Multi-Track Control Space System
Significant Model

Prepared for
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This technical report has been reviewed and is approved for publication. Publication of this report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

FOR THE COMMANDER

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**ABSTRACT**

The equations of motion of a multi-body gimbal servo system for simulation purposes are derived using a free body momentum formulation that is maintained throughout the development, thereby retaining the original variables defined in the beginning of the model formulation. By avoiding undesirable algebraic manipulation and transformation of the original variables, a simplification is achieved in evolving and understanding a complex computer simulation model that can be debugged and run with enhanced insight to the physical process being simulated.
In the formulation, translational motion is permitted for each gimbal frame, but a translational momentum conservation constraint for the system of "rigidly" connected bodies is invoked to avoid solution of translational equations of motion for each body.

A four-body example is presented where the bodies are simply connected in a series configuration, such as would represent a spacecraft with controlled gimballed appendages.
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1. INTRODUCTION

Connected multi-body dynamics problems of all kinds have been encountered since the invention of the wheel and its application to a moving platform. The precise formulation of the equations of motion for more than two connected bodies brought little attention, except for the compound pendulum, until the space age, probably because techniques and means for the solution to the equations for two bodies with more than one or two degrees of freedom were generally unavailable or too unwieldy and impractical to use. In the last two decades, tremendous advances have been made in both the formulation methods and computer solution to multi-body/multi-degree of freedom problems. Today, it is not uncommon to obtain computer solutions for multi-body systems including flexible modes with more than twenty degrees of freedom or more than fifty state variables. Even so, such computer programs evolve to usefulness only after the problem equations formulation, computer coding of the equations, and debugging of the problem. The general effectiveness of the resulting multi-body dynamics and control simulation is usually assessed in terms of the formulation and coding and debugging effort, the computer run time, the ease of an analyst to become familiar with the program after it has been developed, and the ease of modifying the program to reflect changes and additions to the design of the system. With these ideas in mind, the multi-body dynamics formulation for a spaceborne gimbal system was undertaken anew.

The approach initially favored was one wherein the "primitive"* Newtonian/Eulerian equations for translational and rotational motion are retained throughout. This would have required retention of all 6 degrees of freedom for each body and the necessity of integrating more state variable equations than are actually needed. Because computer run time is roughly proportional to the number of state variables and to the highest frequency

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*We use Jerkovsky's definition: By "primitive" equations and variables, we mean equations and variables that refer to each body as a separate and distinct body without regard to how it fits into the multi-body configuration.
mode in the system, we can reduce run time considerably by reducing the
degrees of freedom of each body to 3 by not solving the translational
equations of motion by integration. However, holonomic constraint equations,
which are essentially algebraic, that constrain the gimbal geometry to con-
serve translational momentum of the gimbal system, have been introduced in
this paper which supplant the translational differential equations of
motion. This is basically the approach initially taken by Russell(2) in his
so-called "primed momentum" formulation method. We are then left with 3
rotational degrees of freedom for each body which are permitted for each axis
of a Cartesian coordinate frame associated with that body. The Cartesian
reference frames are retained from the beginning in order to permit insight,
visibility, and a feel for what is dynamically happening throughout the
formulation and simulation. Generalized coordinates formulations such as are
used in Lagrange, D'Alembert, Hamilton Boltzmann-Hamel, Gibbs, and Kane(3)
derivations of the equations of motion are avoided on purpose as is the
transformation operator formalism of Jerkovsky.(1,4) This is because a)
insight seems to get lost, and b) either hand reduction by matrix inversion of
the constraint equations to reduce computer solution time is needed, or this
problem is left to the computer whereby run time suffers due to matrix
inversions that are performed in each integration step. These other
approaches admittedly have the potential of shorter computer run times than
are obtained using a momentum formulation because they reduce the number of
degrees of freedom for each body to an absolute minimum, e.g., the equations
for a single bearing axis gimballed body only needs one degree of "free"
rotational freedom, the other two degrees of relative freedom being "hard"
constrained or completely nulled via constraint equations.

In the alternative approach taken here, relative motion about the
constrained motion axes of the Cartesian frames is permitted, but only
slightly. The small to infinitesimal relative motions allowed require
additional solutions of angular motions transverse to the bearing axis but we
eliminate the need for hand or computer solution of "hard" constraint
equations which generally involve matrix inversions. An advantage obtained is
that the true transverse axis dynamics can be revealed and their effect on
system stability evaluated. However, because gimbal bearing transverse axis stiffnesses can be large, the highest associated eigenvalue frequency can be considerably higher than the highest control loop component bandwidth thereby requiring shortening of the integration step size merely to get the constraint satisfied. This shortcoming can be alleviated after one ascertains that high frequency mode instability is not a problem by artificially reducing transverse axis stiffness or increasing transverse axis viscous friction in a way that lowers the transverse axis characteristic frequency to the vicinity of the control loop bandwidths.

The multi-body equations presented in this paper are derived for bodies in an open chain tree configuration typical of a spacecraft with two or more gimballed elements. The rotational equations of motion of one of the bodies are general for open chain tree configurations as are the translational momentum constraint equations. The equations and matrix-scalar block diagrams are given for an example 4-body system comprised of a 3 axis controlled spacecraft, a single axis controlled solar array, and a 2-axis gimbal controlled payload with a gimbal yoke that has mass and inertia.
Referring to Figure 1, we imagine the gimbaled body with a mass element \( dm \) at \( \mathbf{R} \) relative to an inertial frame \( I \) to be gimbaled about one of the axes of a frame \( G \) whose basis vectors are parallel to those of frame \( g \) at the c.m. of the body. Frame \( g \) is displaced an amount \( \mathbf{r}_I \) from the frame \( G \) and both frames are fixed in the body and therefore rotate with the body at its inertial angular rate \( \mathbf{\omega} \). Then the moment of momentum of this body about the origin of the frame \( g \), or in other words, about its center of mass is

\[
\mathbf{H}_g = \int \mathbf{r} \times \mathbf{R} \, dm
\]

\[
= \int \mathbf{r} \times (\mathbf{R}_I + \mathbf{\omega} \times \mathbf{r}) \, dm
\]

\[
\mathbf{H}_g = \int \mathbf{r} \times (\mathbf{\omega} \times \mathbf{r}) \, dm - \mathbf{R}_I \times \int \mathbf{r} \, dm
\]

The first term on the right side of (3) is \( \mathbf{I}_g \cdot \mathbf{\dot{\omega}} \) where \( \mathbf{I}_g \) is the inertia dyadic or inertia matrix with respect to the frame \( g \), and the integral in the second term is equal to the gimbaled body mass \( M \) times the radius vector from the frame \( g \) to the c.m. which is the null vector and hence is zero. Thus (3) is written as

\[
\mathbf{H}_g = \mathbf{I}_g \cdot \mathbf{\dot{\omega}}
\]
Figure 1. Model of the $i$th Body
3. EQUATIONS OF MOTION

The equation of translational motion for the gimballed body of Figure 1 is

\[ M_i \ddot{R}_i = \dot{P}_i - \dot{P}_{(i-1)} \]  
(5)

In the figure, we have assumed two gimbal points on the \(i^{th}\) body. The subscript \(i\) identifies the gimbal frame associated with that body and the interbody force and torque at that point, whereas the subscript \((i-1)\) identifies the adjacent connected body and its associated gimbal that it shares with the \(i^{th}\) body and its interbody force and torque. The negative of the interbody force and torque acting on the \((i-1)\) body are felt by reaction on the \(i^{th}\) body.

