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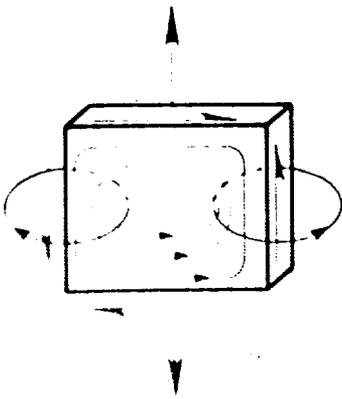
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Topical Report

21 TWO PAPERS ON EDDY CURRENT CALCULATIONS  
IN THIN PLATES .

K.Y. Yuan, J.F. Abel, F.C. Moon  
M.A. Morjaria, and S. Mukherjee

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Departments of Structural Engineering  
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Report 7, Part 1

EDDY CURRENT CALCULATIONS IN THIN CONDUCTING PLATES  
USING A FINITE ELEMENT-STREAM FUNCTION CODE

K.Y. Yuan, J.F. Abel, and F.C. Moon

presented at COMPUMAG-3, the  
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20. Abstract (continued)

*at* density, the code also calculates induced temperatures due to Joule heating and the magnetic forces on the plate. The results are compared to infrared measurements of induced currents in rectangular plates. Extension of the code to transient problems using both fast Fourier transform and direct integration methods is in progress.

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# EDDY CURRENT CALCULATIONS IN THIN CONDUCTING PLATES USING A FINITE ELEMENT-STREAM FUNCTION CODE\*

K-Y. Yuan, J.F. Abel, F.C. Moon

**Abstract** - A stream function or vector potential for the current density vector is used to develop a finite element code for calculating induced currents in thin conducting plates. While two-dimensional, the code includes self field effects for harmonic fields for skin depths on the order of the plate thickness and larger. The solution of the resulting integro-differential equation, using a Galerkin method, leads to a complex, nonsymmetric, fully populated global matrix. In addition to the current density, the code also calculates induced temperatures due to Joule heating and the magnetic forces on the plate. The results are compared to infrared measurements of induced currents in rectangular plates. Extension of the code to transient problems using both fast Fourier transform and direct integration methods is in progress.

## INTRODUCTION

The goal of ongoing research at Cornell is to develop numerical methods for the coupled analysis of forces, currents, and stresses in thin elastic structures exposed to time-dependent magnetic fields. In particular the project is directed toward time dependent magnetic forces and motions of elastic plates and shells. Both applied mathematical analysis and finite element numerical computer techniques are being used. Verification of the newly developed computer codes is carried out by direct comparison of the calculated results with experimental data and is carried out at each stage of the development. The experiments employ a new infrared scanning technique to visualize the induced eddy current patterns in the structures.

Most techniques for calculating stresses due to electromagnetic fields require a three-dimensional treatment even if the structure is one or two-dimensional such as a beam, plate, or shell [1,2]. In this research a stream function for the induced current in the conducting structure has been used which reduces the magnetic problem to a two-dimensional one and allows one to use conformable finite element meshes for the magnetic and elastic deformation fields.

Flat plate. The derivation for the case of a flat plate has been given by Moon [3]. The details are not repeated here; only the essential relationships are recapitulated.

If the mid-plane of the plate is chosen as the xy plane, one obtains

$$\underline{I} = \nabla \times (\psi \underline{n}) = -\underline{n} \times \nabla \psi \quad (1)$$

and

$$\nabla^2 \psi = \sigma h \frac{\partial}{\partial t} (B_z^0 + B_z^I) \quad (2)$$

For the determination of  $B_z^I$ , the current stream function  $\psi$  is assumed to be defined throughout three-

dimensional space, and constant across the sheet thickness. Application of the Biot-Savart law and the divergence theorem yields  $B_z^I$  for the midplane of the plate ( $z=0$ ):

$$B_z^I(x,y,0) = \frac{\mu}{h} \psi - \int \frac{\psi(x',y') d(\text{area}')}{\text{area} [(x-x')^2 + (y-y')^2 + (\frac{h}{2})^2]^{3/2}} \quad (3)$$

$\psi$  has been set to zero on the boundary of the plate in the above derivation. When the plate is multiply connected, different constant values of  $\psi$  will be assigned on each interior boundary. These boundary conditions, together with the known  $B_z^0$  and (3), are then used in (2) for the determination of the stream function  $\psi$ .

## Formulation

For steady state, harmonic current in a flat plate plate, (2) may be nondimensionalized into the following form

$$\nabla^2 \phi - 12\pi R \phi + iR \int \frac{\phi(\xi,\eta)}{\text{area} [(x-\xi)^2 + (y-\eta)^2 + 1]^{3/2}} d\xi d\eta = 12\pi R \phi(x,y) \quad (4)$$

in which the coordinates are nondimensionalized with respect to half the thickness ( $\frac{h}{2}$ );  $\psi = (\frac{hB}{2\mu}) \phi e^{i\omega t}$ ;  $B_z^0 = (\frac{B}{2}) \phi e^{i\omega t}$ ; the magnetic Reynolds number is  $R = \omega \mu \sigma h^2 / 8\pi$  which is related to skin depth  $\delta$  through  $R = \frac{1}{4\pi} (\frac{h}{\delta})^2$ ; and  $B$  is the reference magnetic field.

