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**In this report, the authors approximated the distribution of certain power of
the likelihood ratio test statistic for sphericity of the complex multivariate
normal distribution with a beta type series in the null and nonnull cases.
Applications of these results to multiple time series are also discussed.**

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ASYMPTOTIC DISTRIBUTION OF THE LIKELIHOOD RATIO
TEST STATISTIC FOR SPHERICITY OF COMPLEX
MULTIVARIATE NORMAL DISTRIBUTION*

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1. Introduction

It is known (e.g. see Brillinger (1974)) that certain suitably defined estimates of the spectral density matrix of the Gaussian, stationary multivariate time series are approximately distributed as complex Wishart matrices. So, the problems of inference on the covariance matrices of the complex multivariate normal distributions are closely related to the problems of inference on the spectral density matrices. For a review of the literature on complex multivariate distributions and their applications in time series, the reader is referred to Krishnaiah (1976).

Motivated by the applications in the area of inference on multiple time series, we investigate asymptotic expressions in the null and nonnull cases for the distribution of certain power of the likelihood ratio statistic for testing the hypothesis that the variables are independent and have a common variance. These expressions are in terms of beta series. In the case of null distribution, it is found that the accuracy of the approximation by taking the first term alone in the asymptotic series is sufficient for practical purposes. Here, we note that Krishnaiah, Lee and Chang (1976) approximated the null distribution of certain power of the likelihood ratio test statistic for sphericity with Pearson's Type I distribution. But this approximation is based upon empirical study. In the analogous real case, Khatri and Srivastava (1974) derived the nonnull asymptotic distribution in terms of chi-square series. In the final section of this paper, we discuss

the applications of our results to the area of inference on multiple time series.

2. Preliminaries

In this section, we define some notation and give some lemmas which are needed in the sequel.

The Mellin's integral transform of a function $f(x)$ of real variable x defined for $x > 0$ is

$$M\{f(\cdot)|t\} = \int_0^{\infty} x^{t-1} f(x) dx \quad (2.1)$$

where t is a complex variate (Titchmarsh (1937)).

Lemma 1. Let $y^{c-1} f(y)$ be a measurable function in $(0, \infty)$ and

$$\int_0^{\infty} y^{c-1} f(y) dy < \infty .$$

Also let $f(y)$ be of bounded variation in the neighborhood of the point $y = x$. Then

$$\frac{1}{2}\{f(x+0) + f(x-0)\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M\{f(\cdot)|t\} x^{-t} dt$$

where $M\{f(\cdot)|t\}$ for $t = c + iv$ exists.

In the sequel, we shall assume that $f(x)$ is absolutely continuous in the interval $(0,1)$. Hence $f(x)$ is of bounded variation in the neighborhood of x of interest. Furthermore,

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} M\{f(\cdot)|t\} x^{-t} dt. \quad (2.2)$$

Note that when $f(x) = (1-x)^{b-1}$, $0 < x < 1$. Then

$$M\{f(\cdot)|t\} = \int_0^1 x^{t-1} (1-x)^{b-1} dx = \frac{\Gamma(t)\Gamma(b)}{\Gamma(t+b)}, \quad (2.3)$$

for $\text{Real}(t) > 0$, and $\text{Real}(b) > 0$. Hence

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(t)}{\Gamma(t+b)} x^{-t} dt = \frac{(1-x)^{b-1}}{\Gamma(b)} \quad (2.4)$$

for $c > 0$.

Lemma 2. Let $\phi(t) = \int x^t p(x) dx$ be the moment function of a random variable x with density $p(x)$. If

$$\phi(t) = O(t^{-\nu})$$

with $\text{Real}(t)$ tending to ∞ , then $\phi(t)$ can be expanded as a factorial series of the form

$$\phi(t) = \sum_{i=0}^{\infty} R_i \Gamma(t+a)/\Gamma(t+a+\nu+i) \quad (2.5)$$

where a is any constant (Nair (1940)).

Lemma 3. Let the series $\sum_{i=1}^{\infty} \alpha_i x^i$ converge to the function $g(x)$ in the neighborhood of $x = 0$ (or be its asymptotic expansion when $x = 0$). Then

$$e^{g(x)} = 1 + \sum_{i=1}^{\infty} \beta_i x^i \quad (2.6)$$

where the coefficients β_j satisfy the recurrence relation

$$\beta_j = \frac{1}{j} \sum_{k=1}^j k \alpha_k \beta_{j-k}, \quad \beta_0 = 1. \quad (2.7)$$

We use the following notations as defined in James (1964). The complex multivariate gamma function $\tilde{\Gamma}_p(a)$ is given by

$$\tilde{\Gamma}_p(a) = \pi^{p(p-1)/2} \prod_{i=1}^p \Gamma(a-i+1).$$

The complex multivariate hypergeometric coefficient is given by

$$[a]_{\kappa} = \prod_{i=1}^p (a-i+1)_{k_i}$$

where

$$(a)_k = a(a+1)\dots(a+k-1)$$

$\kappa = (k_1, \dots, k_p)$ is a partition of the integer k such that $k_1 \geq \dots \geq k_p \geq 0$ and $k = k_1 + \dots + k_p$. The transpose and conjugate of a complex matrix B are denoted by B' and \bar{B} respectively. Also, let $\tilde{C}_{\kappa}(A)$ denote the zonal polynomial of a Hermitian matrix A , (i.e., $A = \bar{A}'$). In addition,

$$\begin{aligned} {}_r\tilde{F}_q(a_1, \dots, a_r; b_1, \dots, b_q; A) &= \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[a_1]_{\kappa} \dots [a_r]_{\kappa} \tilde{C}_{\kappa}(A)}{[b_1]_{\kappa} \dots [b_q]_{\kappa} k!} \\ {}_r\tilde{F}_q(a_1, \dots, a_r; b_1, \dots, b_q; A, B) &= \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[a_1]_{\kappa} \dots [a_r]_{\kappa}}{[b_1]_{\kappa} \dots [b_q]_{\kappa}} \frac{\tilde{C}_{\kappa}(A) \tilde{C}_{\kappa}(B)}{\tilde{C}_{\kappa}(I_p) k!} \end{aligned}$$

where $a_1, \dots, a_r, b_1, \dots, b_q$ are real or complex constants. Throughout this paper, $\text{etr } B$ denotes the exponential of the trace of B .

Lemma 4. Let $\Sigma: p \times p$ be a Hermitian matrix.

