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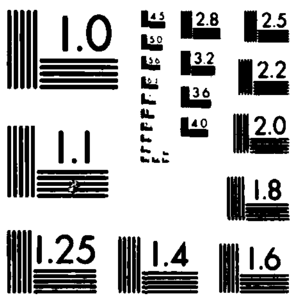

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NUMERICAL SOLUTION OF DEGENERATE VARIATIONAL INEQUALITY ARISING IN THE FLUID FLOW THROUGH POROUS MEDIA

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NUMERICAL SOLUTION OF DEGENERATE VARIATIONAL INEQUALITY ARISING IN THE FLUID FLOW THROUGH POROUS MEDIA

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ABSTRACT

In this paper ~~we~~ propose a numerical method for a degenerate variational inequality arising in the axisymmetric porous flow well problems which have been studied ~~in Cryer and Zhou [1981]. We~~ use the finite element method to discretize the problem, and ~~we~~ establish the convergence of the solution of the discrete problem to the solution of the degenerate variational inequality. The solution of the physical problem depends upon the unknown discharge q. A rapidly convergent numerical method for finding q is obtained.

AMS (MOS) Subject Classifications: 35J70, 35R35, 65N30, 76S05

Key Words: Porous flow; numerical methods; degenerate variational inequalities; convergence; finite elements.

Work Unit Number 3 - Numerical Analysis and Computer Science

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SIGNIFICANCE AND EXPLANATION

Most steady-state porous flow free boundary problems may be reduced to elliptic, variational or quasi-variational inequalities for which the numerical solutions have been studied by many authors. Some axisymmetric problems lead to another kind of variational inequality, namely degenerate variational inequalities. We give a numerical method for a degenerate variational inequality arising in the axisymmetric porous flow well problems, and study the convergence of the solution of the discrete problem. Numerical examples show that the method is efficient.

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NUMERICAL SOLUTION OF DEGENERATE VARIATIONAL INEQUALITY
ARISING IN THE FLUID FLOW THROUGH POROUS MEDIA

C. W. Cryer* and S. Z. Zhou**

1. Introduction

Most steady-state porous flow free boundary problems may be reduced to elliptic, variational or quasi-variational inequalities for which the numerical solutions have been studied by many authors (see, for instance, Baiocchi and Capelo [1978], Oden and Kikuchi [1979], and their references). In some axisymmetric problems there appears another kind of variational inequalities - degenerate variational inequalities. In this paper we propose a numerical method for a type of degenerate variational inequality arising in the axisymmetric porous flow well problems which have been studied in Cryer and Zhou [1981]. We use the finite element method to obtain the discrete problem. The convergence of the solution of the discrete problem to the solution of the degenerate variational inequality is proved. The solution of the physical problem depends upon the unknown discharge q. We give a numerical method for finding q which has been found to converge rapidly.

Here we recall some notations and results about weighted Sobolev spaces V^1 and V^2. (See Chang and Jiang [1978], Zhou [1980].)

A - a bounded domain in (r,z)-plane with a locally Lipschitz boundary Γ, and with $r > 0$.

$C^\infty(R^2)$ - the space of functions infinitely differentiable in (r,z)-plane.

$C^\infty(\bar{A}) = \{v : v$ is defined in A and has extension in $C^\infty(R^2)\}$

$C_0^\infty(A;\Gamma^*) = \{v \in C^\infty(\bar{A}) : v = 0$ in some neighborhood of $\Gamma^*\}$, where $\Gamma^* \subset \Gamma$. If $\Gamma^* = \Gamma$ then denote $C_0^\infty(A;\Gamma^*)$ by $C_0^\infty(A)$.

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$L^2(A,r) = \{v : v$ real measurable in A, $\|v\|_{L^2(A,r)} < \infty\}$, where $\|v\|_{L^2(A,r)} = \int_A r|v|^2 drdz$.

$V^0(A) = L^2(A,r)$, $\|v\|_{V^0(A)} = \|v\|_{L^2(A,r)}$.

$V^1(A) = \{v : \partial^\alpha v \in L^2(A,r),\ |\alpha| \le 1\}$, $\|v\|_{V^1(A)} = \sum_{|\alpha|\le 1} \|\partial^\alpha v\|_{V^0(A)}$.

$V^2(A) = \{v : \partial^\alpha v \in L^2(A,r),\ |\alpha| \le 2;\ \frac{1}{r}\frac{\partial v}{\partial r} \in L^2(A,r)\}$.

$\|v\|_{V^2(A)} = \sum_{|\alpha|\le 2} \|\partial^\alpha v\|_{V^0(A)} + \|\frac{1}{r}\frac{\partial v}{\partial r}\|_{V^0(A)}$.

$V_0^1(A)$ – the closure of $C_0^\infty(A)$ in $V^1(A)$.

$V_0^1(A;\Gamma^*)$ – the closure of $C_0^\infty(A;\Gamma^*)$ in $V^1(A)$.

The following propositions can be easily shown.

<u>Proposition 1.1</u>. If $\Gamma^* \cap \{r = 0\} = \emptyset$ and $\mathrm{meas}(\Gamma^*) > 0$ then there exists a constant C such that

$$\|v\|^2_{V^1(A)} \le C \int_A [(\frac{\partial v}{\partial r})^2 + (\frac{\partial v}{\partial z})^2] r dr dz, \quad \forall\, v \in V_0^1(A;\Gamma^*) \quad .$$

<u>Proposition 1.2</u>. $V^1(A)$, $V^2(A)$ and $V_0^1(A;\Gamma^*)$ are Banach spaces, where we take $\|v\|_{V_0^1(A;\Gamma^*)} = \|v\|_{V^1(A)}$.

<u>Proposition 1.3</u>. $C^\infty(\bar{A})$, $C_0^\infty(A;\Gamma^*)$ are respectively dense in $V^1(A)$, $V_0^1(A;\Gamma^*)$.

<u>Proposition 1.4</u>. If $\Gamma^* \cap \{r = 0\} = \emptyset$ then there exists a unique linear continuous operator $tr : V^1(A) \to L^2(\Gamma^*)$ such that $tr\ v = v$ on Γ^* for any $v \in C^\infty(\bar{A})$. Moreover, tr is a compact operator.

2. Continuous Problem

Let D be a L-shaped domain such as that shown in Figure 1. Set $\Omega_1 = \{(r,z) : 0 < r \le R_0,\ 0 < z < h_0\}$.

The original variational formulation of the physical problem is as follows (Cryer and Zhou [1981]).

