INFINITE EXCESSIVE AND INVARIANT MEASURES

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1. Formulation of Results

1.1. In the paper [9] the following problem was considered. Given a contraction semigroup $T_t$ on a Borel space $D$ and an excessive measure $\nu$, when is it possible to find another contraction semigroup $\tilde{T}_t$ such that $\tilde{T}_t \geq T_t$ and $\nu$ is invariant with respect to $\tilde{T}_t$? The most restrictive condition under which this problem was solved is the finiteness of the excessive measure $\nu$. This condition excludes such an interesting case as the semigroup $T_t$ generated by the transition function of Wiener's process and the Lebesgue measure $\nu$. In the present paper we extend the results of [9] to all quasi-finite null-excessive measures $\nu$.

Definition. Let $T_t$ be a semigroup. A measure $\nu$ is called null-excessive with respect to $T_t$ if for each $\Gamma \subset D$, subject to $\nu(\Gamma) < \infty$

$$\nu_{T_t}(\Gamma) + 0 \text{ as } t \to \infty.$$ 

An excessive measure $\nu$ is called quasi-finite with respect to $T_t$ if for some $s > 0$ the difference between $\nu$ and $\nu_{T_t}$ is a finite measure.

The principal part of the proof of the main result is the same as that of [9]. We consider the transition function $p$ which generate $T_t$, then we construct a stationary Markov process $(w(s), P)$ with the transition function $p$ and the one-dimensional distribution $\nu$. (Actually the process $w(\cdot)$ has random

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birth and death times and the measure $P$ is infinite.) We add a single point $V$ to the space $\mathcal{D}$ and we look for a stationary Markov process $(x_t, \tilde{P})$ with the state space $\mathcal{E} = \mathcal{D} \cup V$ such that

1.1.a. The birth time of $x_t$ is equal to $-\infty$ and the death time of $x_t$ is equal to $\infty$.

1.1.b. The one-dimensional distribution of $\tilde{P}$ is equal to $\nu$.

1.1.c. $p(t, x, \Gamma) = \tilde{P}_x \{ x_t \in \Gamma; \forall s < t \}$.

A process $(x_t, \tilde{P})$ satisfying 1.1.a-1.1.c is called a covering process for $(w(s), P)$ (see [8] for a more detailed discussion). If the measure $\nu$ is infinite then so has to be $\tilde{P}$, and we cannot apply the results of [8] for the construction of $(x_t, \tilde{P})$. In order to extend the results of [9] to infinite measures $\nu$ we have to develop the whole theory anew. Accordingly, all the definitions and notations of [7] and [8] will be used without explicit mentioning.

In the second part of this section we give precise formulations of the main results and give the conditions under which they are proved. In Section 2 we prove the existence of $(0, \Pi)$-generated random set $M$ for any measure $\Pi$ which is the Levi measure of an increasing process with independent increments. (In [7] such sets were constructed only for $\Pi$ having the first moment.) Here the most important tool is the theorem of B. Maisonneuve in [6], which enables us to find an invariant distribution for the "jump process" of the process with independent increments. Using this result, we prove the existence of a covering process for any stationary Markov process with a quasi-finite one-dimensional distribution.
(Section 3). Section 4 is devoted to the construction of a semigroup $T_t$ with respect to which $v$ is invariant.

In the case when the proof is similar to the one given in [7], [8], or [9], we shall only outline it, without going into details.

As always the same letter is used for a measure and the integral with respect to this measure. The word "function" stands for a nonnegative bounded measurable function.

1.2. Let $D$ be a Borel space and $T_t$, $t \geq 0$, be a semigroup in the Banach space of bounded measurable functions on $D$ (we say for brevity that $T_t$ is a semigroup on $D$). The semigroup $T_t$ is called a positivity preserving normal contraction semigroup if

1.2.A. For any $t \geq 0$ and each function $g \geq 0$

$$T_t g \geq 0.$$  

1.2.B. For each $x \in D$

$$T_t 1(x) < 1,$$ and $\lim_{t \to 0} T_t 1(x) = 1$.

1.2.C. If $f(x_0) = 0$ then $T_0 f(x_0) = 0$.

A semigroup $T_t$ is called continuous if

1.2.D. For each $x \in D$

$$T_t 1(x)$$ is a continuous function of $t > 0$.

A positivity preserving normal contraction semigroup is denoted $S$-semigroup. If $S$-semigroup $T_t$ satisfies 1.2.E below, then $T_t$ is called
dying or SD-semigroup; if $T_t$ satisfies 1.2.E', then $T_t$ is called conservative or SC-semigroup.

1.2.E. For each $x \in D$

$$\lim_{t \to +\infty} T^t(x) = 0.$$

1.2.E'. $T^1 \equiv 1$ for each $t > 0$.

(Note that 1.2.E' implies 1.2.D).

If $T_t$ and $T_{t'}$ are two semigroups on $D$ and for each function $g$

(1.2.1) \[ T_{t'} g \leq T_t g, \]

then we say that $T_{t'}$ is larger than $T_t$, or $T_{t'}$ is an enhancing of $T_t$.

We write $T_t = T_{t'}$ a.e.$\mu$ if for any function $g$ for $\mu$-almost all $x$ $T_t g(x) = T_{t'} g(x)$.

In this paper we are going to prove the following theorems.

**Theorem 1.** Given a continuous SD-semigroup $T_t$ and a quasi-finite null-excessive measure $\nu$, one can find a SC-semigroup $\tilde{T}_t$ which is larger than $T_t$ and for which $\nu$ is invariant.