The equation of rotational motion about the point \(g\), or center of mass of the body is (5-8)

\[ \dot{\dot{r}}_T - \dot{r}_Q \times \dot{r}_Q = \ddot{H}_g \]  
(6)

where

\[ \dot{\dot{r}}_T = \dot{\dot{r}}_1 - \dot{\dot{r}}_{(i-1)} \]  
(7)

and

\[ \dot{r}_Q \times \dot{r}_Q = (\dot{r}_1 \times \dot{r}_1) - (\dot{r}_{A1} \times \dot{r}_{(i-1)}) \]  
(8)

where the last expression has been introduced for convenience in simplifying the form of the resulting equations. In the usual approach we would substitute appropriate combinations of (5) into (8) which process is lengthy and replete with terms. To show the general idea of the equation development, we further simplify by introducing
\[
\dot{\mathbf{M}}_{\mathbf{Q}} \dot{\mathbf{r}} = \ddot{\mathbf{r}}, \tag{9}
\]

so that
\[
\dot{\mathbf{r}} = \dot{\mathbf{r}} - \dot{\mathbf{r}} \times \mathbf{M}_{\mathbf{Q}} \dot{\mathbf{r}}, \tag{10}
\]

Following the concept introduced by Russell\(^{(2)}\) we write (10) as
\[
\dot{\mathbf{r}} - \dot{\mathbf{r}} \times \mathbf{M}_{\mathbf{Q}} \dot{\mathbf{r}} = \frac{d}{dt} (\dot{\mathbf{r}} \times \mathbf{M}_{\mathbf{Q}} \dot{\mathbf{r}}) - \dot{\mathbf{r}} \times \mathbf{M}_{\mathbf{Q}} \ddot{\mathbf{r}}, \tag{11}
\]

Employing (11) in (6) we have
\[
\ddot{\mathbf{r}}_{\mathbf{T}} = \ddot{\mathbf{r}}_{\mathbf{g}} + \frac{d}{dt} (\dot{\mathbf{r}}_{\mathbf{Q}} \times \mathbf{M}_{\mathbf{Q}} \dot{\mathbf{r}}) - \dot{\mathbf{r}}_{\mathbf{Q}} \times \mathbf{M}_{\mathbf{Q}} \ddot{\mathbf{r}}_{\mathbf{Q}}, \tag{12}
\]

The vector derivatives indicated in (12) are relative to an inertial reference. In order to solve a vector differential equation computationally, it needs to be scalarized in a convenient frame of reference (see Appendix). We wish to use the frame \(\mathbf{G}\) in our computations and so we find the components of the inertial vector derivatives of \(\ddot{\mathbf{h}}_{\mathbf{g}}\) and \(\ddot{\mathbf{r}}_{\mathbf{Q}}\) which have their bases in that rotating frame. Using Appendix formula (A-15) and scalar component matrix notation,
\[
\ddot{\mathbf{h}}_{\mathbf{gI}} = \ddot{\mathbf{h}}_{\mathbf{gG}} + \ddot{\mathbf{w}}_{\mathbf{G}} \times \ddot{\mathbf{h}}_{\mathbf{gG}}, \tag{13}
\]
\[
\ddot{\mathbf{r}}_{\mathbf{QI}} = \ddot{\mathbf{r}}_{\mathbf{QG}} + \ddot{\mathbf{w}}_{\mathbf{G}} \times \ddot{\mathbf{r}}_{\mathbf{QG}}, \tag{14}
\]

Because the position vectors in (8) are fixed in the body fixed frame \(\mathbf{G}\), \(\ddot{\mathbf{r}}_{\mathbf{QG}} \equiv 0\).

The vectors \(\ddot{\mathbf{r}}_{\mathbf{T}}\) and \(\ddot{\mathbf{r}}_{\mathbf{Q}}\) may have been originally known and expressed in terms of their components in an inertial frame \(\mathbf{I}\), however their components in the \(\mathbf{G}\) frame are straightforwardly obtained by transforming the vector components from the inertial frame \(\mathbf{I}\) to the gimbal frame \(\mathbf{G}\) as required, e.g.,
and

\[ \dot{\vec{R}}_{QG} = \vec{C}_I \cdot \dot{\vec{R}}_{QI} \]

We now employ (13) and (14) in (12) to obtain the matrix-scalar equation expressed in the \( G \) frame.

\[ \dot{\vec{T}}_{TG} = \vec{H}_G + \vec{\omega}_G \times \vec{H}_G + \frac{d}{dt}(\vec{r}_Q \times \vec{M}_Q \vec{R}_Q)_G \]

\[ + \vec{\omega}_G \times (\vec{r}_Q \times \vec{M}_Q \vec{R}_Q) - (\vec{\omega}_G \times \vec{r}_Q \vec{R}_Q) \times \vec{M}_Q \vec{R}_Q \]

where

\[ \vec{H}_G = \vec{H}_G - (\vec{I}_G \cdot \vec{\omega}_G) = \vec{I}_G \cdot \vec{\omega}_G \]

because the frame \( g \) is parallel to the frame \( G \). Equation (7) can be written as

\[ \dot{\vec{T}}_{TG} = \vec{H}_{PG} + \vec{\omega}_G \times \vec{H}_{PG} - \vec{T}_Q \]

where

\[ \vec{H}_{PG} = \vec{H}_G + \vec{H}_Q \]

and

\[ \vec{H}_Q = \vec{r}_Q \times \vec{M}_Q \vec{R}_Q \]
The quantity $\vec{H}_{PG}$ is defined as Russell's\(^{(2)}\) "primed" angular momentum, which includes the momentum $\vec{H}_G$ due to rotation about the center of mass plus a term $\vec{H}_Q$ resulting from the translational motion of the body center of mass due to gimbaling. The $\vec{H}_Q$ term is referred to as the "swing" momentum.

The term $T_Q$ in (18) is a torque that is related to the acceleration term $(\vec{\omega}_G \times \vec{r}_{QG}) \times \vec{H}_G$ and is referred to as the "swing" torque. The swing torque and the time derivative of the swing momentum comprise the interbody torque, i.e., the net torque acting on the $i^{th}$ body due to motion of the ensemble of bodies.