Then

$${}_0\tilde{F}_0(-\Sigma^{-1}, L) = e^{-\text{tr}L/q} {}_0\tilde{F}_0\left(\frac{1}{q} M, L\right) \quad (2.8)$$

where $M = I - q\Sigma^{-1}$, $L: p \times p$ is a positive definite Hermitian matrix, q is a constant and

$${}_0\tilde{F}_0(B, D) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_{\kappa}(B) \tilde{C}_{\kappa}(D)}{\tilde{C}_{\kappa}(I_p) k!} \quad (2.9)$$

Proof: By using the splitting formula (James (1964)), we know that

$${}_0\tilde{F}_0(-\Sigma^{-1}, L) = \int_{U(p)} {}_0\tilde{F}_0(-\Sigma^{-1} UL\bar{U}', I) dU \quad (2.10)$$

where

$${}_0\tilde{F}_0(-\Sigma^{-1} UL\bar{U}', I) = e^{-\text{tr } \Sigma^{-1} UL\bar{U}'},$$

dU is the invariant Haar measure on the unitary group $U(p)$ normalized to make the total measure unity. Since

$$\text{tr } \Sigma^{-1} UL\bar{U}' = \frac{1}{q} \text{tr } L - \text{tr} \left(\frac{1}{q} M UL\bar{U}' \right), \quad (2.11)$$

where $M = I - q\Sigma^{-1}$. We prove the result by using Eq. (2.11) in Eq. (2.10).

Lemma 5. Let V be a $p \times p$ positive definite Hermitian matrix, and $T: p \times p$ be an arbitrary complex symmetric matrix. Then

$$\begin{aligned} & \int_{\bar{V}'=V>0} \exp(-\text{tr}V/q) |V|^{b-p} (\text{tr } V)^m \tilde{C}_\kappa(TV) dV \\ &= \tilde{\Gamma}_p(b, \kappa) \Gamma(bp+k+m) \tilde{C}_\kappa(T) q^{bp+k+m} / \Gamma(bp+k) \end{aligned} \quad (2.12)$$

where

$$\tilde{\Gamma}_p(b, \kappa) = \pi^{\frac{1}{2}p(p-1)} \prod_{i=1}^p \Gamma(b-i+1+k_i), \quad \kappa = (k_1, \dots, k_p).$$

Proof. We know (see Khatri (1966)) that

$$\int_{\bar{V}'=V>0} \exp(-\text{tr } V) |V|^{b-p} \tilde{C}_\kappa(VT) dV = \tilde{\Gamma}_p(b, \kappa) \tilde{C}_\kappa(T).$$

Substituting $Z^{\frac{1}{2}}VZ^{\frac{1}{2}}$ for V in the above equation with Jacobian $|Z|^p$ (Khatri (1965)) where $Z^{\frac{1}{2}}$ is a Hermitian positive definite matrix, we have

$$\int_{\bar{V}'=V>0} \exp(-\text{tr } ZV) |V|^{b-p} \tilde{C}_{\kappa}(VT) dV = \tilde{\Gamma}_p(b, \kappa) \tilde{C}_{\kappa}(TZ^{-1}) |Z|^{-b}.$$

Now take $Z = (\frac{1}{q} - x)$

$$\int_{\bar{V}'=V>0} \exp(-\text{tr}(\frac{1}{q} - x)V) |V|^{b-p} \tilde{C}_{\kappa}(VT) dV = \tilde{\Gamma}_p(b, \kappa) \tilde{C}_{\kappa}(T) (\frac{1}{q} - x)^{-(bp+k)}$$

Equating the coefficient of $x^m/m!$ for both sides of the above equation we obtain (2.12).

Lemma 6. For any integer r , variate x and Hermitian positive definite V , we have

$$\sum_{k=r}^{\infty} \sum_{\kappa} \frac{x^k \tilde{C}_{\kappa}(V)}{(k-r)!} = x^r (\text{tr } V)^r \text{etr}(xV) \quad (2.13)$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{x^k \tilde{a}_1(\kappa) \tilde{C}_{\kappa}(V)}{k!} = (x^2 \text{tr } V^2 - x \text{tr } V) \text{etr}(xV) \quad (2.14)$$

$$\begin{aligned} \sum_{k=r}^{\infty} \sum_{\kappa} \frac{x^k \tilde{a}_1(\kappa) \tilde{C}_{\kappa}(V)}{(k-r)!} &= \{x^{r+2} \text{tr } V^2 (\text{tr } V)^{r-x^{r+1}} (\text{tr } V)^{r+1} \\ &+ 2r x^{r+1} \text{tr } V^2 (\text{tr } V)^{r-1} - r x^r (\text{tr } V)^r \\ &+ r(r-1)x^r \text{tr } V^2 (\text{tr } V)^{r-2}\} \text{etr}(xV) \end{aligned} \quad (2.15)$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{x^k \tilde{a}_2(\kappa) \tilde{C}_{\kappa}(V)}{k!} = \{x^4 (\text{tr } V^2)^2 + 4x^3 \text{tr } V^3 - 2x^3 \text{tr } V \text{tr } V^2$$

$$+ 3x^2(\text{tr } V)^2 - 4x^2 \text{tr } V^2 + x \text{tr } V \} \text{etr}(xV) \quad (2.16)$$

$$\sum_{k=0}^{\infty} \sum_{\kappa} \frac{x^k \tilde{a}_2(\kappa) \tilde{C}_{\kappa}(V)}{k!} = \{ 2x^3 \text{tr } V^3 + 3x^2(\text{tr } V)^2 - 3x^2 \text{tr } V^2 + 2x \text{tr } V \} \text{etr}(xV) \quad (2.17)$$

where

$$\tilde{a}_1(\kappa) = \sum_{j=1}^p k_j (k_j - 2j) \quad (2.18)$$

$$\tilde{a}_2(\kappa) = 2 \sum_{j=1}^p k_j (k_j^2 - 3jk_j + 3j^2)$$

and κ was defined earlier.

The above lemma was proved by Hayakawa (1972).

3. Asymptotic Null Distribution of the Likelihood Ratio Test Statistic for Sphericity

In this section, we derive an asymptotic expression for the null distribution of the likelihood ratio statistic for testing the hypothesis of sphericity for complex multivariate normal distribution. The expression obtained is in the form of a beta series.

Let \underline{Z} : $p \times 1$ be distributed as a complex multivariate normal with mean vector $\underline{\mu}$ and covariance matrix Σ . The density of \underline{Z} in this case is known (see Wooding (1956)) to be

$$f(\underline{Z}) = \frac{1}{\pi^p |\Sigma|} \text{etr}[-\Sigma^{-1}(\underline{Z}-\underline{\mu})(\bar{\underline{Z}}-\bar{\underline{\mu}})'] \quad (3.1)$$

