primary are <u>secondary</u>.

Figure 5 shows some examples of primary, secondary, and superfluous sensors in a two dimensional case. Points representing the clarification vectors of each sensor are connected where they are adjacent.

The distinction which primary sensors bear is given by

Theorem 10: For each primary sensor j_o, if

$$\langle \vec{q}(0), \vec{a}_{j_o} \rangle < \langle \max_j \langle \vec{q}(0), \vec{a}_j \rangle \tag{94}$$

then sensor j will never appear along the resulting $\vec{z}(t), \vec{q}(t)$ trajectory.

Proof: Since $\vec{z}(0) = \vec{0}$, (85) becomes

$$\langle \vec{q}(0), \vec{n}_{jjo}(0) \rangle > 0 \tag{95}$$

when (94) holds. Since, for sufficiently small δ,

$$\langle \vec{z}(\delta), \vec{a}_j(\delta) \rangle = \delta \langle \vec{q}(0), \vec{a}_j(0) \rangle \tag{96}$$

at time δ $\vec{z}$ selects some $j \neq j_o$ as optimal <u>and</u> $\bar{q}(\delta) \approx \vec{q}(0)$ implies $\vec{q}(\delta)$ satisfies (85) strictly. Since the switch surface from each j adjacent

to j_o us closed from j to j_o, $\vec{z}(t)$ will never move into the sector corresponding to j_o and hence j_o will not be selected. □

Corollary 10A: A primary sensor j_o may be activated only if it is uniquely selected at $t = 0$, or if a singular arc is followed which lies in some switch curve on the boundary of sector j_o.

Proof: With theorem 10 eliminating j_o from consideration if not selected by $\vec{q}(0)$ in (94), these are the only remaining possibilities. □

A more powerful result can also be obtained.

Definition: A <u>primary face</u> of the control polytope is a set of mutually adjacent primary sensors.

Corollary 10B; If a schedule involves the sensors of a primary face, it does so by commencing with a singular arc mixing them. Until the terminal condition is met, the schedule will successively drop the primary sensors from the mix, and invoke a sequence of secondary sensors.[6]

Proof: By corollary 10A, if any of the sensors on the face are to be active, they must be included in a singular mix at $t = 0$. Since they are mutually adjacent, it is possible (albeit not necessary) that they all appear in the mix. Since singular arcs are confined to lie on switch surfaces, which are not dense in $\mathbb{R}^{2N}$, control must pass to exactly one primary sensor unless

[6] A graphic view of this solution in three dimensions involves imagining the control polytope rolling along a plane. The plane is one orthogonal to $\vec{z}(t)$; as $\vec{z}(t)$ evolves, the orientation of the polytope changes with respect to the plane. The plane initially includes an entire (primary) face of the polytope; as time goes on the polytope will roll off that face and only a subset of the vertices (an edge) will intersect the plane. Later, one of these will leave and only the last primary vertex will touch the plane; successive points of contact will be secondary vertices. Since the point of contact with the plane is indeed the maximal projection of the polytope on $\vec{z}(t)$, this is optimal motion.

the terminal condition happens to lie on a switch surface. Since the other sensor cannot appear intermittently (although the values of the mix variables $\vec{u}(t)$ may vary), they must drop out successively until one is left. Since no other sensors can be activated until $\vec{z}$ leaves the final primary sector and enters a secondary sector, nothing can be added to the mix until secondary sensors appear. By corollary 10A, no more primary sensors may appear. □

Conclusion: This section has addressed the optimal control problem formulated in section III from the original scheduling problem. The result is a two point boundary value problem, which in reverse time is

$$\begin{aligned} \dot{\vec{z}}(t) &= \vec{q}(t) & \vec{z}(0) &= \vec{0} \\ \dot{\vec{q}}(t) &= \vec{a}(\vec{u}^*(t)) \cdot \vec{q}(t) & \vec{q}(T) &= \vec{c} \cdot \vec{b} \end{aligned} \tag{97}$$

where

$$\vec{a}(\vec{u}^*(t)) = \sum_j \vec{a}_j(t)\, u_j^*(t) \tag{98}$$

and

$$\langle \vec{z}(t), \vec{a}(\vec{u}^*(t)\rangle \geq \langle \vec{z}(t), \vec{a}(\vec{u}(t))\rangle \tag{99}$$

for any $\vec{u}(t)$ admissible. In addition, structural characteristics of sensors, which may be easily tested, are available to constrain the set of possible sensor sequences which may be optimal. In fact, the general structure culminating in theorem 10 and its corollaries provides most of the solution for a large class of problems.

V. Computation of Optimal Schedules

Returning from the optimal control problem above to the original scheduling problem involves reinterpreting the previous results in the context originally developed in sections II and III.

Begin by returning[7] to forward time, so (97) becomes

$$\dot{\vec{z}}(t) = -\vec{q}(t) \qquad \vec{z}(T) = \vec{0} \tag{100}$$
$$\dot{\vec{q}}(t) = -\vec{a}(\vec{u}(t))\ \vec{q}(t) \qquad \vec{q}(0) = \vec{c} \cdot \vec{b}$$

as the two point boundary value problem. (98,99) are static conditions and remain unchanged. The solution $\vec{u}(t)$ specifies the fraction of effort to be devoted to each sensor at time t. This section will discuss the numerical solution of these equations using structural knowledge obtained in section IV; specialization to the cases of primary and stationary sensors yields more specific techniques.

<u>Numerical solution:</u> Solution of the equations (100) can be achieved by a number of techniques, such as an iterative strategy which refines guesses of the unknown boundary conditions at each end of the time interval [18]. While the suitability of various techniques will depend heavily on the dynamics of the $\vec{a}_i(t)$, the following procedure is suggested for those problems where these coefficients vary slowly.

1. Initial guess: $\vec{z}(t) \equiv 0, \qquad 0 \leq t \leq T$
2. Integrate the equation for $\vec{q}$ forward in time, selecting $\vec{u}(t)$ using an approximation based on the partitions induced on $\vec{q}$ space by (83)

$$\vec{u}(t) = \arg\max_{\vec{u}} \ \langle \vec{q}(t), \vec{a}(\vec{u}(t)) \rangle \tag{101}$$

   to get an initial guess of $\vec{q}(T)$, denoted $\hat{\vec{q}}(0;T)$
3. Integrate (100) forward using $\hat{\vec{q}}(k;T)$ as terminal condition $\vec{q}(T)$;

---

7 i.e., by reversing the transformation (51)

store the switch schedule. Obtain $\vec{z}(k;0)$ as an estimate of the initial condition on $\vec{z}$.

4. Integrate (100) backwards using $\vec{z}(0) = \vec{z}(k;0)$ and the schedule obtained in (3); obtain $\vec{q}(k+1;T)$ and repeat 3 and 4.

The advantages of this technique are that only the switch times (and mixes over singular arcs) need be stored from iteration to iteration, rather than either the entire $\vec{z}$ or $\vec{q}$ trajectories. For certain classes of problems it converges in one step; these are discussed below. No other general properties of this solution are known at this point.