Equations (18) and (19) are the equations describing the angular motion of one of the gimballed bodies in a multi-body system. The solution to these equations is depicted in block diagram form in Figure 2 where $\vec{T}_T$, $\vec{T}_Q$, and $\vec{H}_Q$ are regarded as independent variable inputs in the figure and the computed components $\vec{\omega}_G$ of $\vec{\omega}$ in the gimbal frame $G$ are the output. The inverse $I_g^{-1}$ of the inertia matrix $I_g$ which is determined relative to the frame $g$ is a physical parameter as are the mass $M$ and the components of $\vec{r}$ in the gimbal frame.

Equations (18) and (19) further simplify to familiar forms if $\vec{r}_Q = 0$ or if $\vec{r}_Q = 0$ which makes the center of mass inertially fixed. The terms $\vec{H}_Q$ and $\vec{T}_Q$ generally will not be zero in a multi-body system and the presence of $\vec{r}_Q$ which is related to the gimballed mass velocity contributes in some cases very significantly to the angular momentum of the body and therefore is very important to consider. Its effect is amplified by the vector $\vec{r}_Q$ which is related to the distance from the gimbal points to the center of mass of the body.

\[ \vec{T}_Q = (\vec{\omega} \times \vec{r}_Q) \times M \vec{r}_Q \] (21)
Figure 2. Single Gimbaled Body Equation of Motion

\[ \mathbf{T}_Q = (\mathbf{\omega} \times \mathbf{\dot{R}}_Q) \times M_R \mathbf{\dot{R}}_Q \]

\[ \mathbf{\ddot{H}}_Q = \mathbf{T}_Q \times M_R \mathbf{\dot{R}}_Q \]
4. FOUR BODY SYSTEM DYNAMICS

The translational equations of motion given by (5) are written out, for example, for the four body chain configuration shown in Figure 3,

\[ M_1 R_1 = \ddot{F}_1 \]  
(22)

\[ M_2 R_2 = \ddot{F}_2 - \ddot{F}_1 \]  
(23)

\[ M_3 R_3 = \ddot{F}_3 - \ddot{F}_2 \]  
(24)

\[ M_4 R_4 = -\ddot{F}_3 \]  
(25)

4.1 INTERBODY TORQUES

The interbody torques resulting from interbody forces are now determined. It is easily seen that equations (8) and (5) or equations (22-25) yield for the \( i \)th body in the chain

\[
\begin{align*}
\frac{d}{dt}\left[r_i \times \ddot{F}_i \right] &= r_i \times \sum_{k=1}^{i} M_k R_k - r_i \times \sum_{k=1}^{i-1} M_k R_k \\
&= r_i \times \dddot{M}_i R_i - \rho_i \times \sum_{k=1}^{i-1} M_k R_k \\
&= \frac{d}{dt}\left[r_i \times \dddot{M}_i R_i - \rho_i \times \sum_{k=1}^{i-1} M_k R_k \right] \\
&- \left[r_i \times \dddot{M}_i R_i - \rho_i \times \sum_{k=1}^{i-1} M_k R_k \right]
\end{align*}
\]

(26)
where
\[ \mathbf{\rho}_i = \mathbf{r}_{Ai} - \mathbf{r}_i \]  

(27)

Comparing (26) with corresponding terms in (17) and (18) it is apparent that the first term in (26) is after scalarizing to the \( i \)th gimbal frame

\[ \mathbf{\bar{H}}_{Q1} = \mathbf{\bar{r}}_1 \times \mathbf{M}_1 \mathbf{\bar{R}}_1 - \mathbf{\bar{\rho}}_1 \times \sum_{k=1}^{(i-1)} \mathbf{M}_k \mathbf{\bar{C}}_k \times \mathbf{\bar{R}}_k \]  

(28)

where as in (20) we have defined the quantity \( \mathbf{\bar{H}}_Q \) as the swing momentum term \( (\mathbf{r}_Q \times \mathbf{M}_Q \mathbf{\bar{R}}_Q) \). In particular for the four body chain of Figure 3, using (28) we obtain,

\[ \mathbf{\bar{H}}_{Q1} = \mathbf{\bar{r}}_1 \times \mathbf{M}_1 \mathbf{\bar{R}}_1 \]  

(29)

\[ \mathbf{\bar{H}}_{Q2} = \mathbf{\bar{r}}_2 \times \mathbf{M}_2 \mathbf{\bar{R}}_2 - \mathbf{\bar{\rho}}_2 \times \mathbf{M}_1 \mathbf{\bar{R}}_1 \]  

(30)

\[ \mathbf{\bar{H}}_{Q3} = \mathbf{\bar{r}}_3 \times \mathbf{M}_3 \mathbf{\bar{R}}_3 - \mathbf{\bar{\rho}}_3 \times (\mathbf{M}_1 \mathbf{\bar{R}}_1 + \mathbf{M}_2 \mathbf{\bar{R}}_2) \]  

(31)

or

\[ \mathbf{\bar{H}}_{Q3} = \mathbf{\bar{r}}_3 \times \mathbf{M}_3 \mathbf{\bar{R}}_3 - \mathbf{\bar{\rho}}_3 \times \mathbf{M}_4 \mathbf{\bar{R}}_4 \]  

(32)

\[ \mathbf{\bar{H}}_{Q4} = \mathbf{\bar{r}}_4 \times \mathbf{M}_4 \mathbf{\bar{R}}_4 \]  

(33)

The second term in (26) is the swing torque

\[ \mathbf{T}_{Q1} = \left[ \mathbf{\bar{r}}_1 \times \mathbf{M}_1 \mathbf{\bar{R}}_1 - \mathbf{\bar{\rho}}_1 \times \sum_{k=1}^{(i-1)} \mathbf{M}_k \mathbf{\bar{R}}_k \right] \]  

(34)
which becomes after scalarizing to the ith gimbal frame,

\[
\vec{T}_{Q1} = (\dot{\omega}_1 \times \vec{r}_1) \times M_1 \vec{R}_1 - (\dot{\omega}_1 \times \vec{\rho}_1) \times \sum_{k=1}^{(i-1)} M_k C_k^{\vec{R}_k} (35)
\]

For the four-body system of Figure 3, the swing torques on each body are found to be

\[
\vec{T}_{Q1} = (\dot{\omega}_1 \times \vec{r}_1) \times M_1 \vec{R}_1 
\]

(36)

\[
\vec{T}_{Q2} = (\dot{\omega}_2 \times \vec{r}_2) \times M_2 \vec{R}_2 - (\dot{\omega}_2 \times \vec{\rho}_2) \times M_1 C_1^2 \vec{R}_1 
\]

(37)

\[
\vec{T}_{Q3} = (\dot{\omega}_3 \times \vec{r}_3) \times M_3 \vec{R}_3 - (\dot{\omega}_3 \times \vec{\rho}_3) \times (M_1 C_1^3 \vec{R}_1 + M_2 C_2^3 \vec{R}_2) 
\]