Next, let $\underline{Z}_1, \dots, \underline{Z}_N$ be N independent observations on \underline{Z} and let

$$A = \sum_{t=1}^N (\underline{Z}_t - \underline{Z}_{\cdot})(\bar{\underline{Z}}_t - \bar{\underline{Z}}_{\cdot})' = (A_{ij}) \quad (3.2)$$

where $\bar{\underline{Z}}$ denotes the conjugate of \underline{Z} , and

$$\underline{NZ}_{\cdot} = \sum_{t=1}^N \underline{Z}_t$$

We are interested in testing the hypothesis H where $H: \Sigma = \sigma^2 I_p$ and σ^2 is unknown. The hypothesis H can be decomposed as $H = H_1 \cap H_2$ where H_1 is the hypothesis that Σ is diagonal matrix and H_2 is the hypothesis that the diagonal elements of Σ are equal given H_1 is true. The likelihood ratio test statistic for testing H_1 and H_2 are known to be λ_1 and λ_2 respectively where

$$\lambda_1 = \frac{|A|^n}{\prod_{i=1}^p A_{ii}^n} \quad (3.3)$$

$$\lambda_2 = \frac{\prod_{i=1}^p A_{ii}^n}{(\text{tr}A/p)^{pn}} \quad (3.4)$$

where $n = N-1$. The likelihood ratio test for H is

$$\lambda = \lambda_1 \lambda_2. \quad (3.5)$$

Now, let $w = \lambda^{1/n}$. Then

$$w = \frac{\prod_{i=1}^p \ell_i}{\left(\sum_{i=1}^p \ell_i/p\right)^p} \quad (3.6)$$

where $\ell_1 \geq \dots \geq \ell_p$ are the eigenvalues of A . The moments of w under the hypothesis H are known to be

$$E(w^h) = [p^{ph}/\tilde{\Gamma}_p(n)] \Gamma(np) \tilde{\Gamma}_p(n+h) / \Gamma(np+ph) \quad (3.7)$$

Next, let $u = w^{1/s}$ where w is given by Eq. (3.6) and s is a constant to be chosen to govern the rate of convergence for the resultant series. The null h^{th} moment of u is obtained by replacing h with h/s in Eq. (3.7). By using the Mellin's inversion transform (see Eq. (2.2)), the density of u becomes

$$f(u) = \frac{K(p,n)}{2\pi i} \int_{c-i\infty}^{c+i\infty} u^{-h-1} p^{ph/s} \frac{\prod_{i=1}^p \Gamma(n + \frac{h}{s} - i + 1)}{\Gamma(np + p \frac{h}{s})} dh \quad (3.8)$$

and $K(p, n) = \Gamma(np) / [\prod_{i=1}^p \Gamma(n - i + 1)]$. Set $m = n - \delta$,
 $d = c + ms$ and $m + \frac{h}{s} = \frac{t}{s}$, where δ is also a converging
 factor to be chosen for the resultant series. Then, we have

$$f(u) = K(p, n) p^{-pm} u^{sm-1} \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} u^{-t} \phi(t) dt \quad (3.9)$$

and

$$\phi(t) = p^{pt/s} \frac{\prod_{i=1}^p \Gamma(\frac{t}{s} + \delta - i + 1)}{\Gamma(p\frac{t}{s} + \delta p)} \quad (3.10)$$

By the use of the formula for the asymptotic expansion of
 gamma function

$$\begin{aligned} \log \Gamma(x + b) &= \log \sqrt{2\pi} + (x + b - \frac{1}{2}) \log x - x \\ &- \sum_{r=1}^{\infty} (-1)^r \frac{B_{r+1}(b)}{r(r+1) x^r} \end{aligned} \quad (3.11)$$

for b bounded and $B_r(b)$ is the Bernoulli polynomial of degree
 r . So, we have

$$\log \phi(t) = \log(2\pi)^{\frac{p-1}{2}} s^v p^{-\delta p + \frac{1}{2}} + \log t^{-v} + \sum_{r=1}^{\infty} \frac{A_r}{t^r} \quad (3.12)$$

where

$$v = (p^2 - 1)/2 \quad (3.13)$$

$$A_r = \frac{(-1)^r s^r}{r(r+1)} \left[\frac{B_{r+1}(\delta p)}{p^r} - \sum_{i=1}^p B_{r+1}(\delta - i + 1) \right] \quad (3.14)$$

Hence

$$\phi(t) = (2\pi)^{\frac{p-1}{2}} s^v p^{-\delta p + \frac{1}{2}}$$

$$\times t^{-\nu} \left[1 + \sum_{r=1}^{\infty} \frac{Q_r}{t^r} \right]. \quad (3.15)$$

The coefficient Q_r can be obtained by the recursive equation Eq. (2.7),

$$Q_r = \frac{1}{r} \sum_{\ell=1}^r \ell A_{\ell} Q_{r-\ell}, \quad Q_0 = 1. \quad (3.16)$$

Since $\phi(t) = O(t^{-\nu})$, we can write $\phi(t)$ as follows by applying Lemma 2:

$$t^{-\nu} \left\{ 1 + \sum_{r=1}^{\infty} \frac{Q_r}{t^r} \right\} = \sum_{i=0}^{\infty} R_i \frac{\Gamma(t+a)}{\Gamma(t+a+\nu+i)} \quad (3.17)$$

and a is a constant to be chosen to govern the rate of convergence for the resultant series. Using Eq. (3.11) to expand the gamma function on the right hand side of Eq. (3.17), we obtain

$$\log \frac{\Gamma(t+a)}{\Gamma(t+a+\nu+i)} = -(\nu+i) \log t + \sum_{j=1}^{\infty} \frac{A_{ij}}{t^j} \quad (3.18)$$

where

$$A_{ij} = \frac{(-1)^j}{j(j+1)} [B_{j+1}(\nu+a+1) - B_{j+1}(a)]. \quad (3.19)$$

Thus

$$\frac{\Gamma(t+a)}{\Gamma(t+a+\nu+i)} = t^{-(\nu+i)} \left[1 + \sum_{j=1}^{\infty} \frac{C_{ij}}{t^j} \right] \quad (3.20)$$

and C_{ij} can be recursively computed by Eq. (2.7) as

$$C_{ij} = \frac{1}{j} \sum_{\ell=1}^j \ell A_{i\ell} C_{i,j-\ell}; \quad C_{i0} = 1. \quad (3.21)$$

Substituting Eq. (3.20) in Eq. (3.17) and equating the coefficient of same powers of t , R_i is determined explicitly as

$$\sum_{j=0}^i R_{i-j} C_{i-j,j} = Q_i, \quad R_0 = 1. \quad (3.22)$$

Now using Eq. (3.17), (3.15) in Eq. (3.9) and noting that the term by term integration is valid since a factorial series is uniformly convergent in a half-plane (Doetsch (1971)), we have

$$f(u) = K(p,n) K_1 u^{sm-1} \sum_{j=0}^{\infty} R_j \frac{1}{2\pi i} \int_{d-i\infty}^{d+i\infty} u^{-t} \frac{\Gamma(t+a)}{\Gamma(t+a+v+j)} dt \quad (3.23)$$

where $K_1 = (2\pi)^{\frac{p-1}{2}} s^v p^{-p(m+\delta)+\frac{1}{2}}$. Now using Eq. (2.4) in the above integral, we have

$$f(u) = K(p,n) K_1 \sum_{j=0}^{\infty} R_j u^{sm+a-1} (1-u)^{v+j-1} / \Gamma(v+j), \quad 0 \leq u \leq 1. \quad (3.24)$$