**Theorem 2.** If $T_t$ and $\nu$ satisfy the conditions of Theorem 1 and in addition $\nu$ is an extreme excessive measure then $\tilde{T}_t$ is unique up to the measure $\nu$.

2. **Regenerative Sets with Infinite Underlying Measures**

2.1. Let $(\Omega, F)$ be a measurable space and $Q$ be a measure on $F$ (not necessarily finite). A subset $M \subset T \times \Omega$ is called a random
set (r.s.) if it is \( \mathcal{B} \times \mathcal{F} \)-measurable and \( M(\omega) \) is nonempty for a.e. \( \omega \).

(Here \( T \) is the real line \( ]-m,\infty[ \) and \( \mathcal{B} \) is its Borel \( \sigma \)-field.)

A r.s. \( M \) is called closed (closed from the right, perfect, discrete, etc.) if for a.e. \( \omega \), \( M(\omega) \) is closed (closed from the right, perfect, discrete, etc.). Only closed random sets will be considered in the sequel.

We refer the reader to [7] for the definitions of the associated random process \( z_t \), sets \( M_t, M_{\leq}, M_{>}, M^t, \) etc.; the definitions of regenerativity, translation invariancy, as well as the definitions of \( (a,\Pi) \)-processes, \( (a,\Pi) \)-generated set.

A r.s. \( M \) is said to have a \( \sigma \)-finite distribution (or \( M \) is a \( \sigma \)-f-set) if

2.1.A The process \( z_t \) has \( \sigma \)-finite one-dimensional distributions.

For example, consider any increasing process with independent increments with the Lebesque initial distribution (i.e. initial distribution uniform on \( T \)). The range of this process is a r.s. whose distribution is not \( \sigma \)-finite. Let us take now any \( \sigma \)-finite measure \( \nu \) with support on \([0,1]\) and let \( \Pi \) be a unit measure concentrated in the point 1.

If we consider the range of the \((0,\Pi)\)-process with initial distribution \( \nu \), then this r.s. has a \( \sigma \)-finite distribution.

Any measure \( \Pi \) on \( ]0,\infty[ \) subject to

\[
(2.1.1) \quad \int_0^\infty x^\wedge \Pi(dx) < \infty
\]

may be considered as the Levi measure of an increasing process with independent increments (subordinator), and any subordinator has the Levi measure
satisfying (2.1.1). The range of any subordinator is a right regenerative set; all translation invariant sets of such type with finite underlying distributions are described in [7]. These sets are in one-to-one correspondence with the ranges of all \((a,\mathbb{N})\)-processes with \(\mathbb{N}\) having the first moment. It is possible to perform a similar analysis for all r.r.t.i. of-sets, but we restrict our attention only to the theorem of existence.

**Theorem 2.1.1.** For any \(a > 0\) and any measure \(\mathbb{N}\) on \([0,\infty[\) subject to (2.1.1) there exists a t.i. \((a,\mathbb{N})\)-generated of-set \(M\). The set \(M\) is left regenerative and moreover, \(-M\) has the same distribution as \(M\).

Let the complement of \(M\) be the union of disjoint open intervals \([\gamma,\delta[\). Then for any function \(f\) on \(T \times T\)

\[
Q\{f(\gamma,\delta)\} = \int_{-\infty}^{\infty} f(s, s + y)\mathbb{N}(dy)ds .
\]

2.2 For simplicity of calculations we shall consider only the case of \(a = 0\). The modification of the proof for \(a > 0\) is trivial. Let \(y_t\) be a \((0,\mathbb{N})\)-process and \(Q_y\) be its transition probabilities. We denote by \(\sigma^t\) the first hitting time of \([\gamma,\delta[\) by \(y_t\); and by \(Y^t = (U^t, V^t) = (\sigma^t - \gamma, \sigma^t - \delta)\) we denote the "jump" process of \(y_t\) (see [7] Section 2). \(V^t\) as well as \(Y^t\) is a Markov process. Let

\[
q(s, x; t, \Gamma) = \begin{cases} l_T(x) & \text{if } x \geq t \\ Q_x(V_t \in \Gamma) & \text{if } x < t , \Gamma \subset T 
\end{cases}
\]

be the transition function of the process \(V_t\). Let \(\Pi(x_-, \pi_\bullet), \lambda_\bullet\), etc. be the measures and the kernels defined in Section 2 of [7]. Denote
By the theorem of Maisonneuve (see [6], Th. (3.2)) the family $\mu_t$ is an entrance law with respect to $\nu$. Note that $\mu_t(\Gamma) = \mu_0(\Gamma - t)$. Consider the Markov process $(v_t, Q)$ with the one-dimensional distributions $v_t$ and with the transition function $\nu$. (The measure $Q$ is finite iff $\mu_0$ is a finite measure.) The existence of such a Markov process is proved in [5]. The same way as in Lemma 6.2 of [7], we can show that $v_t$ is a stochastically continuous increasing process; hence there exists a right-continuous version of it. Consider the random set $M$ which is the range of $v_t$ (i.e. the closure of the set of values of $v_t$). We are going to prove that $M$ is the set we are looking for.