(38)

or

\[
\vec{T}_{Q3} = (\dot{\omega}_3 \times \vec{r}_3) \times M_3 \vec{R}_3 - (\dot{\omega}_3 \times \vec{\rho}_3) \times M_4 C_4^3 \vec{R}_4 
\]

(39)

\[
\vec{T}_{Q4} = (\dot{\omega}_4 \times \vec{r}_4) \times M_4 \vec{R}_4 
\]

(40)

We can now proceed with the determination of the \( \vec{R}_1 \) terms in (28) in order to calculate the solution of equations (18) and (19) shown in Figure 2. In the material that follows, the subscripts denote the mass and gimbal numbers. The subscript G has been suppressed and replaced merely with the gimbal number i. The total torque \( \vec{T}_{T1} \) on body 1 will be found to be comprised of the torque from the ith gimbal elements (torque motor, bearing friction, etc.) plus the reaction torque from the (i-1)th gimbal elements.

4.2 MOMENTUM - KINEMATIC CONSTRAINTS

Figure 3 schematically shows several bodies connected together. A main body is shown which could be the main part of a spacecraft which has a gimbal system that points a payload, e.g., the main body \( M_3 \) could be the spacecraft, \( M_4 \) its solar array, \( M_2 \) a gimbal, and \( M_1 \) the payload. The axes \( G_1 \) and \( G_2 \) would
then be the payload gimbal axes and $G_3$ the solar array gimbal axis. In the following analysis, the subscript number denotes the gimbal frame number and the associated gimballed body number. Refer to the appendix for definitions of vectors, scalars, matrix-scalar expressions and transformations.

It is obvious that the gimbal points move, i.e., the $\dot{R}_{0i}$ change as the gimballed bodies are articulated. We now proceed to find the $\dot{R}_{04}$.

The center of mass of the system is chosen as the origin of the inertially fixed reference frame corresponding to the $X_I Y_I Z_I$ frame in Figure 1. This restricts the analysis in this paper to the situation of no external forces that would cause the system c.m. to undergo translational motion. Pure moments about the system c.m. would be permissible. The definition of the center of mass is employed; i.e.,

$$\sum_{i=1}^{4} M_i \dot{R}_i = 0 \quad (41)$$

If we differentiate (41) we get

$$\sum_{i=1}^{4} \ddot{M}_i \dot{R}_i = 0 \quad (42)$$

which is a statement of the conservation of translational momentum of the system. Referring to Figure 3, the following vector relations are found:

$$\dot{R}_1 = \dot{R}_{01} + \dot{r}_1$$
$$\dot{R}_2 = \dot{R}_{02} + \dot{r}_2$$
$$\dot{R}_3 = \dot{R}_{03} + \dot{r}_3$$
$$\dot{R}_4 = \dot{R}_{04} + \dot{r}_4$$
In general, for the chain configuration

\[ \dot{\mathbf{R}}_i = \dot{\mathbf{R}}_{0i} + \dot{\mathbf{r}}_i \]  

(43)

Also,

\[ \dot{\mathbf{R}}_{02} = \dot{\mathbf{R}}_{01} + \dot{\mathbf{p}}_2 \]

\[ \dot{\mathbf{R}}_{03} = \dot{\mathbf{R}}_{02} + \dot{\mathbf{p}}_3 \]

where

\[ \dot{\mathbf{p}}_2 \] = vector from GI to C2, etc.

so in general

\[ \dot{\mathbf{R}}_{01} = \dot{\mathbf{R}}_{0(1-1)} + \dot{\mathbf{p}}_i \]  

(44)

or

\[ \dot{\mathbf{R}}_{01} = \dot{\mathbf{R}}_{01} + \sum_{k=2}^{i} \dot{\mathbf{p}}_k \]  

(45)

Equations (43) and (44) are depicted in Figures 1 and 3. Differentiating (43), we get for the 1\textsuperscript{th} body

\[ \dot{\mathbf{R}}_1 = \dot{\mathbf{R}}_{01} + \dot{\omega}_1 \times \dot{\mathbf{r}}_1 \]  

(46)

An \((\dot{\omega}_1 \times \dot{\mathbf{R}}_{01})\) term is not present because \(\dot{\mathbf{R}}_{01}\) vector is not based in the 1\textsuperscript{th} body moving/rotating coordinate frame but rather is based in the inertial frame centered at the system c.m. On the other hand, the \(\dot{\mathbf{r}}_1\) vector is based and fixed in the 1\textsuperscript{th} body frame so that its time derivative is identically zero and the \((\dot{\omega}_1 \times \dot{\mathbf{r}}_1)\) term arises from the \(\dot{\omega}_1\) angular rate of the 1\textsuperscript{th} body. Similarly, differentiating (44) and (45) we get
\[ \dot{\mathbf{R}}_{01} = \dot{\mathbf{R}}_{01(1-1)} + \dot{\omega}_1 \times \dot{\mathbf{p}}_1 \]  
\quad (47)

and

\[ \dot{\mathbf{R}}_{01} = \dot{\mathbf{R}}_{01} + \sum_{k=2}^{i} (\omega_k \times \dot{\mathbf{p}}_k) \]  
\quad (48)

Substituting (48) into (46) and the result into (42) we obtain

\[ \dot{\mathbf{M}}_{R01} + \sum_{i=1}^{N} M_1 (\dot{\omega}_1 \times \dot{\mathbf{r}}_1) + \sum_{i=1}^{N} M_1 \left( \sum_{k=2}^{i} (\omega_k \times \dot{\mathbf{p}}_k) \right) = 0 \]  
\quad (49)

which can also be expressed as

\[ \dot{\mathbf{M}}_{R01} + \sum_{i=1}^{N} M_1 (\dot{\omega}_1 \times \dot{\mathbf{r}}_1) + \sum_{i=2}^{N} \left( \sum_{k=1}^{i-1} M_k \right) (\dot{\omega}_1 \times \dot{\mathbf{p}}_1) = 0 \]  
\quad (50)