Thus the c.d.f of u in terms of incomplete beta functions

$I_x(\cdot, \cdot)$ is

$$\text{Prob}(u \leq x) = K(p,n) K_1 \sum_{j=0}^{\infty} R_j I_x(sm+a, v+j) \frac{\Gamma(sm+a)}{\Gamma(sm+a+v+j)} \quad (3.25)$$

where

$$I_x(\alpha, \beta) = \int_0^x \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1} (1-u)^{\beta-1} du. \quad (3.26)$$

Further expansion of $K(p,n) = \frac{\Gamma(mp+\delta p)}{\prod_{i=1}^p \Gamma(m+\delta-i+1)}$ and

$\frac{\Gamma(sm+a)}{\Gamma(sm+a+v+j)}$ gives us that

$$\log K(p,n) = \log \left[(2\pi)^{\frac{1-p}{2}} p^{p(m+\delta)-\frac{1}{2}} m^v \right] + \sum_{r=1}^{\infty} \frac{A_r^*}{m^r}$$

where

$$A_r^* = \frac{(-1)^r}{r(r+1)} \left[\sum_{i=1}^p B_{r+1}(\delta-i+1) - \frac{B_{r+1}(\delta p)}{p^r} \right] = -\frac{A_r}{s^r} \quad (3.27)$$

and

$$K(p, n) = (2\pi)^{\frac{1-p}{2}} p^{p(m+\delta)-\frac{1}{2}} m^v \left[1 + \sum_{r=1}^{\infty} \frac{Q_r^*}{m^r} \right]$$

where

$$Q_r^* = \frac{1}{r} \sum_{\ell=1}^r \ell A_\ell^* Q_{r-\ell}^*, \quad Q_0^* = 1 \quad (3.28)$$

$$\log \frac{\Gamma(sm+a)}{\Gamma(sm+a+v+j)} = \log (ms)^{-(v+j)} + \sum_{r=1}^{\infty} \frac{A_{jr}^*}{m^r} \quad (3.29)$$

$$A_{jr}^* = \frac{(-1)^r}{s^r r(r+1)} [B_{r+1}(a+v+j) - B_{r+1}(a)] = \frac{A_{jr}}{s^r} \quad (3.30)$$

and

$$\frac{\Gamma(sm+a)}{\Gamma(sm+a+v+j)} = (ms)^{-(v+j)} \left[1 + \sum_{r=1}^{\infty} \frac{C_{jr}^*}{m^r} \right] \quad (3.31)$$

$$C_{jr}^* = \frac{1}{r} \sum_{\ell=1}^r \ell A_{j\ell}^* C_{j, r-\ell}^* = \frac{C_{jr}}{s^r}, \quad C_{j0}^* = 1 \quad (3.32)$$

Hence, Eq. (3.25) is of the form

$$\begin{aligned} \text{Prob}(u \leq x) &= \left(1 + \sum_{r=1}^{\infty} \frac{Q_r^*}{m^r} \right) \sum_{j=0}^{\infty} R_j I_x(sm+a, v+j) (ms)^{-j} \\ &\times \left(1 + \sum_{r=1}^{\infty} \frac{C_{jr}^*}{(sm)^r} \right) \\ &= I_x(sm+a, v) + \sum_{i=1}^{\infty} \frac{1}{m^i} G_i \end{aligned} \quad (3.33)$$

where

$$G_i = \sum_{j=0}^i R_{i-j} I_x(sm+a, v+i-j) \sum_{\ell=0}^j \frac{Q_\ell^* C_{i-j, j-\ell}}{s^{i-\ell}}. \quad (3.34)$$

The exact c.d.f. can be calculated using Eq. (3.33). When n is small, a suitable choice of δ will make m large in order to expedite the convergence of the series in Eq. (3.33).

We will now examine the first few terms of Eq. (3.33) when n is large. We know that

$$G_1 = I_x(sm+a, v) \left(\frac{C_{01}}{s} + Q_1^* \right) + R_1 I_x(sm+a, v+1)/s \quad (3.35)$$

$$G_2 = I_x(sm+a, v) \left(\frac{C_{02}}{s^2} + \frac{Q_1^* C_{01}}{s} + Q_2^* \right) + R_1 I_x(sm+a, v+1) \left(\frac{C_{11}}{s^2} + \frac{Q_1^*}{s} \right) + R_2 I_x(sm+a, v+2)/s^2 \quad (3.36)$$

$$G_3 = I_x(sm+a, v) \left(\frac{C_{03}}{s^3} + \frac{Q_1^* C_{02}}{s^2} + \frac{Q_2^* C_{01}}{s} + Q_3^* \right) + R_1 I_x(sm+a, v+1) \left(\frac{C_{12}}{s^3} + \frac{Q_1^* C_{11}}{s^2} + \frac{Q_2^*}{s} \right) + R_2 I_x(sm+a, v+2) \left(\frac{C_{21}}{s^3} + \frac{Q_1^*}{s^2} \right) + R_3 I_x(sm+a, v+3)/s^3. \quad (3.37)$$

From Eqs. (3.27) and (3.28), we obtain

$$Q_1^* = A_1^* = -\frac{1}{2} \left[\sum_{i=1}^p B_2(\delta-i+1) - \frac{B_2(\delta p)}{p} \right]. \quad (3.38)$$

Using Eqs. (3.14) and (3.16), we obtain

$$Q_1 = A_1 = -sA_1^* = -sQ_1^* \quad (3.39)$$

Also, from Eq. (3.22), we have

$$R_1 + C_{01} = Q_1 \quad (3.40)$$

Eqs. (3.19) and (3.21) give

$$C_{01} = A_{01} = -\frac{1}{2} [B_2(v+a) - B_2(a)]. \quad (3.41)$$

Using the Bernoulli polynomial $B_2(x) = x^2 - x + \frac{1}{6}$ in

Eqs. (3.38) and (3.41) and set

$$\delta_0 = (2p^2 + 1)/6p \quad (3.42)$$

$$a_0 = (1-v)/2 = (3-p^2)/4 \quad (3.43)$$

We obtain

$$A_1^* = Q_1^* = C_{01} = 0 \quad (3.44)$$

$$A_1 = Q_1 = R_1 = 0.$$

Using Eqs. (3.14), (3.16), (3.19), (3.21), (3.22)

and (3.44) we obtain

$$R_2 + C_{02} = Q_2 \quad (3.45)$$

$$C_{02} = A_{02} = \frac{1}{6} [B_3(v+a_0) - B_3(a_0)] \quad (3.46)$$

$$Q_2 = A_2 = \frac{s^2}{6} \left[\frac{B_3(\delta_0 p)}{p^2} - \sum_{i=1}^p B_3(\delta_0 - i + 1) \right]. \quad (3.47)$$

By equating Eq. (3.46) and Eq. (3.47) and using $B_3(x)$

$= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$, we obtain,

$$s^2 = \frac{3p^2}{4} \frac{(p^6 - 3p^4 - p^2 + 3)}{(2p^6 - 6p^4 + 3p^2 + 1)}. \quad (3.48)$$

With the above choice of s^2 , $R_2 = 0$.

