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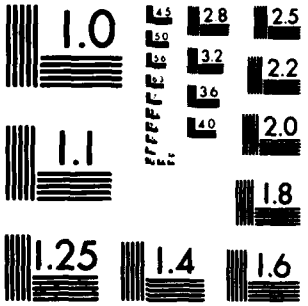

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BORSUK-ULAM THEOREMS FOR ARBITRARY S^1 ACTIONS AND APPLICATIONS

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ABSTRACT

An S^1 version of the Borsuk-Ulam Theorem is proved for a situation where Fix S^1 may be nontrivial. The proof is accomplished with the aid of a new relative index theory. Applications are given to intersection theorems and the existence of multiple critical points is established for a class of functionals invariant under an S^1 symmetry.

AMS(MOS) Subject Classification: 55B25, 58E05

Key Words: Equivariant cohomology, index theory, Borsuk-Ulam theorem, intersection theorems, minimax, critical point, Hamiltonian system.

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SIGNIFICANCE AND EXPLANATION

Minimax arguments from the calculus of variations serve as an important tool in establishing the existence of nonlinear vibrations of discrete mechanical systems as modelled by Hamilton's equations. In these arguments one obtains the solutions of the differential equations as critical points of an associated Lagrangian by minimaxing the Lagrangian over appropriate classes of sets. Intersection theorems such as are proved in this paper play a crucial role in this process. In addition to obtaining some intersection theorems ~~we~~ illustrate their use by proving an existence theorem for multiple critical points of a functional invariant under an S^1 symmetry group.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

BORSUK-ULAM THEOREMS FOR ARBITRARY S^1 ACTIONS AND APPLICATIONS

E. R. Fadell, S. Y. Husseini and P. H. Rabinowitz

Introduction.

One of the variants of the Borsuk-Ulam Theorem states that if Ω is a bounded neighborhood of 0 in $\mathbb{R}^n$ which is symmetric with respect to the origin and is a continuous odd map of $\partial\Omega$ into a proper subspace of $\mathbb{R}^n$, then f has a zero on $\partial\Omega$ [1]. An extension of this result to an infinite dimensional setting for a class of Fredholm maps was carried out by Granas [2] and more quantitative versions of the result which provide lower bounds for a topological measure of the size of $f^{-1}(0)$ $\partial\Omega$ both in finite and infinite dimensions have been given by Holm and Spanier [3]. (See also [4]).

Our main goal in this paper is to obtain analogous results when Ω is invariant and f equivariant with respect to an S^1 rather than a $\mathbb{Z}_2$ action. For a class of such actions which are fixed point free on $\mathbb{R}^{2n}$ $\{0\}$, available tools such as the index theory of [5] lead to an S^1 version of the Borsuk-Ulam Theorem by merely repeating the arguments of [3] or [4]. However when the action is not free, this approach fails. Nevertheless we will show how a relative index theory related to the cohomological index theory of [5] can be used to obtain a Borsuk-Ulam Theorem for a class of non-free S^1 actions. In §1 some properties of this index theory will be developed in a restricted setting. (A more systematic development will be carried out in a future paper). Some S^1 versions of the Borsuk-Ulam Theorem will be proved in §2. A special case is the following analogue of the $\mathbb{Z}_2$ result:

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Theorem: Let S^1 act linearly on $\mathbb{R}^\ell \times \mathbb{R}^{2k}$ (i.e. via a group of unitary operators) so that Fix $S^1 = \mathbb{R}^\ell \times \{0\}$. Suppose that A is an annulus in $\mathbb{R}^\ell \times \mathbb{R}^{2k}$ and $f: A \to \mathbb{R}^\ell \times \mathbb{R}^{2k'}$, $k' < k$, is an equivariant map to a proper invariant subspace with $f|A^G = \mathrm{id}$, $A^G = (\mathrm{Fix}\ S^1) \cap A$. If Ω is a closed, bounded invariant neighborhood of the origin with $\partial\Omega \subset A$, then $f^{-1}(0) \cap \partial\Omega \neq \phi$, i.e., f has zeroes on $\partial\Omega$.

The motivation for this paper was a study of the existence of periodic solutions of Hamiltonian systems of ordinary differential equations [6]. Minimax arguments from the calculus of variations furnish an important tool for treating such questions. In these arguments one obtains solutions of the differential equations as critical points of an associated functional by minimaxing the functional over appropriate classes of sets. Intersection theorems play a crucial role in getting estimates, in particular lower bounds, on minimax values and in showing they are indeed critical values of the functional. In §3 the Borsuk-Ulam results will be used to obtain some new intersection theorems. To illustrate their use an abstract critical point theorem (see Theorem 3.14 below) in an S^1 setting related to a $\mathbb{Z}_2$ result from [7] will be proved.

§1. A Relative Index

Throughout this section $G = S^1$, the group of complex numbers of norm 1. Suppose X is a paracompact G-space, i.e. $G(=S^1)$ acts on X. As usual, the fixed point set under this action is given by

$$X^G = \{x \in X : gx = x \text{ for all } g \in G\}.$$

Also, for $x \in X$, the isotropy subgroup at $x \in X$ is given by

$$G_x = \{g \in G : gx = x\}.$$

Note, that because $G = S^1$, G_x is a finite cyclic group when $x \notin X^G$.

Following [5], let $\eta = (E_G, p, B_G)$ denote the universal G-bundle, e.g. $E_G = S^\infty$, $B_G = CP^\infty$, see [5]. Set

$$E_G(X,X^G) = E_G \times_G (X,X^G) = (E_G \times_G X, B_G \times X^G)$$

where G acts <u>freely</u> on $E_G \times X$ and $E_G \times X^G$ by the action $g(e,x) = (ge,gx)$ and $E \times_G X$, $B_G \times X^G$ are the resulting orbit spaces. The projection $E_G \times X \to E_G$ induces a fiber bundle pair

$$(q,q_0) : E_G(X,X^G) \to B_G$$

with fiber (X,X^G). $q: E \times_G X \to B_G$ is, in fact, the classifying map of the principal G-bundle, $\eta : E_G \times X \to E_G \times_G X$ and $q_0 : B_G \times X^G \to B_G$ is simply projection on B. Now, using Alexander-Spanier cohomology H^* with rational coefficients, we set

$$H_G^*(X,X^G) = H^*(E_G(X,X^G))$$

and

$$H_G^*(X) = H^*(E_G(X)) = H^*(E_G \times_G X).$$

The relative cup product

$$H_G^*(X) \times H_G^*(X,X^G) \longrightarrow H_G^*(X,X^G)$$

then gives $H_G^*(X,X^G)$ a module structure over the ring $\Lambda = H^*(B_G) = H_G^*$ (point) with left action given by

$$\lambda \cdot z = q^*(\lambda) \cup z, \quad \lambda \in \Lambda, \quad z \in H_G^*(X,X^G).$$

We recall also the fact that $\Lambda = \mathbb{Q}[\alpha]$, where $\mathbb{Q}[\alpha]$ is the polynomial algebra on a single generator $\alpha \in H^2(B_G)$.

Definition (1.1) The relative α-index $\mathrm{Ind}_\alpha(X,X^G)$ of the pair (X,X^G) is the smallest integer s such that α^s annihilates $H_G^*(X,X^G)$, i.e.

$$\mathrm{Ind}_\alpha(X,X^G) = \min_s\{\alpha^s \cdot x = 0 \text{ for all } x \in H_G^*(X,X^G)\}.$$

Proposition (1.2). When $X^G = \emptyset$, the relative index coincides with the α-index in [5], i.e.

$$\mathrm{Ind}_\alpha(X,\emptyset) = \mathrm{Index}_\alpha^* X$$

and hence in this special situation all the basic properties of Index_α^* in [5] hold for $\mathrm{Ind}_\alpha(X,\emptyset)$.