The vectors in (50) should be expressed in terms of their components in the same coordinate frame. Thus, we scalarize (50)

\[ \dot{\mathbf{M}}_{R01} + \sum_{i=1}^{N} M_1 \mathbf{C}_1 \cdot (\dot{\omega}_1 \times \dot{\mathbf{r}}_1) + \sum_{i=2}^{N} \left( \sum_{k=1}^{i-1} M_k \right) \mathbf{C}_k \cdot (\dot{\omega}_1 \times \dot{\mathbf{p}}_1) = 0 \]  
\quad (51)

where we have arbitrarily chosen the first body as the computational coordinate frame and the transformation matrices \( \mathbf{C}_i \) are employed to transform the vectors \( \dot{\omega}_1, \dot{\mathbf{r}}_1 \) and \( \dot{\mathbf{p}}_1 \) expressed in the \( i \)th body coordinates, back to the first body coordinates. As an example, for 4-bodies in series (51) is used to find \( \dot{\mathbf{R}}_{01} \).
\[-\dot{\mathbf{R}}_{01} = M_1 (\ddot{\mathbf{w}}_1 \times \ddot{\mathbf{r}}_1) + M_2 \mathbf{c}_2 \cdot (\ddot{\mathbf{w}}_2 \times \ddot{\mathbf{r}}_2) \]
\[+ M_3 \mathbf{c}_3 \cdot (\ddot{\mathbf{w}}_3 \times \ddot{\mathbf{r}}_3) + M_4 \mathbf{c}_4 \cdot (\ddot{\mathbf{w}}_4 \times \ddot{\mathbf{r}}_4) \]
\[+ (M_2 + M_3 + M_4)\mathbf{c}_2 \cdot (\ddot{\mathbf{w}}_2 \times \ddot{\mathbf{r}}_2) \]
\[+ (M_3 + M_4)\mathbf{c}_3 \cdot (\ddot{\mathbf{w}}_3 \times \ddot{\mathbf{r}}_3)\]

(52)

(The \(\ddot{\mathbf{w}}_1\) and \(\ddot{\mathbf{w}}_4\) terms are noted to be zero.) After \(\dot{\mathbf{R}}_{01}\) is computed, the remaining \(\dot{\mathbf{R}}_{01}\) needed to complete the solution for the constraints are found from (47). These \(\dot{\mathbf{R}}_{01}\) are needed in the solution of the equations for momentum and motion which are computationally performed in each of the gimballed bodies respective coordinate frames and so they are scalarized in the 1th frame.

From (46), and (47) we obtain the scalarized result

\[
\dot{\mathbf{R}}_1 = \mathbf{c}_1 \cdot \dot{\mathbf{R}}_{01} + \ddot{\mathbf{w}}_1 \times (\ddot{\mathbf{r}}_1 + \ddot{\mathbf{r}}_1) \]

(53)

and from (46) and (48)

\[
\dot{\mathbf{R}}_1 = \mathbf{c}_1 \cdot \dot{\mathbf{R}}_{01} + \ddot{\mathbf{w}}_1 \times \ddot{\mathbf{r}}_1 + \sum_{k=2}^{i} \mathbf{c}_k \cdot (\ddot{\mathbf{w}}_k \times \ddot{\mathbf{r}}_k) \]

(54)

Equation (53) is obviously simpler than (54) and so it is the preferred method of calculating the \(\dot{\mathbf{R}}_1\) (i.e., in succession).
Figure 4 shows the block diagram representation for the solution of the $R_i$ terms in equation (19) for the $i$th body angular momentum term $H_{Q_i}$ using equations (51) and (53). It is noted in Figure 3 that the 4th and last body is assumed to have the same reference gimbal point as the 3rd body so that $R_{04} = R_{03}$. The first and last bodies in the chain in Figure 3 have only a single gimbal point and so their values for $\bar{P}_i$ are zero or undefined.
Figure 4. Momentum-Kinematic Constraints
The solution of the equations of motion of a single body in the ensemble of bodies as depicted in Figure 2 can be carried out as previously noted using equations (18), (19), (28), (35), (51) and (53) yielding the body angular rates. The incremental angular motions that can be obtained by integration of the body angular rates, accounting for kinematics, are those angular motions about the gimballed \( i \)th body \( G_i \) axes. Practically speaking, for gimbal motion freedom, one axis is unrestrained in the sense that large relative angular motion may occur (about the gimbal bearing axis) between the gimballed body and the neighboring body to which it is attached. The other two body axis motions relative to the attached body are restrained by the bearings to be small or near zero. One approach in writing simulation equations is to assume these transverse gimbal axes motions to be zero. This requires that "hard" constraint relations be solved algebraically in each integration step of the simulation. The analytical determination of the hard constraint relations is avoided in the approach taken here. Instead, the constraints for the bearing transverse axes are implemented by feeding back reaction torques about these axes that are proportional to small relative transverse axes angular motions. In effect then, the constraint relations appear as restraint gimbal stiffnesses. In fact, the stiffnesses used in the simulation can be actual bearing transverse axes stiffnesses or perhaps the stiffnesses associated with the lowest frequency modes of vibration about these axes.

One can take the point of view that the so-called unrestrained axis or the bearing "free" axis is also restrained by the reaction and control torques relations for that axis. These torques include, for example, motor electromagnetic torque, motor cogging torque, motor magnetic hysteresis torque, bearing and motor solid friction torque, and viscous friction torque.

With these notions then, the transverse axes motions that would be oscillatory with stiffness feedback only, can also be dampened using realistic structural damping relations. If the simulation is expected to give good fidelity response data at the usually higher structural frequencies, the
damping introduced should be realistic. If the simulation is not expected to provide high fidelity results except at frequencies well below the structural frequencies, it is anticipated that high damping can be artificially introduced for the bearing transverse axis modes. This will allow the use of larger integration step sizes in a digital simulation and consequently will be more economical in computer run time. The restrained body dynamics equations are shown for the $i^{th}$ body in block diagram form in Figure 5.
Figure 5. Gimbaled Body Restraint Dynamics
6. FOUR BODY SERVO SYSTEM EXAMPLE

6.1 GIMBALS/COORDINATES

An example of a multi-body gimbal servo system is illustrated in Figure 6. A block diagram of the "complete" dynamics equations for this example is shown in Figure 7.

The first body is the pointed experiment payload of mass $M_1$ which has a gimbal coordinate frame $X_1, Y_1, Z_1$ fixed in the payload with $Z_1$ along the line-of-sight of the experiment, $X_1$ is the cross-track gimbal axis, and $Y_1$ completes the triad. The center of mass of $M_1$ is offset from the $G_1$ gimbal center an amount $\vec{r}_1$.

The second body with mass $M_2$ is the gimbal yoke. The yoke gimbal center point $G_2$ is assumed to be midway between the cantilevered yoke bearings that provide free rotation of the yoke about the $Y_2$ axis. The yoke $X_2$ axis moves with the yoke and is parallel to the experiment $X_1$ gimbal axis. The $Z_2$ axis completes the yoke orthogonal frame. The center of mass of the yoke is offset from the $G_2$ gimbal point an amount $\vec{r}_2$.

The third body is the spacecraft and its gimbal frame $G_3$ is arbitrarily assigned to the same point as the solar array frame, $G_4$, i.e., it could also have been assigned to the same point as the yoke frame point $G_2$. As shown in Figures 4 and 6, the $G_3$ and $G_4$ points coincide. The spacecraft $Y_3$ axis is parallel to the yoke $Y_2$ axis and coincides with the solar array $Y_4$ axis. The spacecraft $X_3$ axis is fixed in the spacecraft parallel to one of its edges and also nominally parallel to the spacecraft orbital velocity vector. The $Z_3$ axis points in the direction of orbit nadir and completes the right handed triad.

