MULTIVARIATE ANALYSIS AND ITS APPLICATIONS

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MULTIVARIATE ANALYSIS AND ITS APPLICATIONS

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This report covers the work done under the contract D49620-79-C-0161 during the period of June 1, 1979 - Dec. 14, 1981. All this work is reported in various papers and the abstracts of these papers are attached. The work in these papers is supported completely or partially by the above contract.
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Inequalities concerning bivariate and multivariate distributions in statistics are surveyed, as well as historical background. Subjects treated include inequalities arising through positive and negative dependence, Boole, Bonferroni and Prachar inequalities, convex symmetric and inequalities; stochastic ordering; stochastic majorization and inequalities obtained by majorization, Chebyshev and Kolmogorov type inequalities; multivariate normal inequalities; and applications to simultaneous inference, unbiased testing and reliability theory.


Three measures of divergence between vectors in a convex set of k-dimensional real vector space have been defined in terms of certain types of entropy functions, and their convexity property studied. Among other results, a classification of the entropies of degree s is obtained by the convexity of these measures. These results have applications in information theory and biological studies.


The paper is devoted to unification of probability spaces through the introduction of a quadratic differential metric in the parameter space of the probability distributions. For this purpose, a s-entropy functional is defined on the probability space and its variation along a direction of the tangent space of the parameter space is taken as the metric. The distance between two probability distributions is computed as the goodle
distance induced by the metric. The paper also deals with three measures of divergence between probability distributions and their inter-relationships.


In an earlier work, the authors introduced a divergence measure, called the first order Jensen difference, or in short, \( \beta \)-divergence, which is based on entropy functions of degree \( \beta \). This provided a generalization of the convexity of mutual information based on Shannon's entropy (corresponding to \( \beta = 1 \)). It can be shown that the first order divergence is a convex function only when \( \beta \) is restricted to some range. In this paper we define higher order Jensen differences and show that they are convex functions only when the underlying entropy function is of degree \( \beta \). A statistical application requiring the convexity of higher order Jensen differences is investigated.


We discuss the construction of differential metrics in probability spaces through entropy functions and examine their relationship with the information metric introduced by Rao using the Fisher information metric in the statistical context of classification and discrimination, and the classical Bregman metrics. It is suggested that the scalar and Ricci curvature associated with the Fisher-information metric may yield results in statistical inference analogous to those of Bregman curvature.

A class of homogeneous polynomials $P_i(x)$ with a general number $r$ symmetric matrix arguments $x_1, \ldots, x_l$ is invariant under the orthogonal group. It is proposed through the theory of polynomial representations of the linear group. The class of polynomials is a generalization of the invariant polynomials with smaller numbers of matrix arguments developed previously or in the case of these matrix arguments. Functional properties and relations of the polynomials $P_i(x)$ are shown, and some applications in multivariate distribution theory are indicated. The complex analogue of the above results is also discussed.


We consider simultaneous confidence regions for some hypotheses on ratios of the distribution coefficients of the linear distribution function that the population mean and variance covariance matrix are unknown. This problem, involving hypotheses on ratios, yields the so-called "pseudo" confidence regions valid conditionally on subsets of the parameter space. We obtain the explicit formulae of the regions and give further discussion on the validity of these regions. Illustrations of the pseudo confidence regions are given.


In this paper, it is shown that some distributions
of the Weiss–Portion and latest results arising in the
multivariate analysis. Fisher's discriminant problem can
be explicitly expressed in terms of the invariant poly-
nomials under the calendar argument, and to D. H. Bailey,
estimating the total polynomial of matrices argument.

6) (Jahnke, R. (1968). Table of the coefficients under
Institute for Statistical and Applications, Depart-

The problem of finding confidence regions (CR) for a vector of selected values on the selection of a linear functional relationship (LFR) is investigated. Here, an LFR is selected and an error model of the linear regression model is considered to be multi-linear, nonlinear, and positive definite. The results in this section are based on the results of existing works.

For a given level of confidence \( t \), it is given by a suitable boundary, which leads to the so-called "general" confidence region (GCR) valid conditionally in subsets of the predictor space. The discussion is focused on the "honest mean" confidence region (HCR) close to the solution of a linear problem. The few conditions necessary for a GCR to exist are shown to be the conditionally consistent conditions concerning the multivariate LFR. The conditions that these conditions hold appear to have "nonsensical circumstances" in most practical situations. Hence, an only valid confidence approximation \( t \) is given. Some simulation results are presented.