Now

$$\frac{C_{02}}{s_0} = \frac{Q_2}{s_0} = \frac{A_2}{s_0} = -A_2^* = -Q_2^* \quad (3.49)$$

where s_0 is the positive root of (3.48). Hence

$$\frac{C_{02}}{s_0} + Q_2^* = 0. \quad (3.50)$$

Now in Eq. (3.22), we have

$$R_3 = Q_3 - C_{03}. \quad (3.51)$$

From Eqs. (3.14) and (3.16), we have

$$Q_3 = A_3 = -s_0^3 A_3^* = -s_0^3 Q_3^* \quad (3.52)$$

and hence

$$R_3 = -s_0^3 \left(Q_3^* + \frac{C_{03}}{s_0} \right) \quad (3.53)$$

Using the identities given above, it is seen that

$$G_1 = G_2 = 0 \quad (3.54)$$

$$G_3 = \frac{R_3}{s_0} \left(I_x(s_0 m_0 + a_0, \nu+3) - I_x(s_0 m_0 + a_0, \nu) \right) \quad (3.55)$$

where

$$m_0 = n - \delta_0. \quad (3.56)$$

Thus by so choosing δ_0 , a_0 and s_0 , we have the c.d.f. of u in asymptotic form as

$$\begin{aligned} \text{Prob}(u \leq x) &= I_x(s_0 m_0 + a_0, \nu) \\ &+ \frac{1}{m_0} \frac{R_3}{s_0} \left[I_x(s_0 m_0 + a_0, \nu+3) \right. \\ &\left. - I_x(s_0 m_0 + a_0, \nu) \right] + O(m_0^{-4}) \quad (3.57) \end{aligned}$$

Note that when $p = 2$, then $s_0 = 1$, $a_0 = -1/4$, $\delta_0 = 3/4$, $v = 3/2$. So, Eq. (3.10) becomes

$$\phi(t) = 2^{2t} \frac{\Gamma(t+\delta_0) \Gamma(t+\delta_0-1)}{\Gamma(2(t+\delta_0))} \quad (3.58)$$

Using the duplicating formula that

$$\Gamma(2z) = \pi^{-1/2} 2^{2z-1} \Gamma(z+1/2) \quad (3.59)$$

we have

$$\phi(t) = 2^{1-2\delta_0} \pi^{1/2} \frac{\Gamma(t-(1/4))}{\Gamma(t+(5/4))} \quad (3.60)$$

But, using Eqs. (3.15) and (3.17), we have

$$\phi(t) = \pi^{1/2} 2^{1-2\delta_0} \left(\frac{\Gamma(t-(1/4))}{\Gamma(t+(5/4))} + \sum_{i=1}^{\infty} R_i \frac{\Gamma(t-(1/4))}{\Gamma(t+(5/4)+i)} \right). \quad (3.61)$$

Hence

$$\sum_{i=1}^{\infty} R_i \frac{\Gamma(t-(1/4))}{\Gamma(t+(5/4)+i)} = 0.$$

For $p = 2$, Eq. (3.57) becomes

$$\text{Prob}(u \leq x) = I_x(n-1, 3/2) \quad (3.62)$$

which is the exact distribution of w (see Nagarsenker and Das (1975)).

Krishnaiah, Lee and Chang (1976) approximated the distributions of certain powers of the likelihood ratio statistics for testing several hypotheses in multivariate analysis with Pearson's type I distribution. The accuracy of this approximation was found to be quite good. In the sequel, we will refer to the above approximation as KLC approximation.

Table 1 gives a comparison of the accuracy of the approximations by taking the first term and the first two terms respectively in (3.57).

In the table, the constant \tilde{w} is defined by

$$c = \exp(-\tilde{w}/2) \quad \text{where } P[w \leq c] = \alpha$$

and the values of \tilde{w} are taken from the tables of Krishnaiah, Lee and Chang (1976). When the first term in the expansion (3.57) is used, the value of $P(w \leq c)$ is denoted by α_1 whereas the corresponding value obtained by using the first two terms is denoted by α_2 .

TABLE 1

Significance Level Associated with the Asymptotic Expression for the Likelihood Ratio Test for Sphericity

n	p=3, $\alpha=0.05$			p=6, $\alpha=0.05$		
	\tilde{w}	α_1	α_2	\tilde{w}	α_1	α_2
10	1.745	.0499	.0500	6.512	.0476	.0486
15	1.115	.0500	.0500	3.895	.0495	.0498
20	.820	.0499	.0499	2.791	.0499	.0500
28	.576	.0500	.0500	1.924	.0499	.0500

TABLE 1 (Continued)

n	p=3, $\alpha=0.01$			p=6, $\alpha=0.01$		
	\tilde{w}	α_1	α_2	\tilde{w}	α_1	α_2
10	2.263	.0100	.0100	7.531	.0092	.0095
15	1.445	.0100	.0100	4.491	.0098	.0099
20	1.062	.0100	.0100	3.216	.0099	.0100
28	.746	.0100	.0100	2.216	.0100	.0100

The above table indicates that the first term in the asymptotic expression alone gives a good approximation. KLC approximation is based upon approximating certain power of w with Pearson's Type I distribution by using empirical methods whereas the approximation given in this paper is analytic in nature.

4. Asymptotic Nonnull Distribution of the Likelihood Ratio Test Statistic For Sphericity

In this section, we derive the asymptotic distribution of the likelihood ratio test statistic for sphericity of the complex multivariate normal distribution under the following sequences of local alternative hypotheses:

$$(i) (I - q\Sigma^{-1}) = V/m, \quad (ii) (I - q^{-1}\Sigma) = W/m$$

where V and W are fixed matrices as $m \rightarrow \infty$ and $0 < q < \infty$. The expressions obtained are in terms of beta series. In the analogous real cases, Khatri and Srivastava (1974) obtained asymptotic expressions in terms of chi-square series. To derive the asymptotic distribution in the complex case, we need the following lemma in the sequel.

Lemma 7. Let $A: p \times p$ be distributed as a complex Wishart $CW_p(A; n, \Sigma)$. Then the non-null h^{th} moment of w defined in Eq. (3.6) is

$$E(w^h) = \frac{p^{ph}}{\Gamma_p(n)} |q\Sigma^{-1}|^n \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_{\kappa}(M)}{k!} \frac{\Gamma(np+k)}{\Gamma(np+k+ph)} \tilde{\Gamma}_p(n+h, \kappa) \quad (4.1)$$

where $M = I - q\Sigma^{-1}$.