Proof. When $X^G = \emptyset$, $H_G^*(X,X^G) = H_G^*(X)$ and we have a unit $1 \in H_G^0(X)$. Then, since

$$\alpha^s \cdot 1 = 0 \iff \alpha^s = 0$$

we have the desired result.

Remark (1.3). The basic properties of the relative index $\mathrm{Ind}_\alpha(X,X^G)$ in general require some modification and will be taken up in a future work. They will be studied in the category of pairs (X,A) where X is a G-space. A an invariant subspace and G an arbitrary compact Lie group. The sequel will not require these modified properties. However, the following one is of interest since in [5], $X^G \neq \emptyset$ always implies $\mathrm{Index}_\alpha^* X = \infty$.

Proposition (1.4). If X is a finite dimensional separable metric space, then

$$\mathrm{Ind}_\alpha(X,X^G) < \infty.$$

Proof. The natural projection $E \times X \to X$ induces a map

$$\varphi : E_G(X,X^G) \to (X/G,X^G)$$

which induces isomorphisms [15]

$$\varphi^* : H^*(X/G,X^G) \xrightarrow{\approx} H_G^*(X,X^G)$$

(because $G = S^1$ and coefficients are rational). When X is a finite dimensional separable metric space, so is X/G and hence $H^n_G(X, X^G)$ vanishes for n sufficiently large.

We now proceed to calculate an important example. Suppose that $G = S^1$ acts linearly on $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$, the n-fold cartesian product of the real line. Introduce a G-equivariant inner-product on $\mathbb{R}^n$ (by the "unitary trick") and assume that $G = S^1$ is a subgroup of $SO(n,\mathbb{R})$, the group of rotations of $\mathbb{R}^n$. The fixed-point set $(\mathbb{R}^n)^G = \{x \in \mathbb{R}^n | \sigma x = x, \text{ for all } \sigma \in G\}$ is a linear subspace $\mathbb{R}^\ell$ of dimension, say ℓ. Then clearly the orthogonal complement $(\mathbb{R}^\ell)^\perp$ is G-invariant and of even-dimension, say $2n'$. Thus we obtain the G-equivariant decomposition

$$\mathbb{R}^n \simeq \mathbb{R}^\ell \times \mathbb{R}^{2n'}$$

with $\mathbb{R}^\ell \times \{0\} = (\mathbb{R}^n)^G$. Note that if $x \notin \mathbb{R}^\ell \times \{0\}$ then the isotropy subgroup $G_x \subset G$ is a finite cyclic subgroup of G. We wish to compute $\mathrm{Ind}_\alpha(S^{n-1}, (S^{n-1})^G)$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$,

$$(S^{n-1})^G = S^{n-1} \cap (\mathbb{R}^\ell \times \{0\}) = S^{\ell-1}$$

and $S^{\ell-1}$ is the unit sphere in $\mathbb{R}^\ell \times \{0\}$.

<u>Proposition</u> (1.5). $\underline{\mathrm{Ind}_\alpha}(S^{n-1}, S^{\ell-1}) = n'$, <u>where</u> $n = \ell + 2n'$.

Note that if $\ell = 0$, then Proposition (1.5) is just Proposition (7.4) of [5]. We shall prove Proposition (1.5) in stages. First, let us consider the fibration

$$q : E_G(S^{n-1}) \to B_G$$

with fiber S^{n-1}.

<u>Lemma</u> (1.6): <u>There exists an isomorphism of</u> $H^*(B_G)$<u>-modules</u>

$$\varphi : H^*(B_G) \otimes_{\mathbb{Q}} H^*(S^{n-1}) \to H^*_G(S^{n-1})$$

<u>given by</u> $\varphi(u \otimes x) = q^*(u) \cup \theta(x)$, <u>where</u> $\theta : H^*(S^{n-1}) \to H^*_G(S^{n-1})$ <u>is a cohomology extension of the fiber. Let</u> $\tilde{\gamma}$ <u>denote a generator of</u> $H^{n-1}(S^{n-1})$

and $\gamma = \varphi(1 \otimes \bar{\gamma})$. Then, for certain integers r and t and $c_i \in Q$,

$$\gamma^2 = \gamma(c_1 \alpha^r + c_2 \alpha^t \otimes \bar{\gamma}).$$

Proof. This is an immediate consequence of the Leray-Hirsch Theorem [15] as soon as we establish the triviality of the (rational) spectral sequence of the fibration $S^{n-1} \to E \times_G S^{n-1} \to B_G$. The differential operators $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ vanish for $2 \leq r \leq n-1$ and in fact the only possible non-trivial differential operator occurs when $r = n$. Consider the diagram

$$\begin{tikzcd} E_G(S^{\ell-1}) \arrow[rr] \arrow[dr, "q'"'] & & E_G(S^{n-1}) \arrow[dl, "q"] \\ & B_G & \end{tikzcd}$$

Since $E_G(S^{\ell-1}) = S^{\ell-1} \times B_G$, $(q')^*$ and hence q^* is injective. If the differential operators

$$d_n^{p,n-1} : E_n^{p,n-1} \to E_n^{p+n,0}$$

were not trivial, then the edge homomorphism diagram

$$\begin{tikzcd} H^{p+n}(B_G) = E_n^{p+n,0} \arrow[rr] \arrow[dr, "q^*"'] & & E_{n+1}^{p+n,0} = E_\infty^{p+n,0} \arrow[dl] \\ & H_G^{p+n}(S^{n-1}) & \end{tikzcd}$$

would produce a non-trivial kernel for q^*. Thus all d_r's are trivial.

Remark (1.7). If $(n-1)$ is odd, then $\gamma^2 = 0$ above.

We consider now the exact sequence of the pair $E_G(S^{n-1}, S^{\ell-1})$

$$\cdots \to H_G^q(S^{n-1}, S^{\ell-1}) \xrightarrow{j^*} H_G^q(S^{n-1}) \xrightarrow{i^*} H_G^q(S^{\ell-1}) \to \cdots .$$

Lemma (1.8). i^* *is a monomorphism.*

Proof. First observe that j^* and i^* are $H^*(B_G)$-homomorphisms. Suppose $i^*(x) = 0$. Then, for some element $u \in H_G^*(S^{n-1}, S^{\ell-1})$ $j^*(u) = x$. But since $H_G^r(S^{n-1}, S^{\ell-1})$ vanishes for r large, $j^*(\alpha^r(u)) = 0 = \alpha^r(j^*u) = 0$. But $H^*(B_G)$ operates freely on $H_G^*(S^{n-1})$ and hence $j^*(u) = 0 = x$.

Lemma (1.9). $H_G^*(S^{n-1}, S^{\ell-1})$ *is generated over* $H^*(B_G)$ *by a single element* β *with* $\alpha^{n'}\beta = 0$, *i.e.*

$$H_G^*(S^{n-1}, S^{\ell-1}) = \{\beta, \alpha\beta, \cdots, \alpha^{n'-1}\beta \mid \alpha^{n'}\beta = 0\}.$$

Proof. Consider the short exact sequence of $H^*(B_G)$-modules

$$0 \to H_G^q(S^{n-1}) \xrightarrow{i^*} H_G^q(S^{\ell-1}) \xrightarrow{\delta^*} H_G^{q+1}(S^{n-1}, S^{\ell-1}) \to 0$$

and $H^*(B_G)$-isomorphisms

$$\varphi : H^*(B_G) \otimes H^*(S^{n-1}) \to H_G^*(S^{n-1})$$
$$\psi : H^*(B_G) \otimes H^*(S^{\ell-1}) \to H_G^*(S^{\ell-1}).$$

Let $\bar{\gamma}$ and $\bar{\beta}$ denote the fundamental class of S^{n-1} and $S^{\ell-1}$, respectively, and $\gamma = \varphi(1 \otimes \bar{\gamma})$, $\beta = \psi(1 \otimes \bar{\beta})$. Then, we have two cases:

$$i^*(\gamma) = \psi\left(c\alpha^{n'} \otimes \beta + d\,\alpha^{\frac{n-1}{2}}\right) \text{ if } n-1 \text{ is even}$$

and

$$i^*(\gamma) = \psi(c\,\alpha^{n'} \otimes \beta) \qquad \text{if } n-1 \text{ is odd.}$$

In either case $\psi(\alpha^{n'} \otimes \beta)$ is in the image of i^* while the elements $\psi(\alpha^j \otimes \beta)$, $j < n'$, are not. This gives the desired result.