This paper gives sufficient conditions that certain statistics have a common distribution under a wide class of underlying distributions. The tests and methods are the primary technical tools used to establish the theoretical results. These results are applied to MANOVA problems, problems involving canonical correlations, and certain statistics associated with the complex normal distribution.

In this paper, the author gave a review of the literature on various developments in certain aspects of multivariate distribution theory when the underlying distribution is not multivariate normal.


Let $x$ and $y$ be statistically independent non-

central Wishart matrices. Then the distribution of $x^T y^T$ is shown to be double central $k$-elliptical. In this paper, asymptotic distributions of certain functions of the
eigenvalues of $x^T y^T$ are derived. Applications of these results in the case of inference of tests and other cases are discussed.


In this paper, the authors obtained asymptotic ex-

pressions for the null and non-null distributions of the

likelihood ratio test statistic for multiple covariance

when the underlying distribution is complex multivariate

normal. The authors also derived asymptotic null distribu-
tion of the likelihood ratio test statistic for homogeneity

of the covariance matrices of several complex multivariate

normal populations. Asymptotic expressions for the

null case were also obtained for the general case when we have two populations. The

expressions obtained in this paper are in terms of cor-

relations.

In this paper, the authors give asymptotic expressions for the joint distribution of the functions of the elements of the sample covariance matrix and sample correlation matrix in the noncentral cases when the underlying distribution is multivariate normal. Accuracy of the above expressions is also studied. Also, asymptotic expressions are given for functions of the elements of the sample covariance matrix for nonnormal populations. Finally, some applications of the above results are discussed.


A discrete survival model is considered where an unobservable random variable is subjected to destruction so that what is observed and recorded is only the unobserved part of it. Assuming the destruction process is represented by the Negative-Poisson distribution, a characterization of the negative binomial distribution is obtained. Utilizing the complete is property of the negative binomial distribution, a characterization of the Negative-Poisson distribution is derived. Several other characterization theorems are also proved concerning these probability distributions.


A discrete model is considered where the original observation is subjected to partial destruction according to
the generalized Markov-Polya damage model. A character-
zation of the generalized Polya-Eggenberger distribution
is given in the context of the Rao-Rubin condition. Sev-
eral other characterization theorems are also proved con-
cerning these probability distributions.


This paper gives necessary and sufficient condition
for the null distribution of a test statistic to remain
the same in the class of left $O(n)$-invariant distributions.
Secondly, it is shown that in certain special cases, the
usual MANOVA tests are still uniformly most powerful in-
variant in a class of left $O(n)$-invariant distributions.

Kariya, T. (1981). A new concept of second order effici-
ency and its application to a missing data problem. Tech-
nical Report No. 81-16. Institute for Statistics and Ap-
plications, Department of Mathematics and Statistics,
University of Pittsburgh.

In this paper, following the framework of Kariya,
Krishnaiah and Rao (1981), a finite sample concept of
second order efficiency is defined and it is applied to
the analysis of the problem of estimating bivariate normal
parameters with extra data on the first variate. Only the
modified maximum likelihood estimator is shown to be sec-
ond order efficient.

Kariya, T., Krishnaiah, P. R. and Rao, C. R. (1981). In-
ference on the parameters of multivariate normal population
when some data is missing. In Developments in Statistics,

In this paper, the authors considered the problems of
estimation and tests of hypotheses on the parameters of the
multivariate normal population when the data is incom-
plete. Special emphasis is made for the case of the bivariate nor-
mal when the variances of the variables are equal. The
authors also obtained certain expressions for the asymptotic
distributions of a wide class of test statistics useful in
testing various hypotheses on the parameters of the multi-
variate normal when some of the data is missing.
**Kariya, T., Sinha, B. K. and Subramanyam, K. (1981).**

*Nearly efficient estimators based on order statistics.*


In this paper, based on 3 or 5 order statistics out of a order statistics, nearly asymptotically efficient estimators are obtained for Cauchy, logistic and normal distributions.

**Kariya, T., Sinha, B. K. and Subramanyam, K. (1981).**

*First, second and third order efficiencies of the estimators for a common mean.* To be given for typing.

Based on Kariya, Krishnaiah and Rao (1981) and Kariya (1981), this paper considers efficiencies of several estimators proposed in the problem of estimating a common mean of K univariate normal populations. Only the Graybill-Deal (1959) estimator is shown to be third order efficient.