This provides a technique to apply to complex problems with little structure; special cases with strong structure can lead to much simpler solutions.

<u>Primary Sensors</u>: The corollaries to theorem 10 shed light on the special case of primary sensors, where $\vec{z}(t)$ is forced to lie within a closed primary sector as long as $\vec{q}(t)$ satisfies the condition (85) for each sensor pair defining the sector; i.e. $\vec{q}(t)$ lies in a convex region in $\vec{q}$ space. If a pair $\vec{z}(0)$, $\vec{q}(0)$ can be found so that these satisfy all constraints

$$< \vec{z}(0, \vec{n}_{jl}(0)> \geq 0 \tag{102}$$

$$< \vec{q}(0), \vec{n}_{jl}(0) > + < \vec{z}(0), \dot{\vec{n}}_{jl}(0)> \geq 0$$

for a primary sensor j and all adjacent sensors l, we can conclude that this initial condition corresponds to the terminus of generic singular arc as described in corollary 10A (in reverse time). In particular, a non-negative $\dot{\vec{z}}(0)$ satisfying (102) without equality, for the known $\dot{\vec{q}}(0)$

guarantees that sensor i is the optimal sensor[8] to apply to time 0!

Inequalities (102-103) define 2M' linear constraints which the nonnegative $\vec{z}(0)$ must satisfy, where M' is the number of adjacent sensors. If a solution is found with one or more of these constraints satisfied with equality, then the optimal sensor selection at time 0 is mixed.

This special case immediately suggests using an open loop feed back strategy, where at each point in time the initial sensor is recomputed using (102 - 103) repeatedly. If singular controls should arise, they may be approximated by a pure choice of a sensor; receiving an observation will update the conditional probabilities, hence $\vec{c}$, and hence $\vec{q}$ which will fall on the same boundary at the next instant of time with negligible probability. This suggests a discrete time implementation, with the sensor choice held fixed over the sample period δ, and δ satisfies (45).

<u>Stationary Primary Sensors</u>: If a particular primary sensor and it adjacent sensors are stationary,

$$\dot{\vec{n}}_{j\ell}(t) = 0 \tag{103}$$

in (102) and hence $\vec{z}(0)$ does not affect the satisfaction of (102) at all. Since the primary sensor j is not superflous, a $\vec{z}(0)$ can always be found to satisfy (102); $\vec{q}$ satisfies (103) iff

$$\langle \vec{q}(0), \vec{a}_j \rangle \geq \langle \vec{q}(0), \vec{a}_1 \rangle \tag{104}$$

Thus if $\vec{a}_j$ is chosen to achieve (104) and $\bar{q}(0)$ lies in the corresponding primary sector defined by the stationary version of (85)

$$\langle \vec{q}(0), \vec{n}_{j\ell} \rangle \geq 0 \tag{105}$$

then sensor j is optimal to apply at t = 0.

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8 The question of uniqueness can not be addressed here, but it is conjectured that such an i is unique.

In particular, if all sensors are primary, the optimal openloop feedback law can be computed using only (104). Also, the entire schedule may be computed in these cases using (104) to select the sensor as $\vec{q}$ is integrated forward in time (and this corresponds to steps 1 and 2 of the numerical algorithm given above, partially justifying it as a suggested approach to solving nearly stationary problems).

The interpretation of this rule in terms of the control polytope is that $\vec{q}(0)$ represents the need for information, and one selects the sensor which best matches that need in terms of its projection on $\vec{q}(0)$.

Conclusion: The major structural component which helps determine $\vec{u}(0)$ is that of primary sensor. By exploiting the implications of a schedule which has $\vec{z}$ and $\vec{q}$ in a single sector, it becomes easy to determine the initial sensor to be selected whithout finding the entire schedule, and this greatly encourages on-line implementations.

VI. Examples

Three practical examples illustrate both the range of applications for which these results may be used, and the use of the various pieces of structure developed in section IV in solving problems.

Surveillance: Consider a one-dimensional surveillance problem, where a single physical sensor is to be pointed at a number K of discrete bins, each of which may or may not contain an object. The signal received from each bin (e.g. a radar reflection or an acoustic emission) takes on a fixed level dependent only on the state of the bin and is corrupted by Gaussian noise. The optimal pointing schedule for obtaining a clear picture of the

contents of the bins is to be determined.

Each of the basic underlying hypotheses for this problem specifies the subset of bins which are occupied by objects. The number of hypotheses grows esponentially with K, and leads to cumbersome problems unless the decomposition properties of the Cartesian obscurity function (18) are exploited. Let

$\pi_{io}(t)$ = conditional probability that bin i is occupied

$\pi_{i1}(t)$ = conditional probability that bin i is unoccupied

so

$$v_i(\vec{\pi}) = (\pi_{io}\,\pi_{i1})^{1/2} \tag{106}$$

distinguish between the states of bin i.

Pointing the sensor at bin j produces a Gaussian random variable with variance σ_j^2 and mean

m_{jo} if j occupied

m_{j1} if j unoccupied

so, from appendix C, the clarification coefficients are

$$a_{ij} = \begin{cases} 0 & i \neq j \\ \dfrac{(m_{jo} - m_{j1})^2}{\sigma_j^2} & i = j \end{cases} \tag{107}$$

It is immediate that no sensor is superfluous, and that all sensors are adjacent. (The control polytope has vertices only on each coordinate axis). Any $\gamma \geq \frac{1}{a_{jj}}$ satisfies

$$\gamma\,\vec{a}_j(\vec{a}_j - a_k) \geq \vec{a}_j - \vec{a}_k \tag{108}$$

so all sensors are primary. Since this is a stationary problem, it is optimal to select the sensor at time zero which maximizes

$$< \vec{q}(0), \vec{a}_j > = a_{jj}\, q_j(0) = a_{jj}\, b_j\, c_j \tag{109}$$

where c_j is the obscurity initially in bin j

$$c_j = (\pi_{jo}(0)\ \pi_{j1}(0))^{1/2} \tag{110}$$

The interpretation of this role is to first look at the bin with highest clarification, importance, and initial uncertainty. When all bins are equally important and have identical priors, this rule selects first that bin about which greatest clarification can be obtained, and postpones looking into bins with lowest signal to noise ratio until last.

Implementation of the open loop feedback law is quite simple in this case: maintain a list of bins ordered by

$$a_{jj}\, b_j (\pi_{jo}(t)\ \pi_{j1}(t))^{1/2} \tag{111}$$

At each time sample, point at the bin at the head of the list, receive $y_i(t)$, update π_{jo} and π_{j1}, reinsert bin j into the appropriate spot in the list, and repeat.

<u>Search</u>: Suppose there are K bins as above, with a sensor which can point at one at a time, but <u>at most one</u> bin contains a target. Now there are K+1 hypotheses, describing the location of the object as well as H_o representing the absence of the object.

The Cartesian obscurity function can no longer be used since an observation of one bin provides information which updates all probabilities π_k.

An alternative is the uniform obscurity function (13).

$$\sum_{k_1=0}^{K-1} \sum_{k_2=k_1+1}^{K} (\pi_{k_1} \pi_{k_2})^{1/2} \tag{112}$$

which separates all hypotheses equally.