Proof: The distribution of the eigenvalues $\ell_1, \ell_2, \dots, \ell_p$ of A is (James (1964))

$$K(p, n, \Sigma) {}_0\tilde{F}_0(-\Sigma^{-1}, L) |L|^{n-p} \prod_{i < j} (\ell_i - \ell_j)^2 \prod_{i=1}^p d\ell_i \quad (4.2)$$

where

$$K(p, n, \Sigma) = \frac{|\Sigma|^{-n} \pi^{p(p-1)}}{\tilde{\Gamma}_p(n) \tilde{\Gamma}_p(p)}$$

and $L = \text{diag} (\ell_1, \ell_2, \dots, \ell_p)$

Using Lemma 4, Eq. (4.2) can be written as

$$K(p, n, \Sigma) e^{-\text{tr } L/q} {}_0\tilde{F}_0(M/q, L) |L|^{n-p} \prod_{i < j} (\ell_i - \ell_j)^2 \prod_{i=1}^p d\ell_i \quad (4.3)$$

where $M = I - q\Sigma^{-1}$.

Multiply Eq. (4.3) by w^h , also expand ${}_0\tilde{F}_0(M/q, L)$ as in Eq. (2.9) and perform the transformation $L = UV\bar{U}'$ where U is unitary and V is Hermitian positive definite matrix. The Jacobian of the transformation is

$$J(V; L, U) = \prod_{i < j} (\ell_i - \ell_j)^2 h_2(U)$$

and the integration on U is

$$\int_{U(p)} h_2(U) dU = \pi^{p(p-1)} / \tilde{\Gamma}_p(p)$$

(see Khatri (1965)).

We have

$$E(w^h) = \frac{p^{ph} |\Sigma^{-1}|^n}{\tilde{\Gamma}_p(n)} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_{\kappa}(M)}{q^k \tilde{C}_{\kappa}(I) k!} \int_{V=\bar{V}'>0} \exp(-\text{tr } V/q) |V|^{n+h-p} (\text{tr } V)^{-ph} \times \tilde{C}_{\kappa}(V) dV.$$

Applying Lemma 5 to the integral, we obtain Eq. (4.1). Note that for $q = 1$, we get the expression as in Pillai and Nagasenker (1971).

We will first derive the asymptotic distribution of w when $(I - q\Sigma^{-1}) = V/m$. Let $u = w^{1/s_0}$ where s_0 is the positive square root of the right side of (3.48) and w was defined by (3.6) From the non-null h^{th} moment of w given in Eq. (4.1),

we use the inverse Mellin transform on $E(u^h) = E(w^{h/s_0})$,
to get the density of u as

$$f(u) = K_*(p, n, \Sigma) \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_{\kappa}(M)}{k!} \Gamma(np+k) \cdot \psi(u) \quad (4.4)$$

where

$$\psi(u) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} u^{-h-1} p^{ph/s_0} \frac{\prod_{i=1}^P \Gamma(n+(h/s_0)+k_i-i+1)}{\Gamma(np+k+(ph/s_0))} dh \quad (4.5)$$

and

$$K_*(p, n, \Sigma) = |q \Sigma^{-1}|^n / \prod_{i=1}^P \Gamma(n-i+1),$$

$$M = I - q \Sigma^{-1} = V/m_0. \quad (4.6)$$

Following the same argument as in Section 3 we obtain

$$\begin{aligned} \text{Prob}(u \leq x) &= |q \Sigma^{-1}|^n \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\tilde{C}_{\kappa}(M)}{k!} m_0^k \left\{ 1 + \sum_{i=1}^{\infty} \frac{Q_i^*(k)}{m_0^i} \right\} \\ &\times \sum_{\alpha=0}^{\infty} R_{\alpha}(k) I_X(s_0 m_0 + a_0, v + \alpha) (m_0 s_0)^{-\alpha} \left[1 + \sum_{j=1}^{\infty} \frac{C_{\alpha j}^*}{m_j^j} \right] \quad (4.7) \end{aligned}$$

where we obtain the following expressions analogous to
(3.22), (3.16), (3.14), (3.28) and (3.27):

$$\sum_{j=0}^i R_{i-j}(k) C_{i-j, j} = Q_i(k); \quad R_0(k) = 1 \quad (4.8)$$

$$Q_r(k) = \frac{1}{r} \sum_{\ell=1}^r \ell A_{\ell}(k) Q_{r-\ell}(k); \quad Q_0(k) = 1 \quad (4.9)$$

and

$$A_r(k) = \frac{(-1)^r s_0^r}{r(r+1)} \left[\frac{B_{r+1}(\delta_0 p + k)}{p^r} - \sum_{i=1}^p B_{r+1}(\delta_0 + k_i - i + 1) \right] \quad (4.10)$$

$$Q_r^*(k) = \frac{1}{r} \sum_{\ell=1}^r \ell A_\ell^*(k) Q_{r-\ell}^*(k), \quad Q_0^*(k) = 1 \quad (4.11)$$

$$A_r^*(k) = \frac{(-1)^r}{r(r+1)} \left[\sum_{i=1}^p B_{r+1}(\delta_0 - i + 1) - \frac{B_{r+1}(\delta_0 p + k)}{p^r} \right]. \quad (4.12)$$

The asymptotic expansion of $|q \Sigma^{-1}|^n$ gives

$$\begin{aligned} |q \Sigma^{-1}|^n &= |I - V/m_0|^{(m_0 + \delta_0)} \\ &= \exp[-(m_0 + \delta_0) \left(\sum_{r=1}^{\infty} \frac{\text{tr} V^r}{r m_0^r} \right)] \\ &= \exp(-\text{tr} V) \left[1 - \frac{1}{m_0} (\delta_0 \text{tr} V + \frac{1}{2} \text{tr} V^2) \right. \\ &\quad \left. - \frac{1}{m_0^2} \left\{ \left(\frac{\delta_0}{2} \text{tr} V^2 + \frac{1}{3} \text{tr} V^3 \right) \right. \right. \\ &\quad \left. \left. - \frac{1}{2} (\delta_0 \text{tr} V + \frac{1}{2} \text{tr} V^2)^2 \right\} \right. \\ &\quad \left. + O(m_0^{-3}) \right] \quad (4.13) \end{aligned}$$

and

$$\tilde{C}_\kappa(M) m_0^k = \tilde{C}_\kappa(V/m_0) m_0^k = \tilde{C}_\kappa(V) \quad (4.14)$$

since the zonal polynomial $\tilde{C}_\kappa(M)$ is homogeneous of degree k .