**Kariya, T., Sinha, B. K. and Krishnaiah, P. R. (1981).**

*Some properties of left orthogonally invariant distributions.*


In this note, we have considered generalizations of some of the results on spherical distributions to the case of left spherical matrix variate distributions. First, it is shown that the independence of n rows of an n x p left spherical random matrix implies multivariate normality. Secondly, most results in Eaton (1981) are extended to the matrix variate case.

**Kariya, T., Sinha, B. K. and Subramanyam, K. (1981).**

*Berkson’s bioassay problem – revisited.*


Consider Berkson’s problem of estimating θ on the basis of independent random variables $X_i$ having the binomial distribution $\text{B}(n_i, p_i(θ))$ ($i = 1, \ldots, k$), where
\[ \tau(\theta) = [1 + \exp(-\theta - \beta d_1)]^{-1}, \] and \( \beta \) and \( d_1 \)'s are known. 

Extensive asymptotic and nonasymptotic comparisons of two particular estimates of \( \theta \), the MLE \( \hat{\theta}_M \) and the Rao-Blackwellized version \( \hat{\theta}_B \) of Berkson's estimator \( \hat{\theta}_B \), have been made in the literature. Our subject in this paper is three fold: (a) to derive a simple necessary condition for \( \hat{\theta}_B \) to dominate \( \hat{\theta}_M \) uniformly in the MSE criterion; (b) to propose three new estimators and compare them with \( \hat{\theta}_B \) and \( \hat{\theta}_M \); (c) to show that \( \hat{\theta}_B \) and \( \hat{\theta}_M \) are the first order efficient no matter what \( \tau(\theta) = F(\theta + \beta d_1) \) may be so long as \( F \) is strictly increasing.


Kantorovich gave an upper bound to \((x'Vx)(x'V^{-1}x)\) where \( x \) is an \( n \)-vector of unit length and \( V \) is an \( n \times n \) positive definite matrix. Bloomfield, Watson and Knott found the bound to \(|X'VX X'V^{-1}X|\), and Khatri and Rao to the trace and determinant of \( X'V Y Y'V^{-1}X \) where \( X \) and \( Y \) are \( n \times k \) matrices such that \( X'X = Y'Y = 1 \). In the present paper we establish bounds for traces and determinants of \( X'V Y Y'V^{-1}X \) and \( X'BYY'CX \) when \( X \) and \( Y \) are matrices of different orders. A review of previous results on generalizations of Kantorovich inequality and a number of new results of independent interest are also given.


This paper presents four new statistical measures of monotone relationship derived from the concept of monotone correlation. A nonlinear optimization algorithm is employed to evaluate these new measures, as well as the monotone correlation, for ordinal contingency tables. A computer program to implement the algorithm is developed, and is applied to several insightful examples to provide further understanding of the usefulness of these measures.
MONCOR is described. MONCOR computes the concordant monotone correlation, discordant monotone correlation, isoconcordant monotone correlation, isodiscordant monotone correlation and their associated monotone variables. Data input can be finite discrete bivariate probability mass functions or ordinal contingency tables, both of which must be given in matrix form. The well-known British Mobility data are used to illustrate the input and output options available in MONCOR.

Let \( N, X_1, X_2, \ldots \) be non-constant independent random variables with \( X_1, X_2, \ldots \) being identically distributed and \( N \) being nonnegative and integer valued. It is shown that the independence of \( \frac{1}{N} \sum_{i=1}^{N} X_i \) and \( N - \frac{1}{N} \sum_{i=1}^{N} X_i \) implies that the \( X_i \)'s have a Bernoulli distribution and \( N \) has a Poisson distribution. Other related characterization results are considered.

This paper reviews the present state-of-the-art on computations of functions of the roots of several random matrices which arise in multivariate statistical analysis. Many of the available tables useful in statistical analysis of multivariate data are given in the Appendix.

In this paper, the authors discussed various tests for studying the structure of interactions in two-way classification model with one observation per cell. These tests basically involve various functions of the eigenvalues of the Wishart matrix.


A general solution of the functional equation

$$\int_0^\infty f(x+y)d\mu(y) = f(x)$$

where $f$ is a nonnegative function and $\mu$ is a positive Borel measure on $[0, \infty]$ is shown to be $f(x) = p(x) \exp(\lambda x)$ where $p$ is a periodic function with every $y \in \text{supp} \mu$, the support of $\mu$, as a period. The solution is applied in characterizing Pareto, exponential and geometric distributions by properties of integrated lack of memory, record values, order statistics and conditional expectation.