<u>Problem (PPW)</u>. Let R_0, R_1, h_0, h_w, H be numbers such that $R_1 > R_0 > 0$, $H > h_w > h_0 > 0$. Find functions $\varphi(r)$ and $u(r,z)$ such that

$$\varphi \in C^0([R_0,R_1]),\ \varphi(R_0) \ge h_w,\ \varphi(R_1) = H \tag{2.1}$$

$$\varphi \text{ is strictly increasing} \tag{2.2}$$

$$u \in V^1(\Omega) \cap C^0(\bar{\Omega}) \quad . \tag{2.3}$$

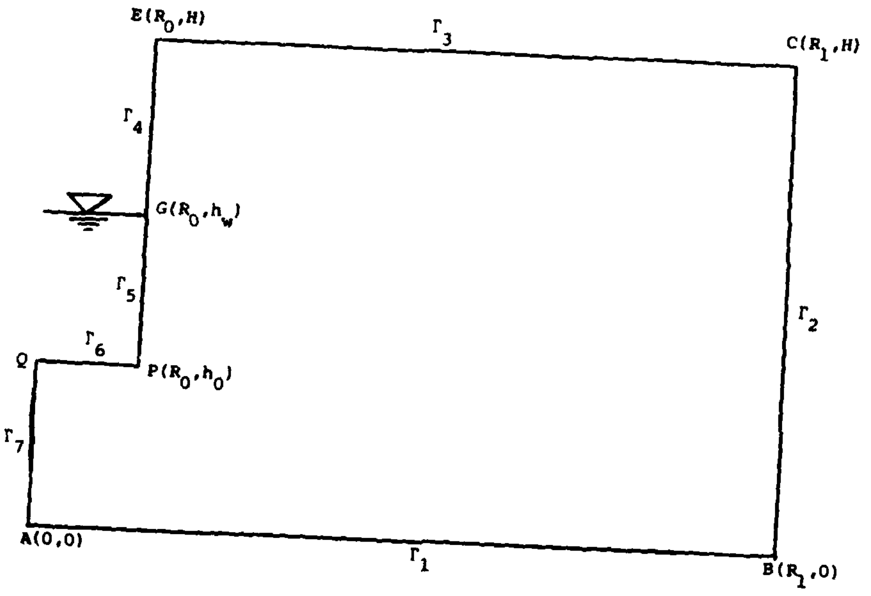

Figure 1

$$\begin{aligned} u &= H \quad \text{on } \Gamma_2 \\ &= z \quad \text{on } \Gamma_0 \cup (\Gamma_4 \cap \partial\Omega) \\ &= h_w \quad \text{on } \Gamma_5 \cup \Gamma_6 \end{aligned} \tag{2.4}$$

$$\int_\Omega r\nabla u \cdot \nabla v \, dr dz = 0 \quad \text{for all} \quad v \in K_1 \tag{2.5}$$

where

$$\Gamma_0 = \{(r,z) : z = \varphi(r),\ R_0 < r < R_1\} \ ,$$

$$\Omega = \{(r,z) : 0 < z < \varphi(r),\ R_0 < r < R_1\} \ , \tag{2.6}$$

$$K_1 = \{v \in V^1(\Omega) : v = 0 \ \text{ on } \ \Gamma_2 \cup \Gamma_3 \cup (\Gamma_4 \cap \partial\Omega) \cup \Gamma_5 \cup \Gamma_6\} \ .$$

By using a kind of Baiocchi transform

$$\begin{aligned} \tilde{u} &= u(r,z) && \text{in } \bar{\Omega} \\ &= z && \text{in } \bar{D}\setminus\bar{\Omega} \end{aligned} \tag{2.7}$$

$$w(r,z) = \int_0^z [\tilde{u}(r,t) - t]\,dt \quad \text{in } \bar{D} \ . \tag{2.8}$$

Cryer and Zhou [1981] have derived the following degenerate variational inequality with a real parameter q.

<u>Problem (PPW1)</u>. Find $w_q \in K_q^{**}$ such that

$$\begin{aligned} &\int_D r\nabla w_q \cdot \nabla(v - w_q)\,dr\,dz \\ &\quad \geq (h_w - h_0)\int_0^{R_0} (v - w_q)\big|_{z=h_0} r\,dr + \int_D (v - w_q)\,r\,dr\,dz, \ \forall\, v \in K_q^{**} \end{aligned} \tag{2.9}$$

where

$$K_q^{**} = \{v \in V^1(D) : v \geq 0 \ \text{ in } \ D,\ v \leq g_q(r,H) \ \text{ in } \ D\setminus\Omega_1,\ v = g_q \ \text{ on } \ \Gamma_D\} \ , \tag{2.10}$$

$$\Gamma_D = \bigcup_{i=1}^{5} \Gamma_i \ , \tag{2.11}$$

$$\begin{aligned} g_q(r,z) &= 0 && \text{on } \Gamma_1 \\ &= Hz - \frac{z^2}{2} && \text{on } \Gamma_2 \\ &= \frac{H^2}{2} + q\,\ln\frac{r}{R_1} && \text{on } \Gamma_3 \cup \Gamma_4 \\ &= \frac{H^2}{2} + q\,\ln\frac{R_0}{R_1} - \frac{(h_w - z)^2}{2} && \text{on } \Gamma_5 \ . \end{aligned} \tag{2.12}$$

For Problem (PPW1) there is a equivalent form which is more convenient with regard to the numerical solution.

Problem (PPW2). Find $w_q \in K_q^{**}$ such that

$$J(w_q) = \min_{v \in K_q^{**}} J(v) \tag{2.13}$$

where

$$J(v) = \int_D r|\nabla v|^2 drdz - 2(h_w - h_0) \int_0^{R_0} v|_{z=h_0} rdr - 2\int_D v\, rdrdz \quad . \tag{2.14}$$

Cryer and Zhou [1981] have proved the following results.

Proposition 2.1. For any q with $0 < q < q_0$, where

$$q_0 = \frac{H^2 - (h_w - h_0)^2}{2\, \ln(R_1/R_0)} \quad , \tag{2.15}$$

(PPW1) has a unique solution.

Proposition 2.2. Let w_q be the solution of (PPW1). Then there exist α, β and $F^*(q)$ with α, β real numbers, $\beta \neq 0$, F^* a continuous bounded function, such that the following two conditions are equivalent:

$$w_q \in C^1(\bar{D}) \cap V^2(D) \tag{2.16}$$

$$F^*(q) - \beta q - \alpha = 0 \quad . \tag{2.16'}$$

We call w_q a regular solution of (PPW1) if (2.16) is valid.

