The Lower Bounds on the Additive Complexity of Bilinear Problems in Terms of Some Algebraic Quantities

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Abstract. The lower bounds on the additive complexity of a bilinear problem are expressed in terms of the rank of the problem and also as a minimum number of elementary steps for the transformation of the identity matrix into a strongly regular one.

Key Words. Additive complexity, bilinear algorithms, tensor rank.

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As is well known, the basic part of the theory of algebraic computational complexity had been shaped by 1966; cf. [1,2,3]. In particular, until very recently the lower bounds on the additive complexity, $C(\pm)$, of intensively studied linear and bilinear arithmetic algorithms for arithmetic computational problems (such as DFT and matrix and polynomial multiplication, MM, PM) have relied on the active operation-basic substitution argument due to [1,2,3]; cf. also [4]. Consequently, those bounds have not exceeded $D$, the dimension of the problems that is the total number of input variables and outputs. In the present paper we consider another algebraic approach that generalizes the ingenious method of [5]. This enables us to reduce the problem to estimating the ranks of multidimensional tensors that we associate with the given computational problems. The successful solution of a similar problem in [6] gives some ground for optimism in the attempts to establish nonlinear lower bounds on $C(\pm)$ along this line. We also present another direction to attack the problem which reduces it to the study of a strong regularity of matrices; see Definition 2 and the Theorem below.

Notation. $I, J, K$ are positive integers. $v_h = (V)_h$, $\mu_{js} = (\mu)_{js}$ are the entry $h$ of a vector $V$ and the entry $(j, s)$ of a matrix $\mu$, respectively. $F$ is a field of constants. $X$ is a vector of indeterminates, $x_i$, $i = 0, 1, \ldots, I - 1$. $L(X, F)$ is the set of all homogeneous linear forms of $x_0, \ldots, x_{I-1}$ with the coefficients from $F$.

Any $K \times J$ matrix, $\mu = \mu(X)$ with the entries from $L(X, F)$ defines a bilinear arithmetic problem that is the set of bilinear forms $\{b_k(X, Y)\}$ whose $Y$-coefficients form the matrix $\mu(X)$; cf. [7,8]. A bilinear arithmetic algorithm, $A$, that solves such a problem can be represented as a chain of matrices $(\mu(0), \mu(1), \ldots, \mu(C))$ (cf. [5,7,8]) such that $\mu(0)$ is the $J \times J$ identity matrix, $\mu$ is a submatrix of $\mu(C)$, each $\mu(q)$ is a $J \times (J + q)$ matrix such that

$$\mu(q + 1) = (\mu(q) | \ Y(q + 1)) \quad \text{for } q = 0, 1, \ldots, C - 1, \tag{1}$$

where for all $j$ either

$$(Y(q + 1))_{js} = L(q)(\mu(q))_{js} \quad \text{for some } s = s(q) \leq q + J \tag{2}$$

or

$$(Y(q + 1))_{js} = (\mu(q))_{jp} + \delta(\mu(q))_{js} \quad \text{for some } p = p(q) \leq q + J, \quad s = s(q) \leq q + J. \tag{3}$$

In (3), $\delta = 1$ or $\delta = -1$. In (3) either $L(q) \in F$ or otherwise: $L(q) \in L(X, F)$ and $(\mu(q))_{js} \in F$ for $s = s(q)$ and for all $j$. $C_A(\pm)$ designates the number of $q$ such that (3) holds.

Definition 1 (cf. [9,10]). Given $P(X)$, a homogeneous polynomial in $x_0, \ldots, x_{I-1}$ of degree $d$, then $r(P(X))$, the rank of $P(X)$, is the minimum integer $r \geq 0$ such that

$$P(X) = \sum_{s=1}^{r} \prod_{h=1}^{d} r_{ph}(X), \quad L_{ph}(X) \in L(X, F).$$

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Let $D = D(M(X))$ designate the set of all minors of a matrix $M(X)$ with the entries from $L(X,F)$, $r(M) = \max_{m \in D} r(m)$. (We say that $r(M)$ is the rank of the bilinear problem associated with the matrix $M(X)$.) Then the next lemma is easily verified.

**Lemma 1.** Equation (2) implies that $r(\mu(q+1)) = r(\mu(q))$, Equation (3) implies that $r(\mu(q+1)) \leq 2r(\mu(q))$.

**Corollary.** Given a bilinear algorithm, $A$ (cf. (1)-(3)), for the bilinear problem defined by a matrix $\mu = \mu(X)$, then $C_A(\pm) \geq \log_2 r(\mu)$.

Hence, $\log r(\mu) = \Omega(n \log n)$ for the general $n \times n$ Toeplitz matrix $(J = K = n, \mu_{jk} = x_{j-k+n-1}, j, k = 0, 1, \ldots, n - 1)$ would imply nonlinear lower bounds on the complexity of PM and DFT.

**Remark.** If $P_n(X) = P_n(X_1, \ldots, X_n)$ is an $n$-linear form in $n$ vectors of indeterminates, $X_1, \ldots, X_n$, then the polylinear rank, $R_P(X)$, can be defined as the minimum integer $R$ such that

$$P_n(X) = \sum_{g=1}^{R} \prod_{h=1}^{n} L_{gh}(X_h), \quad L_{gh}(X_h) \in L(X_h,F).$$

As is obvious, $R(P_n(X)) \geq r(P_n(X))$. $R(P_2(X_1, X_2))$ equals the "usual" rank of the matrix of coefficients of the bilinear form $P_2(X_1, X_2)$. $R(P_3(X_1, X_2, X_3))$ equals the multiplicative complexity of the three bilinear computational problems associated with $P_3(X_1, X_2, X_3)$, (cf. [11,12]). If $\mu(X)$ is an $n \times n$ matrix with row-vectors of indeterminates $X_1, X_2, \ldots, X_n$, then $\log_2 R(per \mu(X)) \leq n$ (cf. [13]). Because of the latter estimate the inequality $\log_2 r(M) > n$ seems to be either false or very hard to prove even in the case of a general $n \times n$ matrix $\mu$.

Despite the latter remark, we hope that the reader will be challenged to look for a better modification of the above approach and for new methods for establishing lower bounds on $C(\pm)$. Here is another example of natural approaches to this problem.

**Definition 2.** A matrix is strongly regular if it contains no singular submatrix. Given a $J \times s$ matrix $\mu$ and a field $F$ then the elementary additive augmentation step consists of adding a new column-vector to $\mu$ which is a linear combination with the coefficients from $F$ of two columns of $\mu$. $C_{\pm}(J)$, the regularisation number of order $J$ is the minimum number of elementary additive augmentation steps required to transform the $J \times J$ identity matrix into a matrix that has a strongly regular $J \times J$ submatrix.

**Theorem.** Let $Y$ be the $J$-dimensional vector of indeterminates, $\mu$ be a $J \times J$ matrix over $F$ that has a strongly regular $s \times s$ submatrix. Then the additive complexity of the evaluation of $\mu Y$ is at least $C_{\pm}(s)$.

In particular, the general Toeplitz matrices are strongly regular. Hence any nonlinear lower bound on $C_{\pm}(s)$ would imply a nonlinear lower bound on $C(\pm)$ in the cases of PM and DFT.

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