Proof of Proposition (1.5). Apply the previous lemma.

Proposition (1.5) is the key to the following Theorem. (cf. Proposition (3.9) of [5].)

Theorem (1.10): *Suppose that* G *acts on* $\mathbb{R}^n = \mathbb{R}^\ell \times \mathbb{R}^{2n'}$ *as described above, and let* Ω *be a closed, bounded invariant neighborhood of* 0 *in* $\mathbb{R}^n$. *Then*

$$\mathrm{Ind}_\alpha(\partial\Omega, (\partial\Omega)^G) > n'.$$

Theorem (1.10) is an immediate consequence of the following form of the Piercing Property [5]. Notice the conclusion is $>$ and not equality.

<u>Proposition</u> (1.11) (<u>Piercing Property</u>). <u>Let</u> X <u>denote a paracompact</u> G-<u>space and</u> $X \times I$ <u>the corresponding</u> G-<u>space with action</u> $g(x,t) = (gx,t)$, $g \in G$, $x \in X$, $t \in I$. <u>Let</u> $A_0 = X \times \{0\}$, $A_1 = X \times \{1\}$. <u>Suppose</u>

$$X \times I = B_0 \cup B_1, \quad A_0 \subset B_0, \quad A_1 \subset B_1$$

<u>where</u> B_0 <u>and</u> B_1 <u>are closed invariant subsets. Then, if</u> $C = B_0 \cap B_1$,

$$\mathrm{Ind}_\alpha(C,C^G) > \mathrm{Ind}_\alpha(X,X^G).$$

<u>Proof.</u> The proof is essentially the same as in [5]. Let $B = B_0 \cup B_1$. Then the inclusion maps

$$\begin{tikzcd} A_0 \arrow[rr, "k_0"] \arrow[dr, "j_0"'] & & B_0 \arrow[dl, "i_0"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} A_1 \arrow[rr, "k_1"] \arrow[dr, "j_1"'] & & B_1 \arrow[dl, "i_1"] \\ & B & \end{tikzcd} \qquad \begin{array}{l} \ell_0 : C \subset B_0 \\ \ell_1 : C \subset B_1 \end{array}$$

induce $H^*(B_G)$-homomorphisms, $k_0^* : H_G^*(B_0,B_0^G) \to H_G^*(A,A_0^G)$, j_0^*, i_0^*, etc.

Consider the Mayer-Vietoris sequence

$$\cdots \to H_G^q(B,B^G) \xrightarrow{\zeta} H_G^q(B_0,B_0^G) \oplus H_G^q(B_1,B_1^G) \xrightarrow{\eta} H_G^q(C,C^G) \to$$

where $\zeta = (i_0^*,-i_1^*)$, $\eta = \ell_0^* + \ell_1^*$. If $\ell_0^*(x) = 0$, then $\eta(x,0) = 0$ and $i_0^*(y) = x$, $i_1^*(y) = 0$ for some y. But then $j_1^*(y) = 0$. Since $j_1 : (A_1,A_1^G) \to (B,B^G)$ is an equivariant homotopy equivalence, $j_1^* : H_G(B,B^G) \to H_G(A_1,A_1^G)$ is an isomorphism and hence $y = 0$. Thus $x = 0$ and ℓ_0^* is a monomorphism. Now, let $\gamma : B \to A_0$ denote projection (which is equivariant) and $\gamma_C : C \to A_0$, $\gamma_0 : B_0 \to A_0$ the corresponding restrictions. Notice that we do not conclude that γ_C implies $\mathrm{Ind}_\alpha(C,C^G) \leq \mathrm{Ind}_\alpha(A_0,A_0^G)$. However, in the diagram

$$\begin{tikzcd} H_G^q(B_0,B_0^G) \arrow[rr, "\ell_0^*"] & & H_G^q(C,C^G) \\ & H_G^q(A_0,A_0^G) \arrow[ul, "\gamma_0^*"] \arrow[ur, "\gamma_C^*"'] & \end{tikzcd}$$

γ_0^* is also a monomorphism which forces γ_C^* to be a monomorphism of $H^*(B_G)$-modules. Thus if $\alpha^s \cdot \gamma_C^*(c) = 0$, then $\alpha^s c = 0$ and since $(A_0,A_0^G) \equiv (X,X^G)$,

$$\mathrm{Ind}_\alpha(C,C^G) \geq \mathrm{Ind}_\alpha(X,X^G).$$

<u>Proof of (1.10).</u> Let A denote an annulus in $\mathbb{R}^n$ centered at the origin with $\partial\Omega \subset \mathrm{int}\, A$. Then, $A = B_0 \cup B_1$ where $B_0 = A \cap \Omega$, $B_1 = A \cap (\mathbb{R}^n \setminus \mathrm{int}\, \Omega)$ and $\partial\Omega = B_0 \cap B_1$. Thus, applying (1.11) we have

$$\mathrm{Ind}_\alpha(\partial\Omega,\ (\partial\Omega)^G) \geq n'.$$

We next consider special forms of the monotonicity and additivity properties which will be useful in the next section.

<u>Proposition</u> (1.12). <u>Let</u> X <u>and</u> Y <u>denote paracompact</u> G-<u>spaces and</u> $f:X \to Y$ <u>an equivariant map. If</u>

$$f^* : H_G^*(Y,Y^G) \to H_G^*(X,X^G)$$

<u>is surjective, then</u> $\mathrm{Ind}_\alpha(X,X^G) \leq \mathrm{Ind}_\alpha(Y,Y^G)$.

<u>Proof.</u> Suppose $\mathrm{Ind}_\alpha(Y,Y^G) = s$. Then, $f^*(\alpha^s y) = \alpha^s f^*(y) = 0$. Since every element of $H_G^*(X,X^G)$ is of the form $f^*(y)$, α^s annihilates $H_G^*(X,X^G)$ and the result follows.

<u>Proposition</u> (1.13). <u>Let</u> X <u>denote a paracompact</u> G-<u>space and</u> A <u>and</u> B <u>paracompact subsets whose interiors cover</u> X. <u>Suppose further that</u> $B^G = \emptyset$ <u>(so that all the fixed points of</u> X <u>are in</u> A). <u>Then,</u>

$$\mathrm{Ind}_\alpha(X,X^G) \leq \mathrm{Ind}_\alpha(A,A^G) + \mathrm{Ind}_\alpha(B,\emptyset).$$

Proof. Suppose $\mathrm{Ind}_\alpha(A,A^G) = a$, $\mathrm{Ind}_\alpha(B,\emptyset) = b$ where both a and b are finite. Consider the diagram

$$\begin{tikzcd} E_G(A) \arrow[dr, "i_1"] & & \\ & E_G(X) \arrow[r, "f"] & B_G \\ E_G(B) \arrow[ur, "i_2"'] & & \end{tikzcd}$$

where i_1 and i_2 are induced by inclusions and f is the classifying map for $E \times G \to E_G(X)$. Then, if $\beta = i_1^* f^*(\alpha)$ and $\gamma = i_2 f^*(\alpha)$ we have β^a annihilates $H_G^*(A,A^G)$ and $\gamma^b = 0$. Take any element

$$x \in H_G^p(X,X^G)$$

and consider the exact sequences $(X^G = A^G)$

$$\cdots \to H_G^r(X,A) \xrightarrow{j_1^*} H_G^r(X,X^G) \to H_G^r(A,A^G) \to$$

$$\cdots \to H_G^s(X,B) \xrightarrow{j_2^*} H_G^s(X) \to H_G^s(B) \to .$$

Then, $f^*(\alpha^a) \cup x$ and $f^*(\alpha^b)$ pull back under j_1^* and j_2^*, respectively. Then the following diagram

$$\begin{tikzcd} H_G^*(X,A) \otimes H_G^*(X,B) \arrow[r, "\cup"] \arrow[d] & H_G^*(X,X) \arrow[d] \\ H_G^*(X,X^G) \otimes H_G^*(X) \arrow[r, "\cup"] & H_G^*(X,X^G) \end{tikzcd}$$

shows that $f^*(\alpha^{a+b}) \cup x = 0$ and $\mathrm{Ind}_\alpha(X,X^G) \leq a+b$.