The finite element Galerkin method is used to solve (4).  $\phi$  is approximated globally and locally by

$$\phi = \sum_{k=1}^G M_k \phi_k \quad \phi^E = \sum_{k=1}^6 N_k^E \phi_k^E \quad (5)$$

in which  $G$  is the total number of nodal points,  $E$  denotes the  $E$ th element,  $M_k$  are the quadratic global interpolation functions generated from the local element shape functions  $N_k^E$ . Six-node triangular elements are used here. The local element shape functions are all quadratic in this case. The element algebraic equations are

$$\sum_{k=1}^6 K_{jk}^E \phi_k^E + i \sum_{k=1}^6 P_{jk}^E \phi_k^E + i \sum_{k=1}^6 Q_{jk}^E \phi_k^E = iR_k^E \quad (6)$$

in which

$$K_{jk}^E = \sum_{i=1}^6 \int_{A^E} j_{j,i} N_{j,i}^E N_{k,i}^E dA^E \quad (7a)$$

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The authors are currently with the University of Wisconsin, Madison, WI 53706 (Yuan), and Cornell University, Ithaca, NY 14853 (Abel, Moon), respectively.

$$P_{jk}^E = 2\pi R \int_{A^E} N_j^E N_k^E dA^E \quad (7b)$$

$$Q_{jk}^E = -R \iint_{\text{area}} M_k(\xi, \eta) W_j^E(\xi, \eta) d\xi d\eta \quad (7c)$$

$$R_k^E = -2\pi R \int_{A^E} N_j^E g dA^E \quad (7d)$$

The weighting function  $W_j^E$  is given by

$$W_j^E(\xi, \eta) = \int_{A^E} \frac{N_j^E(x, y)}{[(x-\xi)^2 + (y-\eta)^2 + 1]^{3/2}} dx dy \quad (8)$$

Because of the numerical difficulty associated with the sharp variation of the Kernel function in (8), the weighting function is calculated analytically within the element and numerically outside the element. (7c) is then integrated entirely by numerical quadrature.

Since  $N_j^E$  is quadratic, six basic integrations with numerators 1,  $x$ ,  $y$ ,  $x^2$ ,  $xy$ , and  $y^2$  in (8) need to be performed. To simplify the integration, each element is first mapped onto a standard triangle, which is independent of the coordinates of the nodes of the element. The integrations are then performed. The expressions obtained are then used for the calculations of (8) for all the elements.

The resulting global matrix is complex, fully populated, and nonsymmetric for general nonuniform meshes.

#### CAPABILITIES OF THE PROGRAM EDDY2

A fortran program EDDY2 has been developed based on the formulation of (5)-(8). It calculates the local and nonlocal solutions of stream function, eddy current, temperature, and pressure. As of this writing, the image solution and two-dimensional graphic output capabilities remain to be implemented. Uniform magnetic field and fields due to any number of magnetic dipoles can be handled. Magnetic fields generated from some types of coils of interest will be added.

#### Input

The geometry of the plate and the description of the external magnetic field are the two basic forms of data needed by program EDDY2. The total numbers of nodal points, load cases, and elements need to be specified. Coordinate and boundary condition must be given for each input node. Intermediate nodes may be generated for any groups of nodal points that are uniformly spaced. Element information may also be generated. Although only six-node triangles are included in the present version of EDDY2, the program has been structured so that other types of elements may be added. Element group information and the master card for each element group need, therefore, to be inputted too.

The program allows for different orders of numerical integrations. Six- and seven-point formula are now provided. The order may be specified on the master element group card.

#### Output

Values of the stream function, eddy current, temperature induced in a half cycle of the current, and time-averaged magnetic pressure are produced as output. The stream function is calculated at the nodes of the finite element mesh. Current, temperature, and pressure are evaluated at the centroid of each element. The stream function and current are calculated in complex form. The modulus and phase angle of the current are evaluated in the interest of spectral analysis for the calculation of transient currents.

#### Examples

Two problems analyzed by EDDY2 are shown below. Figure 1 shows the stream function contours for a 4 to 1 rectangular plate excited by a harmonic uniform field with magnetic Reynolds number  $R = 0.0012$ . Figures 2 and 3 give the stream function contours and isotherms for a notched plate subjected to a harmonic uniform field with  $R = 0.001$ . Only half of the plate is shown in both problems because of symmetry. In the analysis, however, the whole plates are analyzed without taking advantage of the symmetry conditions.

The CPU time is 145 seconds with 32 elements in the first problem using the IBM 370-168 computer. Most of the time is used in evaluating the nonlocal integration terms in (6). The computational time can obviously be reduced by using the symmetry conditions and by optimizing the program. Also, the quadratic rectangular element with its much simpler expressions for the nonlocal integration terms will substantially reduce the CPU time.