A general solution of the integrated Cauchy functional equation

$$\int_0^\infty f(x+y)d\mu(y) = f(x) \text{ a.e. for } x \ (\infty, \infty)$$

where $f$ is a locally integrable positive function and $\mu$ is a positive Borel measure on $\mathbb{R}$ is shown to be

$$f(x) = p_1(x)e^{\lambda_1 x} + p_2(x)e^{\lambda_2 x} \text{ a.e.}$$

where $p_1$ and $p_2$ are positive periodic functions with every $u \in \text{supp} \mu$, the support of $\mu$, as a period. A variant of the Choquet-Deny theorem on $\mu$-harmonic functions is given.
In the present paper, certain random damage models are examined, such as the Generalized Markov-Polya and the Quasi-Binomial, in which an integer-valued random variable $N$ is reduced of $N_A$. If $N_B$ is the missing part, where $N = N_A + N_B$, the covariance between $N_A$ and $N_B$ is obtained for some general classes of distributions, such as the C.P.S.D. and M.P.S.D. for the random variable $N$. A characterization theorem is proved that under the generalized Markov-Polya damage model, the random variables $N_A$ and $N_B$ are independent if, and only if, $N$ has the Generalized Polya-Eggenberger distribution. This generalizes the corresponding result for the Quasi-Binomial damage model and the generalized Poisson distribution. Finally, some interesting identities are obtained using the independence property and the covariance formulas between the numbers $N_A$ and $N_B$.


In this paper, the role of Minimum Mean Square Error as a general criterion for estimation of parameters is critically examined. It is shown that smaller mean square error does not necessarily imply greater concentration of the estimator around the true value. The empirical Bayes method for simultaneous estimation of parameters introduced by Fisher is shown to provide a good ranking of populations for individual populations.

This paper provides a comprehensive review of the methods of estimation of variance components. Conditions for identifiability and estimability of variance components have been given. Methods of minimum variance unbiased estimation in the normal case, MINQE in the general case and ML estimation in the normal case have been discussed.


Three general methods for obtaining measures of diversity within a population and dissimilarity between populations are discussed. One is based on an intrinsic notion of dissimilarity between individuals and others make use of the concepts of entropy and discrimination. The use of a diversity measure in apportionment of diversity between and within populations is discussed.


It is shown that estimators obtained by MSE (minimizing the mean square error) may not have optimum properties with respect to other criteria such as PN (probability of nearness of the true value in the sense of Pitman) or PC (probability of concentration around the true value). In particular, a detailed study is made of estimators obtained by shrinking the minimum variance unbiased estimators to reduce the MSE. It is suggested that because of mathematical convenience and some intuitive considerations, MSE could be used as a primitive postulate to derive estimators, but their acceptability should be judged on more intrinsic criteria such as PN and PC.

Formulas for multiple, partial and canonical correlations are generally expressed in terms of the elements of the inverse covariance matrix of the variables. These are not valid when the covariance matrix is singular. Appropriate formulas using g-inverse are developed.

The results depend on some basic lemmas on the idempotent matrix \( A A^* \) where \( A A^* = A \) (g-inverse as defined by Rao, 1962) and on the spectral decomposition of a hermitian matrix with respect to a non-negative definite matrix.


Inequalities are obtained for expressions of the type \[ |X^* A Y Y^* A^* X|, |X^* A Y Y^* B Y^* B^* X|, |X^* D X^*|/|X^* B X^*|, \]
\[ |X^* C^2 X|/|X^* B X^*| \] etc., for variations in matrices \( X, Y \) given matrices \( A, B, C \) when \( X \) and \( Y \) are vectors, we have the Kantorovich inequality. The inequalities when \( X \) and \( Y \) are matrices have applications in determining the efficiency of estimators of several parameters in linear models.


The paper refers to the expanding frontier of knowledge in statistics and current controversies, and outlines the role that the International Statistical Institute can play in directing future research to make statistics a socially meaningful and viable science. Suggestions are made about the training of statisticians with
a proper blend of theoretical knowledge and skill in
applications, development of statistical courses for
specialists in other disciplines, the role of govern-
ment statisticians, and the use of computers in sta-
tistical research. Some examples are given to high-
light the difficulties involved in defining the ef-
ciciency of an estimator and the possible dangers in
the statistical use of methods developed by academic
statisticians in practical work.

in separated families - two parameter case. Technical
Report No. 81-32, Institute for Statistics and Ap-
plications, Department of Mathematics and Statistics,
University of Pittsburgh.