Remark (1.14). (a) If $G = S^0 = \{\pm 1\}$, then using cohomology with coefficients in $\mathbb{Z}_2$ instead of $\mathbb{Q}$, one proves in a similar fashion to the above that

$$\mathrm{Ind}_\alpha(\partial\Omega, (\partial\Omega)^G) \geq n - \ell = 2n',$$

where $R^n = R^\ell \times R^{2n'}$.

(b) If $G = S^3$, the unit quaternions and $G_x \subset G$ is finite for $x \notin R^\ell \times \{0\}$, then one can prove, using rational cohomology, that

$$\mathrm{Ind}_\alpha(\partial\Omega, (\partial\Omega)^G) \geq n',$$

where $R^n = R^\ell \times R^{4n'}$. The general situation is much more complicated.

(c) If G is a finite cyclic group of prime order, then using cohomology with coefficients in $\mathbb{Z}_p$, one can prove that

$$\mathrm{Ind}_\alpha(\partial\Omega, (\partial\Omega)^G) \geq n',$$

where $R^n = R^\ell \times R^{2n'}$. If G does not have prime order, the problem is more complicated.

We close this section with the following result which was not included in [5] and will also prove useful.

<u>Proposition</u> (1.15). <u>When</u> X <u>is compact and</u> $X^G = \emptyset$, <u>then</u>

$$\mathrm{Ind}_\alpha(X,\emptyset) = \mathrm{Index}^*_\alpha X < \infty.$$

<u>Proof.</u> For $x \in X$, the orbit $Gx = G/G_x$, where G_x is the isotropy group at x. G_x is a finite cyclic group and $\mathrm{Index}^*_\alpha(Gx) = 1$ ([5], Proposition 6.12). A suitable neighborhood $N(Gx)$ of this orbit also has $\mathrm{Index}^*_\alpha N(Gx) = 1$. Since X is compact, a finite number of such neighborhoods cover X and subadditivity ([5], Proposition 6.6) completes the proof.

§2. Zeros of Equivariant Maps.

Suppose that $G\ (=S^1)$ acts on $\mathbb{R}^a \approx \mathbb{R}^\ell \times \mathbb{R}^{2a'}$ and $\mathbb{R}^b = \mathbb{R}^\ell \times \mathbb{R}^{2b'}$ as a group of rotations in such a way that (as before)

$$(\mathbb{R}^a)^G = \mathbb{R}^\ell \times \{0\}, \quad (\mathbb{R}^b)^G = \mathbb{R}^\ell \times \{0\}.$$

Suppose we have an equivariant map

$$f : S^{a-1} \to \mathbb{R}^b \setminus \{0\}$$

such that the restriction $f_0 : S^{\ell-1} \to \mathbb{R}^\ell \setminus \{0\}$ has the property that

$$f_0^* : H^{\ell-1}(\mathbb{R}^\ell \setminus \{0\}) \to H^{\ell-1}(S^{\ell-1})$$

is an isomorphism. Then, in the following diagram

$$\begin{array}{ccc} H_G^{\ell-1}(\mathbb{R}^\ell \setminus \{0\}) & \xrightarrow{\delta} & H_G^\ell(\mathbb{R}^b \setminus \{0\}, \mathbb{R}^\ell \setminus \{0\}) \\ (f_0)_G^* \Big\downarrow & & \Big\downarrow f_G^* \\ H_G^{\ell-1}(S^{\ell-1}) & \xrightarrow{\delta} & H_G^\ell(S^{a-1}, S^{\ell-1}) \end{array}$$

$(f_0)_G^*$ is an isomorphism and f_G^* takes the generator (over $H^*(B_G)$) to a generator and f_G^* is surjective. Applying the monotonicity property (1.12) we have

$$a' = \mathrm{Ind}_\alpha(S^{a-1}, S^{\ell-1}) \leq \mathrm{Ind}_\alpha(\mathbb{R}^b \setminus \{0\}, \mathbb{R}^\ell \setminus \{0\}) = b'$$

Thus, we have the following result.

<u>Proposition (2.1). Suppose</u> $f : S^{a-1} \to \mathbb{R}^b$ <u>is an equivariant map, where</u> $\mathbb{R}^a$ <u>and</u> $\mathbb{R}^b$ <u>are as above. If</u>

$$f_0^* : H^{\ell-1}(\mathbb{R}^\ell \setminus \{0\}) \to H^{\ell-1}(S^{\ell-1})$$

<u>is an isomorphism, then</u> $f^{-1}(0) \neq \emptyset$, <u>i.e.</u> f <u>possesses at least one zero.</u>

We will now extend this result to maps $f : \partial\Omega \to \mathbb{R}^b$ where Ω is a closed, bounded invariant subset of $\mathbb{R}^a$. Let A denote an annulus in $\mathbb{R}^a$ (as usual, the region between two concentric spheres centered at the origin) and let Ω denote a closed, bounded invariant neighborhood of the origin with

$\partial\Omega \subset A$, where $\partial\Omega$ is the boundary relative to $\mathbb{R}^a$. We assume once and for all that

$$f : A \to \mathbb{R}^b$$

is a given equivariant map.

<u>Proposition</u> (2.2). <u>Suppose</u> $f(A^G) \subset \mathbb{R}^\ell \setminus \{0\}$ <u>and</u>

$$(f|A^G)^* : H^*(\mathbb{R}^\ell \setminus \{0\}) \to H^*(A^G)$$

<u>is an isomorphism. Then, if</u> $a > b$, $Z = f^{-1}(0) \cap \partial\Omega \neq \emptyset$.