Using the same sensor model as above, with

$$m_{jo} = m_o \qquad m_{j1} = m_1 \qquad \sigma_j^2 = \sigma^2 \tag{113}$$

for simplicity gives clarification coefficients

$$a(k_1\ k_2)_j = \begin{cases} \dfrac{(m_1 - m_o)^2}{\sigma^2} & \text{if } j = k, \text{ or } q \\ 0 & \text{else} \end{cases} \tag{114}$$

where $j = 1,\ldots,K$. Thus pointing the sensor at bin j affects all terms involving π_j by a factor of

$$a = \frac{(m_1 - m_o)^2}{\sigma^2} \tag{115}$$

and all others not at all.

Here again, no sensor is superfluous (since $a(0,j)_j = a$ uniquely for control j) and all are adjacent. Also a $\gamma \geq \frac{1}{a}$ ensures

$$\gamma\ a(k_1,k_2)_j\ (a(k_1,k_2)_j - a_{(k_1,k_2)\ell}) \geq (a_{(k_1,k_2)j} - a_{(k_1,k_2)\ell}) \tag{116}$$

for all ℓ, so every switch curve is closed in both directions and all sensors are primary. Again (104) gives the optimal sensor; it maximizes

$$\sum_{k_1=0}^{K-1} \sum_{k_2=k_1+1}^{K} a_{(k_1,k_2)j}\ (\pi_{k_1} \pi_{k_2})^{1/2} \tag{117}$$

or using (14)

$$a\sqrt{\pi_j} \sum_{\substack{k=0 \\ k\neq j}}^{K} \sqrt{\pi_k} \tag{118}$$

Thus the sensor is pointed at the bin which has maximum obscurity between it and all other hypotheses. Note that although v^n defines a $\frac{1}{2}k(k+1)$ dimensional space in which the clarification coefficients lie, and there are only k sensors, the primary sensor structure still proves useful.

<u>Testing</u>: An object consists of two parts, each of which can be normal or flawed. Radiation can be directed through the object in three directions (figure 6) and the attenuation of the radiation measured. After normalization the attenuation coefficient is 1 if both parts are normal, ϕ_1 if the beam passes through a flawed first part, and ϕ_2 if through a flawed second part (assume $\phi_1, \phi_2 < 1$).

While the uniform obscurity function could be used here, its six terms can be reduced to four using

$$\begin{aligned} v(\pi) &= (\pi_o\,\pi_1)^{1/2} + (\pi_o\,\pi_2)^{1/2} + (\pi_1\,\pi_3)^{1/2} + (\pi_2\,\pi_3)^{1/2} \\ &= v_1(\vec{\pi}) + v_2(\vec{\pi}) + v_3(\vec{\pi}) + v_4(\vec{\pi}) \end{aligned} \tag{119}$$

where

π_o = probability of no flaw

π_1 = probability 1 is flawed

π_2 = probability 2 is flawed

π_3 = probability both are flawed.

Assuming the observations are corrupted by zero mean, unit variance Gaussian

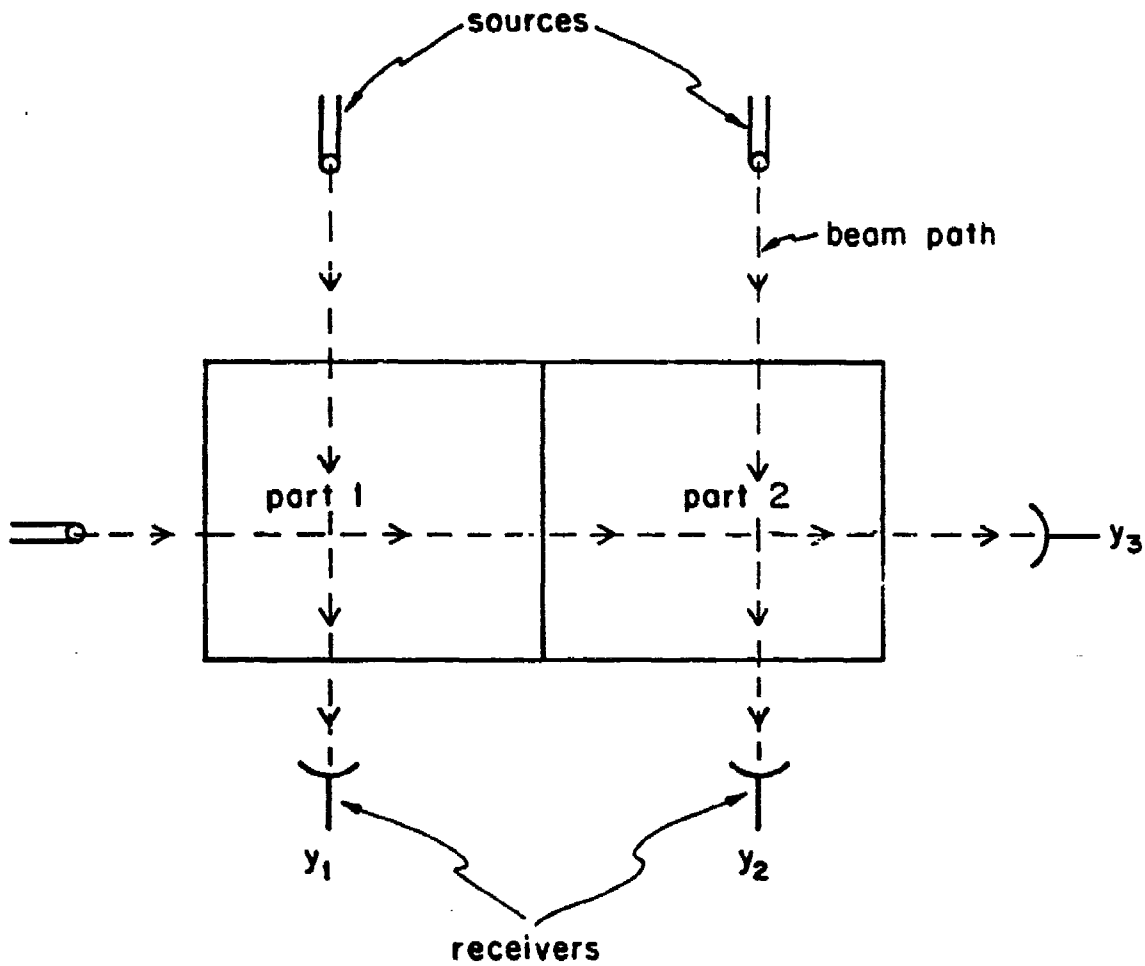

FIGURE 6

Testing a Two Part Object

sensor j

	a_{ij}	1	2	3
	v_1	$(1-\phi_1)^2$	0	$(1-\phi_1)^2$
term i	v_2	0	$(1-\phi_2)^2$	$(1-\phi_2)^2$
	v_3	$(1-\phi_1)^2$	0	$\phi_1^2(1-\phi_2)^2$
	v_4	0	$(1-\phi_2)^2$	$\phi_2^2(1-\phi_1)^2$

TABLE 1: Clarification coefficients for Testing Example

noise, and ignoring scattering and other effects, the clarification coefficients for this problem are given in Table 1.