Furthermore the use of formula for the Bernoulli polynomial

$$B_n(x+b) = \sum_{r=0}^n \binom{n}{r} B_{n-r}(x) b^r \quad (4.15)$$

gives

$$\begin{aligned}
 A_1(k) &= \frac{(-1)s_0}{2} \left[\frac{B_2(\delta_0 p+k)}{p} - \sum_{i=1}^p B_2(\delta_0 + k_i - i + 1) \right] \\
 &= A_1 - \frac{1}{2} s_0 \left[\frac{k^2}{p} - \left(1 + \frac{1}{p}\right)k - \tilde{a}_1(\kappa) \right] \quad (4.16)
 \end{aligned}$$

$$\begin{aligned}
 A_2(k) &= \frac{s_0^2}{6} \left[\frac{B_3(\delta_0 p+k)}{p^2} - \sum_{i=1}^p B_3(\delta_0 + k_i - i + 1) \right] \\
 &= A_2 + \frac{s_0^2}{6} \left[\frac{k^3}{p^2} + \frac{3B_1(\delta_0 p)k^2}{p^2} + \left(\frac{3B_2(\delta_0 p)}{p^2} - 3\delta_0^2 - 3\delta_0 - \frac{1}{2} \right)k \right. \\
 &\quad \left. - \frac{1}{2} \tilde{a}_2(\kappa) - \left(3\delta_0 + \frac{3}{2}\right) \tilde{a}_1(\kappa) \right]. \quad (4.17)
 \end{aligned}$$

From eqs. (4.11) and (4.12)

$$Q_1^*(k) = A_1^*(k) = A_1^* + \frac{1}{2p} (2 B_1(\delta_0 p)k + k^2) \quad (4.18)$$

$$\begin{aligned}
 Q_2^*(k) &= \frac{1}{2} [A_1^*(k)^2 + 2 A_2^*(k)] \\
 &= Q_2^* + \frac{1}{8p^2} (2 B_1(\delta_0 p)k + k^2)^2 \\
 &\quad - \frac{1}{6p^2} (3 B_2(\delta_0 p)k + 3 B_1(\delta_0 p)k^2 + k^3) \quad (4.19)
 \end{aligned}$$

where A_1 , A_2 , A_1^* , Q_2^* are as in Eq. (3.14), (3.27), (3.28), $\tilde{a}_1(\kappa)$ and $\tilde{a}_2(\kappa)$ are defined in Eq. (2.18). Note that in Eq. (3.44), (3.49)

$$A_1 = A_1^* = 0, \quad Q_2^* + C_{02}^* = Q_2^* + \frac{C_{02}}{s_0^2} = 0 \quad (4.20)$$

Also from Eqs. (4.8), (4.9)

$$Q_1(k) = A_1(k)$$

$$Q_2(k) = \frac{1}{2} (A_1(k)Q_1(k) + 2A_2(k))$$

$$R_1(k) = Q_1(k) - C_{01} \quad (4.21)$$

$$R_2(k) = Q_2(k) - C_{02} - (Q_1(k) - C_{01}) C_{11}.$$

Substitute these identities in Eq. (4.7) and then use Eqs. (2.13) - (2.17) for the summation over k . By neglecting the higher order terms of m_0 , we obtain an asymptotic expression of $\text{Prob}(u \leq x)$ as

$$\begin{aligned} & \text{Prob}(u \leq x) \\ &= I_x(s_0 m_0 + a_0, v) + \frac{1}{m_0} d_1 [I_x(s_0 m_0 + a_0, v) - I_x(s_0 m_0 + a_0, v+1)] \\ &+ \frac{1}{m_0^2} \sum_{i=1}^3 \alpha_i I_x(s_0 m_0 + a_0, v+i-1) + O(m_0^{-3}) \end{aligned} \quad (4.22)$$

$$\text{where } d_1 = \frac{1}{2p} (\text{tr } V)^2 - \frac{1}{2} \text{tr } V^2$$

$$\alpha_1 = \delta_0 d_1 + \frac{d_1^2}{2} + d_2 ; d_2 = \frac{1}{3p^2} (\text{tr } V)^3 - \frac{1}{3} \text{tr } V^3$$

$$\alpha_2 = \left(\frac{v+1}{2s_0} - 2\delta_0 - \frac{1}{p} - \frac{2 \text{tr } V}{p} \right) d_1 - d_1^2$$

$$\alpha_3 = \left(\frac{1}{p} + \delta_0 - \frac{v+1}{2s_0} \right) d_1 + \frac{d_1^2}{2} + d_3$$

$$d_3 = \frac{2}{3p^2} (\text{tr } V)^3 + \frac{1}{3} \text{tr } V^3 - \frac{1}{p} \text{tr } V \text{tr } V^2$$

We now discuss the asymptotic distribution of w when $I - q^{-1}\Sigma = W/m_0$. In this case, we have

$$q \Sigma^{-1} = (I - W/m_0)^{-1} = I + \frac{W}{m_0} (I - \frac{W}{m_0})^{-1}$$

$$M = I - q \Sigma^{-1} = -\frac{W}{m_0} (I - \frac{W}{m_0})^{-1} \quad (4.23)$$

We now substitute Eq. (4.23) in Eq. (4.4) and expand. It may be noted that when compared with Eq. (4.6), $-W(I-W/m_0)^{-1}$ plays the same role as V in those equations and hence the non-null asymptotic expression for the present case can be obtained by treating $-W(I-W/m_0)^{-1}$ as V in Eq. (4.22). Further expanding $-W(I-W/m_0)^{-1}$ and neglecting the higher order terms of m_0 gives

$$\text{Prob}(u \leq x)$$

$$= I_x(s_0 m_0 + a_0, v) + \frac{1}{m_0} d_1 [I_x(s_0 m_0 + a_0, v) - I_x(s_0 m_0 + a_0, v+1)]$$

$$+ \frac{1}{m_0^2} \sum_{i=1}^3 \alpha_i I_x(s_0 m_0 + a_0, v+i-1) + O(m_0^{-3}) \quad (4.24)$$

$$\text{where } d_1 = \frac{1}{2p} (\text{tr } W)^2 - \frac{1}{2} \text{tr } W^2$$

$$\alpha_1 = \delta_0 d_1 + \frac{d_1^2}{2} + d_2 ; d_2 = \frac{1}{p} \text{tr } W \text{tr } W^2 - \frac{1}{3p^2} (\text{tr } W)^3 - \frac{2}{3} \text{tr } W^3$$

$$\alpha_2 = (\frac{v+1}{2s_0} - 2\delta_0 - \frac{1}{p}) d_1 - d_1^2 + d_3 ; d_3 = \frac{1}{p^2} (\text{tr } W)^3 - \frac{2}{p} \text{tr } W \text{tr } W^2 + \text{tr } W^3$$

$$\alpha_3 = (\frac{1}{p} + \delta_0 - \frac{v+1}{2s_0}) d_1 + \frac{d_1^2}{2} + d_4$$

$$d_4 = \frac{1}{p} \text{tr } W \text{tr } W^2 - \frac{1}{3} \text{tr } W^3 - \frac{2}{3p^2} (\text{tr } W)^3$$

5. Applications in Inference on Multiple Time Series

In this section, we discuss the applications of the results of Sections 3 and 4 in drawing inference on the spectral density matrix of the multiple time series.