<u>Proof.</u> Suppose $f|\partial\Omega : \partial\Omega \to \mathbb{R}^b \setminus \{0\}$ and consider the diagram

$$\begin{tikzcd}
 & H_G^{\ell-1}(\mathbb{R}^\ell \setminus \{0\} \arrow[r, "\delta"] \arrow[d, "(f|(\partial\Omega)^G)^*"] & H_G^\ell(\mathbb{R}^b \setminus \{0\}, \mathbb{R}^\ell \setminus \{0\}) \arrow[d, "(f|\partial\Omega)^*"] \\
(f|A^G)^* \arrow[ur] \arrow[dr] & H_G^{\ell-1}((\partial\Omega)^G) \arrow[r, "\delta"] & H_G^\ell(\partial\Omega, (\partial\Omega)^G) \\
 & H_G^{\ell-1}(A^G) \arrow[r, "\delta"] \arrow[u, "i^*"] & H_G^\ell(A, A^G) \arrow[u, "j"]
\end{tikzcd}$$

The G-space A is equivariantly homotopic to S^{a-1}, and $\mathbb{R}^b \setminus \{0\}$ to S^{b-1}. Choose

$$\bar{\beta}_1 \in H_G^{\ell-1}(\mathbb{R}^\ell \setminus \{0\}), \quad \bar{\beta}_2 = (f|A^G)^*(\bar{\beta}_1) \in H_G^{\ell-1}(A^G)$$

so that $\beta_1 = \delta\bar{\beta}_1$ and $\beta_2 = \delta\bar{\beta}_2$ are generators over $H^*(B_G)$ of $H_G^*(\mathbb{R}^b \setminus \{0\}, \mathbb{R}^\ell \setminus \{0\})$ and $H_G^*(A, A^G)$, respectively. Then, if $\bar{\beta}_3 = i^*(\bar{\beta}_2) = (f|(\partial\Omega)^G)^*(\bar{\beta}_1)$, $\beta_3 = \delta\bar{\beta}_3$ generates a subalgebra isomorphic by j^* to $H_G^\ell(A, A^G)$. Since $\beta_3 = (f|\partial\Omega)^*(\beta_1)$ we have $\alpha^{b'} \cdot \beta_3 = 0$. On the other hand, $\alpha^{a'} \cdot \beta_3 = 0$ <u>with a' minimal.</u> Thus, $a' \leq b'$ and $a \leq b$. Thus, $a > b$ implies $Z = f^{-1}(0) \cap \partial\Omega \neq \emptyset$.

We now proceed to determine the index of the zero set Z.

<u>Theorem</u> (2.3). <u>Under the hypotheses of</u> (2.2), <u>if</u> $Z = f^{-1}(0) \cap \partial\Omega$,

$$\mathrm{Ind}_\alpha Z \geq a' - b'.$$

Proof. We know $Z \neq \emptyset$ by Proposition (2.2), although this fact is not necessary for the argument. Suppose $s = \mathrm{Ind}_\alpha Z < a' - b'$. Then, Z is a closed subset of $\partial\Omega$ and hence of $\partial\Omega \setminus (\partial\Omega)^G$. Since $Z^G = \emptyset$, $\mathrm{Index}^*_\alpha Z = \mathrm{Ind}\, Z$, where Index^*_α is the index theory in [5]. Thus, since Index^*_α is continuous, we have a closed neighborhood N of Z (in $\partial\Omega$) such that $N^G = \emptyset$, $\mathrm{Ind}_\alpha N = s$ and a closed set $M \subset \partial\Omega$ such that $\mathrm{int}M \cup \mathrm{int}N = \partial\Omega$, $M \cap Z = \emptyset$ and $M^G = (\partial\Omega)^G$. We know that $H^*_G(\partial\Omega,(\partial\Omega)^G)$ contains an $H^*(B_G)$-submodule $\{\beta, \alpha\beta, \alpha^2\beta, \cdots, \alpha^{a'-1}\beta : \alpha^{a'}\beta = 0\}$. The inclusion map

$$j : (M,M^G) \to (\partial\Omega,(\partial\Omega)^G)$$

induces

$$j^* : H^*(\partial\Omega,(\partial\Omega)^G) \to H^*(M,M^G)$$

and we set $\beta_M = j^*(\beta)$. Let t denote the smallest integer such that $\alpha^t\beta_M = 0$. The diagram

$$\begin{tikzcd}
H^{\ell-1}_G(\mathbb{R}^\ell \setminus \{0\}) \arrow[r, "\delta"] \arrow[d, "(f|M^G)^*"', "\approx"] & H^\ell(\mathbb{R}^b \setminus \{0\},\mathbb{R}^\ell \setminus \{0\}) \arrow[d, "(f|M)^*"] \\
H^{\ell-1}_G(M^G) \arrow[r, "\delta"] & H^\ell(M,M^G) \\
H^{\ell-1}_G((\partial\Omega)^G) \arrow[r, "\delta"] \arrow[u, "\mathrm{id}"] & H^\ell(\partial\Omega,(\partial\Omega)^G) \arrow[u]
\end{tikzcd}$$

tells us that $t \leq b'$. On the other hand using the argument for the additivity property (1.13) and the diagram

$$\begin{tikzcd}
H^*(\partial\Omega,N) \arrow[d] & \oplus & H^*(\partial\Omega,M) \arrow[d] & \to & H^*(\partial\Omega,\partial\Omega) \arrow[d] \\
H^*(\partial\Omega) \arrow[d] & \oplus & H^*(\partial\Omega,(\partial\Omega)^G) \arrow[d] & \to & H^*(\partial\Omega,(\partial\Omega)^G) \\
H^*(N) & \oplus & H^*(M,M^G) \arrow[urr] & &
\end{tikzcd}$$

we see that $\alpha^{s+t}\beta = 0$ so that

$$a' \leq s+t < (a'-b') + b' = a'$$

which is a contradiction. Thus, $\mathrm{Ind}_\alpha Z \geq a' - b'$.

<u>Corollary</u> (2.4). If $f: A \to \mathbb{R}^b$ is the identity map on A^G and Z is as above, then

$$\mathrm{Ind}_\alpha Z \geq a' - b'.$$

3. Applications.

In this section some applications of Theorem 2.3 will be given. First some finite and infinite dimensional intersection theorems will be obtained. Then these results will be used to prove an abstract critical point theorem in an S^1 setting. In the applications below we will be dealing with situations in which $X \cap \text{Fix } S^1 = \phi$. Hence $X^G = \phi$ and $\text{ind}_\alpha(X, X^G) = \text{Ind}_\alpha(X, \phi) = \text{Index}_\alpha^* X$ ([5]). As a convenience we list the properties of Ind_α from [5] (and (1.14)) we will require below, and at the same time suppress the α. We also let $C^k(X,Y)$ denote the set of k times continuously Frechet differentiable mappings from X to Y.

<u>Lemma 3.1: Let</u> E <u>be a Hilbert space and let</u> S^1 <u>act on</u> E. <u>Let</u> $\mathcal{E}$ <u>denote the family of equivariant subsets of</u> $E \setminus \text{Fix } S^1$. <u>Then</u> $\text{Ind}: \mathcal{E} \to \mathbb{N} \cup \{\infty\}$ <u>possesses the following properties: For</u> $X, Y \in \mathcal{E}$,

1°. <u>If</u> $h \in C(X,Y)$ <u>is equivariant,</u> $\text{Ind } X \leq \text{Ind } Y$.

2°. $\text{Ind}(X \cup Y) \leq \text{Ind } X + \text{Ind } Y$.

3°. <u>If</u> X <u>is compact, then</u> $\text{Ind } X < \infty$ <u>and there exists a</u> $\delta > 0$ <u>such that</u> $\text{Ind } X = \text{Ind } N_\delta(X)$ <u>where</u> $N_\delta(X) = \{x \in E \mid \|x - X\| \leq \delta\}$.

Our first intersection result is in a Euclidean space setting. Let B_R^j denote the closed ball of radius R centered about 0 in $\mathbb{R}^\ell \times \mathbb{R}^{2j'}$ where $j = \ell + 2j'$. Whenever $j' < k'$, we will consider $\mathbb{R}^{2j'}$ as a subspace of $\mathbb{R}^{2k'}$ via $\mathbb{R}^{2j'} \cong \mathbb{R}^{2j'} \times \{0\} \subset \mathbb{R}^{2j'} \times \mathbb{R}^{2(k'-j')} \cong \mathbb{R}^{2k'}$.