**Lemma 2.2.1.** The set $M$ is a translation invariant right-regenerative $(0,1)$-generated set with the associated process $z_t$ having the one-dimensional distributions

\[
(2.2.1) \quad v_t(\Gamma) = \int_\Gamma x dx, \quad \Gamma \subset \mathbb{R}^t.
\]

**Proof:** Fix $s \in T$. Consider a $(0,1)$-process $y_s$ with initial distribution $\mu_s$. Let $V_s = y_s$. By the construction of $(v_t, Q)$ the process $v_t$, $t \geq s$ has the same finite-dimensional distributions as $V_t$, $t \geq s$. Both processes are right-continuous, therefore their ranges have equal distributions. But the range of $V_s$ is equal to that of $y_s$, ...
and that proves that M is \((0,\mathbb{N})\)-generated (right-regenerativity is a consequence of this fact).

By the construction, the process \(v_t - t\) is Markov with stationary transition function and stationary one-dimensional distributions (equal to \(\mu_0\)). Hence M is a t.i. set. Any \((0,\mathbb{N})\)-generated set is thin; as a result, for the t.i. set M we have

\[
Q(t \in M) = 0.
\]

Since M is \((0,\mathbb{N})\)-generated

\[
Q(z^+_s \in \Gamma | z_s^+ = x) = Q_z(z^+_s \in \Gamma),
\]

a.e. Q on the set \(\{z^+_s < t\} \), \(\Gamma \subset \mathbb{R}_t \times \mathbb{R}^t\).

To prove (2.2.1) we can consider only bounded sets \(\Gamma\) of the form \(\Lambda_1 \times \Lambda_2\). By virtue of (2.2.2), \(Q(z_t = (t,t)) = 0\); consequently we may take \(\Lambda_1 < t\) and \(\Lambda_2 > t\). Since \(\Lambda_1\) is bounded there exists \(s\) such that \(\Lambda_1 > s\). The distribution of \(z^+_s\) is equal to that of \(v_s\), and we can write

\[
Q(z_t \in \Gamma) = \int_{\mathbb{R}_s^+} \mu_s(dx)Q(z_t \in \Gamma | z_s^+ = x)
= \int_{\mathbb{R}_s^+} \mu_s(dx)Q(Y_t \in \Gamma).
\]

The last equality in (2.2.3) due to the fact that \(\{z_s^+ < t\} = \{z_t^- > s\}\). By virtue of Lemma 2.1 of [7] the right hand side of (2.2.3) is equal to...
Let $y_t^* = -y_t$ and let $Q_t^*, \lambda_t^*, \Pi^*(x; \Gamma), v_t^*$, etc. be defined as in Section 6 of [7]. Performing the same transformations as in Lemma 6.6 of [7], we get that (2.2.4) equals

$$
(2.2.5) \quad \int_{\Delta_2}^{\Delta_1} \int_{\lambda_t^*(dy; R_s)}^{\lambda_t^*(dy)} \Pi^*(y; R_s) \, dx.
$$

By virtue of Lemma 2.1 of [7]

$$
\int_{s}^{x} \lambda_t^*(dy; R_s) = Q_t^*[y^*_t; \sigma_s^* < s] = 1,
$$

here $\sigma_s^*$ is the first hitting time of $[0, s]$. Hence (2.2.5) is equal to (we use (6.11) of [7])

$$
(2.2.6) \quad \int_{\Delta_2}^{\Delta_1} \int_{\lambda_t^*(dx; \Delta_1 \times \Delta_2)}^{\Delta_1} \Pi_t^*(x; \Delta_1 \times \Delta_2) \, dx.
$$

**Corollary:** The distribution of $-M$ is equal to that of $M$.

**Proof:** Since $M$ is a $(0, \Pi)$-generated set, $z_t$ is a Markov process with the transition function $p$ given by (5.2) of [7]. As a result, $v_t$ is an entrance law with respect to $p$. Let $v_t^*$ be defined as in Lemma 6.5 of [7]. Formula (2.2.6) shows that $v_t^* = v_{-t}$. Repeating the proof of Lemma 6.6 in our case, we get that $z_t$ has backward transition function $p^*$. Then we must argue in the same way as in Lemma 6.7 of [7].
Lemma 2.2.2. The set $M$ satisfies (2.1.2).

Proof: In (2.1.2) we may consider only the functions $f$ such that

\[ f(x,y) = 0 \text{ if } x > y. \]  

Put $R_{st} = \{(x,y): x < s, y > t\}$, $f_{st} = f_{R_{st}}^1$. For $\Lambda = r_1, r_2, \ldots, r_k$, set $R_\Lambda = R_{r_1} \cup R_{r_2} \cup \ldots \cup R_{r_k}$, $f_\Lambda = f_{R_\Lambda}^1$. If $r_1, r_2, \ldots, r_k \ldots$ is a sequence of all rational numbers and $\Lambda(n) = \{r_1, \ldots, r_n\}$, then $f_\Lambda(n) + f$ for any function $f$ subject to (2.2.7). Trivial computations show that the function $f_\Lambda(n)$ is a linear combination of the functions $f_{st}$, $s < t$. Since both sides of (2.1.2) are stable under linear operations and monotone passage to the limit, we have to verify (2.1.2) only for the functions $f_{st}$, $s < t$.

\[
Q[\sum_{\gamma} f_{st}(\gamma, \delta)] = Q[\sum_{\gamma} f_{st}(\gamma, \delta) 1_{\delta > t}]
\]

\[
= Q(f_{st}(z_t))
\]

\[
= v_t(f_{st})
\]

\[
= \int_{-\infty}^{t} \Pi (f_{st}) dx
\]

\[
= \int_{-\infty}^{\infty} \Pi (f_{st}) dx
\]

\[
= \int_{-\infty}^{\infty} f_{st} (x, x + y) \Pi(dy)
\]

3.1. Consider a (generalized) stationary Markov process \((x_t, \overline{P})\), that is a process satisfying the definition given in Section 1.2 of [8]. Assume that \(x_t\) is conservative, i.e. \(\overline{P}(a \not\rightarrow) = \overline{P}(\beta \not\leftarrow) = 0\). Suppose that the state space \(E\) of this process is divided into two sets \(D\) and \(V\) in such a way that

\[(3.1.1)\quad M = \{t: x_t \in V\}\] is closed a.e. \(\overline{P}\).