Structurally, no sensors are superfluous; corollary 9A assures that the switch surface between 1 and 2 is closed bidirectionally, and the surface from 1 to 3 and from 2 to 3 are closed. However, only one surface from 3 to the others can be closed; if $\phi_1 > \phi_2$, the boundary from 3 to 1 fails the test (76-78).

For a specific example, let $\phi_1 = 1/2$, $\phi_2 = 1/4$. Then (76-78) requires

$$\gamma \begin{bmatrix} \frac{1}{4} & \cdot & 0 \\ 9/16 & \cdot & 9/16 \\ 9/64 & \cdot & -7/64 \\ 1/64 & \cdot & 1/64 \end{bmatrix} \geq \pm \begin{bmatrix} 0 \\ 9/16 \\ -7/64 \\ 1/64 \end{bmatrix}$$

for the surface from 3 to 1 to be closed, and no $\gamma > 0$ exists for this. However, the one from 3 to 2 is closed, so both sensors 2 and 3 are primary sensors. If $\vec{q}(0)$ lies in either of these, the respective sensor is optimal at $t = 0$; if not, it lies in the secondary sector 1. Even detailed, point-by-point analysis from theorem 9 shows $\vec{q}(t)$, hence $\vec{z}(t)$, can only cross this boundary from 3 to 1 and there can be no singular arcs.

Thus, if $\vec{q}(0)$ lies in sector 1, i.e.

$$\langle\vec{q}(0), \vec{a}_1\rangle > \langle\vec{q}(0), \vec{a}_2\rangle, \ \langle\vec{q}(0), \vec{a}_1\rangle > \langle\vec{q}(0), \vec{a}_3\rangle \tag{120}$$

the most general schedule possible is

a) use 1 alone for T_1 seconds, then

b) use 3 alone for T_2 seconds, then

c) singular arc between 1 and 3 for T_3 seconds,

where $T_1 + T_2 + T_3 = T$ and all are nonnegative. The mix necessary to sustain (c) can be found easily; solutions for T_i are found by integrating the $\vec{z}$ and $\vec{q}$ equations explicitly and using the terminal condition on $\vec{q}$, and the condition that $\vec{z}$ must cross the boundary between 1 and 2 at exactly time T_1.

Thus the existence of secondary sensors makes a problem more complex; in particular, the choice of sensor 1 or 3 in this case will depend not only on $\vec{q}(0)$, but also on T since the above equations are nonlinear in the T_i's.

VII. Summary and Future Directions

This paper has addressed the general problem of selecting a sequence of sensor observations to take in order to acquire information to test the truth of each of several hypotheses. The principal contributions are:

a) a problem formulation with a cost functional possessing both desirable qualitative properties and useful analytic structure;

b) the evaluation of the information provided by a sensor in terms of a vector of clarification coefficients,

c) classification of sensors into superfluous, primary, and secondary sensors;

d) structural considerations such as adjacency and closure which limit the set of possible sequences;

e) a numerical technique for finding a schedule in the general case, and

f) special conditions under which the first sensor in the schedule may be determined easily.

However, there are a number of extensions to be considered. These include

a) Optimal stopping: by including a "null" sensor with clarification coefficients all zero, and introducing additional penalty for the time non-null sensors are used, one can consider terminating the sequence when enough information is collected (see appendix G for a binary example related to [20]).

b) Terminal cost: the penalty function here assumes that only obscurity over the interval of interest is to be considered; an additional term penalizing obscurity left at time T would fit nicely into the framework (and result in a boundary condition on $\vec{z}$ which is not the origin).

c) Cost linear in $\vec{\pi}$: Costs of this form are also eigenfunctions of the Bayes update, with eigenvalue 1. By making the coefficients of these terms dependent on the sensor choice, however, one can model the fact that selection of a particular sensor might be undesirable if a particular hypothesis were true.

d) Sensor dynamics: The dynamic optimization problem resulting from this formulation can also be augmented with state variables describing sensor dynamics (e.g. position and velocity of a sensor platform), with controls affecting them. This would allow modelling of dynamic constraints which prohibit instantaneous switching between sensors.

e) Correlated observation processes: This formulation requires the observations to be independent increment processes when conditioned on the hypotheses. Relaxing this condition to allow them to be, say

noise filtered through a linear system where the system parameters depend on the hypothesis,would extend the scope of applications. However, sensors would be selected in this case not only to distinguish among hypotheses, but also to acquire good state estimates to aid this distinguishing.

f) Dynamic hypotheses: If the hypothesis changes over time, as would the state of a Markov chain, one could model a number of dynamic detection and identification problems. Conceptually this can be placed in the current framework by regarding each state sequence as a hypothesis with time varying observation statistics. However, the state space structure should be exploitable to reduce the complexity.

Many other opportunities exist, but those seem to be both important and relatively closely related to the results here.

APPENDIX A: NOTATION

Symbol	Quantity	Defined (eq.)
a_{ij}	clarification coefficient for term i by sensor j	(40)
$\vec{a}_j$	vector of a_{ij}	(56)
$\vec{a}$, $\vec{a}(\vec{u}(t))$	convex combination of $\vec{a}j$ selected	(50)
α_{ij}	eigenvalue of mode i under update by sensor j	(25)
$\underline{A}$	matrix of all a_{ij}	(57)
b_i	incremental cost of obscurity of form in term i	(8)
$\vec{b}$	vector of b_i's	(61)
c_i	terminal cost coefficient ($v_i(\cdot)$ at $t = 0$)	(31)
$\vec{c}$	vector of c_i's	(61)
$c(\cdot)$	general terminal cost function	(E-1)
$d(\cdot)$	general running cost function	(E-1)
δ	sample interval between schedule points	(43)
$f_i(\cdot)$	general state transit function	(E-2)
γ	nonnegative scale factor	(Thm.6)
H_k	basic hypothesis k	(1)
$\hat{H}$	maximum a posteriori estimate of H	(14)
i	index over terms in obscurity function	(8)
j	index over sensors or controls	(24)
K	number of hypotheses	(1)

k	index over hypotheses	(1)
L	number of subhypotheses	(19)
ℓ	index over subhypotheses	(15)
λ_k	Poisson rate associated with hypothesis k	(B-2)
M	number of sensors	(47)
$\vec{m}_k$	mean of Gaussian distribution associated with hypothesis k	(107) (B-4)
N	number of terms in $v(\vec{\pi})$	(8)
$\vec{n}_{ij}$	normal to switch surface between sensors i and j	(68)
$\underline{P}_k$	covariance matrix associated with hypothesis k	(B-4)
p_i	costate variable	(E-8)
$\vec{p}$	costate vector	(E-3)
p_D	probability of detection	(D-2)
p_F	probability of false alarm	(D-2)
π_k	conditional probability of hypothesis k	(1)
$\vec{\pi}$	conditional distribution on hypotheses	(8)
$\hat{\pi}$	probability of $\hat{H}$	(14)
ϕ_k	attenuation coefficient of hypothesis k	(Table
q_i	modified costate	(55) (E-
$\vec{q}$	modified costate vector	(60)
$\hat{\vec{q}}(k;T)$	k^{th} iteration on boundary condition on $\vec{q}$	(101)
r_k	coefficient in convex combination	(7) (63)
σ	standard deviation	(107)
T	terminal time of problem	(22)
t	time index (continuous and discrete)	(2)
t'	reverse time variable	(51)