Let S^1 act on $\mathbb{R}^\ell \times \mathbb{R}^{2k'}$ via a group of unitary operators such that $\text{Fix } S^1 = \mathbb{R}^\ell \times \{0\}$ and $\{0\} \times \mathbb{R}^{2j'}$ is an invariant subspace of $\{0\} \times \mathbb{R}^{2k'}$ for $j' < k'$. Suppose $h \in C(B_R^k, \mathbb{R}^\ell \times \mathbb{R}^{2m'})$ where $m' \geq k'$, $h(x) = x$ on $(\mathbb{R}^\ell \times \{0\}) \cap B_R^k$, and h is equivariant. Let $\rho < R$ and consider $h^{-1}(B_\rho^m)$. This is an invariant neighborhood of 0 in $\mathbb{R}^\ell \times \mathbb{R}^{2k'}$. Let Ω denote the component of $h^{-1}(B_\rho^m)$ which contains 0. Then $\partial\Omega$ is an

invariant set. For $j' < k'$, let $P_{\ell+2j'}$ denote the orthogonal projector of $\mathbb{R}^\ell \times \mathbb{R}^{2m'}$ onto $\mathbb{R}^\ell \times \mathbb{R}^{2j'}$ and consider $f \equiv P_{\ell+2j'}h$. Then $f \in C(\partial\Omega, \mathbb{R}^\ell \times R^{2j'})$, $f(x) = x$ for $x \in (\mathbb{R}^\ell \times \{0\}) \cap \Omega$, and f is equivariant. Thus as an immediate consequence of Theorem 2.3 we have

<u>Proposition 3.2:</u> $\mathrm{Ind}(f^{-1}(0) \cap \partial\Omega) \geq k' - j'$.

With a bit more structure on h we have

<u>Proposition 3.3:</u> <u>If</u> $k \in C(B_R^k, \mathbb{R}^\ell \times \mathbb{R}^{2m'})$ <u>where</u> $m' \geq k'$, $h(x) = x$ <u>for</u> $x \in (\mathbb{R}^\ell \times \{0\}) \cup \partial B_R^k$, <u>and</u> h <u>is equivariant, then for any</u> $\rho < R$ <u>and</u> $j' < k'$,

$$\mathrm{Ind}(h(B_R^k) \cap \partial B_\rho^m \cap (\{0\} \times \mathbb{R}^{2(m'-j')})) \geq$$

(3.4)

$$\mathrm{Ind}(\{x \in B_R^k | h(x) \in \partial B_\rho^m \cap (\{0\} \times \mathbb{R}^{2(m'-j')})\} \geq k' - j'.$$

<u>Proof:</u> We need only observe that $h(x) = x$ on ∂B_ρ^m implies Ω is strictly interior to B_ρ^m. Hence if $x \in \partial\Omega$, $h(x) \in \partial B_\rho^m$ via the maximality of Ω. Now (3.4) follows from 1° of Lemma 3.1 and Proposition 3.2.

Next we will prove two infinite dimensional extensions of Proposition 3.2. For what follows let E be a separable infinite dimensional Hilbert space and suppose S^1 acts on E, the action being given by a group G of unitary operators on E. Further assume $\mathrm{Fix}\, G \equiv E_0$ is a finite dimensional subspace of E with $\dim E_0 = \ell$ and we can choose an orthogonal basis $\{v_j\}$ in E such that $E_0 = \mathrm{span}\{v_1, \cdots, v_\ell\}$, $E_m = \mathrm{span}\{v_1, \cdots, v_{\ell+2m}\}$, E_m is an invariant subspace of E, and $E = \overline{\cup E_m}$. For brevity when the above are satisfied, we say E, S^1 satisfy (*).

As an example, consider $W^{1,2}(S^1)$ the Hilbert space of 2π periodic functions under the norm

$$\|q\|^2 = \int_0^{2\pi} (|\frac{dq}{dt}|^2 + |q|^2)dt.$$

There is a natural S^1 action on $W^{1,2}(S^1)$. Indeed for $q \in E$ and $\theta \in [0,2\pi)$ let $T_\theta q(t) = q(t+\theta)$ and $G = \{T_\theta | \theta \in [0,2\pi)\}$. Then $G \cong S^1$, and Fix $G \cong R$ and consists of the constant functions. Moreover E, S^1 satisfy (*).

Below B_r denotes the closed ball in E of radius r centered at the origin and $F^\perp$ denotes the orthogonal complement of a subspace F.

<u>Theorem 3.5:</u> <u>Let</u> E, S^1 <u>satisfy</u> (*) <u>and let</u> $\varphi \in C(B_R \cap E_k, E)$ <u>be</u> G <u>equivariant with</u> $\varphi = \mathrm{id}$ <u>on</u> $(E_0 \cap B_R) \cup (E_k \cap \partial B_R)$. <u>Then for any</u> $\rho < R$, <u>and</u> $j < k$,

$$\mathrm{Ind}(\ (B_R \cap E_k) \setminus \varphi^{-1}(\partial B_\rho \cap E_j^\perp) >$$

(3.6)

$$> \mathrm{Ind}(\{x \in B_R \cap E_k | \varphi(x) \in \partial B_\rho \cap E_j^\perp\}) > k-j.$$

<u>Proof:</u> Let P_m denote the orthogonal projector of E onto E_m. Then for $m > k$, $P_m\varphi \in C(B_R \cap E_k, E_m)$ and is an equivariant map. Moreover $P_m\varphi = \mathrm{id}$ on $(E_0 \cap B_R) \cup (E_k \cap \partial B_r)$. Hence by Proposition 3.3,

(3.7) $\quad \mathrm{Ind}(\{x \in B_R \cap E_k | \varphi(x) \in \partial B_\rho \cap E_m \cap E_j^\perp) > k-j.$

Let K_m denote the argument of Ind in (3.7) and let

$$K = \{x \in B_R \cap E_k | \varphi(x) \in \partial B_\rho \cap E_j^\perp\}.$$

Then K is compact and $K \cap E_0 = \phi$. Hence by 3° of Lemma 3.1, $\mathrm{Ind}\, K < \infty$ and there exists a $\delta > 0$ such that $\mathrm{Ind}\, K = \mathrm{Ind}\, N_\delta(K)$. We claim $K_m K_m \subset N_\delta(K)$ $N_\delta(K)$ for m large. If so by 1° of Lemma 3.1, $\mathrm{Ind}\, K_m < \mathrm{Ind}_\delta N\ (K) = \mathrm{Ind}\, K$ and (3.7) and 1° of Lemma 3.1 imply (3.6). If our claim were false, for all large m, there is an $x_m \in K_m \setminus N_\delta(K)$. Since $B_R \cap E_k$ is compact, we can assume $x_m \to \hat{x} \in (B_R \cap E_k) \setminus \mathrm{int}\, N_\delta(K)$ as $m \to \infty$. Since

$$\|\varphi(\hat{x}) - P_m\varphi(x_m)\| < \|\varphi(\hat{x}) - P_m\varphi\ \varphi(\hat{x})\| + \|P_m(\varphi(\hat{x}) - \varphi(x_m))\| \to 0$$

as $m \to \infty$, $\varphi(\hat{x}) \in \partial B_j \cap E^\perp$, i.e. $\hat{x} \in K$. But $\hat{x} \notin \mathrm{int}\, K$, a contradiction. The proof is complete.

Remark 3.8: The above sort of situation occurs in the study of second order Hamiltonian systems [6]. Next we will study a case in which the domain of φ is not necessarily compact. This occurs in dealing with general Hamiltonian systems.