The solar array $X_4$ axis is parallel to an edge of the array which is maintained normal to the sun line. The $Z_4$ axis completes the array frame $G_4$ triad.
Figure 6. Example of a Multi-Body Gimbal Servo System
Figure 7. Simulation Model for a Spacecraft With Gimballed Experiment 
and Solar Array

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6.2 CONSTRAINTS AND RESTRAINTS

The constraint relations in Figure 4 apply to this example and in effect maintain the gimbal points geometry so that the distances between gimbals are maintained constant (e.g., \( \mathbf{\hat{p}}_2 \) = vector from \( g_1 \) to \( g_2 \) = constant).

The "restraint" provisions that keep the angular rotation about the gimbal transverse axis small are the realistic stiffnesses about the experiment \( Y_1 \) and \( Z_1 \) axes, the yoke \( X_2 \) and \( Z_2 \) axes, and the solar array \( X_4 \) and \( Z_4 \) axes. The stiffnesses enter into the equations of restrained motion; for example, as indicated in Figure 5 for the \( i^{th} \) body, and Figure 8 for the complete system, in the blocks labeled \( \mathbf{K}_i \) which are \( 3 \times 3 \) stiffness matrices. The diagonal elements are the stiffnesses about the \( X_2, Y_2, Z_2 \) axes. The off-diagonal elements are assumed to be negligible. Since the yoke \( Y_2 \) axis is unrestrained except for solid friction, its stiffness is zero or small compared to the \( X_2 \) and \( Z_2 \) stiffness elements in \( \mathbf{K}_2 \). Data for the stiffness elements can be obtained from design data or test data.

The other restraint provisions are applied somewhat realistically such as viscous friction in all axes in the \( \mathbf{B}_2 \), \( 3 \times 3 \), matrix in Figure 5. The diagonal elements are the viscous friction coefficients operating on the \( X_2, Y_2, Z_2 \) relative angular velocities. The off-diagonal terms are assumed negligible. The controlled \( Y_2 \) axis viscous friction coefficient can be found from test data. The \( X_2 \) and \( Z_2 \) axis friction coefficients can be assigned values that make the roots of the characteristic equation of the \( \mathbf{L}_2^{-1}X_2/S^2 \) + \( \mathbf{Y}_2^{-1}B_2/S \) loops taken together for the \( X_2 \) and \( Z_2 \) channels overdamped such that the low frequency root is not much higher than the \( Y_2 \) control bandwidth (say for the rate control mode). If the undamped roots of the aforementioned characteristic equation lie within the control bandwidth, then an attempt should be made to use as realistic a friction model as possible for the transverse axes. Some viscous friction may be present for transverse axis motion of the gimbal axis relative to the adjacent spacecraft body through the action of lubrication as the viscous media. Perhaps a better choice of a model for incorporating damping is to use a simplified solid friction model\(^{(9)}\) that results in realistic solid friction hysteretic damping. The model is an integral function of the relative angular rate \( \Delta \omega_2 \) given by
where $T_{Fj}$ is the $j$ axis component of solid friction torque that retards the motion, $a_j$ is the solid friction model rest slope, and $T_{Cj}$ is the coulomb friction torque level. Even though it is hysteretic, the model behaves like viscous friction for medium amplitudes of angular oscillatory motion but the behavior transitions to non-linear "structural damping" for very low amplitudes of oscillation. Structural damping virtually disappears near zero amplitude motion.

6.3 CONTROLLED AXES

The controlled gimbal axes are the experiment $X_1$ axis, the yoke $Y_2$ axis, and the solar array $Y_4$ axis.

6.4 SENSORS

Rebalance rate gyros (in $F_1$ and $F_2$ of Figure 7) are mounted with input axes along the $X_1$ and $Y_2$ axes and sense the inertial angular rates of the experiment and yoke about these axes. Angle pickoffs (in $F_1$ and $F_2$ of Figure 7) are also mounted on these axes and measure the relative Euler angles $\phi_2, \phi_3$. A sun sensor (in $F_4$) is mounted on the solar array and measures the angle between the sun vector and the $Y_4Z_4$ plane. An earth-sensor (in $F_3$) looks at the earth in the $Z_3$ direction and measures spacecraft pitch (about $Y_3$) and roll (about $X_3$) attitude errors. The spacecraft attitude control system also has rebalance rate gyros (in $F_3$) and a star sensor mounted on the spacecraft body (not shown in Figure 7).

6.5 ACTUATORS

The controlled gimbal axes are driven relative to the adjacent body by reacting against the adjacent body with D.C. torque motors. The spacecraft primary attitude control mode during experiment gimballing utilizes reaction wheels. The gimbal torque motor actuators are elements in the $\bar{Q}_1$ and $\bar{Q}_2$ blocks in Figure 7. The torque motor elements of $\bar{Q}_1$ can be considered to be
Laplace transform transfer functions to account for dynamical effects of the torque motors if necessary. The spacecraft reaction wheels (not their housings) are actually additional bodies in the multi-body system and should be modelled like the other bodies. However, a simplification is justified if the reaction wheel momentum is small enough that it does not cause significant gyroscopic effects on the system. In this case, the reaction wheels can be considered to be devices which apply torque to the spacecraft through the dynamics of the wheel motor drive without the need to consider the effects of wheel speed on the system momentum. The wheels as torque generating devices are depicted in block $Q_3$ of Figure 7. The elements of the $Q_3$ matrix may be dynamics transfer functions.

6.6 COMPENSATION

The blocks labelled $p$ contain position control loop compensation transfer functions and the blocks labelled $r$ contain the rate control loop compensation transfer functions. In the case of the experiment, the $\theta$ and $\phi$ pointing position commands as well as the $\omega_{lx}$ and $\omega_{zy}$ rate commands are generated external to the control loops. Switchover from position to rate control is done to provide better tracking performance because of lower noise and higher loop bandwidth using the rate loop only. The usual proportional plus integral or proportional plus rate compensation is employed in the various compensators. The simplified diagram in Figure 7, for example, shows the solar array control without gyros or tachometer feedback and so proportional plus rate type compensation is used for this control loop.
7. SUMMARY

The equations of motion of a multi-body gimbal servo system for simulation purposes were derived using a simplistic gimballed body momentum formulation that was maintained throughout, thereby retaining the original variables defined in the beginning of the model formulation. The avoidance of unnecessary algebraic manipulation and the introduction of new variables through transformation of the original variables is seen as a simplification in evolving and understanding a complex computer simulation model that can be debugged and run with enhanced insight to the physical process being simulated.