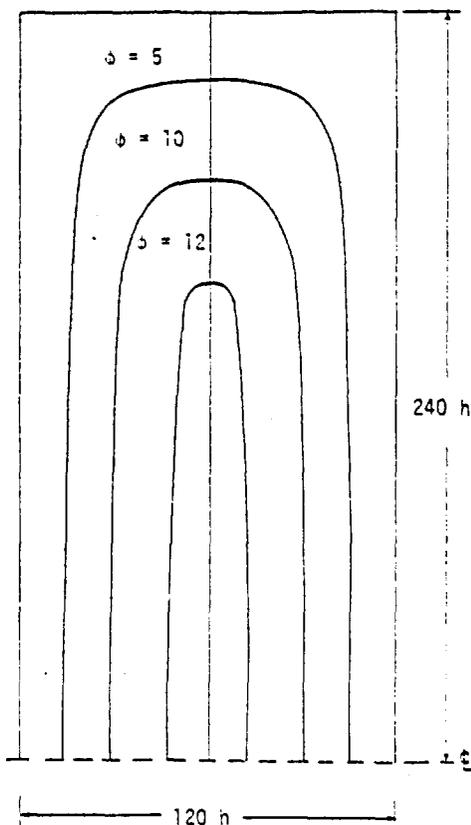


Fig. 1 Stream function contours for a long rectangular plate excited by a harmonic uniform field ( $R = 0.0012$ ).

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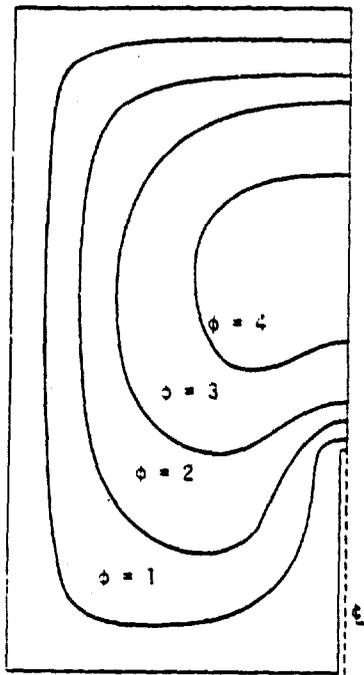


Fig. 2 Stream function contours for a notched plate excited by a harmonic uniform field (notchwidth =  $2h$ ,  $R = 0.001$ ).

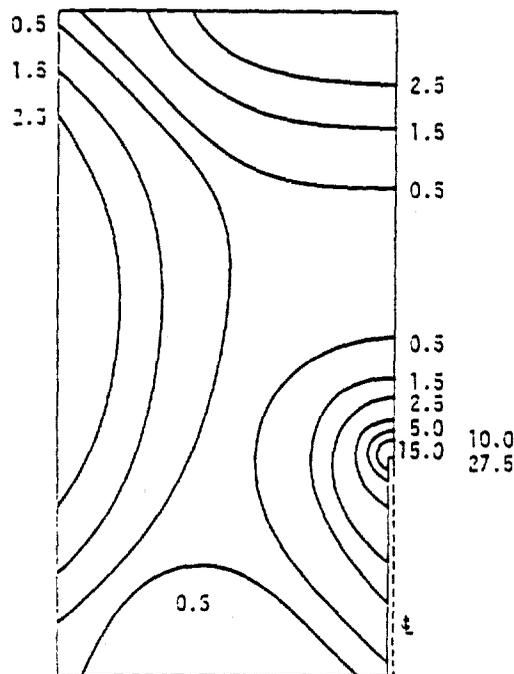


Fig. 3 Isotherms for a Notched Plate Excited by a Harmonic Uniform Field (notchwidth =  $2h$ ,  $R = 0.001$ ).

Report 7, Part 2

A BOUNDARY INTEGRAL METHOD FOR EDDY CURRENT FLOW  
AROUND CRACKS IN THIN PLATES

M.A. Morjaria, S. Mukherjee, and F.C. Moon

presented at COMPUMAG-3, the  
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→ The effect of the relative position and length of the crack, with respect to the plate width, on the eddy current density near the tips of the crack is given special attention. These results may be useful to simulate eddy current flow detection phenomena.



# A BOUNDARY INTEGRAL METHOD FOR EDDY CURRENT FLOW AROUND CRACKS IN THIN PLATES

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## ABSTRACT

A boundary element method which employs a Green's function for a crack has been developed to calculate the induced eddy current flow around cracks in thin conducting plates. The theoretical equations employ a stream function for the current density vector and is equivalent to the electric field vector potential method. A low frequency or large skin depth approximation leads to a Poisson equation for steady harmonic inductor fields. Induced currents around a crack in a square plate due to a uniform inductor field for various crack positions and sites have been calculated in this paper.

The effect of the relative position and length of the crack, with respect to the plate width, on the eddy current density near the tips of the crack is given special attention. These results may be useful to simulate eddy current flow detection phenomena.

## INTRODUCTION

The boundary element method (BEM) (also called the boundary integral equation method) has emerged as an important computational technique for electrodynamic problems. Wu et al [1] and Ancelle et al [2] have addressed magnetostatic problems by the BEM while Trowbridge [3] has considered problems by the magnetic potential method. Very recently, Salon and Schneider [4] have solved problems of eddy current flow in long prismatic conductors by the BEM based on the electric potential approach.

In this paper, we describe a powerful boundary element technique for calculating induced eddy current flows in conducting plates with through cracks using the electric potential approach. The BEM has the important advantage that only the boundary of a body (rather than the entire domain) needs to be discretised in a numerical solution procedure.