Proposition 2.3. If $w_{\bar{q}}$ is a regular solution of (PPW1) then $0 < \bar{q} < q_0$.

Proposition 2.4. There exists at least one regular solution w_q for Problem (PPW1).

Proposition 2.5. For the regular solution $w_{\bar{q}}$ of (PPW1) we have

$$\frac{\bar{q}}{r} > \frac{\partial w_{\bar{q}}}{\partial r} > 0, \quad \frac{\partial w_{\bar{q}}}{\partial z} > 0 \quad \text{in } D \tag{2.17}$$

$$\begin{aligned} \frac{\partial w_{\bar{q}}}{\partial n} &= h_w - h_0 \qquad \text{on } \bar{\Gamma}_6 \\ &= 0 \qquad \text{on } \bar{\Gamma}_7 \quad . \end{aligned} \tag{2.18}$$

<u>Proposition 2.6</u>. If u is a solution of (PPW) then w defined by (2.8) is a regular solution of (PPW1) corresponding

$$q = \bar{q} = r \int_0^H \frac{\partial \bar{u}(r,t)}{\partial r}\, dt \quad .$$

Conversely, if $w_{\bar{q}}$ is a regular solution of (PPW1) then the functions $\varphi_{\bar{q}}$, $u_{\bar{q}}$ defined as follows, is the solution of (PPW):

$$\Omega_{\bar{q}} = \Omega_1 \cup \{(r,z) \in D : r > R_0,\ w_{\bar{q}} < g_{\bar{q}}(r,H)\}$$

$$\varphi_{\bar{q}}(r) = \sup\{z : (r,z) \in \Omega_{\bar{q}}\}, \quad R_0 < r < R_1$$

$$\varphi_{\bar{q}}(R_0) = \lim_{r \to R_0+0} \varphi_{\bar{q}}(r), \quad \varphi_{\bar{q}}(R_1) = \lim_{r \to R_1-0} \varphi_{\bar{q}}(r)$$

$$\bar{u}_{\bar{q}} = \frac{\partial w_{\bar{q}}}{\partial z} + z \quad \text{in } \bar{D}$$

$$u_{\bar{q}} = \bar{u}_{\bar{q}}\big|_{\Omega_{\bar{q}}} \quad .$$

<u>Remark 2.1</u>. Physically, $2\pi\bar{q}$ is the "discharge" while u is the "hydraulic head".

3. <u>Numerical Approximation of (PPW2)</u>

In this section we consider the approximate problem of Problem (PPW2): Given q and h with $0 < q < q_0$, $h > 0$, find $w_q^h \in K_q^h$ such that

$$J(w_q^h) = \min_{v \in K_q^h} J(v) \tag{3.1}$$

where K_q^h is a convex closed nonempty subset of V^h, and $\{V^h\}$ is a family of finite dimensional subspaces of $V^1(D)$. It is not required that $K_q^h \subset K_q^{**}$. So far, for space V^2 there is no approximation theorem similar to that for usual Sobolev space H^2. Hence we can not prove the convergence theorem for our problem by using usual methods such as those in Falk [1974], Brezi and Sacchi [1976], Cryer and Fetter [1977].

Let $\{T_h\}$ be a family of triangulations of the domain D. The set of interior gridpoints will be denoted by D_h, and the set of boundary gridpoints by ∂D_h. Set $\Gamma_{Dh} = \partial D_h \cap \Gamma_D$. For each T_h and for each triangle $T \in T_h$ we set

$$\rho(T) = \text{diameter of } T$$

$$\theta(T) = \text{minimum angle of } T$$

$$\theta_h = \min_{T \in T_h} \theta(T) \quad .$$

We assume that

$$\lim_{h \to 0} \max_{T \in T_h} \rho(T) = 0 \quad , \tag{3.2}$$

and that there exists a positive constant θ_0 independent of h such that (Zlamal [1968])

$$\theta_h \geq \theta_0 \quad . \tag{3.3}$$

As V^h we take the space of linear finite elements corresponding to T_h in $V_0^1(D;\Gamma_1)$. Set

$$K_q^h = \{v \in V^h : v \geq 0 \text{ on } D_h,\ v \leq g_q(r,H) \text{ on } D_h - \Omega_1,\ v = g_q \text{ on } \Gamma_{Dh}\} \quad . \tag{3.4}$$

We need the following basic theorem (see, for instance, Glowinski [1980, Th. 5.2 in Ch. I]).

<u>Theorem 3.1</u>. Let V be a real Hilbert space, $a(\cdot,\cdot)$ - a bilinear, continuous, symmetric and coercive form on $V \times V$, $f(\cdot)$ - a continuous, linear functional on V, K - a closed, convex, nonempty subset of V, $\{K^h\}$ - a family of closed, convex, nonempty subsets of V with $K^h \subset V^h$. Assume that

(i) If $\{v^h\}$ is bounded in V and $v^h \in K^h$ then the weak cluster points of $\{v^h\}$ belong to K;

(ii) There exists a set $X \subset V$ with $\bar{X} = K$ such that $\forall\, v \in X$ there exists $\{v^h\}$ satisfying that $v^h \in K^h$ and that $\lim_{h \to 0} v^h = v$ strongly in V.

Then

$$\lim_{h \to 0} \|u^h - u\|_V = 0$$

where u^h and u are respectively the solutions of the problems

$$J(u) = \min_{v \in K} J(v)$$

and

$$J(u^h) = \min_{v \in K^h} J(v)$$

provided that $J(v) = a(v,v) - 2f(v)$.

In our case we will take $C^1(\bar{D}) \cap K_q^{**}$ as the set X in the above theorem.

From now on we assume that $0 < q < q_0$.

<u>Lemma 3.2</u>. $\overline{C^1(\bar{D}) \cap K_q^{**}} = K_q^{**}$ in $V^1(D)$.

<u>Proof</u>. Set $Y = C^1(\bar{D}) \cap K_q^{**}$. By Cryer and Zhou [1981, (3.28)] there exists a function $v_q \in Y$, hence Y is nonempty.