u_j	fraction of time spent using sensor j	(47)
$\vec{u}$	input (control) vector	(50)
u_o	point of symmetry in detector	(D-4)
$\vec{u}^*$	optimum value of $\vec{u}$	(54)
$v(\vec{\pi})$	obscurity function	(8)
$v_i(\vec{\pi})$	i^{th} term in obscurity function	(8)
$v^c(\pi)$	Cartesian obscurity function	(18)
$v^u(\pi)$	uniform obscurity function	(13)
V	cost-to-go function	(26)
x_i	state variable, amplitude of mode of cost function	(27)
x	state vector	(E-1)
x_{io}	initial condition on x_i	(E-2)
$\vec{y}_j$	random output obtained when sensor j selected	(2)
z_i	alternate state variable	(55) (E-10)
$\vec{z}$	alternate state vector	(59)
$\hat{\bar{z}}(k;0)$	k^{th} iteration on boundary condition for $\bar{z}$	(101)

APPENDIX B: CLARIFICATION COEFFICIENTS FOR GAUSSIAN AND POISSON PROCESSES

This appendix contains explicit values for $a_{ij}(t)$ for observations which are Poisson or vector Gaussian processes. Let the two hypotheses in the obscurity function be H_1 and H_2; a_{ij} is defined as

$$a_{ij}(t) = -\ln \int_{-\infty}^{\infty} [p(\vec{y}|H_1)\ p(\vec{y}|H_2)]^{1/2} d\vec{y} \tag{B-1}$$

where this is interpreted as a Steiljes integral if the functions are of denumerable support.

<u>Poisson process</u>: If $y(t)$ is the number of events in the tth sample interval resulting from a Poisson process (such as photons striking a detector) which is characterized by rate $_k(t)$ under each hypothesis, then

$$p(y(t) = n|H_k) = \frac{(\lambda_k(t)n\ e^{-\lambda_k}}{n!} \tag{B-2}$$

(B-1) becomes

$$a_{ij}(t) = \frac{1}{2}(\sqrt{\lambda_1} - \sqrt{\lambda_2})^2 \tag{B-3}$$

<u>Gaussian process</u>: If $\bar{y}(t)$ is Gaussian with mean $\vec{M}_k(t)$ and covariance $\underline{P}_k(t)$ under each hypothesis, direct evaluation of (B-1) , completing the square in the exponent, and using a well known matrix equality [19] gives

$$a_{ij}(t) = \frac{1}{2}\ln\det\left(\frac{\underline{P}_1(t) + \underline{P}_2(t)}{2}\right) - \frac{1}{4}\ln\det\underline{P}_1(t) \tag{B-4}$$

$$- \frac{1}{4}\ln\det\underline{P}_2(t) + \frac{(\vec{m}_1(t) - \vec{m}_2(t))^T}{2}(\underline{P}_1(t) + \underline{P}_2(t))^{-1}\frac{(\vec{m}_1(t) - \vec{m}_2(t))}{2}$$

which is easily interpreted as the signal to noise ratio if $\underline{P}_1(t) = \underline{P}_2(t)$.

APPENDIX C: DIRECT DERIVATION OF CLARIFICATION COEFFICIENTS

Clarification coefficients for the continuous time processes of Appendix B can be derived directly without recourse to the definition involving an approximation to discrete time such as (51). It is restrictive in that the independence of samples over time (when conditioned on the hypothesis)(2) must be replaced with the assumption that $\vec{y}(t)$ is an independent increments process with

$$p(\vec{y}(t+\delta) - \vec{y}(t)) \tag{C-1}$$

known for all t, τ and is independent of $\vec{y}(t)$ and all other disjoint intervals of time.

<u>Poisson processes</u>: Let (C-1) be that of a Poisson counting process, so

$$p(\vec{y}(t+\delta) - y(t) = n|H_k) = \frac{(\lambda_k(t)\ \delta)^n\ e^{-\lambda_k \delta}}{n!} \tag{C-2}$$

where δ is small compared with the derivatives of $\lambda_k(t)$. The eigenvalue (25) associated with a term distinguishing H_1 from H_2 is

$$\alpha_{12}(t) = e^{-\frac{1}{2}(\sqrt{\lambda_1}(t)\ -\sqrt{\lambda_2}(t))^2\ \delta} \tag{C-3}$$

by direct computation. As $\delta \to 0$, the state equation

$$x(t+\delta) = \alpha_{12}(t)\ x(t) + b\delta \tag{C-4}$$

can be written as

$$\frac{x(t+\delta) - x(t)}{\delta} = \frac{\alpha_{12}(t) - 1}{\delta}\ x(t) + b \tag{C-5}$$

In the limit

$$\dot{x}(t) = -\frac{1}{2}(\sqrt{\lambda_1}(t) - \sqrt{\lambda_2}(t))^2\, x(t) + b \tag{C-6}$$

or indeed

$$\dot{x}(t) = -[-\ln \alpha_{12}(t)]\, x(t) + b \tag{C-7}$$

$$= -a_{12}(t)\, x(t) + b \tag{C-8}$$

<u>Gaussian processes</u>: Here the distribution (C-1) for a small interval is Gaussian with mean $\vec{m}(t)\delta$ and covariance $\underline{P}_x(t)\delta$. The eigenvalue associated with a term distinguishing H_1 and H_2 for samples at t and $t+\delta$ is

$$\alpha_{12}(t) = \left[\frac{\det^2\left(\frac{P_1(t)\delta + P_2(t)\delta}{2}\right)^{-1}}{}\right]^{1/4} \times$$

$$e^{\delta\left(\frac{\vec{m}_1(t) - \vec{m}_2(t)}{2}\right)^T \left[\underline{P}_1(t) + \underline{P}_2(t)\right]^{-1} \left(\frac{\vec{m}_1(t) - \vec{m}_2(t)}{2}\right)} \tag{C-9}$$

which, after limiting as in (C-4, C-8) gives $a_{12}(t)$ as (B-4)

APPENDIX D: APPLICATION TO OPTIMAL DETECTOR DESIGN

The problem of detecting a known signal in noise has been well studied [1]. Essentially, it is a binary hypothesis testing problem - H_1 assumes no signal present and H_2 corresponds to the existence of a signal. The detector observes the signal, if any, corrupted by noise and makes a decision as to whether H_1 or H_2 is the actual case.

There are a variety of criteria for judging detector performance: Bayes,Risk Neyman-Pearson, etc. All result in a detector which computes the posterior probabilities of H_1 and H_2 given the observed waveform, then compares their ratio to a fixed threshold. As the threshold is varied, the probability of falso alarm p_F (choosing H_2 when H_1 is true) and the probability of detection p_D (choosing H_2 when H_1 is true) vary. These criteria give formulae for setting the threshold in some optimal way.