There have been some attempts to model eddy current flow around annular cracks in rods and in plates by replacing cracks by slots (see for example Ref. [5]). However, we have shown that the induced current in the vicinity of a crack leads to a singularity of current density at the crack tips [6,7]. This high concentration allows one to use eddy current testing devices such as active and passive search coils to detect the presence of cracks. It also results in a temperature hot spot which can be detected by infrared scanning [6,8]. The boundary element technique introduced by the authors [6,7] and described here allows one to model exactly the singular nature of current

density at crack tips of thin plates. This technique can handle any arbitrary shape of the plate and general magnetic fields.

In this paper we discuss application of the BEM to eddy current flow in a cracked square plate due to an uniform inductor field applied normal to the plate. A number of crack sizes to plate size configurations has been considered. Also, effect of the relative position of a crack tip to the plate edge on the induced eddy current distribution has been investigated.

## GOVERNING EQUATIONS

A thin plate with a crack in it is shown in Fig. 1. The plate is made of a conducting material of conductivity  $\sigma$ . The plate boundary can be arbitrary and its thickness (uniform) is  $h$ . The thin line crack is of length  $2a$  and can have arbitrary orientation relative to the boundary of the plate. The coordinate system for the problem is also shown in Fig. 1. The origin of coordinates lies at the center of the crack and at the midsurface of the plate.

An external, oscillatory magnetic field,  $B^0$ , is applied which induces a current density  $J$  in the plate. It is assumed that the current density is uniform across the plate thickness and that the skin depth (which is inversely proportional to the square root of the frequency) is large compared to the plate thickness.

A stream function (or electric potential) formulation is used in this problem. The stream function,  $\psi(x_1, x_2)$ , is defined as

$$\mathbf{J} = \nabla \times (\psi \mathbf{k}) = -\mathbf{k} \times \nabla \psi \quad (1)$$

This equation guarantees the conservation of charge equation  $\nabla \cdot \mathbf{J} = 0$  for charge free regions.

Using Ohm's law the governing differential equation for the stream function is obtained as [6,7]

$$\nabla^2 \psi = \sigma \frac{\partial}{\partial t} (B_3^0 + B_3^1) \quad (2)$$

In the above,  $B_3^1$  is the self magnetic field due to the current  $\mathbf{J}$ . It has been shown in ref. [9], however, that for a sinusoidal applied field, with the skin depth much greater than the thickness of the plate,  $B_3^1$  can be neglected relative to the applied field  $B_3^0$ . This assumption simplifies the problem, and, with  $B_3^0 = \hat{B}_3^0 e^{i\omega t}$  (with  $i = \sqrt{-1}$  and  $\omega$  the frequency), the spatial part of  $\psi$  satisfies a two-dimensional nonhomogeneous Poisson's equation

$$\nabla^2 \psi = L_{\omega\sigma} \hat{B}_3^0 = f(x_1, x_2) \quad (3)$$

The boundary condition requires that the current must be tangential to the plate boundary. Thus  $\psi$  is required to be constant on the boundaries  $\partial C_1$  and  $\partial C_2$ . On one boundary, the value of  $\psi$  is set to zero, while on the other boundary  $\psi = C$  and  $C$  is obtained from the assumption that the net flux flowing through the crack boundary is zero. This leads to the condition

$$\oint_{\partial C_1} \mathbf{J} \cdot d\mathbf{s} = 0 \quad (4)$$

where  $\mathbf{t}$  is an unit tangent to  $\partial C_1$  and  $s$  is the distance measured along a boundary in the anticlockwise sense. This formulation assumes that no current flows across the crack or crack tip and leads to a singularity of the  $\mathbf{J}$  field at a crack tip. This is analogous to the stress singularity in fracture mechanics. It is possible that some leakage of current occurs across a crack tip and thus relieves the singularity in actual conductors. Possible leakage of current is not considered in this paper. (It is noted here that infrared scans of eddy current flow around cracks do indeed show a large increase in temperature at the crack tips, indicating high current density at the crack tips [6].)

In summary, the boundary conditions on  $\psi$ , used in this formulation, are

$$\psi = 0 \text{ on the crack boundary } \partial C_1 \quad (5)$$

$$\frac{d\psi}{ds} = 0 \text{ on the outside boundary } \partial C_2 \quad (6)$$

$$\oint_{\partial C_1} \frac{d\psi}{dn} ds = 0 \quad (7)$$

These boundary conditions, together with the field equation (3), constitute a well posed problem.

## BOUNDARY ELEMENT FORMULATION

### Integral equations

An integral equation formulation for Poisson's equation (3) can be written as (Fig. 1) [6,7]

$$2\pi\psi(p) = \oint_{\partial C_2} K(p, Q) G(Q) ds_Q + \int_A K(p, q) f(q) dA_q \quad (8)$$

This is a single layer potential formulation where  $G$ , a source strength function on the outside boundary, must be determined from the boundary condition on it (equation 9). The points  $p$  (or  $P$ ) and  $q$  (or  $Q$ ) are source and field points, respectively, with capital letters denoting points on the boundary of the body and lower case letters denoting points inside the body. The area of the body  $B$  is denoted by  $A$ .