Set

$$Y_1 = \{v \in V^1(D) : v \geq -v_q \text{ in } D, v \leq g_q(r,H) - v_q \text{ in } D \setminus \Omega_1, v = 0 \text{ on } \Gamma_D\}$$

$$Y_2 = Y_1 \cap C_0^\infty(D;\Gamma_D) \quad .$$

Then by the argument similar to that of Lemma 2.4 in Glowinski [1980, Ch. II] we can prove that

$$\bar{Y}_2 = Y_1 \tag{3.5}$$

$\forall\, v \in K_q^{**}$ we have $v^* = v - v_q \in Y_1$. By (3.5) there exists a sequence $v_n^* \in Y_2$ such that $v_n^* \to v^*$ strongly in V^1. Thus $v_n = v_n^* + v_q \in Y$ and $v_n \to v$ strongly in V^1. So we obtain that

$$\overline{C^1(\bar{D}) \cap K_q^{**}} \supset K_q^{**} \quad .$$

Using the well-known subsequence argument and proposition 1.4 we can easily obtain that

$$\overline{C^1(\bar{D}) \cap K_q^{**}} \supset K_q^{**} \quad .$$

Q.E.D.

The following lemma may be easily proved.

<u>Lemma 3.3</u>. If $T \in T_h$ and $v^h \in V^h$ then

$$\int_T v^h \, dr dz = \frac{\text{meas } (T)}{3} \sum_{i=1}^{3} v^h(M_{iT}) \quad .$$

where M_{iT} $(i = 1,2,3)$ are the vertices of T.

Now we can prove the convergence theorem.

Theorem 3.4. Problem (3.1) has a unique solution w_q^h which converges to the solution w_q of Problem (PPW2) in $V^1(D)$ as $h \to 0$.

Proof. Let $V = V^h$, $a(v_1,v_2) = \int_D r\nabla v_1 \cdot \nabla v_2 \, drdz$, and $f(v) = \int_D rv drdz + (h_w - h_0) \int_0^{R_0} v|_{z=h_0} rdr$. Then it is easy to see that V is a Hilbert space with inner product

$$(v_1,v_2) = \int_D r(v_1v_2 + \nabla v_1 \cdot \nabla v_2) drdz \quad ,$$

that k_q^h is a closed, convex, nonempty subset of V, that $a(\cdot,\cdot)$ is a symmetric, continuous, coercive (by proposition 1.1), bilinear form on $V \times V$, and that $f(\cdot)$ is a linear, continuous (by proposition 1.4) functional on V. By the well-known theorem (Stampacchia [1964], or Lions and Stampacchia [1967] we know that Problem (3.1) has a unique solution w_q^h.

Now we prove the convergence of w_q^h by using Theorem 3.1. Let $V = V_0^1(D;\Gamma_1)$, $K = K_q^{**}$. It is sufficient to verify the conditions (i) and (ii) because it is obvious that the rest of the conditions are satisfied.

Verification of (i): Let $\{v^h\}$ be a sequence such that $v^h \in K_q^h$, and

$$v^h \to v \text{ weakly in } V_0^1(D;\Gamma_1) \quad . \tag{3.6}$$

We prove that $v \in K_q^{**}$. First we prove that

$$v \le g_q(r,H) \text{ in } D\backslash\Omega_1 \quad . \tag{3.7}$$

Define $g_q(r,z) = g_q(R_0,z)$ in Ω_1. Let g^h be the piecewisely linear interpolation of $g_q(r,H)$. Then it is easy to see that

$$g^h \to g_q(r,H) \text{ in } C^0(\bar{D}) \quad . \tag{3.8}$$

For any $\psi \in C_0^\infty(D\backslash\Omega_1)$ with $\psi \ge 0$ we define $\psi = 0$ in Ω_1 and

$$\psi^h = \sum_{T\in\mathcal{T}_h} \psi(G_T)\Phi(T)$$

where $\Phi(T)$ is the characteristic function of T, and G_T is the centroid of T. It can be shown that

$$\psi^h \to \psi \text{ in } C^0(\bar{D}) \quad . \tag{3.9}$$

Denote by T_h^* The union of triangles T which are contained in $\bar{D}\,\Omega_1$. Then we have

$$\int_{D\setminus\Omega_1} (v^h - g^h)\psi^h drdz = \int_{T_h^*} + \int_{D\setminus(\Omega_1\cup T_h^*)}$$

$$= \sum_{T\subset T^*} \int_T + \int_{D\setminus(\Omega_1\cup T_h^*)}$$

$$= \sum_{T\subset T^*} \psi(G_T)\int_T (v^h - g^h)drdz + \int_{D\setminus(\Omega_1\cup T_h^*)} (v^h - g^h)\psi^h drdz$$

$$= \sum_{T\subset T^*} \frac{\psi(G_T)\text{meas}(T)}{3} \sum_{i=1}^{3} (v^h(M_{iT}) - g^h(M_{iT})) + \int_{D\setminus(\Omega_1\cup T_h^*)} \quad \text{(by lemma 3.3)}$$

$$\leq \int_{D\setminus(\Omega_1\cup T_h^*)} (v^h - g^h)\psi^h drdz \quad (\text{since } v^h \leq g^h \text{ in } D^h - \Omega_1) \quad . \tag{3.10}$$

Noting that $\text{meas}(D\setminus(\Omega_1 \cup T_h^*)) \to 0$ as $h \to 0$ we obtain by (3.6), (3.8), (3.9) and (3.10) that

$$\int_{D\setminus\Omega_1} (v - g_q(r,H))\psi \, drdz \leq 0, \quad \forall\, \psi \in C_0^\infty(D\setminus\Omega_1) \text{ with } \psi \geq 0$$

which implies that (3.7) is valid.

By similar argument we obtain that

$$v \geq 0 \text{ in } D \quad . \tag{3.11}$$

Finally, it is easy to see that

$$v^h \to g_q(r,z) \text{ in } C^0(\bar{\Gamma}_D) \quad .$$

On the other hand, it follows from proposition 1.4 that $v^h \to v$ strongly in $L^2(\Gamma_D)$. Thus we have

$$v = g_q \text{ on } \Gamma_D \quad . \tag{3.12}$$

This equation as well as (3.11) and (3.7) means that $v \in K_q^{**}$.

<u>Verification of (ii)</u>: Let $X = C^1(\bar{D}) \cap K_q^{**}$. By lemma 3.2 we have $\bar{X} = K_q^{**}$. For any $v \in X$ we take piecewisely linear interpolation v^h of v. Then $v^h \in K_q^h$. By a result of Feng [1965] we have (note that (3.2) and (3.3))

$$v^h \to v, \frac{\partial v^h}{\partial r} \to \frac{\partial v}{\partial r}, \frac{\partial v^h}{\partial z} \to \frac{\partial v}{\partial z} \text{ in } L^\infty(D)$$

as $h \to 0$. Therefore we have that $v^h \to v$ strongly in V^1.

Q.E.D.