This appendix proposes yet a third way to select the threshold as an example of the application of the binary hypothesis testing result of section IV. Suppose the detector is used for data compression - reducing its entire observed waveform to a binary choice between H_1 and H_2. This choice is to be communicated to some other point where it will be used as a basis for some decisions and hence minimum obscurity is desired. The result of section IV says that the clarification coefficient of the detector should be maximized; here, the variable which can be selected is the threshold in the detector.

Recall that maximizing the clarification coefficient is equivalent to minimizing

$$\sum_{k=1}^{z} \left(p(\hat{H}_k|H_1)\ p(\hat{H}_k|H_2)\right)^{1/2} \qquad \text{(D-1)}$$

where H is the output of the detector. In terms of p_D and p_F this is

$$(1 - p_F(u)\ (1 - p_D(u)))^{1/2} + (p_F(u)\ p_D(t))^{1/2} \tag{D-2}$$

where $p_F(u)$ and $p_D(u)$ are the false alarm and detection probabilities when u is used as the threshold. Assuming differentiability, a necessary condition for u^*, the optimum threshold is that

$$d_{p_D}/d_u \Big|_{u^*} = - \left. \frac{p_D(1 - p_D)(1 - p_F))^{1/2} - (1 - p_D)(p_D\ p_F)^{1/2}}{p_F\ (1 - p_D)(1 - p_F))^{1/2} - (1 - p_F)(p_D\ p_F)^{1/2}} \right|_{u^*} \tag{D-3}$$

Consider u_o, the t value of u for which $p_D(u) = 1 - p_F(u)$ (which exists for $p_D(\cdot)$ and $p_F(\cdot)$ are continuous.) If enough symmetry is present so that

$$\left.\frac{dp_D}{du}\right|_{u_o} = \left.\frac{dp_F}{du}\right|_{u_o} \tag{D-4}$$

then the above condition is satisfied. For the case of a signal in additive white Gaussian noise, this holds and yields, incidentally, the minimum probability of error rule.

In summary, this appendix illustrates how the techniques developed here can be applied to specification of sensor parameters as well as sensor selection. The resulting condition for setting detector thresholds is interesting in that it arises naturally from an information theoretic approach, and depends only on internal sensor characteristics without recourse to prior probabilities or to cost parameters.

APPENDIX E: DERIVATIOIN OF NECESSARY CONDITIONS

Given an optimal control problem: minimize

$$c(\vec{x}(t)) + \int_0^T d(\vec{x}(t), \vec{u}(t))dt \tag{E-1}$$

with

$$\dot{x}_1 = f_i(\vec{x}(t), \vec{u}(t), t) \qquad x_i(0) = x_{io} \qquad i=1,2,\ldots,N \tag{E-2}$$

and with fixed terminal time T and no terminal constraints on $\vec{x}$, Pontryagin principle gives the following necessary conditions.

The Hamiltonian function is defined as

$$H(\vec{x},\vec{p},\vec{u},t) = \vec{p}^T(t)\ f(\vec{x},\vec{u},t) + d(\vec{x},\vec{u},t) \tag{E-3}$$

wwhere $\vec{p}(t)$ is an N-dimensional <u>costate</u> vector. The necessary conditions to be satisfied are

1. $\dot{x}_1(t) = f_i(x^*(t), \vec{p}^*(t)\ \vec{u}^*(t), t) \qquad x_i^*(0) = x_{io}$ (E-4)

2. $\dot{p}_i^*(t) = -\frac{\partial}{\partial x_i} H(\vec{x}^*(t), \vec{p}^*(t), \vec{u}^*(t), t) \qquad \vec{p}_i^*(T) = \frac{\partial}{\partial x_i} c(\vec{x}(T))$ (E-5)

3. $H(\vec{x}^*(t), \vec{p}(t), \vec{u}(t), t) \leq H(\vec{x}^*(t), \vec{p}(t), \vec{u}(t), t)$ for all $\vec{u}(t)$ (E-6)

In the problem (52,53) these become:

$$H(\vec{x}(t), \vec{p}(t), \vec{u}(t), t) = \sum_{i=1}^{N} -p(t)\ x_i(t)\ a_i(\vec{u}(t)) + p_i(t)\ b_i$$

1. $\dot{x}_i^*(t) = -a_i(\vec{u}^*(t))\ x_i^*(t) + b_i \qquad x_i^*(0) = 0$ (E-7)

2. $\dot{p}_i(t) = a_i(\vec{u}(t)\ p_i(t) \qquad p_i^*(T) = c_i$ (E-8)

3. $\sum_{i=1}^{N} p_i^*(t)\ x_i^*(t)\ a_i(\vec{u}^*(t)) \geq \sum_{i=1}^{N} p_i^*(t)\ x_i^*(t)\ a_i\ (\vec{u}(t))$ for all $\vec{u}(t)$ (E-9)

The Hamiltonian condition (E-9) suggests the change of variable

$$z_i(t) = p_i(t)\ x_i(t) \tag{E-10}$$

which satisfies (using (E-7, E-8))

$$\dot{z}_i(t) = b_i\ p_i(t) \qquad z_i(0) = 0 \tag{E-11}$$

Define also

$$q_i(t) = b_i\ p_i(t) \tag{E-12}$$

which (E-8) gives as satisfying

$$\dot{q}_i(t) = a_i(\vec{u}^*(t))\ q_i(t) \qquad q_i(T) = b_i\ c_i \tag{E-13}$$

Finally, the Hamiltonian condition itself becomes

$$\sum_{i-1}^{N} z_i(t)\ a_i(\vec{u}^*(t) \geq \sum_{i=1}^{N} z_i(t)\ a_i(\vec{u}(t)) \quad \text{for all } \vec{u}(t) \tag{E-14}$$

(E-11, E-13, E-14) are the conditions which appear as (54-56).

Finally, the total cost is

$$\sum_{i=1}^{N} c_i\ x_i(T) = \sum_{i=1}^{N} p_i(T)\ x_i(T)$$

$$= \sum_{i=1}^{N} z_i(T) \tag{E-15}$$

APPENDIX F: PROOF OF THEOREM 9

Statement: At time t,

$$\langle \vec{z}(t), \vec{n}_{12}(t) \rangle = 0 \tag{F-1}$$

$$\langle \vec{q}(t), \vec{n}_{12}(t) \rangle + \langle \vec{z}(t), \dot{\vec{n}}_{12}(t) \rangle = 0 \tag{F-2}$$

and $\vec{z}_i(\tau)$, $\vec{q}_i(\tau)$ are the trajectories following this point when sensor i is applied, $i = 1,2$, $\tau \varepsilon [t, t+\delta]$. There are four conclusions to be drawn concerning the relationship of $\vec{z}_1(t) - \vec{z}_2(t)$ and $\vec{n}_{12}(t)$, the normal to the switch curve, as well as on the existence of singular controls.

$$\text{a)} \quad \langle \vec{z}_1(\tau), \vec{n}_{12}(\tau) \rangle \geq \langle \vec{z}_2(\tau), \vec{n}_{12}(\tau) \rangle \tag{F-3}$$

Interpretation: Application of sensor 1 moves $\vec{z}(\tau)$ toward the sector in which sensor 1 is optimal at a greater rate than does application of sensor 2.