Let $\underline{X}'(t) = (X_1(t), \dots, X_p(t))$ ($t = 1, 2, \dots, T$) be a real stationary Gaussian multivariate time series with mean vector 0 and covariance matrix $R(\tau) = E\{X(t) X'(t+\tau)\}$. The spectral density matrix $F(\omega)$ at the frequency ω is given by

$$F(\omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} \exp(-i\tau\omega) R(\tau) \quad (5.1)$$

A well known estimate (e.g. see Wahba (1971)) of $F(\omega)$ is

$\hat{F}(\omega) = (\hat{f}_{ij}(\omega))$ where

$$\hat{f}_{ij}(\omega) = \frac{1}{(2n_1+1)} \sum_{r=-n_1}^{n_1} I_{ij}(\omega + \frac{2\pi r}{T}) \quad (5.2)$$

$$I_{ij}(\omega) = Z_i(\omega) \bar{Z}_j(\omega)$$

$$Z_j(\omega) = \frac{1}{(2\pi T)^{\frac{1}{2}}} \sum_{t=1}^T X_j(t) \exp(-it\omega).$$

where \bar{Z} denotes the conjugate of Z . Then, it is known (see Goodman (1963) and Wahba (1968)) that $(2n_1+1)\hat{F}(\omega)$ is distributed approximately as central complex Wishart matrix with $(2n_1+1)$ degrees of freedom and with $E(\hat{F}(\omega)) = F(\omega)$. So, we can test the hypothesis $H: F(\omega) = \lambda(\omega) I_p$ in a similar fashion as in the case of the test for sphericity of complex multivariate normal distribution. So, analogous

to the likelihood ratio test for sphericity, we propose the following procedure for testing H . We accept or reject H according as

$$T \geq c_1 \quad (5.3)$$

where

$$P[T \geq c_1 | H] = (1-\alpha) \quad (5.4)$$

and

$$T = \frac{|\hat{F}(\omega)|}{(\text{tr } \hat{F}(\omega)/p)^p} \quad (5.5)$$

So, the approximate asymptotic distribution of T^{1/s_0} in the null case is obtained from (3.57) by replacing n with $(2n_1+1)$ and s_0 is the square root of the right side of (3.48). Now, consider the alternative hypothesis A_1 and A_2 where $A_1: (I - q(F(\omega))^{-1}) = V(\omega)/m$ and $A_2: (I - q^{-1}F(\omega)) = W(\omega)/m$ where $m = 2n_1 + 1 - \delta$, δ is a constant, and $V(\omega)$ and $W(\omega)$ are fixed matrices as $m \rightarrow \infty$.

In the above cases, approximate asymptotic expressions for the distribution functions of T^{1/s_0} can be obtained by replacing n with $(2n_1+1)$ on the right sides of (4.22) and (4.24) respectively.

Next consider k frequencies $\omega_1, \dots, \omega_k$ where $\omega_\ell = 2\pi j_\ell/T$ and $j_\ell = (\ell-1)(2n_1+1) + (n_1+1)$. Then, it is known (e.g., see Wahba (1971)) that $\hat{F}(\omega_1), \dots, \hat{F}(\omega_k)$ are distributed independently and approximately as central complex Wishart matrices with $(2n_1+1)$ degrees of freedom and $E(\hat{F}(\omega_j)) = F(\omega_j)$. We now discuss procedures for testing H_1, \dots, H_k and H_0 simultaneously

against A_1, \dots, A_k and A_0 where

$$H_i: F(\omega_i) = \lambda(\omega_i)I_p, H_0 = \bigcap_{i=1}^k H_i$$

$$A_i: F(\omega_i) \neq \lambda(\omega_i)I_p, A_0 = \bigcup_{i=1}^k A_i$$

We accept or reject H_i according as

$$T_i \gtrless c_i$$

where

$$\begin{aligned} & P[T_i \geq c_i; i=1, 2, \dots, k | H_0] \\ & \approx \prod_{i=1}^k P[T_i \geq c_i | H_i] = (1-\alpha). \end{aligned} \quad (5.6)$$

$$T_i = \frac{|\hat{F}(\omega_i)|}{(\text{tr } \hat{F}(\omega_i)/p)^p} \quad (5.7)$$

The total hypothesis H_0 is accepted if and only if all the individual hypotheses H_i are accepted. We can compute approximate values of α for given values of c_i using the results of Section 3. The power functions of the above test under certain local alternatives can be computed using the results of Section 4. We can also test the hypothesis H_0 by using $T_0 = T_1 \dots T_k$ as test statistic. Approximate null distribution of this statistic can be obtained by following the same lines as in Section 3.

REFERENCES

- [1] Brillinger, D. R. (1974). Time Series: Data Analysis and Theory. Holt, Rinehart and Winston, New York.
- [2] Doetsch, G. (1971). Guide to the Applications of the Laplace and Z-transformations. Van Nostrand-Reinhold, New York.
- [3] Goodman, N. R. (1963). Statistical analysis based on a certain multivariate complex Gaussian distribution (an introduction). Ann. Math. Statist., 34, 152-176.
- [4] Hayakawa, T. (1972). The asymptotic distributions of the statistics based on the complex Gaussian distributions. Ann. Inst. Statist. Math., 24, 231-244.
- [5] James, A. T. (1964). Distributions of matrix variates and latent roots derived from normal samples. Ann. Math. Statist., 35, 475-501.
- [6] Khatri, C. G. (1965). Classical statistical analysis based on a certain multivariate complex Gaussian distribution. Ann. Math. Statist., 36, 98-114.
- [7] Khatri, C. G. (1966). On certain distribution problems based on positive definite quadratic functions in normal vectors. Ann. Math. Statist., 37, 468-479.
- [8] Khatri, C. G. and Srivastava, M. S. (1974). Asymptotic expansions of the non-null distributions of likelihood ratio criteria for covariance matrices. Annals of Statistics 2, 109-117.
- [9] Krishnaiah, P. R. (1976). Some recent developments on complex multivariate distributions. J. Multivariate Anal., 6, 1-30.
- [10] Krishnaiah, P. R., Lee, J. C. and Chang, K. C. (1976). The distributions of the likelihood ratio statistics for tests of certain covariance structures of complex multivariate normal populations. Biometrika 63, 543-549.
- [11] Nagarsenker, B. N. and Das, M. M. (1975). Exact distribution of sphericity criterion in the complex case and its percentage points. Communications in Statistics 4, 363-374.
- [12] Nair, U. S. (1940), Application of factorial series in the study of distribution laws in statistics. Sankhya 5, 175.

- [13] Pillai, K. C. S. and Nagarsenker, B. N. (1972). On the distributions of a class of statistics in multivariate analysis. J. Multivariate Anal., 2, 96-114.
- [14] Rao, C. R. (1951). An asymptotic expansion of the distribution of Wilks' Λ criterion. Rev. Inst. Internat. Statist., 33 177-230.
- [15] Titchmarch, E. C. (1937). Introduction to the Theory of Fourier Integrals. Oxford University Press, London.
- [16] Wahba, G. (1968). On the distributions of some statistics useful in the analysis of jointly stationary time series. Ann. Math. Statist., 39, 1849-1862.
- [17] Wahba, G. (1971). Some tests of independence for stationary multivariate time series. J. Roy. Statist. Soc. Ser. B. 33, 153-166.
- [18] Wooding, R. A. (1956). The multivariate distribution of complex normal variables. Biometrika 43, 212-215.

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