Remark 3.1. The condition (3.3) may be replaced by a weaker condition $\theta_h^* \leq \pi - \theta_0$, where $\theta_h^* = \max_{T \in T_h} \theta^*(T)$. $\theta(T)$=maximum angle of T. (Cf. Feng [1965].)

The Problem (3.1) is a quadratic programming problem which can be computed using S.O.R. with projection (Cryer [1971], Glowinski [1971]). The iterative process is convergent (see, for instance, Glowinski et al. [1976, p. 70]). We have used Carré's scheme (Carré [1961]) to choose the relaxation factor ω.

4. Numerical Method for Regular Solution

Numerical experiments indicate that $w_q^h \to w_q$ as $h \to 0$ for q with $0 < q < q_0$, just as theorem 3.4 claims. But w_q does not satisfy (2.17) if q does not correspond to the regular solution. Here we give a method for searching for regular solution and corresponding values of q.

The basic idea is as follows. For the solutions (PPW1), their derivatives are continuous everywhere in $\bar{D}$ except at the reentrant point P. The character that a regular solution possesses is that its derivatives are continuous even at the point P. Hence we may use (2.18) at P to find regular solutions. By the way, equation (2.40) in Baiocchi et al. [1973] may also be derived by the same idea. Now we turn to the concrete computation.

We choose a subset S of the family $\{T_h\}$ such that $\forall\, T_h \in S$ there is a element $T^* \in T_h$ which has a vertical edge and P is an endpoint of this edge. Denote by h^* and P' respectively the length and the other endpoint of the edge. Thus we have the discrete form of (2.18) at the point P:

$$f(q) = w_q^h(P) - w_q^h(P') - h^*(h_w - h_0) = 0 \quad . \tag{4.1}$$

This equation can be numerically solved by, for example, the secant method which is given by

$$q_{n+1} = q_n - \frac{q_n - q_{n-1}}{f(q_n) - f(q_{n-1})} f(q_n), \quad n = 2,3,\dots \quad . \tag{4.2}$$

The initial values q_1, q_2 must be subject to the condition $0 < q < q_0$. Then we compute $w^h_{q_1}, w^h_{q_2}$ by using S.O.R. with projection, and $f(q_1), f(q_2)$ by using (4.1), and q_3 by using (4.2). This process is repeated until prescribed accuracy is reached.

5. Numerical Examples

We adopt the triangulation of D used by Cryer and Fetter [1979] (see Figure 2). Suppose that m denotes the number of subdivisions of D in z-direction, n the number of subdivisions of $D\backslash\Omega_1$ in r-direction. Then the coordinates of the gridpoints are given by

$$z_j = (j-1)H/m, \quad 1 \le j \le m+1$$

$$r_i = R_0 \exp[(i-k-1)/n \cdot \ln(R_1/R_0)], \quad i = k+1,\ldots,n+k+1$$

$$r_i = (i-1)R_0/k, \quad i = 1,\ldots,k$$

where

$$k = [R_0/\Delta] \quad ,$$

$$\Delta = R_0[(R_1/R_0)^{1/n} - 1] \quad .$$

For $v^h \in K^h_q$ let $U_{ij} = v^h(r_i,z_j)$, vector $U = \{U_{ij}\}$. Then

$$J(v^h) = \sum_{R(i,j)} J_{R(i,j)}(v^h)$$

where $R(i,j)$ are the rectangles

$$R(i,j) = \{(r,z) : r_i \le r \le r_{i+1}, z_i \le z \le z_{i+1}\} \quad .$$

It is easy to compute

$$J_{R(i,j)}(v^h) = U^T A_{R(i,j)} U + 2b^T_{R(i,j)} U \quad .$$

The matrix $A_{R(i,j)}$ and the vector $b_{R(i,j)}$ are almost same as that in (7.11) and (7.12) of Cryer and Fetter [1977]. The only difference is that for $j = k_1 - 1$ we must add respectively $(ho-hw)\Delta x \cdot x1/2$ and $(ho-hw)\Delta x \cdot x_U/2$, which correspond to the line integral in $J(v)$ now, to $b_{R(i,j)}(i,j+1)$ and $b_{R(i,j)}(i+1,j+1)$, where k_1 is the value of j corresponding $z = ho$.

For the S.O.R. with projection we only note that there are two constraints in our problem - $v^h \ge 0$ in D_h and $v^h \le g_q(r,H)$ in $D_h - \Omega_1$.

The equation (4.1) now becomes

$$f(q) = U(kk,k_1) - U(kk,k_1 - 1) - (h_w - h_0)H/m = 0$$

where $kk = k+1$.

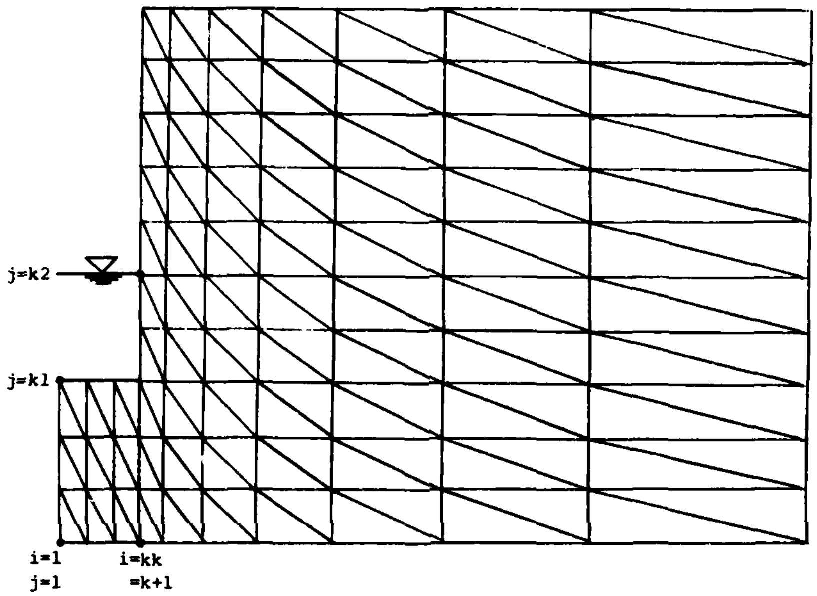

Figure 2

Example 1. $R_0 = 4.8$, $R_1 = 76.8$, $h_0 = 9$, $h_w = 12$, $H = 48$;

discretization: (1) $m = 8$, $n = 12$; $kk = 4$

(2) $m = 16$, $n = 24$; $kk = 9$

(3) $m = 32$, $n = 48$; $kk = 17$

(4) $m = 64$, $n = 96$; $kk = 35$

stopping test:

$$\max_{i,j} |U^{(\ell+1)}(i,j) - U^{(\ell)}(i,j)| < 10^{-6} \quad \text{for inner iteration}$$

$$|f(q_s)| < 10^{-6} \quad \text{for outer iteration} \quad .$$

The results are shown in the following table.

discretization	ω (Carré's Scheme)	number of outer iterations	number of inner iterations	q	h_s
(1)	1.4993	2	40	399.26	30
(2)	1.7291	4	90	371.83	33
(3)	1.8419	4	160	362.75	31.5
(4)	1.9336	4	380	358.66	30.75

where h_s is the approximate value of $\varphi_{\bar{q}}(R_0)$.