Proof: Since $\vec{q}(t) \geq 0$ by theorem 6, for any sensors 1 and 2

$$q_i(t) \; (a_{i1}(t) - a_{i2}(t))^2 \geq 0 \tag{F-4}$$

or

$$q_i(t) \; a_{i1}(t) \; (a_{i1}(t) - a_{i2}(t)) \geq q_i(t) \; a_{i2}(t) \; (a_{i1}(t) - a_{i2}(t)) \tag{F-5}$$

Summing over i:

$$\langle \vec{a}_1(t) \cdot \vec{q}(t), \vec{n}_{12}(t) \rangle \geq \langle \vec{a}_2(t) \cdot \vec{q}(t), \vec{n}_{12}(t) \rangle \tag{F-6}$$

Adding

$$2\langle \vec{q}(t), \vec{n}_{12}(t) \rangle + \langle \vec{z}(t), \vec{n}_{12}(t) \rangle \tag{F-7}$$

to both sides gives

$$\frac{d^2}{dt^2} \langle \vec{z}_1(t), \vec{n}_{12}(t) \rangle \geq \langle \vec{z}_2(t), \vec{n}_{12}(t) \rangle \tag{F-8}$$

The condition (F-2) rewritten as

$$\frac{d}{dt} \langle z_1(t), \vec{n}_{12}(t)\rangle = \frac{d}{dt} \langle \vec{z}_2(t)\ \vec{n}_{12}(t)\rangle \tag{F-9}$$

ensures that (F - 3) holds for a short period after t.

b) $\langle \vec{z}_1(\tau), \vec{n}_{12}(\tau)\rangle \geq 0$ iff

$$\langle \vec{a}_1(t) \cdot \vec{q}(t), \vec{n}_{12}(t)\rangle + 2\langle \vec{q}(t), \dot{\vec{n}}_{12}(t)\rangle + \langle \vec{z}(t), \ddot{\vec{n}}_{12}(t)\rangle \geq 0 \tag{F-10}$$

<u>Interpretation</u>: Sensor 1 can drive $\vec{z}$ into the sector where it is solely optimal.

<u>Proof</u>: As above, (F-2) implies that at time t

$$\frac{d}{dt} \langle \vec{z}(t), \vec{n}_{12}(t)\rangle = 0 \tag{F-11}$$

The $\langle \vec{z}_1(\tau), \vec{n}_{12}(\tau)\rangle$ increases if the second derivative at t is positive; (F-10) ensures this.

c) Similar to (b), only driving $\vec{z}(t)$ into the sector where 2 is solely optimal requires the second derivative to be negative.

d) Singular controls exist iff both condition (F-10)and its companion hold.

<u>Interpretation</u>: $\vec{z}(t)$ can be maintained on the switch curve iff sensor 1 alone can move it into its sector, and sensor 2 alone can move it off the curve into the sector where $\vec{a}_2$ is optimal.

<u>Proof</u>: Singular controls require

$$\frac{d^2}{dt^2} \langle \vec{z}(t), \vec{n}_{12}(t)\rangle = 0 \tag{F-12}$$

for some

$$\vec{a}(t) = u_1(t)\ \vec{a}_1(t) + \vec{u}_2(t)\ \vec{a}_2(t) \tag{F-13}$$

with

$$u_1(t),\ u_2(t) \geq 0; \qquad u_1(t) + u_2(t) = 1 \tag{F-14}$$

Expanding (F-12) and substituting (F-13), it is necessary that

$$u_1(t)\ \gamma_1(t) - u_2(t)\ \gamma_2(t) = 0 \tag{F-15}$$

where

$$\gamma_1(t) = \langle \vec{a}_1(t) \cdot \vec{q}(t),\ \vec{n}_{12}(t) \rangle + 2\langle \vec{q}(t),\ \dot{\vec{n}}_{12}(t) \rangle + \langle z(t),\ \dot{\vec{n}}_{12}(t) \rangle \tag{F-16}$$

and

$$-\gamma_2(t) = \langle \vec{a}_2(t) \cdot \vec{q}(t),\ \vec{n}_{12}(t) \rangle + 2\langle \vec{q}(t),\ \dot{\vec{n}}_{12}(t) \rangle + \langle \vec{z}(t),\ \dot{\vec{n}}_{12}(t) \rangle \tag{F-17}$$

(F-15) has the convex sum of two scalars equal to zero; this is possible iff one is nonnegative and the other nonpositive. Since (F-8) implies

$$\gamma_1(t) \geq -\gamma_2(t) \tag{F-18}$$

$\gamma_1(t)$ must be nonnegative and $-\gamma_2(t)$ nonpositive.

<u>Computation of singular controls</u>: The mixture of sensors 1 and 2 which maintain singularity, if they exist, are given by

$$u_1(t) = \frac{\gamma_2(t)}{\gamma_1(t) + \gamma_2(t)} \tag{F-19}$$

$$u_2(t) = \frac{\gamma_1(t)}{\gamma_1(t) + \gamma_2(t)} \tag{F-20}$$

<u>Extention to several controls</u>: With M' singular controls, (F-13) is generalized to

$$\vec{a}(t) = \sum_{j=1}^{M'} u_j\ \vec{a}_j(t) \tag{F-21}$$

and (F-14) to

$$u_j \geq 0 \qquad \sum_{j=1}^{M'} u_j = 1 \tag{F-22}$$

Pairwise constraints give (F-15) as

$$u_i(t) = \frac{\gamma_j(t)}{\gamma_i(t)} u_j(t) \tag{F-23}$$

for all sensor pairs(i,j) in the set. The forms (F-23) give at most M'-1 independent equations in $\vec{u}(t)$; they give exactly M'-1 such constraints when the sensors are linearly independent. Combining these with (F-22) gives the unique control mix to maintain singularity unless the sensors are linearly dependent; in this case, all of the available solutions are dynamically equivalent.

APPENDIX G: EXAMPLE OF OPTIMAL STOPPING

The general problem formulation of this paper can be augmented to allow explicit consideration of stopping (or becoming inactive for a period of time in nonstationary problems). The objective is to capture the notion that the $\vec{\pi}(t)$ can not be expected to reach an extremal distribution in finite time. However, at some point it becomes sufficiently close to an extremal distribution that further effort is unjustified. This can be modelled by augmenting the state vector with a "clock" state keeping track of the time a sensor is in use, and penalizing its final value. The clock is turned off whenever a <u>null sensor</u> is invoked, one which provides no information on any term in the obscurity function.

As this is intended as a simple example, it will consider only a stationary binary hypothesis testing problem (one term in $v(\vec{\pi})$). The development of this problem is identical to the main work up through section III; we pick up from there.