Theorem 3.9: Let E, S^1 satisfy (*) with $E_0^\perp = E^+ \oplus E^-$, where $E^+ = \overline{\cup E_m^+}$ (resp. $E_m^- = \overline{\cup E_m^-}$) and E_m^+, E_m^- are $2m$ dimensional mutually orthogonal invariant subspaces of E. Suppose $F_s \equiv E_0 \oplus E_s^+ \oplus E^-$ and $\varphi \in C(B_R \cap F_k, E)$ with φ equivariant, $\varphi = id$ on $(E_0 \cap B_R) \cup (F_k \cap \partial B_R)$ and $P^-\varphi(x) = \psi(x)P^-x + T(x)$ where P^- is the orthogonal projector of E onto E^-, $\psi \in C(E,[1,\alpha])$ and $T \in C(E,E^-)$ is compact. Then for any $\rho < R$ and $j > k$,

$$\mathrm{Ind}(\varphi(B_R \cap F_k) \cap \partial B_\rho \cap F_j^\perp) > k-j. \tag{3.10}$$

Proof: Let Q_m denote the orthogonal projector of E onto $E_0 \oplus E^+ \oplus E_m^-$. Then $Q_m\varphi$ satisfies the hypotheses of Theorem 3.5 with E_k, E_j and E being replaced by F_k, F_j and Q_m. Hence by (3.6)

$$\mathrm{Ind}(\{x \in B_R \cap Q_m F_k \mid Q_m\varphi(x) \in \partial B_\rho \cap F_j^\perp\}) > k-j. \tag{3.11}$$

Let K_m denote the argument of Ind in (3.11) and let

$$K = \{x \in B_R \cap F_k \mid \varphi(x) \in \partial B_\rho \cap F_j^\perp\}$$

Then K is compact. Indeed if (x_j) is a sequence in K, $x_j \equiv x_j^0 + x_j^+ + x_j^- \in E_0 \oplus E^+ \oplus E^-$. Since (x_j) is bounded and $B_R \cap F_j$ is closed and convex, we can assume x_j converges weakly to $x = B_R \cap F_j$. Since (x_j^0), (x_j^+) lie in finite dimensional subspaces of E, we can assume these sequences converge strongly. Moreover $P^-\varphi(x_j) = 0 = \psi(x_j)x_j^- + T(x_j)$ or

$$x_j^- = -\psi(x_j)^{-1}T(x_j) \tag{3.12}$$

with $1 \le \psi(x_j) \le \alpha$. Thus the boundedness of (x_j), compactness of T, and (3.12) allow us to assume (x_j^-) also converges strongly. Hence $\varphi(x_j) \to \varphi(x) \in \partial B_\rho \cap F_j^\perp$. Since $K \cap E_0 = \phi$, $i(K) < \infty$ by 3° of Lemma 3.1.

Continuing as in the proof of Theorem 3.5 with the additional information given by the form of $P^{-}\varphi$ shows $\operatorname{Ind} K_m < \operatorname{Ind} K$ so (3.11) implies (3.10).

Remark 3.13: Results in the spirit of Theorems 3.5 and 3.9 have been obtained by Benci [9] for mappings φ which are equivariant homeomorphisms of E onto E of the form linear + compact.

Next a critical point theorem will be proved in which Theorem 3.5 plays an important role. If $I : E \to \mathbb{R}$, we say I is a G-invariant functional if $I(gx) = I(x)$ for all $g \in G$, $x \in E$. For $I \in C^1(E,\mathbb{R})$, we say I satisfies the Palais-Smale condition (PS) if any sequence (x_m) along which I is uniformly bounded and $I'(x_m) \to 0$ is a precompact sequence. Here $I'(x)$ denotes the Frechet derivative of I at x.

Suppose I satisfies

(I_1) For all finite dimensional subspaces $\tilde{E}$ of E, there is an $r(\tilde{E}) > 0$ such that $I(x) < 0$ for $x \in \tilde{E}$ and $\|x\| > r(\tilde{E})$.

Choosing $\tilde{E} = E_0$, we see (I_1) and the continuity of I imply that $\sup_{E_0} I < \infty$.

Theorem 3.14: Suppose that E, S^1 satisfy (*) and $I \in C^1(E,\mathbb{R})$ is a G-invariant functional which satisfies (PS) and (I_1). If further I satisfies

(I_2) There is an $m_0 \in \mathbb{N}$ and $\rho > 0$ such that

$$I|_{\partial B_\rho \cap E_{m_0}^{\perp}} > \max(0, \sup_{E_0} I), \tag{3.14'}$$

then I possesses an unbounded sequence of critical values.

Remark 3.15): A somewhat less general result in a $\mathbb{Z}_2$ setting was proved in [7]. One cannot obtain Theorem 3.14 by merely restricting to a subgroup of S^1 of order two and e.g. appealing to [7]. Indeed a special free $\mathbb{Z}_2$ action

was required in [7] and furthermore due to the possible presence of finite isotopy subgroups, restricting to $\mathbb{Z}_2$ may produce a fixed point set. Other $\mathbb{Z}_2$ and S^1 analogues of Theorem 3.14 can be found in Benci [9], [10], and Bahri [11].

The proof of Theorem 3.14 will be accomplished in several steps. Let $R_m \equiv r(E_m)$ obtained via (I_1). Set $D_m = B_{R_m} \cap E_m$. Let $G_m = \{h \in C(D_m,E) | h$ is equivariant and $h(x) = x$ on $(E_0 \cup \partial B_{R_m}) \cap D_m\}$. Define

$$\Gamma_j = \{\overline{h(D_m \setminus Y)} | m > j,\ h \in G_m,\ Y \text{ invariant, } \mathrm{Ind}(Y) \leq m-j\}.$$

Classes of sets somewhat like the Γ_j were used in [5]. The sets Γ_j possesses the following properties

<u>Lemma 3.16:</u> (i) $\Gamma_{j+1} \subset \Gamma_j$.

(ii) If $B \in \Gamma_j$ and Z is an invariant set with $\mathrm{Ind}(Z) \leq s < j$, then $\overline{B \setminus Z} \in \Gamma_{j-s}$.

(iii) If $\chi \in C(E,E)$ is equivalent and $\chi = \mathrm{id}$ on $(E_0 \cup \partial B_{R_m}) \cap D_m$ for all $m > j$, then $\chi : \Gamma_j \to \Gamma_j$, i.e.

$$B \in \Gamma_j \Longrightarrow \overline{\chi(B)} \subset \Gamma .$$

<u>Proof:</u> (i) is immediate by the definition of Γ_j. For (ii), let $B \in \Gamma_j$. Therefore $B = \overline{h(D_m \setminus Y)}$ with $h \in G_m$ and $\mathrm{Ind}\, Y \leq m-j$. Since $\overline{B \setminus Z} = \overline{h(D_m \setminus Y \cup h^{-1}(Z))}$ with $\mathrm{Ind}(Y \cup k^{-1}(Z)) \leq \mathrm{Ind}(Y) + \mathrm{Ind}(h^{-1}(Z)$ $\leq \mathrm{Ind}(Y) + \mathrm{Ind}(Z) = m-j+s = m-(j-s)$ via 2° and 1° of Lemma 3.1, $\overline{B \setminus Z} \in \Gamma_{j-s}$. Lastly (iii) follows since if $B \in \Gamma_j$, $B = \overline{h(D_m \setminus Y)}$, then $\chi \circ h \in G_m$ and $\overline{\chi(h(D_m \setminus Y))} = \overline{\chi \circ h(D_m \setminus Y)}$.

With the aid of these sets Γ_j, we define a sequence of minimax values:

$$c_j = \inf_{B \in \Gamma_j} \sup_{u \in B} I(u), \quad j \in \mathbb{N} \tag{3.17}$$

Since $\Gamma_{j+1} \subset \Gamma_j$, we have $c_{j+1} \geq c_j$.

Lemma 3.18: *For* $j > m_0+1$, $c_j > c_{m_0+1} > \sup_{E_0} I$.