One of the features that helped in this goal was the choice of writing the equations of rotational motion in the body fixed coordinates of the gimballed bodies. The result is the simple body rotational dynamics equation diagram of Figure 2. The \( r^Q \times M^R_Q \) moment of momentum term that is added to the body rotational angular momentum is determined algebraically by evaluating the gimbal interbody forces and utilizing conservation of translational momentum in order to avoid solution of the translational dynamics. Although somewhat involved, this process seems to evolve to solutions that are straightforward as indicated by Figure 4.

The equations of rotational motion are expressed in the gimballed body coordinates and thereby require fairly extensive transformation of angular rates and torques from adjacent bodies, but this is true of other formulation approaches. When a generalized coordinate approach to formulation is taken, the coordinate transformations are hidden or essentially replaced by velocity and momentum transformations. Thus, in this regard, there appears to be no great disadvantage in the approach taken in this paper. However, there seems to be an advantage in keeping the avenues of insight open by retaining the original coordinate transformations throughout.

The equations necessary for the simulation of a 4-body system were presented in matrix-scalar block diagram form in Figures 4 and 7. The elements of the \( 3 \times 3 \) matrices or dyadics need to be defined whereby it will
be found that many are zero and many are Laplace transform transfer functions. The component scalar equations are too detailed to devote additional attention in this report.
REFERENCES


1. **SCALARIZING**

The vector equations in this paper are used in vector-dyadic form until the need arises to scalarize into matrix-scalar forms. For computer time history simulations, we need the matrix-scalar form as an end product to work from and the block diagrams used herein are in that form. In scalarizing, the assumption is made that a vector $\hat{R}$ can be written in terms of its components along some bases $\hat{u}_{Ax}, \hat{u}_{Ay}, \hat{u}_{Az}$. In vector form,

$$\hat{R} = \hat{R}_A \hat{u}_A = R_{Ax} \hat{u}_{Ax} + R_{Ay} \hat{u}_{Ay} + R_{Az} \hat{u}_{Az}$$  \hspace{1cm} (A-1)

Then alternatively in matrix-vector form

$$\hat{R}_A = \{ R_A \} = [ R_{Ax}, R_{Ay}, R_{Az} ] \{ \hat{u}_{Ax}, \hat{u}_{Ay}, \hat{u}_{Az} \}^T$$  \hspace{1cm} (A-2)

$$= [ R_A ] \{ \hat{u}_A \}$$  \hspace{1cm} (A-3)

$$\begin{bmatrix}
R_{Ax} \hat{u}_{Ax} \\
R_{Ay} \hat{u}_{Ay} \\
R_{Az} \hat{u}_{Az}
\end{bmatrix}$$  \hspace{1cm} (A-4)

or in terms of its scalar components in frame $A$ the matrix-scalar form is

$$\overline{R}_A = [ R_A ] = \begin{bmatrix}
R_{Ax} \\
R_{Ay} \\
R_{Az}
\end{bmatrix}$$  \hspace{1cm} (A-5)
The idea of a matrix such as the component matrix \( [R_A] \) operating on \( i \) vector \( \hat{v}_A \) producing a "transformed" vector \( [R_A] \hat{v}_A \) is applied to coordinate transformations where a rotation matrix \( [C_A^B] \) operating on a vector say \( \hat{s}_A \) produces a vector \( \hat{s}_B \). In "matrix vector" form,

\[
[\hat{s}_B] = [C_A^B] [\hat{s}_A]
\]

or

\[
\begin{pmatrix}
\hat{s}_{Bx} \\
\hat{s}_{By} \\
\hat{s}_{Bz}
\end{pmatrix} =
\begin{pmatrix}
C_{Ax}^{Ba} & 0 & 0 \\
0 & C_{Ay}^{Ba} & 0 \\
0 & 0 & C_{Az}^{Ba}
\end{pmatrix}
\begin{pmatrix}
\hat{s}_{Ax} \\
\hat{s}_{Ay} \\
\hat{s}_{Az}
\end{pmatrix}
\]

and in matrix-scalar form

\[
[\hat{s}_B] = [C_A^B] [\hat{s}_A]
\]

or

\[
\hat{s}_B = [C_A^B] \hat{s}_A
\]

or

\[
\begin{pmatrix}
\hat{s}_{Bx} \\
\hat{s}_{By} \\
\hat{s}_{Bz}
\end{pmatrix} =
\begin{pmatrix}
C_{Ax}^{Ba} & 0 & 0 \\
0 & C_{Ay}^{Ba} & 0 \\
0 & 0 & C_{Az}^{Ba}
\end{pmatrix}
\begin{pmatrix}
\hat{s}_{Ax} \\
\hat{s}_{Ay} \\
\hat{s}_{Az}
\end{pmatrix}
\]

The vector dyadic form

\[
\hat{s}_B = [C_A^B] \cdot \hat{s}_A
\]
is represented in matrix scalar form as

$$\vec{S}_B = C_A^{B} \cdot \vec{S}_A \tag{A-12}$$

where the dyadic and matrix $C_A^{B}$ are equivalent.

The vector $\vec{R}$ in (A-1) and (A-3) did not have to be identified with respect to a coordinate frame until its components needed to be known, at which time its bases $\hat{u}_{AX}$, $\hat{u}_{AY}$, $\hat{u}_{AZ}$ have to be specified. In the case of the coordinate transformation, (A-6), the bases of $\vec{S}$ were presumably known to be in frame A before the transformation matrix or dyadic $C_A^{B}$ could be generated to produce that vector's components in the B frame. In this sense, the "matrix vector" form is already in a scalarized form by identifying the A and B frame bases vectors.

2. DERIVATIVES

If a vector derivative operation is required in a fixed inertial frame with bases $u_{AX}$, $u_{AY}$, $u_{AZ}$, the components of $\vec{R}$ are identified from

$$\dot{\vec{R}} = \dot{\vec{R}}_A A = \dot{\vec{R}}_{AX} A_{AX} + \dot{\vec{R}}_{AY} A_{AY} + \dot{\vec{R}}_{AZ} A_{AZ} \tag{A-13}$$

wherein the scalar components are seen to be

$$\dot{\vec{R}}_A = \begin{bmatrix} \dot{R}_{AX} \\ \dot{R}_{AY} \\ \dot{R}_{AZ} \end{bmatrix} \tag{A-14}$$

If the components of a vector derivative of $\vec{R}$ are known in a frame B rotating at an inertial angular rate $\hat{\omega}$ with respect to an inertial frame A, the components of the vector derivative in the inertial frame can be found from the vector formula

$$\dot{\vec{R}}_A = \dot{\vec{R}}_B + \hat{\omega} \times \vec{R}_B \tag{A-15}$$
They are the corresponding quantities of the matrix scalar equation,

\[ \dot{\begin{bmatrix} \text{RA} \end{bmatrix}} = \dot{\begin{bmatrix} \text{RB} \end{bmatrix}} + \omega_B \times \begin{bmatrix} \text{RB} \end{bmatrix} \]  