In an earlier paper Cheek and Subramanyam (1975)
which will be referred to GS (1975) studies some pro-
portions of the ale (maximum likelihood estimator) of
a discrete parameter. An expression for the asymptotic
risk of the ale was obtained in the case of a single
discrete parameter. But in most practical cases one
has one discrete and one or more continuous parameter.
For more examples refer to GS (1975) and Cox (1962).
In such a case an expression for the asymptotic risk
of the ale of the discrete parameter is given in sec-
tion 2. Two examples are discussed. The first example
is that considered by Cox (1962), deciding between two
distributions, Poisson or geometric. Our asymptotic
theory is different from that of Cox. The second ex-
ample is B(n,p), both n and p unknown.

Third order efficiency of the maximum likelihood es-
timator in the multinomial distribution. Technical
Report No. 81-21, Institute for Statistics and Ap-
plications, Department of Mathematics and Statistics,
University of Pittsburgh.

Consider a multinomial population with k cells.
Let \( P_1, \ldots, P_k \) be the population proportions
and \( p_1, \ldots, p_k \) be the sample proportions out of k
sample of size $n$. We restrict our attention to the class of estimators $T_n$ obtained as the solution of the equation:

$$0 = \left( \frac{\partial^2}{\partial \theta^2} \right) E_n(T_n).$$

We show that a c.o. condition for $T_n$ to be second order efficient (SCE) is that $E_n''(1) \neq 0$. In the subclass of SCE estimators let $T_n$ be any estimator such that the corresponding $E_n''(1) \neq 0$. It is proved that there exist bias-adjusted versions of $E_n', T_n$, and $E_n''$ of $E_n$, each with a smaller mean squared error up to $o(n^{-1/2})$ than $E_n$. Here $E_n'$ and $T_n$ have the same bias up to $O(n^{-1/2})$. This generalizes the SCE property of the aln proved earlier by Rao (1981).


The Gini-Simpson index of diversity of a multinomial distribution defined by a vector of probabilities $p = (p_1, \ldots, p_k)$ is $1 - \sum p_i p_j = 1 - p^T p$. In this paper, this index is characterized by a set of postulates based on the concept of distance between distributions. Further, it is suggested that in practical applications involving measures of qualitative variation it is more appropriate to use an index of the type $p^T A p$ where $A$ is a $k \times k$ matrix of assigned distances between attributes characterizing the cells of the multinomial distribution. The Gini-Simpson index corresponds to a special choice of $A$.

The expression $p^T A p$ may also be interpreted as a generalized quadratic entropy. An example is worked out illustrating analysis of diversity and cluster analysis based on the measure $p^T A p$.
A general method of constructing a measure of diversity within a population subject to some postulates is described. It is shown that such a diversity measure, when it satisfies a concavity condition, can be used to analyze diversity in a mixture of populations as due to main effects and interactions of factors, by the levels of which the individual populations are specified. The method is applicable to both quantitative and categorical response data, and provides a generalization of the technique of analysis of variance. Some applications are considered.

The paper reviews some methods of optimizing functions of vectors and matrices subject to some restrictions and develops techniques for solving them without using the calculus of matrix derivatives. Their application to a number of statistical problems in linear estimation and multivariate analysis is illustrated.

Two general methods of obtaining measures of diversity within a population are discussed. One is based on an intrinsic notion of dissimilarity between individuals and the other makes use of the concept of entropy. Some examples are given of the decomposition of diversity within a population in terms of given or conceptual
Analysis for assessment of diversity in a
hierarchically classified set of populations is dis-
cussed.

The concept of analysis of diversity as a gener-
alization of analysis of variance is developed for pop-
ulations classified by combinations of different levels
of choice factors.

SASSOON, A. G. (1980). Nonnegative Cholesky Decomposition
and its application to association of random variables.
Stat. and Biometric Methods 1700-1701.

The concept of a multivariate family of distribu-
tions indexed by a covariance scale parameter \( t \) in
formally defined and examples are given. The multivariate
normal is one such family. Sufficient conditions are
given so that a positive definite matrix has a nonnegative
Cholesky decomposition. These conditions also yield the
association of random variables with a covariance scale
parameter distribution. These results are related to
other results and to Mallows and Preece's stronger
conditions (Statistical Theory of Reliability and Life
Testing: Probability Models, Holt, Rinehart, Winston,
New York, 1975), for the association of the multivariate
normal, nearly \( \sum_{i=1}^{n} t_{i} x_{i} = \sum_{i=1}^{n} t_{i} x_{i} \).

of elliptically symmetric distributions. Accepted sub-
ject to revision in Journal of Multivariate Analysis.