Example 2. $R_0 = 10$, $R_1 = 1130$, $h_0 = 120$, $h_w = 200$, $H = 460$;

discretization: $m = 50$, $n = 100$; $kk = 21$

$\omega = 1.939$

stopping test: same as that in example 1.

The following table shows the convergence of the outer iteration process.

outer iteration	q	f(q)	number of inner iterations
0	100	25380.14648	450
1	200	25206.47851	450
2	14714.1875	43.04459	440
3	14739.0156	0.089335	440
4	14739.0673	0.00000035	440

$h_s = 349.6$ (compared with $h_s = 350$ in Boreli [1955]).

The exact value of q is 14218.462.

The theoretical proof that $w_q^h \to w_{\bar{q}}$ as $h \to 0$ is still an open problem.

Appendix A: An approximation theorem (Feng Kang [1965]).

Theorem. Let T be a triangle such as in Figure 3, P_0, P_1 and P_2 its vertices, P_0P_1 the largest edge with length ρ. Assume that $u \in C^1(\bar{T})$, v is the linear interpolation of u with $v(P_i) = u(P_i)$, $i = 0,1,2$.

Let

$$\omega' = \sup_{\substack{P,P' \in \bar{T} \\ 0 \le \theta \le \pi}} |u_\theta(P) - u_\theta(P')|$$

where u_θ is the directional derivative of u. Then

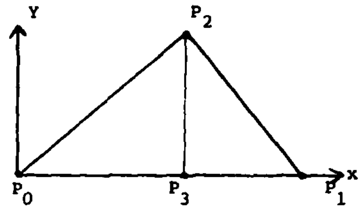

Figure 3

$$\left|\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\right| \le \omega' \tag{A.1}$$

$$\left|\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\right| \le (1 + \operatorname{ctg} \alpha_0)\omega' \tag{A.2}$$

$$|v - u| \le (2 + \operatorname{ctg} \alpha_0)\rho\omega' \tag{A.3}$$

where α_0 is the inner angle with vertex P_0.

Proof: Let P_3 be a point such that $P_3 \in P_0P_1$ and $P_2P_3 \perp P_0P_1$. Denote by ρ_1 the length of P_2P_3. It is easy to see that for any $P \in \bar{T}$ we have

$$v(P) = u(P_0) + \frac{u(P_1) - u(P_0)}{\rho} x + \frac{u(P_2) - \bar{u}(P_3)}{\rho_1} y \tag{A.4}$$

where

$$\bar{u}(P_3) = \sigma u(P_1) + (1-\sigma)u(P_0)$$

$$\sigma = \overline{P_0P_3}/\rho \quad .$$

By the mean value theorem there exist $Q_0 \in P_0P_3$ and $Q_1 \in P_1P_3$ such that

$$u(P_0) = u(P_3) + \int_{P_3}^{P_0} \frac{\partial u}{\partial x}\, dx = u(P_3) - \sigma\rho \frac{\partial u(Q_0)}{\partial x}$$

$$u(P_1) = u(P_3) + \int_{P_3}^{P_1} \frac{\partial u}{\partial x}\, dx = u(P_3) + (1-\sigma)\rho \frac{\partial u(Q_1)}{\partial x} \quad .$$

Hence

$$\bar{u}(P_3) = u(P_3) + \sigma(1-\sigma)\rho\left[\frac{\partial u(Q_1)}{\partial x} - \frac{\partial u(Q_0)}{\partial x}\right] \quad . \tag{A.5}$$

By (A.4) and (A.5) we obtain that

$$\frac{\partial v(P)}{\partial x} = \frac{u(P_1) - u(P_0)}{\rho} = \frac{\partial u(Q)}{\partial x} \qquad \text{for some} \quad Q \in P_0P_1$$

$$\frac{\partial v(P)}{\partial y} = \frac{u(P_2) - u(P_3)}{\rho_1} - \frac{\rho\sigma(1-\sigma)}{\rho_1}\left[\frac{\partial u(Q_1)}{\partial x} - \frac{\partial u(Q_0)}{\partial x}\right]$$

$$= \frac{\partial u(Q^*)}{\partial y} - (1-\sigma)\ \mathrm{ctg}\ \alpha_0\left[\frac{\partial u(Q_1)}{\partial x} - \frac{\partial u(Q_0)}{\partial x}\right] \quad \text{for some} \quad Q^* \in P_2P_3 \quad .$$

Therefore

$$\left|\frac{\partial v(P)}{\partial x} - \frac{\partial u(P)}{\partial x}\right| = \left|\frac{\partial u(Q)}{\partial x} - \frac{\partial u(P)}{\partial x}\right| \le \omega'$$

$$\left|\frac{\partial v(P)}{\partial y} - \frac{\partial u(P)}{\partial y}\right| \le \left|\frac{\partial u(Q^*)}{\partial y} - \frac{\partial u(P)}{\partial y}\right| + \mathrm{ctg}\ \alpha_0 \left|\frac{\partial u(Q_1)}{\partial x} - \frac{\partial u(Q_0)}{\partial x}\right| \le (1 + \mathrm{ctg}\ \alpha_0)\omega'$$

$$|v(P) - u(P)| \le |v(P_0) - u(P_0)| + \left|\int_{P_0}^{P} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\right)dy\right|$$

$$\le (2 + \mathrm{ctg}\ \alpha_0)\omega'\rho \quad .$$

Q.E.D.