We denote by \(]\gamma, \delta[\) an element of the set of all open intervals contiguous to \(M\). For each path \(x_\cdot\) and each \(]\gamma, \delta[\) we associate a trajectory \(w_\delta^\gamma\) in \(D\) by the formula \(w_\delta^\gamma(s) = x_s, \gamma < s < \delta\). The set of all trajectories in \(D\) with random birth time \(\alpha\) and death time \(\beta\) is denoted by \(W\). If \(M\) satisfies 1.2.a of [8] then it is possible to define a measure \(P\) on \(W\) in the following way (\(W\) is endowed with the Kolmogorov \(\sigma\)-field \(G\)).

\[(3.1.2)\quad P(A) = \overline{P}\{\bigcup_{\gamma} w_\delta^\gamma \cap A\} .\]

The process \((w(s), P)\) is called a subprocess in \(D\) of the process \((x_t, \overline{P})\). Let \(\nu, \overline{\nu}, \overline{\nu}_x\) be respectively the one-dimensional distribution, the transition function and the transition probabilities of the process \((x_t, \overline{P})\). The formula (1.2.2) of [8] shows that \(P\) is a Markov measure with the transition function \(\overline{p}\) defined by 1.1.\(\gamma\). If the measure \(\nu\) is \(\sigma\)-finite, then so is \(P\), and if for each \(t\)

\[(3.1.3)\quad \overline{P}(x_t \in V) = 0 ,\]
then the one-dimensional distribution of $P$ is equal to that of $\bar{P}$ (namely to $v$). In the sequel we shall consider only processes $(x_t,\bar{P})$ subject to (3.1.3). Put $\tau_{\bar{s}} = \inf\{t > s: x_t \in V\}; \tau = \tau_0$. If

3.1.A. For each $x \in D$

$$\bar{P}_x(t > t) \to 0 \text{ as } t \to \infty,$$

then for each $x \in D$

(3.1.4) $p(t, x; D) \to 0 \text{ as } t \to \infty.$

If

3.1.B. For any set $\Gamma \subset D$ such that $v(\Gamma) < \infty$

$$\bar{P}(x_s \in \Gamma, \tau > s) \to 0 \text{ as } s \to \infty,$$

then for any set $\Gamma$ such that $P\{w(0) \in \Gamma\} < \infty$

(3.1.5) $P\{w(s) \in \Gamma, \alpha < 0, \beta > s\} \to 0 \text{ as } s \to \infty.$

If

3.1.C. For some $s > 0$

$$\bar{P}(\tau < s) < \infty$$

then

(3.1.6) $P(\alpha \leq 0, 0 < \beta < s) < \infty.$
Let $T_t$ be the semigroup generated by the transition function $p$. Note that (3.1.4) is true iff $T_t$ is a SD-semigroup. The condition (3.1.5) holds iff $\nu$ is null-excessive measure; (3.1.6) is true iff $\nu$ is quasi-finite excessive with respect to $T_t$ measure. If both (3.1.5) and (3.1.6) are satisfied then we say that the process $(w(s),P)$ has a null-quasi-finite one-dimensional distribution.

Let $\Omega$ be the sample space of the process $(x_t,\overline{F})$ and $F$ be the basic $\sigma$-field in $\Omega$ on which the measure $\overline{F}$ is defined, and which is supposed to contain all sets of $\overline{F}$-measure zero. Denote by $F_s$ the completion with respect to $\overline{F}$ of $\sigma(x_u,u<s)$ and by $C_s$ the completion with respect to $\overline{F}$ of the $\sigma$-field generated by the sets

$\{\tau_u < r\}, \; u,r < s$

(If the process $x_t$ is regular, then $C_s \subset F_s$.) We say that the set $D$ is regular for $(x_t,\overline{F})$ if for $t > s$, $C_s \subset F_s$ and $x_t$ are conditionally independent given $x_s$. (This definition certainly assumes $C_s \subset F$).

A Markov process $(x_t^1,Q_1)$ with the state space $E_1 = D \cup V_1$ and a Markov process $(x_t^2,Q_2)$ with the state space $E_2 = D \cup V_2$ are said to be equivalent, if the one-dimensional distributions of both processes are concentrated on $D$ and their finite-dimensional distributions coincide.

The following theorems are similar to Theorems 1 and 2 in [8].

**Theorem 3.1.1.** Let $(w(s),P)$ be a stationary Markov process in the state space $D$ with the transition function $p$, subject to (3.1.4). If the one-dimensional distribution of $P$ is null-quasi-finite, then this process is a subprocess of a conservative stationary Markov process $(x_t,\overline{F})$ satisfying 3.1.A - 3.1.C for which $D$ is a regular set.
Just as in [8] the set of all stationary Markov measures with transition function \( p \) is denoted by \( S(p) \).

**Theorem 3.1.2.** If \((w(s),P)\) satisfies the conditions of Theorem 3.1.1 and if in addition \( P \) is a minimal element of \( S(p) \), then the process \((x_t,\bar{P})\) is unique up to equivalence.