It has been shown [6] that  $\psi$  from equation (8) with the following kernel satisfies the boundary conditions (5) and (7) implicitly.

$$K(p, q) = \text{Re}\{\phi(z, \bar{z}, z_0)\} \quad (9)$$

$$\phi(z, \bar{z}, z_0) = 2\ln(1-r_1/\xi) - 2\ln(1-r_1\bar{\xi}) \quad (10)$$

$$\text{where } r_1 = \frac{z_0 \pm \sqrt{z_0^2 - 4}}{2}, \quad |r_1| \leq 1$$

$$\xi = \frac{z \pm \sqrt{z^2 - 4}}{2}, \quad |\xi| \leq 1$$

$\text{Re}$  denotes the real part of the complex argument,  $z$  and  $z_0$  are the source and field point coordinates, respectively, in complex notation and a superposed bar denotes, as usual, the complex conjugate of a complex quantity.

The remaining boundary condition (6) on the outside surface is satisfied by using a differentiated version of (8) and taking the limit as  $p$  inside  $B$  approaches a point  $P$  on  $\partial C_2$ . Defining

$$H_1 = \text{Im}\left(\frac{\partial\phi}{\partial z} - \frac{\partial\phi}{\partial \bar{z}}\right), \quad H_2 = -\text{Re}\left(\frac{\partial\phi}{\partial z} + \frac{\partial\phi}{\partial \bar{z}}\right) \quad (11)$$

the boundary condition (6) becomes

$$0 = \oint_{\partial C_2} H_1(P, Q) n_1(P) G(Q) ds_Q + \int_A H_2(P, q) n_2(P) f(q) dA_q \quad (12)$$

where  $n_1$  are the components of the unit outward normal to  $\partial C_2$  at some locally smooth point on it.

The current,  $\mathbf{J}$ , at a point inside the body is obtained from equations (1) and (8).

### Discretization of equations and solution strategy

The outer boundary of the body,  $\partial C_2$ , is divided into  $N_2$  straight boundary elements using  $N_b$  ( $N_b = N_2$ ) boundary nodes and the interior of the body,  $A$ , is divided into  $n_1$  triangular internal elements. A discretized version of equation (12) is

$$0 = \sum_{N_2} \int_{\Delta s_1} H_1(P_M, Q) n_1(P_M) G(Q) ds_Q + \sum_{n_1} \int_{\Delta A_1} H_1(P_M, q) n_1(P_M) f(q) dA_q \quad (13)$$

where  $P_M$  is the point  $P$  where it coincides with a node  $M$  at a center of a boundary segment on  $\partial C_2$  and  $\Delta s_1$  and  $\Delta A_1$  are boundary and internal elements respectively.

A simple numerical scheme is used in which the source strengths  $G$  are assumed to be piecewise uniform on each boundary segment with their values to be determined at the nodes which lie at the centers of each segment. Substitution of the piecewise uniform source strengths into equation (13) and carrying out of the necessary integrations, analytically and numerically, leads to an algebraic system of the type

$$\{0\} = [A]\{G\} + \{d\} \quad (14)$$

The coefficients of the matrix  $[A]$  contain boundary integrals of the kernel. The vector  $\{d\}$  contains contributions from the area integrals and the vector  $\{G\}$  the unknown source strengths at the boundary nodes. The dimension of  $\{G\}$  depends only on the number of boundary elements on  $\partial C_2$  and the internal discretization is necessary only for the evaluation of integrals with known integrands.

The solution strategy is as follows. The matrix  $[A]$  and vector  $\{d\}$  in equation (14) are first evaluated by using the appropriate expressions for the kernels and the prescribed function  $f$  in equation (3). Equation (14) is solved for the vector  $\{G\}$ . This value of  $\{G\}$  is now used in a discretized version of equation (8) to obtain the values of the stream function  $\psi$  at any point  $p$ . Finally, the current vector at any point is obtained from equations analogous to (8).

### NUMERICAL RESULTS

In the numerical computations,  $B_3^0$  in Eq. (13) is assumed to be a constant. Eq. (3) can be non-dimensionalized to the form

$$7^2 \psi(x_1, x_2) = 1, \quad \hat{x}_i = x_i/a \quad (15)$$

where

$$\hat{\psi} = \frac{\psi u_0}{14\pi B_3^0 R} \quad R = \frac{2a^2}{\tau_0^2} \text{ and the skin depth}$$

$$\delta = \sqrt{\frac{2}{\omega \sigma u_0}} \quad \hat{j} = \frac{j a u_0}{12\pi B_3^0 R}$$

For the results in this paper  $a = 2$ . A typical mesh for the results for example shown in Fig. 2d has 48 boundary segments uniformly distributed along the upper half (due to symmetry) of the boundary of the plate. In order to evaluate the known area integral in Equation 13, the internal area quadrature was used. It took about 300 c.p.u. secs on IBM 370/168 to obtain the results in Fig. 2d.

The equation (15) is identical to one relating to the torsion of shafts. The BEM was verified by comparing the numerical results for the solution of (15) in a square plate without a crack to known analytical results for the torsion of a shaft. The BEM method has also been checked against a finite element technique developed for eddy current problems [10].