Appendix B: The Computer Program

```
      implicit double precision (a,b,c,d,e,f,h,l,o,p,q,r,s,t,u,y)
      integer ub
      common/rx/r(150)
      common/sy/s(65)
      common/coefo/c0(150,65)
      common/coef1/c1(150,65)
      common/coef2/c2(150,65)
      common/coef3/c3(150,65)
      common/unk/u(150,65)
      common/par/imax,n,m,modit,omega,test,m1,n1,k1,k2,kk,q
      common/par1/y1,y2,y3,b,f2,h2,eps,iter,testa
      data m/8/,n/12/,a/4.8d0/,ab/72.0d0/,y1/48.0d0/,
     q y2/12.0d0/,y3/9.0d0/,eps/1.0d-6/
c calculate the stepsize for the discretization, uniformly for  y,
c nonuniformly for  x
      a1=log(a/a)
      b=a+ab
      b1=log(b/a)
      a1b1=b1-a1
      do 114 ii=1,3
c make the meshs finer
      h1=a1b1/n
      h2=y1/m
      d=a*(exp(h1)-1)
c  d  is the first  dx  in the outside of the well
      k=ifix(a/d)
      kk=k+1
      d1=a/k
c  d1  is the  dx  for uniformly dividing the underneath of
c  the well
      n1=n+kk
      m1=m+1
      do 5 i=1,k
  5   r(i)=(i-1)*d1
      do 6 i=kk,n1
  6   r(i)=a*exp((i-kk)*h1)
      do 10 j=1,m1
```

```
 10     s(j)=(j-1)*h2
c  r(i),s(j)are the coordinates of the mesh points
        k1=dint(y3/h2)+1
        k2=dint(y2/h2)+1
c  k1  corresponds the bottom of the well
c  k2  correspond the water level
        print 15,a,b,y3,y2,y1,n,m,kk,k1,k2,eps
 15     format(2x,3hrw=,f12.4,2x,3hre=,f12.4,2x,2hh=,f12.4,
    q          2x,3hhw=,f12.4,2x,3hhe=,f12.4/
    q          2x,3hnx=,i5,2x,3hny=,i5,2x,3hkk=,i5,2x,3hk1=,i5,2x,
    q          3hk2=,i5,2x,4heps=,f12.8/)
c  d  is now another constant (for saving storage)
        d=d1*(y3-y2)/2
c  calculate the term in the right side corresponding the line
c  integral
        do 50 i=1,k
        c3(i,k1)= c3(i,k1)+(r(i)+d1/3)*d
 50     c3(i+1,k1)=c3(i+1,k1)+(r(i)+2/3)*d
c  calculate the coef's c0,c1,c2  and the right side term corres-
c  ponding the multiple integral
        do 65 i=1,n1-1
        do 60 j=1,m
        if (i .lt. kk .and. j .gt. k1-1) go to 65
        dx=r(i+1)-r(i)
        dy=s(j+1)-s(j)
        t1=(r(i)+dx/3.0d0)*dx*dy/2
        t2=dx**2*dy/36.0d0
        r2=dx*t2+(r(i)+dx/3.0d0)*t1
        r3=-dy*t2/2+(s(j)+dy/3.0d0)*t1
        c0(i,j)=c0(i,j)+t1/dx**2+t1/dy**2
        c1(i,j)=c1(i,j)-t1/dx**2
        c2(i,j)=c2(i,j)-t1/dy**2
        c3(i,j)=c3(i,j)-t1+(r2-r(i)*t1)/dx+(r3-s(j)*t1)dy
        c0(i+1,j)=c0(i+1,j)+t1/dx**2
        c3(i+1,j)=c3(i+1,j)-(r2-r(i)*t1)/dx
        c0(i,j+1)=c0(i,j+1)+t1/dy**2
        c3(i,j+1)=c3(i,j+1)-(r3-s(j)*t1)/dy
        t1p=(r(i)+2*dx/3.0d0)*dx*dy/2
```

```
      r2p=dx*t2+(r(i)+2*dx/3.0d0)*t1p
      r3p=-dy*t2/2+(s(j)+2*dy/3.0d0)*t1p
      c0(i+1,j+1)=c0(i+1,j+1)+t1p/dx**2+t1p/dy**2
      c3(i+1,j+1)=c3(i+1,j+1)-t1p-(r2p-r(i+1)*t1p)/dx
      c3(i+1,j+1)=c3(i+1,j+1)-(r3p-s(j+1)*t1p)/dy
      c0(i,j+1)=c0(i,j+1)+t1p/dx**2
      c1(i,j+1)=c1(i,j+1)-t1p/dx**2
      c3(i,j+1)=c3(i,j+1)+(r2p-r(i+1)*t1p)/dx
      c0(i+1,j)=c0(i+1,j)+t1p/dy**2
      c2(i+1,j)=c2(i+1,j)-t1p/dy**2
 60   c3(i+1,j)=c3(i+1,j)+(r3p-s(j+1)*t1p/dy
 65   continue
c  give the first initial value for  q. calculate the optimum
c  acceleration factor omega using carre's method
      q=100.0d0
      call init
      imax=150
      omega=1.0d0
      call itera
      omega=1.4d0
      do 19 iter=1,20
      call itera
      if (iter .eq. 17) test17=testa
      if (iter .eq. 18) test18=testa
      if (iter .eq. 19) test19=testa
 19   if (iter .eq. 20) test10=testa
      p18=test18/test17
      p19=test19/test18
      p20=test20/test19
      if ((p18-p19)*(p19-p20) .1t. 0.0d0) go to 20
      if (dabs(p18-p19) .1e. dabs(p19-p20)) go to 20
      lamdag=p18-(p19-p18)**2/(p18+p20-2*p19)
      print 17,lamdag
 17   format (20x,7haitken=,f8.4)
      go to 25
 20   lamdag=p20
```

```
 25     sq=sqrt(1.0d0-(lamdag+omega-1.0d0)**2/(lamdag*omega**2))
        omega1=omega0
        omega0=2.0d0/(1.0d0+sq)
        print 26,omega0
 26     format (/20x,7homega0=,f8.4)
        domega=dabs(omega1-omega0)
        if (domega/(2.0d0-omega0) .lt. 0.01d0) go to 45
        omegam=omega0-(2.0d0-omega0)/4
        print 30,omegam
 30     format (20x,7homegam=,f8.4)
        omega=omegam
        do 40 iter=1,20
        call itera
        if (iter .eq. 19) test19=testa
 40     if (iter .eq. 20) test20=testa
        p20=test20/test19
        go to 20
 45     omega=omega0
        print 46,omega
 46     format (/2x,6homega=,f8.4)
        test=1.0d0
        call iterat
        f1=f2
        q2=q
c  give the second initial value for  q
        q=200.0d0
c  outer iteration. secant method for computing  q
        do 79 iter1=1,10
        call init
        test=1.0d0
        call iterat
        q1=q2
        q2=q
        print 68,iter1,q1,q2,f1,f2