3.2. In this section we prove Theorem 3.1.1. Consider the one-dimensional distribution \( v \) of \((w(s),\bar{P})\). It was proved in [3] that

\[
(3.2.1) \quad v = \int_0^s v^s ds,
\]

where \( v^s \) is an entrance law for \( p \). We denote by \( P^* \) a Markov measure on \( G \) with the transition function \( p \) and the one-dimensional distributions \( v^s \). Put

\[
(3.2.2) \quad \Pi(r) = P^*(\beta \in r).
\]

Suppose that the process \((x_t,\bar{P})\) is constructed and \( M \) is defined by \((3.1.1)\). The same heuristic arguments as in Section 3.1 of [8] show that the set \( M \) must be translation invariant \((0,\Pi)\)-generated and all the cuts \( w^s_\delta \) must be conditionally independent, when \( M \) is fixed.

The next three lemmas prove that \( \Pi \), defined by \((3.2.2)\), satisfies \((2.1.1)\).

**Lemma 3.2.1.** For any \( u > 0 \) the measure \( v - vT_u \) is finite.

**Proof:** By our assumptions \( \mu = v - vT_s \) is a finite measure for some \( s > 0 \). For each \( r > 0 \) \( vT_r < \infty \), and for \( u = ks \) we have
Each summand in the right side of (3.2.3) is a finite measure; and so is $v - v_{T_k}s$.

By virtue of (3.2.1)

$$v - v_{T_u} = \int_0^u v \, dt.$$  

Hence if $u < ks$, then $v - v_{T_u} < v - v_{T_k}s$, and the lemma is proved.

Lemma 3.2.1 shows that $v^s(D)$ is finite for $m$-almost all $s > 0$ 
(m is the Lebesgue measure). On the other hand for $t > s$

$v^s(D) = v^t(1) = v^{sT_{t-s}}s v^s(1) = v^s(D)$.

Therefore $v^s(D)$ is finite for all $s > 0$ and is a decreasing function of $s$. Consequently

$$P^s(\beta > s) = P^s(\omega(s) \in D) = v^s(D) < \infty, \quad s > 0.$$  

Formula (3.2.5) shows that the restriction on any interval $[s, \sigma]$ of the measure $P$, defined by (3.2.2), is a finite measure; as a result, $P$ is $\sigma$-finite.

**Lemma 3.2.2.** The measure $P$ defined by (3.2.2) satisfies (2.1.1).

**Proof:** Put $f(\omega) = v^s(D)$.  

(3.2.6) \[ \int_0^1 x \Lambda 1\Pi(dx) = \int_0^1 x\Pi(dx) + f(1) \]
\[ = \int_C dx\Pi(dy) + f(1) \]

where \( C = \{(x,y): x > 0, y > 0, x + y < 1\} \). By Fubini's Theorem (3.2.6) equals

\[ \frac{1}{y} \int_0^1 \{\int_0^{\Pi(dy)} dx + f(1)\} = \int_0^1 (\Pi(R^1) - \Pi(R^2))dx + f(1) \]
\[ = \int_0^1 (f(x) - f(1))dx + f(1) \]
\[ = \int_0^1 f(x)dx = \frac{1}{y} \nu^X(D) \]

By virtue of (3.2.4) the right side of the above formula is equal to \((\nu - \nu T_1)(D)\). Lemma 3.2.1 implies that this expression is finite.

By virtue of Theorem 2.1.1 and Lemma 3.2.2 we are able to construct a \((0,\Pi)\)-generated translation invariant set \( M \), subject to (2.1.2).

Let \( \tilde{\Omega} \) be the corresponding sample space and \( Q \) be the corresponding measure.

**Lemma 3.2.3.** For any function \( f \) on \( T \times T \)

(3.2.7) \[ P(f(\alpha, \beta)) = Q(\sum f(\gamma, \delta)) \]

**Proof:** Let \( P^* \) be defined by (3.2.1) of [8]. By formula (3.2.2) of [8]
By virtue of (2.1.2), the right side of (3.2.8) is equal to the right side of (3.2.7).

Consider a measure \( N \) on \( T \times T \times W \) defined below.

\[
N(T \times \Delta \times A) = P\{a \in \Gamma, \beta \in \Delta, w \in A\}, \quad \Gamma, \Delta \subset T, A \in G.
\]

Put

\[
(3.2.9) \quad N(B) = \bar{N}(B \times W), \quad B \subset T \times T.
\]

**Lemma 3.2.4.** The measure \( N \), defined by (3.2.9) is \( \sigma \)-finite.

**Proof:** If \( \Pi \) satisfies (2.1.1), then for \( t > 0 \)

\[
\Pi(R^+ < \infty).
\]

The support of measure \( N \) is the set \( C = \{(x,y): y > x\} \). The set \( C \) may be represented as a countable union of rectangles \( R = [u,v] \times [r,q] \), where \( u < v < r < q \). For such rectangle \( R \)
Further steps in the construction of \((x_t, P)\) do not differ from the analogous ones in [8]. We take the stochastic \(N\)-quasi kernel \(m(x,y; A)\) which is a Radon-Nikodym derivative of \(\bar{N}(dx \times dy \times A)\) with respect to \(N(dx \times dy)\). Then we define a sequence of stochastic \(Q\)-quasi kernels \(n_k(\omega; A)\) in the same way as it was done in Lemma 3.3.2 of [8]. We put \(\Omega = \bar{\Omega} \times W^\infty\) and define \(\tilde{P}\) on \(\Omega\) by the formula (3.3.3) of [8] (it is necessary only to replace \(\bar{P}\) in the right side of (3.3.3) by \(Q\)). To justify the existence of such a measure \(\tilde{P}\), we use Theorem 3.3.1 of [8], which is true for \(\sigma\)-finite measure \(Q\) as well. We take \(E = D \cup V\), where \(V\) is a singleton, and put \(x_t(\omega) = x_t(\omega, v_1, v_2, \ldots) = v\) if \(t \in M(\bar{\omega})\) and we put \(x_t(\omega) = v_k(t)(t)\) otherwise (see the end of Section 3 of [8] for details).