<u>Problem statement</u>: One of M sensors may be selected at each time; sensor j causes the amplitude $x_1(t)$ of the cost-to-go function to vary as

$$\dot{x}_1(t) = -a_j\, x_1(t) + b \tag{G-1}$$

A penalty on the obscurity remaining at the end of the scheduling interval of the form

$$d(\pi_1(T)\ \pi_2(T))^{1/2} \tag{G-2}$$

would appear in the reverse-time formulation as (see Theorem 3)

$$x_1(0) = d \tag{G-3}$$

(It is necessary to penalize the residual obscurity with a $d \gg b$ or the

schedule will stop at $t = 0$)

To include the time penalty, introduce

$$\dot{x}_o = 1 \qquad x_o(0) = 0 \tag{G-4}$$

and the overall objective is then to minimize

$$c_o\, x_o(T) + c_1\, x_1(T) \tag{G-5}$$

where c_o specifies the weight given to time and c_1 is as before. Finally, introduce the "wait" sensor for which the dynamics become

$$\dot{x}_o = 0 \qquad \dot{x}_1 = 0 \tag{G-6}$$

as neither clarification nor penalty is accrued when it is used.

<u>Necessary Conditions</u>: The maximum principle yields the following necessary conditions for this problem:

Case I: Sensor $j \neq 0$ is optimal

Hamiltonian: $-a_j\, x_1(t)\, p_1(t) + b\, p_1(t) + p_o(t)$

States: $\dot{x}_o(t) = 1$

$\dot{x}_1(t) = -a_j\, x_1(t)$

Costates: $\dot{p}_o = 0$

$\dot{p}_1(t) = a_j\, p_1(t)$

Case II: Sensor 0 is optimal

Hamiltonian: 0

States: $\dot{x}_o(t) = 0$

$\dot{x}_1(t) = 0$

Costates: $\dot{p}_o(t) = 0$

$\dot{p}_1(t) = 0$

Hamiltonian: Choose j_o to achieve the minimum value.

Boundary conditions:

$$x_o(0) = 0 \qquad p_o(T) = c_o$$

$$x_1(0) = d \qquad p_1(T) - c_1$$

<u>Interpretation</u>: The Hamiltonian condition guarantees that if a non-null sensor is selected, it will be the one with greatest a_j, as before. Dropping all others, the decision for sensors is

$$0 \underset{j}{\overset{o}{\lessgtr}} -a_1\, x_1(t)\, p_1(t) + b\, p_1(t) + p_o(t) \tag{G-7}$$

Define

$$z_1(t) = a_1\, x_1(t) + b\, p_1(t) + p_o(t) \tag{G-8}$$

so

$$z_1(t) = p_o(t) \qquad \text{if sensor } j \text{ selected} \tag{G-9}$$
$$= 0 \qquad \text{else}$$

However, $p_o(t)$ is a positive constant, c_o, regardless of schedule. Hence, $z_1(t)$ is monotonically increasing, and the only candidates for optimal schedules can switch from j to 0 at most once, and never from 0 to j.

Let T_1 be the total time the sensor j is on, and T_o the time it is off. Then

$$T_o + T_1 = T \qquad z_1(T_1) = 0 \tag{G-10}$$

give conditions for finding T_1 and T_o. Integrating and substituting boundary conditions gives

$$z_1(T_1) = c_1(b - a_j d)e^{-a_j T_1} + c_o \tag{G-11}$$

which is zero when

$$e^{a_j T_1} = \frac{c_1}{c_o}(a_j d - b) \tag{G -12}$$

Implementation: This rule can be implemented in open loop feedback fashion by noting that sensor j is used at zero iff the solution for T_1 in (G-12) is positive, i.e. iff

$$\frac{c_1}{c_o}(a_j d - b) \geq 1 \tag{G-13}$$

Since

$$c_1 = (\pi_1(0)\ \pi_2(0))^{1/2} = (p(1-p))^{1/2} \leq \frac{1}{2} \tag{G-14}$$

it follows that no sensor is ever used if

$$\frac{1}{c_o(a_j d-b)} > 1/2 \tag{G-15}$$

and j is used only when

$$(p - 1/2)^2 \leq \frac{1}{4} - \frac{1}{c_o^{\ 2}(a_j d-b)^2} \tag{G-16}$$

This gives a decision rule which is fixed over time but shares the structural properties of other stopping strategies [20] in that data continues to be collected when the conditional distribution is near the center of the [0,1] interval, and ceases as it moves toward the boundaries.

References

1. H.L. Van Trees, Detection, Estimation and Modulation Theory: Part 1 Wiley, 1965.

2. M. Marcus and P. Swerling, "Sequential Detection in Radar with Multiple Resolution Elements", IRE Trans. Info. Thy., Vol. IT-8, pp. 237-245, Apr. 1962.

3. B. Gluss, "An Optimum Policy for Detecting a Fault in a Complex System", Operatins Research 7, pp. 468-477, 1959.

4. R. Bellman, H. Sugiyama, and B. Kashef, "Applications of Dynamic Programming and Scab-Rescan Processes to Nuclear Medicine and Tumor Detection", Mathematical Biosciences, Vol. 21, pp. 1-29, 1974.

5. J.M. Dobbie, "A Survey of Search Theory", Operations Research, pp. 525-537, 1968.

6. Y.C. Kan, "A Counterexample for an Optimal Search and Stop Model", Operations Research, Vol. 22, #4, pp. 889-892.

7. N. Blackman, "Prolegomena to Optimum Discrete Search Procedures", Naval Res. Log. Quart., Vol. 6, 1959, pp. 273-281.

8. S.M. Ross, "A Problem in Optimal Search and Stop", Operations Research 17, 1969, pp. 984-992.

9. A.N. Shiryaev, "On the Theory of Decision Functions and Control of a Process of Observation Based on Incomplete Information", Selected Translations in Math. Stat. and Prob., Vol.6, 1966.

10. M. Bowles, Correlation Tracking, Ph.D. Thesis, M.I.T., July 1980.

11. J.L. Doob, Stochastic Proceeses, Wiley, 1953.

12. D.P.Bertsekas, Dynamic Programming and Stochastic Control, Academic Press, 1976.

13. L-T. Fan and C-S.Wang, The Discrete Maximum Principle, Wiley, 1964.

14. H.J. Kusher, "On the Differential Equations Satisfied by the Conditional Probability Densities of Markov Processes", SIAM J. Control, Ser.A., Vol.2, No.1, pp. 106-119, 1964.

15. A.E. Bryson and Y.C. Ho, Applied Optimal Control, Blaisdell, Waltham, Mass. 1969.

16. M. Athans and P.L. Falb, Optimal Control, McGraw-Hill, 1966.

17. V.F. Krotov, "Methods for Solving Variational Problems II: Sliding Regions", Automation and Remote Control, Vol. 24, pp. 539-553, 1963.

18. L.W. Neustadt, "On Synthesizing Optimal Controls", Proc. 2nd IFAC Congress, paper 421, 1963.

19. A.H. Jazwinsky, Stochastic Processes and Applied Estimation, Academic Press, Appendix 7B.

20. A. Wald and J. Wolfowitz, "Optimum Character of the Sequential Probability Ratio Test", Annals Math. Stat., Vol. 20, pp. 165-205, 1949.