Eddy current stream lines ( $\hat{\psi}$  lines) are shown in Figs. 2 and 3 for a square plate with a crack in it. Fig. 2 (a) - (c) shows how the stream lines are affected by varying the size of the plate while keeping the crack size same. Due to symmetry only the upper half of the plate is shown in Fig. 2. Fig. 2 (d) shows the effect of moving the crack towards one of the plate edges. Fig. 3 shows a close up of the stream lines near right crack tip for Fig. 2 (c). The crowding of stream lines near crack tips leads to large gradient of  $\hat{\psi}$  and therefore large induced currents in this region. The local temperature is proportional to the square of the current density ( $\hat{j} \cdot \hat{j}$ ). Figure 4 shows calculated temperature scans along a line slightly above the crack ( $\hat{x}_2 = .0125$ ) for the results shown in Fig. 2. From Figs. 4 (a) - (c) one can conclude that as the crack size increases relative to the plate size the hot spots at crack tips are more significant compared to those at the edges. The effect of moving the crack near the plate edge gives rise to significant hot spots as shown in Fig. 4 (d) and (c). This becomes more apparent when we look at the 'Eddy Current Intensity Factor' defined below. It has been shown [6,7] that the eddy current density squared is inversely proportional to the distance  $r$  from a crack tip. We can define an eddy current intensity factor,  $M_{III}$  as

$$\hat{j}^2 = M_{III} \frac{a}{r}$$

Table 1 shows the calculated values of  $M_{III}$  for the two crack tips for the results shown in Fig. 2. It is seen that the value of  $M_{III}$  remains practically constant for varying plate sizes. However it changes significantly as a crack tip is brought near an edge of the plate.

Table 1. Stress Intensity Factor MIII

$t/a$	$\frac{2x_1^0}{L}$	Right Crack Tip	Left Crack Tip	Figures 2, 4
0.05	1.0	0.125	0.125	(a)
0.10	1.0	0.130	0.130	(b)
0.25	1.0	0.145	0.145	(c)
0.10	0.6	3.96	1.30	(d)
0.10	0.3	15.45	6.93	(e)

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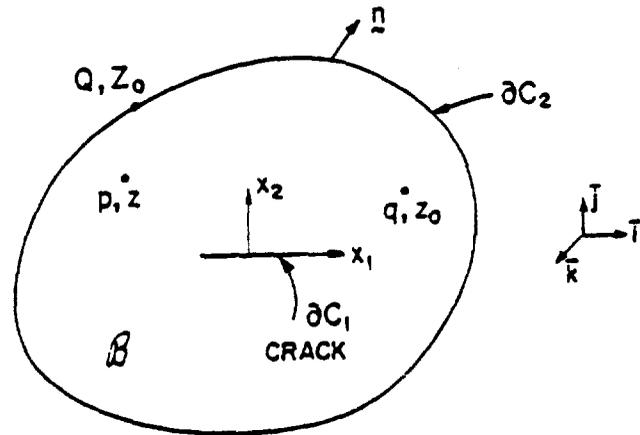


Figure 1. Cracked Plate.

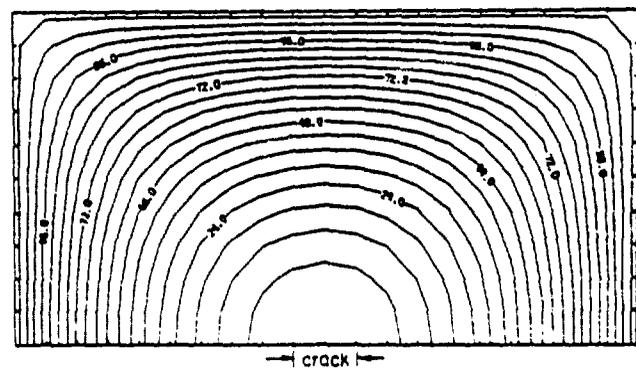


Fig. 2 (a).

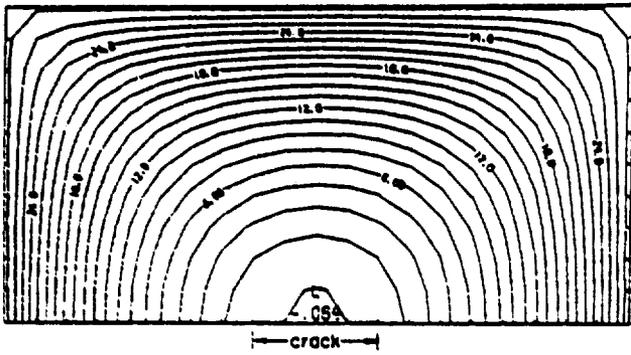


Fig. 2 (b).

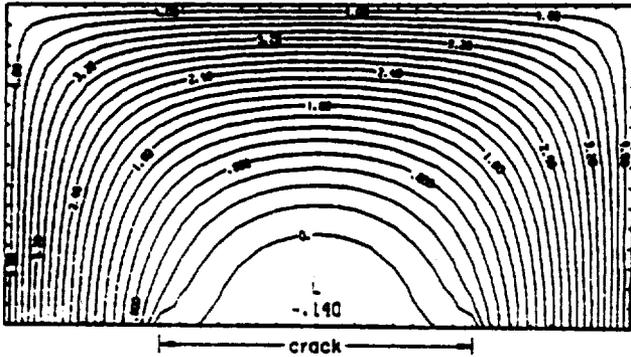


Fig. 2 (c).