**Lemma 3.2.5.** The process \((x_t, \tilde{P})\) is a conservative stationary Markov process. The subprocess in \(D\) of \((x_t, \tilde{P})\) is equal to \((v(s), P)\).
To prove Lemma 3.2.5 we have to repeat without variations all the arguments of Section 4 of [8].

**Lemma 3.2.6.** The set $D$ is a regular set for $(x_t, P)$.

**Proof:** Let $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k, s_1, s_2, \ldots, s_n < s < t$. We need to show that for each $\Gamma, \Gamma_l, \ldots, \Gamma_n, \ell \in \mathbb{E}$ there exists a function $g$ on $E$ such that

$$\mathbb{P}\{x_s \in \Gamma, x_{s_1} \in \Gamma_1, \ldots, x_{s_n} \in \Gamma_n, x_t \in \Delta, \tau_{u_1} < v_1, \ldots, \tau_{u_n} < v_n\}$$

$$= \mathbb{P}\{g(x_s); x_s \in \Gamma, x_{s_1} \in \Gamma_1, \ldots, x_{s_n} \in \Gamma_n, \tau_{u_1} < v_1, \ldots, \tau_{u_n} < v_n\}.$$

For simplicity of calculations we consider only the case of $n = k = 1$, $u < v < s_1$. Since the one-dimensional distributions of $P$ are concentrated on $D$ we may consider only the case in which $\Gamma, \Gamma_1$ and $\Delta$ are subsets of $D$. Put

$$D(s,t) = \{w \in W: \alpha(w) < s < t < \beta(w)\} ,$$

$$E(s,t) = \{w \in W: \alpha(w) < s < \beta(w) < t\} ,$$

$$A = \{w \in W: w(s_1) \in \Gamma_1\} , \quad B = \{w \in W: w(s) \in \Gamma\} ,$$

$$C = \{w \in W: w(t) \in \Delta\} .$$

Denote by $\lambda_1(s,t)$ the indicator of the set $\{y_1 < s < t < \delta_1\}$, by $\delta_1(s,t)$ the indicator of the set $\{y_1 < s < \delta_1 < t\}$, and by $w_i$ the cut off $w_{\delta_1}^{y_1}$, $i = 1, 2, \ldots$. 
(3.2.11) \[ P\{x_{s_1} \in \Gamma_1, x_s \in \Gamma, x_t \in \Delta, \tau_u < \tau_v\} \]

\[ = P\{ \sum_{y_1 < y_2} \delta_1(y_1, \delta_1) \lambda_2(s_1, t)_{ABC}(w_2) + \sum_{y_1 < y_2 < y_3} \delta_1(y_1, \delta_1) \lambda_2(s_1, s_1)_{AB}(w_2) \lambda_3(s, t)_{BC}(w_3) \lambda_4(t, t)_{C}(w_4) \}

\[ + \sum_{y_1 < y_2 < y_3} \delta_1(y_1, \delta_1) \lambda_2(s_1, s_1)_{AB}(w_2) \lambda_3(s, t)_{BC}(w_3) \lambda_4(t, t)_{C}(w_4) \}

The first term in the right hand side of (3.2.11) is equal to

(3.2.12) \[ Q\{ \sum_{y_1 < y_2} m(y_1, \delta_1; E(u, v)) m(y_2, \delta_2; D(s_1, t)AB) \}

\[ = Q\{ \sum_{y} m(y, \delta; D(s_1, t)ABC) \phi(\gamma) \}

\[ = P(\phi(a); ABCD(s_1, t))

\[ = P(\phi(a)p(t - s; w(s), A); AB, a < s_1 < s < \beta) , \]

where

\[ \phi(x) = Q^* \sum_{x_t \in J} m(y_t, y_{t-}; E(u, v)) \] .

(The first equality in (3.2.12) is due to Lemma 6.8 in [7], the second to Lemma 4.1.1 in [8], the last equality is due to the Markov property of \((w(s), P)\).) Similarly, we get that the sum of the second, the third and the fourth term in (3.2.11) equals
(3.2.13) \[ P(\phi(a)\psi(w(s))); AB, a < s_1 < s < \beta \]
\[ + P(\phi_1(a)p(t - s; w(s), \Delta); B, a < s < \beta) \]
\[ + P(\phi_1(a)\psi(w(s))); B, a < s < \beta \]

where

\[ \phi_1(x) = Q^*(x, s, m(y; w(t - s), \Delta; E(u, \nu))) \]

\[ \psi(x) = P_x(\beta \in \Omega)Q_y(\sum_{r \in J, r < Z} m(y_{r-1}; \nu) \Delta(y, y_{r-1}; E(u, \nu))) \]

(Compare to (4.1.5)-(4.1.8) in [8]). Adding (3.2.13) to (3.2.12), we get

(3.2.15) \[ g(x) = p(t - s, x; \Delta) + \psi(x) \]

Lemma 3.2.7. The transition function of \((x, \overline{p})\) is

(3.2.16) \[ \overline{p}(u, x; \Delta) = p(u, x; \Delta) + \int_0^\infty \{ \sum_{r \in J, r < Z} m(y_{r-1}; \nu) \Delta(y, y_{r-1}; E(u, \nu)) \} du \]

Proof: The Kolmogorov-Chapman equation for \(\overline{p}\) was verified in Section 2 of [8].