Proof: Let $B \in \Gamma_{m_0+1}$. Therefore $B = \overline{h(D_m \setminus Y)}$ where $h \in G_m$, $m > m_0+1$ and $\mathrm{Ind}(Y) < m - m_0 - 1$. Let $\rho < R_m$. By Theorem 3.5, if $X = \{x \in D_m | h(x) \in \partial B_\rho \cap E^{\perp}_{m_0}\}$, $\mathrm{Ind}\, X > m - m_0$. Therefore $\mathrm{Ind}\, \overline{X \setminus Y} > m - m_0 - (m - m_0 - 1) = 1$ and $\overline{X \setminus Y} \neq \phi$. Since $h\overline{(X \setminus Y)} \subset W \equiv \overline{h(X \setminus Y)} \cap \partial B_\rho \cap E^{\perp}_{m_0}$ by 2° of Lemma 3.1, $W \neq \phi$. Hence if $w \in B \cap W$

$$\sup_B I > I(w) > \inf_W I > \inf_{\partial B_\rho \cap E^{+}_{m_0}} I. \tag{3.19}$$

Since (3.19) holds for all $B \in \Gamma_{m_0+1}$,

$$c_{m_0+1} > \inf_{\partial B_\rho \cap E^{+}_{m_0}} I \tag{3.20}$$

and the lemma follows from (3.14')

To continue we need a variant of a standard "Deformation Theorem". Let $K_c = \{x \in E | I(x) = c \text{ and } I'(x) = 0\}$ and $A_c = \{x \in E | I(x) \leq c\}$.

Lemma 3.21: *Let* $I \in C^1(E,R)$ *be* G-*invariant and satisfy* (PS). *Then for any* $c \in R$, $\bar{\varepsilon} > 0$, *and invariant neighborhood* $\mathcal{O}$ *of* K_c, *there exists* $\varepsilon \in (0,\bar{\varepsilon})$ *and* $\eta \in C([0,1] \times E, E)$ *such that:*

1° $\eta(t,x) = x$ *if* $x \in I^{-1}([c-\bar{\varepsilon}, c+\bar{\varepsilon}])$

2° $\eta(1, A_{c+\varepsilon} \setminus \mathcal{O}) \subset A_{c-\varepsilon}$

3° *If* $K_c = \phi$, $\eta(1, A_{c+\varepsilon}) \subset A_{c-\varepsilon}$

4° $\eta(t,x)$ *is* G-*equivariant for each* $t \in [0,1]$.

Proof: The result without the G equivariance or invariance statements can be found in [12] or [13]. To obtain 4° also we need only modify these proofs by averaging over G to obtain a G equivariant pseudogradiant vector field as in [9] or [14].

Now we can show that the c_j's are indeed critical values of I together with a multiplicity statement:

<u>Lemma 3.22:</u> <u>For each</u> $j \geq m_0+1$, c_j <u>is a critical value of</u> I. <u>Moreover if</u> $c_{j+1} = \cdots = c_{j+p} \equiv c$, $\mathrm{Ind}(K_c) \geq p$.

<u>Proof:</u> It suffices to prove the stronger multiplicity assertion. Thus suppose $\mathrm{Ind}(K_c) \leq p-1$. BY (PS), K_c is compact. It is also invariant. Hence by 3° of Lemma 3.1, there is a $\delta > 0$ such that $i(N_\delta(K_c)) = i(K_c)$. By Lemma 3.21 with $\mathcal{O} = \mathrm{int}\, N_\delta(K_c)$ and $\bar{\varepsilon} = \frac{1}{2}(c_{m_0+1} - \max(\sup_{E_0} I, 0)) > 0$, there is an $\varepsilon \in (0,\bar{\varepsilon})$ and mapping η such that

$$\eta(1, A_{c+\varepsilon} \setminus N_\delta(K_c)) \subset A_{c-\varepsilon}. \tag{3.23}$$

Choose $B \in \Gamma_{j+p}$ such that

$$\sup_B I \leq c+\varepsilon \tag{3.24}$$

By (ii) of Lemma 3.16, $\overline{B \setminus N_\delta(K_c)} \in \Gamma_{j+1}$. By our choice of $\bar{\varepsilon}$, $\eta(1,\cdot) = \mathrm{id}$ on $(E_0 \cup \partial B_{R_m}) \cap D_m$ for all $m \in \mathbb{N}$. Hence by (iii) of Lemma 3.16,

$\overline{\eta(1, \overline{B \setminus N_\delta(K_c)})} \equiv Q \in \Gamma_{j+1}$. Hence

$$\sup_Q I \geq c_{j+1} \tag{3.25}$$

while by (3.23)-(3.24),

$$\sup_Q I \leq c_{j+1} - \varepsilon, \tag{3.26}$$

a contradiction.

<u>Remark 3.27:</u> Note that for $j \geq m_0+1$, $K_{c_j} \cap E_0 = \phi$ via Lemma 3.18. Hence our multiplicity statement is not due to any contribution from Fix$G$.

To complete the proof of Theorem 3.14, we will show

<u>Lemma 3.28:</u> $c_j \to \infty$ <u>as</u> $j \to \infty$.

Proof: Since $c_{j+1} > c_j$, if $c_j \not\to \infty$ as $j \to \infty$, $c_j \to \bar{c} < \infty$. Arguing as in [7], let $K = \{x \in E \mid c_{m_0+1} \leq I(x) \leq \bar{c}$ and $I'(x) = 0\}$. By (PS) K is compact and by Remark 3.27 and 3° of Lemma 3.1 $\mathrm{Ind}(K) < \infty$ and for some $\delta > 0$, $\mathrm{Ind}(N_\delta(K)) = \mathrm{Ind}\, K$. Let $j = \mathrm{Ind}\, K$. By Lemma 3.21 with $c = \bar{c}$ and $\bar{\varepsilon} = \bar{c} - c_{m_0+1}$, there is an $\varepsilon \in (0,\bar{\varepsilon})$ and equivariant $\eta(1,\cdot): A_{\bar{c}+\varepsilon} \setminus \mathrm{int}\, N_\delta(K) \to A_{\bar{c}-\varepsilon}$. Let m be the smallest integer $> m_0+1$ such that $c_m > \bar{c} - \varepsilon$. Let $B \in \Gamma_{m+j}$ such that

$$\sup_B I \leq \bar{c} + \varepsilon$$

As in Lemma 3.22, $\overline{B \setminus N_\delta(K)}$ and $\overline{\eta(1,\overline{B \setminus N_\delta(K)})} \equiv Q$ belong to Γ_m. Consequently

$$c_m \leq \max_Q I \leq \bar{c} - \varepsilon < c_m ,$$

a contradiction.

Remark 3.29: *In applications [6], one generally has* $I(x) = \mathcal{Q}(x) + b(x)$ where $\mathcal{Q}$ is a quadratic form with $\mathcal{Q}$ positive definite on $E_k^\perp$ for some $k > 0$ and b is weakly continuous with $b(0) = 0$. Hence for any fixed $\rho > 0$, $\mathcal{Q}|_{\partial B_\rho \cap E_m^\perp} > \beta_m \rho^2$ for $m > k$ and β_m bounded away from 0 while $b|_{\partial B_\rho \cap E_m^\perp} \to 0$ as $m \to \infty$ since $x_m \in \partial B_\rho \cap E_m^\perp$ implies x_m converges weakly to 0 and therefore $b(x_m) \to 0$ by the weak continuity of b. Thus (3.15) is satisfied. Hypotheses (I_1) is satisfied if b is "superquadratic" i.e. grows more rapidly than quadratically in an appropriate sense. See [7].

Remark 3.30: The novelty of Theorem 3.14 is not so much the conclusion of the theorem, since it is close to results of [9] and [11] but in the minimax characterization it provides for the critical values c_j. This characterization has proved to be useful in some recent perturbation results [6].

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An S^1 version of the Borsuk-Ulam Theorem is proved for a situation where Fix S^1 may be nontrivial. The proof is accomplished with the aid of a new relative index theory. Applications are given to intersection theorems and the existence of multiple critical points is established for a class of functionals invariant under an S^1 symmetry.

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