 68     format (2x,6hiter1=,i3,2x,3hq1=,f14.8,2x,3hq2=,f14.8,2x,
     q 3hf1=,f14.8,2x,3hf2=,f14.8)
        if (dabs(f2) .lt. eps) go to 80
        q=q2-(q2-q1)*f2/(f2-f1)
```

```
 79     f1=f2
c  print the final results
 80     ub=0
        1b=1
        f=f2
        q=q2
        do 110 n0=1,n1-1,7
        if(n1-1-n0)115,85,85
 85     np=min(n1-n0+1,7)
        ub=ub+np
        print 90,(r(i),i=1b,ub)
 90     format(///15x,f15.8,2x,f15.8,2x,f15.8,2x,f15.8,
     q 2x,f15.8,2x,f15.8,2x,f15.8/)
        do 100 j=1, m1,1
        j1=m1-j+1
        print 95, s(j1), (u(i,j1), i=1b,ub)
 95     format(2x,f11.4,2x,f15.8,2x,f15.8,2x,f15.8,2x,f15.8,
     q 2x,f15.8,2x,f15.8,2x,f15.8)
100     continue
        1b=ub + 1
110     continue
115     print 120, iter1,q,f,test
120     format(///5x,6hiter1=,i3,2x,2hq=,f15.8,
     q 2x,2hf=,f15.8,2x,5htest=,f15.8///)
        do 113 i+1,n1
        do 113 j=1,m1
        u(i,j)=0.0d0
        c0(i,j)=0.0d0
        c1(i,j)=0.0d0
        c2(i,j)=0.0d0
113     c3(i,j)=0.0d0
        m=2*m
114     n=2*n
        stop
        end
c  inner iteration. s.o.r. method for computing  u  and  f
        subroutine iterat
        implicit double precision(c,o,t,u,v,q,y,b,f,h,e)
```

```
      common/coefo/c0(9750)
      common/coef1/c1(9750)
      common/coef2/c2(9750)
      common/coef3/c3(9750)
      common/unk/u(9750)
      common/par/imax,n,m,modit,omega,test,m1,n1,k1,k2,kk,q
      common/par1/y1,y2,y3,b,f2,h2,eps,iter
      iter=0
70    iter=iter+1
      modit=mod(iter,10)
      if (modit .eq. 0) test=0.0d0
      do 7 j=2,m
      do 7 i=1,n1-1
      if (i .le. kk .and. j .gt. k1) go to 7
      if (i .eq. kk .and. j .eq. k1) go to 7
      ij=i+imax * (j-1)
      im1j=ij-1
      ip1j=ij+1
      ijm1=ij-imax
      ijp1=ij+imax
c on boundary segment gama 7, the mesh points do not have
c neighbor mesh points at their left side.  We use then  u(1)=0.0d0
c instead of  u(im1j)  in the equations
      if (i .eq. 1) im1j=1
      uold=u(ij)
      unew=-(c3(ij)+c1(ij)*u(ip1j)+c2(ij)*u(ijp1)
     q +c1(im1j)*u(im1j)+c2(ijm1)*u(ijm1))/c0(ij)
      vint=(1.0d0-omega)*uold+omega*unew
      u(ij)=dmax1(vint,0.0d0)
      if (i .gt. kk) u(ij)=dmin1(u(ij),u(i+imax*m))
      if (modit.ne.0) go to 7
      vabs=dabs(u(ij)-uold)
      test=dmax1(test,vabs)
7     continue
      if (iter .ge. 500) go to 74
      if (test .gt. eps) go to 70
74    print 75,iter,test
75    format(//20x,5hiter=,i3,2x,5htest=,f15.8)
```

```
      f2=u(kk+imax*(k1-1))-u(kk+imax*(k1-2))-(y2-y3)*h2
      return
      end
c  intitialize  u. order: gama 2, gama 3, gama 4, gama 5, else-
c  where (linear interpolation, and constant on gama 6)
      subroutine init
      implicit double precision (b,l,q,r,s,u,y,o,t,f,h)
      common/unk/u(150,65)
      common/rx/r(150)
      common/sy/s(65)
      common/par/imax,n,m,modit,omega,test,m1,n1,k1,k2,kk,q
      common/par1/y1,y2,y3,b,f2,h2
      do 201 j=1,m1
201   u(n1,j)=y1*s(j)-s(j)**2/2
      do 202 i=kk,n1
202   u(i,m1)=y1**2/2+q*log(r(i)/b)
      do 203 j=k2,m
203   u(kk,j)=u(kk,m1)
      do 204 j=k1,k2
204   u(kk,j)=u(kk,m1)-(y2-s(j))**2/2
      do 206 j=2,m
      do 205 i=kk+1,n1-1
      lamda=s(j)/y1
205   u(i,j)=u(i,m1)*lamda
206   continue
      do 208 j=2,k1
      do 207 i=1,kk
      lamda=s(j)/y3
207   u(i,j)=u(kk,k1)*lamda
208   continue
      return
      end
c  iteration for carre's method
      subroutine itera
      implicit double precision (a,b,c,d,e,f,h,o,p,q,r,s,t,u,y)
      common/coefo/c0(9750)
      common/coef1/c1(9750)
      common/coef2/c2(9750)
```

```
aon/coef3/c3(9750)
aon/unk/u(9750)
aon/par/imax,n,m,modit,omega,test,m1,n1,k1,k2,kk
aon/par1/y1,y2,y2,b,f1,h2,eps,iter,testa
:a=0.0d0
71 j=2,m
71 i=1,n1-1
l .le. kk .and. j .gt. k1) go to 71
l .eq. kk .and. j .eq. k1) go to 71
imax*(j-1)+i
j=ij-1
j=ij+1
l=ij-imax
l=ij+imax
(i .eq. 1) im1j=1
l=u(ij)
=-(c3(ij)+c1(ij)*u(ip1j)+c2(ij)*u(ijp1)
im1j)*u(im1j)+c2(ijm1)*u(ijm1))/c0(ij)
:=(1.0d0-omega)*uold+omega*unew
j)=dmax1(vint,0.0d0)
(i .gt. kk) u(ij)=dmin1(u(ij),u(i+imax*m))
s=dabs(u(ij)-uold)
:a=testa+vabs
on that we choose this norm for error may be found
 [1961]
urn
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In this paper we propose a numerical method for a degenerate variational inequality arising in the axisymmetric porous flow well problems which have been studied in Cryer and Zhou [1981]. We use the finite element method to discretize the problem, and we establish the convergence of the solution of the discrete problem to the solution of the degenerate variational inequality. The solution of the physical problem depends upon the unknown discharge q. A rapidly convergent numerical method for finding q is obtained.

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