Putting \(t - w = u\) in (3.2.15), one can see that for \(v\)-a.e. \(x\)

\[ p(u, x; \Delta) = g(x) \]

and for the proof of (3.2.16), it is enough to verify equality between \(\psi(x)\), given by (3.2.14), and the second term in the right hand side of (3.2.16) for \(\nu\)-almost all \(x\). Put

\[ \theta(y) = Q_y(\sum_{r \in J} m(y_{r-1}; \nu) \Delta(y, y_{r-1}; E(u, \nu)) \]

\[ y \in T \]
Applying successively the Markov property of \((w(s), P)\), Lemma 4.1.1 of [8] and Lemma 6.8 in [7], we get

\[
(3.2.17) \quad \int \psi(x) \nu(dx) = P(l_p(w(s)))
\]

\[
= P(l_p(w(p)), \theta(s))
\]

\[
= Q(\sum_{\gamma} m(\gamma, \delta; w(s) \in \Gamma) \theta(\delta))
\]

\[
= Q(\sum_{\gamma} m(\gamma, \delta; w(u) \in \Delta) \theta'(\gamma))
\]

where

\[
\theta'(y) = Q(\sum_{\gamma \in G} m(y, y_{\gamma}; w(s) \in \Gamma))
\]

In view of Lemma 4.1.1 in [8], (3.2.17) equals

\[
(3.2.18) \quad P(\theta'(a) l_\Delta(w(u))) = P(\theta'(a) \xi(a); \beta > u)
\]

where

\[
\xi(y) = \frac{1}{y < \alpha} Q(\sum_{\gamma < \delta} m(\gamma, \delta; w(u) \in \Delta) \xi'(\delta))
\]

Applying Lemma 4.1.1 of [8], Lemma 6.8 of [7] and again Lemma 4.1.1 of [8], we get that (3.2.18) equals

\[
(3.2.19) \quad Q(\sum_{\gamma < \delta} \theta'(y) \xi(y)) = Q(\sum_{\gamma < \delta} m(\gamma, \delta; w(s) \in \Gamma) \xi(\gamma, \delta) \lambda_2(u, u))
\]

\[
= Q(\sum_{\gamma} m(\gamma, \delta; w(s) \in \Gamma) \theta'(\delta))
\]

\[
= P(l_p(w(s))) \theta'(\beta)
\]
where

\[ \xi'(y) = q_y \left( \sum_{\gamma < u < \delta} \xi(\gamma) \right). \]

By virtue of Lemma 2.1 in [7]

\[ \xi'(y) = q_y \left\{ \int_0^{\gamma_t} u \xi_t(\gamma_t; R^u) dt \right\} \]

\[ = q_y \int_0^{\gamma_t} \mathbb{P}^* \{ B > u \} dt \]

\[ = q_y \int_0^{\gamma_t} \mathbb{P}^* \{ w(u) \in \Delta \} dt. \]

Substituting the expression for \( \xi'(y) \) in (3.2.19), we see that for any set \( \Gamma \)

\[ (3.2.20) \]

\[ \int_{\Gamma} \psi(x) \nu(dx) = \int_{\Gamma_1} \psi_1(x) \nu(dx), \]

where \( \psi_1(x) \) is the second term in the right hand side of (3.2.16). Formula (3.2.20) implies

\[ \psi(x) = \psi_1(x) \text{ a.e. } \nu. \]

3.3. The proof of Theorem 3.1.2 does not differ from the proof of Theorem 2 in [8]. Lemma 5.2.1 in [8] is true in our case as well.

If \( \mathbb{P}^*(W) < \infty \), then it is necessary to repeat the proofs of Lemmas 5.3.1-5.3.4 in order to arrive to the expression (5.3.10) in [8] for the two-dimensional distributions of the process \((x_t, \mathbb{F})\).
If $P^*(W) = \infty$, then we must consider the local time $\xi_t$ of $(x_t, F)$ at $V$. We have to introduce a filtration $A_t = C_t V F_t$, with respect to which the local time $\xi_t$ is adapted. Repeating Lemmas 5.4.1-5.4.3 and 5.5.1-5.5.2 in our case, we obtain the expression (5.5.3), which is true in the case of infinite underlying distribution $P$ as well. (The proofs of the above lemmas were based on the lemmas and theorems of Chapters 4 and 5 in [7]. The whole theory in [7] was developed under the assumption that the underlying measure is a probability one. Nevertheless everything remains the same in the case of infinite underlying distribution.) Then we have to consider the process $y_s$, which is the right-continuous inverse of the local time $\xi_t$. Repeating the proofs of Lemmas 5.6.1-5.6.3, we get that $y_t - y_s$ and $A_{y_s}$ are independent; therefore $(y_s, F)$ is the process with independent increments, whose Levi's measure can be obtained through $P^*$. Lemmas 5.7.1 and 5.7.2 of [8] are also true in the case of infinite underlying distribution, and they show that the two-dimensional distributions of $(x_t, F)$ are uniquely determined by the measure $P$.