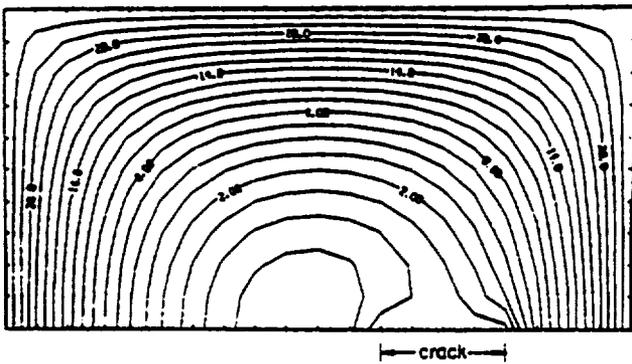


Fig. 2 (d).

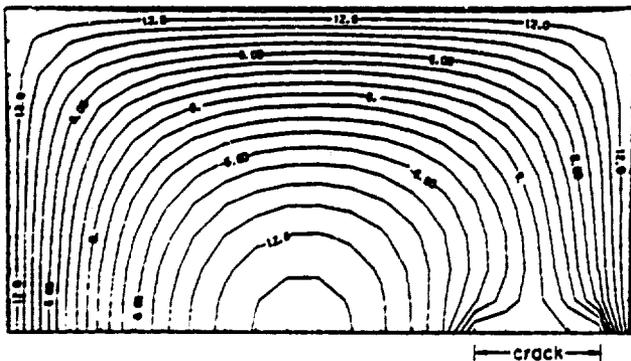


Fig. 2 (e).

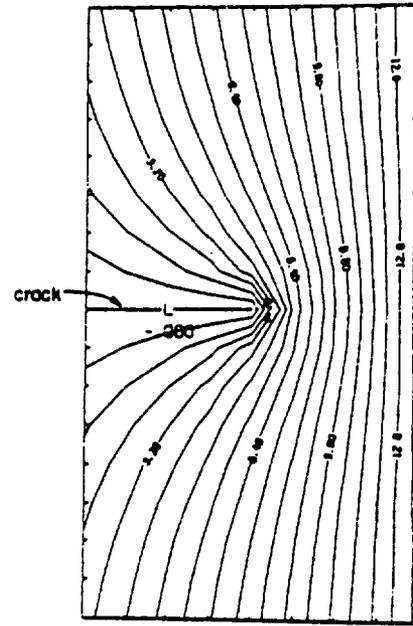


Fig. 3. Close up of Fig. 2 (e).

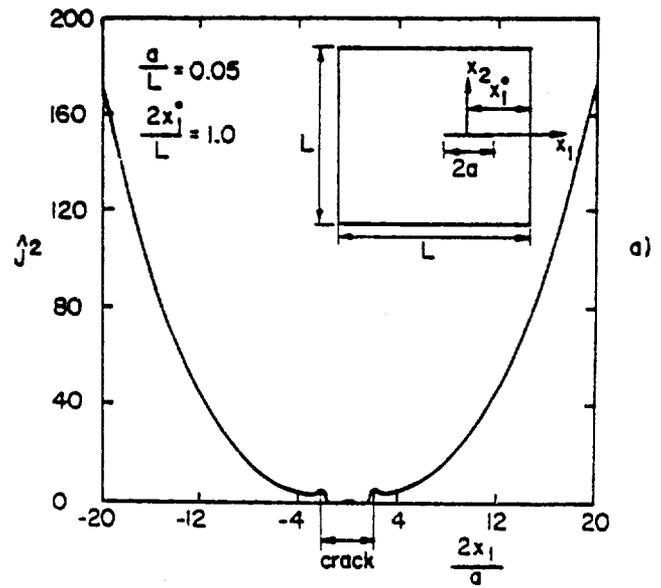


Fig. 4 (a).

Fig. 2. Eddy current stream lines in a square plate with a crack induced by an uniform magnetic field.

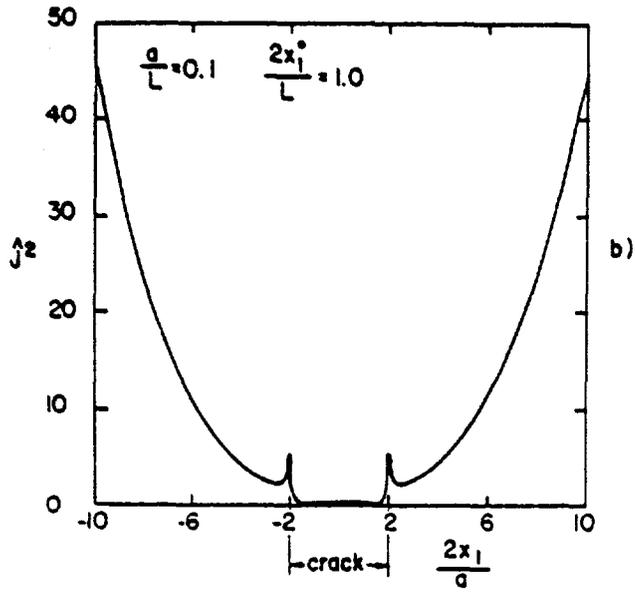


Fig. 4 (b).