(A-16)

where the symbols \( \omega_B \times \) denotes a \( 3 \times 3 \) skew-symmetric matrix operator that operates on the \( 1 \times 3 \) column matrix \( \begin{bmatrix} \text{RB} \end{bmatrix} \) i.e.,

\[
\omega_B \times = \begin{bmatrix}
0 & -\omega_{Bz} & \omega_{By} \\
\omega_{Bz} & 0 & -\omega_{Bx} \\
-\omega_{By} & \omega_{Bx} & 0
\end{bmatrix}
\]  

(A-17)

Sometimes the components of a vector derivative \( \dot{\begin{bmatrix} \text{R} \end{bmatrix}} \) (or any vector for that matter) are known in an inertial frame \( A \) and it is desired to determine its components in another frame \( B \) which may be rotating or not. Here, the components of \( \dot{\begin{bmatrix} \text{R} \end{bmatrix}} \) in the frame \( B \) are [see (A-12)]

\[ \dot{\begin{bmatrix} \text{RB} \end{bmatrix}} = \omega_A \times \dot{\begin{bmatrix} \text{RA} \end{bmatrix}} \]  

(A-18)

It perhaps should be pointed out that \( \omega_A \) in (A-15) is the inertial angular velocity of the \( B \) frame whose components, if need be, can be transformed via formula (A-12)

\[ \begin{bmatrix} \omega_B \end{bmatrix} = \omega_A \times \begin{bmatrix} \omega_A \end{bmatrix} \]  

(A-19)

All of the vectors on the right side of (12) are thereby scalarized in (17) to the same reference frame. Note that if there is more than one subscript to a scalar \( 1 \times 3 \) matrix, the last subscript denotes the reference frame, e.g., \( \begin{bmatrix} \omega_{23} \end{bmatrix} \) indicates the scalar components of \( \omega_2 \) are in frame 3.
3. **TRANSFORMATIONS**

The transformation matrix $[c^B_A]$ or dyadic $n^B_A$ can be a single or multiple axis Euler angle transformation. Here, for single Euler angle rotation transformations,

$$[\phi^B_A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\phi & S\phi \\ 0 & -S\phi & C\phi \end{bmatrix}$$

which is a $\phi$ rotation about $X_A$ axis in the sequence to get to $B$ frame,

$$[\theta^B_A] = \begin{bmatrix} C\theta & 0 & -S\theta \\ 0 & 1 & 0 \\ S\theta & 0 & C\theta \end{bmatrix}$$

is a $\theta$ rotation about $Y_A$ axis in the sequence to get to $B$ frame, and

$$[\psi^B_A] = \begin{bmatrix} C\psi & S\psi & 0 \\ -S\psi & C\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a $\psi$ rotation about $Z_A$ axis in the sequence to get to the $B$ frame, where $X_A$, $Y_A$, $Z_A$ are the axes of the $G_A$ frame. If a gimbal axis set, through which gimbal Euler angles occur, has zero or negligible mass and inertia, and are intersecting, then the transformation (19) can be taken as the product of successive rotational transformations of the massless gimbal set. For example, if the massless gimbal set order of rotation is, first about $X_A$,
second about \( Y_A \), and third about \( Z_A \), then to get from the \( G_A \) frame to the \( G_B \) gimbal frame,

\[
\begin{align*}
\mathbf{e}^B_A &= \mathbf{e}^B_A \cdot \mathbf{e}^B_A \cdot \mathbf{e}^B_A \\
(A-23)
\end{align*}
\]

In the usual case where a gimbal has appreciable or finite mass, and a single bearing axis is employed which may be non-intersecting with other gimbals, then that gimbal should be considered as a separate body and a single angle rotation (about the gimbal axis) transformation is deemed adequate. However, strictly speaking, the small angle rotations about the axes transverse to the bearing axis which can be considered co-centered should be included in the general 3-axis case as in Equation (A-23).

4. **ATTITUDE KINEMATICS**

The gimbal or body angles or parameters that characterize the attitude of the body are computationally obtained from kinematics rate equations. In the case of Euler angles that are used in this paper they are obtained by integration of kinematic rate equations typically of the form

\[
\begin{align*}
\dot{\phi} &= \begin{bmatrix} 1 & \frac{\sin \phi}{\cos \phi} & \frac{-\cos \phi}{\cos \phi} \end{bmatrix} \begin{bmatrix} \omega_x \\
\omega_y \\
\omega_z \end{bmatrix} \\
\dot{\theta} &= \begin{bmatrix} 0 & \tan \phi & -1 \end{bmatrix} \begin{bmatrix} \omega_x \\
\omega_y \\
\omega_z \end{bmatrix} \\
\dot{\psi} &= \begin{bmatrix} 0 & \frac{1}{\cos \phi} & \frac{\tan \phi}{\cos \phi} \end{bmatrix} \begin{bmatrix} \omega_x \\
\omega_y \\
\omega_z \end{bmatrix} \\
(A-24)
\end{align*}
\]

Where the order of Euler angle rotation here is yaw-pitch-roll (i.e., 1\textsuperscript{st} about \( Z \), 2\textsuperscript{nd} about \( Y \), 3\textsuperscript{rd} about \( X \)). The coefficient matrix is non-orthogonal for Euler angle transformations and possesses singularities. The singularity condition is avoided for small angle rotations transverse to the gimbal axis if the first rotation in the sequence is taken to be the gimbal bearing axis. Also, the coefficient matrix is then independent of the bearing axis Euler angle. Thus, if small angle approximations for pitch, \( \theta \), and
roll, $\phi$, are used, which represents the small angular deflections transverse to the bearing Z axis, we get for the above example,

\[
\begin{bmatrix}
\dot{\phi} \\
\dot{\theta} \\
\dot{\psi}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & -\phi \\
0 & \phi & 1
\end{bmatrix}
\begin{bmatrix}
\dot{\omega}_X \\
\dot{\omega}_Y \\
\dot{\omega}_Z
\end{bmatrix}
\]

(A-25)

where, again, it is seen that the coefficient matrix does not contain the bearing axis rotation angle, $\psi$.

It may easily be ascertained that the small angle approximation for the coefficient matrix is not unique and independent of the order of the last two rotations, i.e., pitch-roll vs. roll-pitch. This observation and the assumption that the transverse axis rotations will be truly small and oscillatory, are motivations for setting the small angles in the coefficient matrix to zero. The coefficient matrix is then seen to be the identity matrix.

The spacecraft attitude errors are obtained in a similar fashion, however the attitude errors are referenced to the local orbit frame. Thus, the attitude kinematic rate equations are of the form of equation (A-24) except the rate error vector on the right side of equation (A-24) is

\[
\Delta \dot{\omega} = \dot{\omega}_3 - C_0 \dot{\omega}_0
\]

(A-26)

which is reflected in Figure 7 as the spacecraft attitude error rate.