4. Enhancing of Semigroups

4.1. Now we consider the semigroup $T_t$ and the measure $\nu$ described in Theorem 1.

If $T_t$ is a $S$-semigroup then there exists a transition function $p(t,x;\mathcal{F})$ such that

$$T_tf(x) = p(t,x;f)$$

(see [4], Theorem 2.1). If $T_t$ is a dying semigroup then

$$p(t,x;\mathcal{D}) \to 0 \text{ as } t \to \infty.$$
By the theorem of Kuznecov (see [5]) there exists a stationary Markov process \((w(s), P)\) with random birth and death times whose one-dimensional distribution is equal to \(v\) and transition function is equal to \(p\). (To construct such a process \((w(s), P)\) we need also to specify an excessive function \(h\), but in our case \(h(x) \equiv 1\).) The conditions of Theorem 1 imply that the one-dimensional distribution of \(P\) is null-quasi-finite. By Theorem 3.1.1 there exists a covering process \((x_t, \tilde{P})\) with the state space \(E = D \cup V\).

Let \(\tilde{p}(t,x; D)\) be the transition function of \(\tilde{P}\). The same way as in [9] we can show that the condition 1.2.D implies

\[(\text{4.1.1}) \quad \tilde{p}(t,x; D) = 1 \quad \text{for all } t \text{ and } x \in D .\]

Therefore \(\tilde{p}(t,x; V) \equiv 0\) and \(\tilde{p}(t,x; -)\), considered as a kernel from \(D\) into \(D\), is a transition function. By Theorem 3.1.1 \((x_t, \tilde{P})\) is a stationary conservative process with the one-dimensional distribution \(v\); consequently, \(v\) is invariant with respect to \(\tilde{p}\). The semigroup \(\tilde{T}_t\) generated by \(\tilde{P}\) is the semigroup we are looking for. The properties 1.2.A-1.2.C are automatically satisfied by any semigroup generated by a transition function. The property 1.2.E' is a consequence of (4.1.1); and (1.2.1) follows from the fact that \((x_t, \tilde{P})\) is a covering process for \((w(s), P)\), for which 1.1.\(\gamma\) holds. Lemma 3.2.7 gives us the explicit expression for \(\tilde{T}_t f(x)\) in terms of "internal" characteristics of \(v\) and \(T_t\) (we make trivial transformations in (3.2.16) to obtain the formula below).

\[(\text{4.1.2}) \quad \tilde{T}_t f(x) = T_t f(x) + \int_0^t \int_0^{y_s(f)} \nu_x(dy) .\]
where \( \mu_x \) is a measure on \([0, \infty)\) such that \( \mu[r, u] = T_r \mu(0) - T_u \mu(0) \), the family \( \nu^x \) is an entrance law with respect to \( T_t \) for which (3.2.1) holds, and \((y_s, Q_0)\) is an increasing process with independent increments with translation constant \( 0 \) and the Levi measure \( \Pi \) such that

\[
\Pi[r, u] = \nu^F(D) - \nu^u(D).
\]

It is interesting to compare the formula (4.1.2) with the result of Getoor (see [4], Theorem (8.1)). He solves the inverse problem, namely, he finds an invariant distribution \( \nu \) for the transition function \( \tilde{p} \) given by the formula analogous to (3.2.16). The expression for \( \nu \) he obtains is similar to (3.2.1).

4.2. The proof of Theorem 2 does not differ from the proof of Theorem 2 in [9]. We have to consider a conservative stationary Markov process \((x_t, \mathbb{P})\) with the one-dimensional distribution \( \nu \) and the transition function \( \tilde{p} \) which generates the semigroup \( \tilde{T}_t \) (but in contrast to the situation in [9], now \( \mathbb{P} \) may be an infinite measure). A multiplicative functional \( a_t \) is constructed in such a way that

\[
p(s, x; \Gamma) = \mathbb{F}_x \left( 1_s(x) a_s \right).
\]

Let \( \tilde{\Omega} \) be the sample space of the process \( x_t \). We put \( \tilde{\Omega} = \Omega \times (\mathbb{T})^\infty \) and construct a measure \( Q \) on \( \tilde{\Omega} \) and a family of random variables \( \tau_s(\tilde{\omega}) \) in such a way that

4.2.A. The marginal distribution of \( Q \) on \( \Omega \) is equal to \( \mathbb{F} \).
4.2.B. For \( t > s \) the conditional probability of \( \tau_s \) to be greater than \( t \), given \( w \) is equal to \( a_{t-s}(\theta_s w) \).

4.2.C. For \( t > s \) \( \tau_s = \tau_t \) on the set \( \{ \tau_s > t \} \).

4.2.D. The \( \sigma \)-field \( C_s \cup F_s \) and the pair \( (x_t, \tau_t) \) are conditionally independent, given \( x_s \), where \( C_s \) is the minimal \( \sigma \)-field in \( \tilde{\Omega} \) generated by the sets \( \{ \tau_r < u; r, u \leq s \} \).

The family \( \tau_s \) has the same properties as the family of the first hitting times of a set in the state space. We put \( M(\omega) \) to be the closure of the set of values of the function \( \tau_s(\omega) \). We put \( x^*_t(\omega) \) to be equal to \( x_t(\omega) \) if \( t \in M(\omega) \) and \( x^*_t(\omega) \in V \) otherwise. In the same way as in [9] one can show that \( (x^*_t, Q) \) and \( (x_t, P) \) has the same finite-dimensional distributions and the subprocess in \( D \) of \( (x^*_t, Q) \) is equal to \( (w(s), P) \) where \( P \) is a Markov measure with the one-dimensional distribution \( v \) and the transition function \( p \). Now we need only to apply Theorem 3.2.1 to obtain the final result.
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