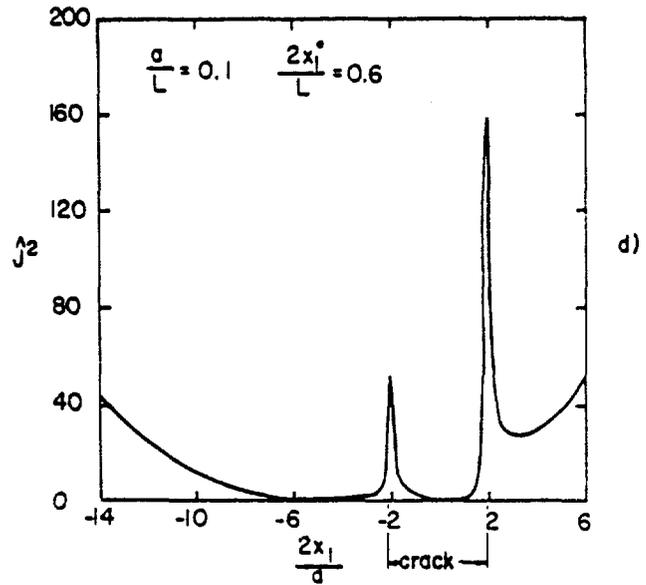


Fig. 4 (d).

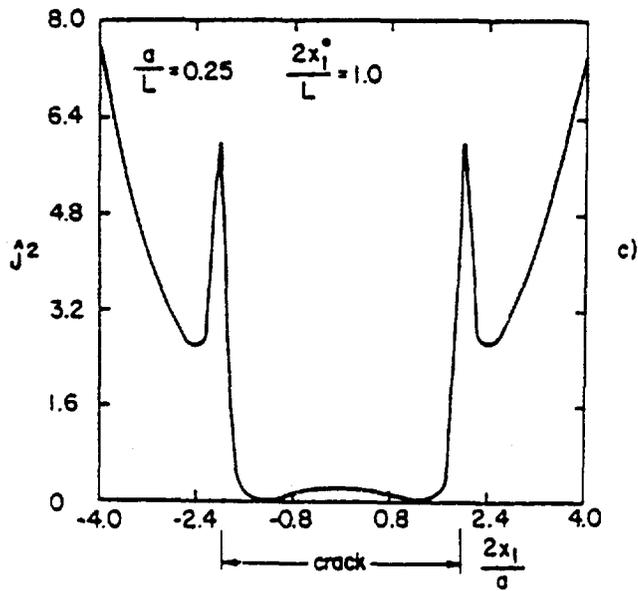


Fig. 4 (c).

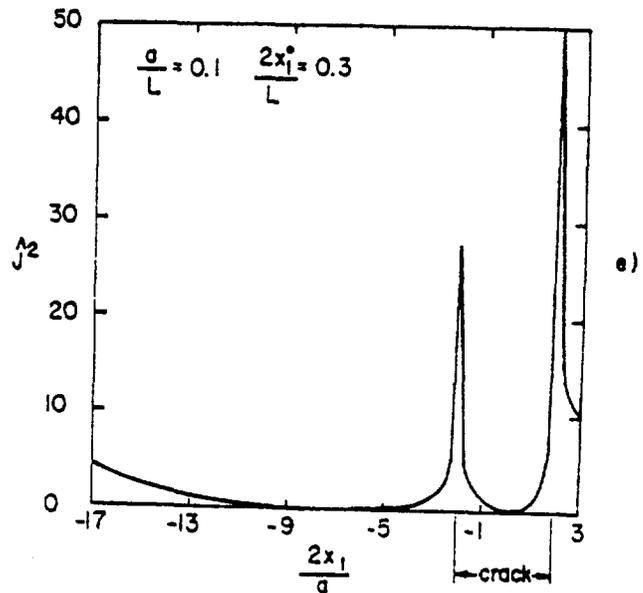


Fig. 4 (e).

Fig. 4. Joule heating intensity ( $j^2$ ) on sections  $\frac{x_2}{a} = 0.0125$  shown in Fig. 2.

COMPOSITE LIST OF TECHNICAL REPORTS  
TO THE  
OFFICE OF NAVAL RESEARCH

NUMERICAL SOLUTIONS FOR COUPLED MAGNETOTHERMOMECHANICS

Task Number NR 064-621

Departments of Structural Engineering and  
Theoretical and Applied Mechanics,  
Cornell University,  
Ithaca, New York 14853

1. K.Y. Yuan, F.C. Moon, and J.F. Abel, "Numerical Solutions for Coupled Magnetomechanics", Department of Structural Engineering Report Number 80-5, February 1980.
2. F.C. Moon and K. Hara, "Detection of Vibrations in Metallic Structures Using Small Passive Magnetic Fields", January 1981.
3. S. Mukherjee, M.A. Morjaria, and F.C. Moon, "Eddy Current Flows Around Cracks in Thin Plates for Nondestructive Testing", March 1981.
4. K.Y. Yuan, F.C. Moon, and J.F. Abel, "Finite Element Analysis of Coupled Magnetomechanical Problems of Conducting Plates", Department of Structural Engineering Report Number 81-10, May 1981.
5. F.C. Moon, "The Virial Theorem and Scaling Laws for Superconducting Magnet Systems", May 1981.
6. K.Y. Yuan, "Finite Element Analysis of Magnetoelastic Plate Problems", Department of Structural Engineering Report Number 81-14, August 1981.
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