Let \( x_{1}, \ldots, x_{p} \) have p.d.f. \( g(x_{1}, \ldots, x_{p}) \). It is
shown that (a) \( x_{1}, \ldots, x_{p} \) are positively lower orthant
dependent if and only if, \( x_{1}, \ldots, x_{p} \) are i.i.d. \( U(0, a^{2}) \); and (b) the p.d.f.\nof \( |x_{1}|, \ldots, |x_{p}| \) in TP\( _{2} \) if and only if, \( \log g(x) \) is convex. Let \( x_{1}, x_{2} \) have p.d.f. \( f(x_{1}, x_{2}) = \frac{1}{2} \left( f_{1}(x_{1}, x_{2}) + f_{2}(x_{1}, x_{2}) \right) \). Necessary and sufficient con-
ditions are given for \( f(x_{1}, x_{2}) \) to be TP\( _{2} \) for fixed cor-
relation \( \rho \). It is shown that if \( f \) is TP\( _{2} \) for all \( \rho > 0 \),
then \( (x_{1}, x_{2}) \) \( \rightarrow \) \( N(0, I) \). Related positive dependence
results and applications are also considered.


A brief review of the historical background and certain known results concerning the multivariate correlation ratio are given. A multivariate correlation ratio of a random vector $Y$ upon a random vector $X$ is defined by

$$
\rho_X(Y) = \frac{\text{cov}(Y, X)}{\text{var}(X)}
$$

where $A$ is a given positive definite matrix. The properties of $\rho_X$ are discussed, with particular attention paid to a "correlation-maximizing" property. A number of examples include the multivariate normal, the elliptically symmetric distributions, and the multilinear. The problem of correlating $\rho_X$ over suitable matrices $X$ is considered and the results that are obtained are related to canonical correlations for the multivariate normal.

(15) Simpson, A. and Smith, B. (1980). Assessing risks through the determination of rare event probabilities. Accepted subject to revision in *Bernoulli*.

We consider the problem in risk assessment of evaluating the probability of occurrence of rare, but potentially catastrophic, events. The lack of historical data due to the short novelty of the event makes conventional statistical approaches inappropriate. The problem is compounded by the complex multivariate dependencies that may exist across potential event sites. In order to evaluate the likelihood of one or more such catastrophic events occurring, we provide an information theoretic model for merging a decision maker's opinion with expert judgment. Also provided is a methodology for the reconciliation of conflicting expert judgments. This merging approach is invariant to the decision maker's viewpoint in the limiting case of exceptionally rare events. These methods are applied to certain case studies.
This article reviews univariate and multivariate stochastic approximation procedures from the Robbins-Monro and Steifel-Moore procedures to early derive stochastic approximation procedures. Subsections of these procedures are also described. Applications of stochastic approximation procedures are also considered.

The problem of estimating the common mean \( \mu \) of two univariate normal populations with unknown and unequal variances is considered from a decision-theoretic point of view. We restrict our attention to an appropriate class of and the three estimators \( C_1, C_2, C_3 \) of unbiased estimates of \( \mu \). It is proved that the usual estimator of \( \mu \), which is the weighted linear combination of the sample means with weights as reciprocals of the sample variances, is admissible in \( C_2 \) and extended admissible in \( C_3 \) but is neither Bayes nor limiting Bayes in \( C_3 \). Admissible estimates in \( C_1 \) and \( C_3 \) are obtained. The losses are always assumed to be squared error.

Let \( x_1, x_2, \ldots \) be independent \( p \)-variate normal vectors with \( E x_i = \mu \), \( \mu = 0, 1, 2, \ldots \) and some \( p \times p \) dispersion matrix \( \Sigma \). Here 0; \( p \times q \) and \( q \) are unknown parameters and \( y_1, y_2 \) are known \( q \times 1 \) vectors. Writing \( 0 = (0; 0) \), \( 1 = (1; 0) \), \( 2 = (2; 0) \), we have constructed invariant confidence sequences for \( \mu \). This uses the basic ideas of Robbins (1951) and generalizes some of his and Lai's (1976) results. In the process
alternative simpler solutions of some of Khan's results (1978) are obtained.