K-CONNECTIVITY IN RANDOM UNDIRECTED GRAPHS

BY

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**Abstract:**
See reverse.
This paper concerns vertex connectivity in random graphs. We present results bounding the cardinality of the biggest $k$-block in random graphs of the $G_{n,p}$ model, for any constant value of $k$. These results generalize those of [Erdős, Rényi, 60] and [Karp, Tarjan, 80] for $k = 1$ and 2. We furthermore prove here that the cardinality of the biggest $k$-block is $\geq n \log n$ with probability $\geq 1 - \frac{2}{n}$ for $p \geq c_1(k)/n$ and $c_1(k) > k + 2$. We also show that if $p \geq c(k) \left( \frac{\log n}{\log \log n} \right)$ with $c(k) > 32k^2$, then the graph $G_{n,p}$ is $k$-connected with probability $\geq 1 - 2^{\sqrt{d}(k)}$, $d(k) > 1$. 
1. **Summary**

This paper concerns vertex connectivity in random graphs. We present results bounding the cardinality of the biggest $k$-block in random graphs of the $G_{n,p}$ model, for any constant value of $k$. These results generalize those of [Erdős, Rényi, 60] and [Karp, Tarjan, 80] for $k=1$ and 2. We furthermore prove here that the cardinality of the biggest $k$-block is $> n\cdot \log n$ with probability $> 1 - n^{-2}$ for $p > c_1(k)/n$ and $c_1(k) > k + 2$. We also show that if $p > c(k) \frac{\log n}{n}$ with $c(k) > 32k^2$ then the graph $G_{n,p}$ is $k$-connected with probability $> 1 - 2n^{-d'(k)}$, $d'(k) > 1$.

2. **Introduction**

A graph $G = (V, E)$ consists of a finite, nonempty set $V$ of vertices together with a prescribed set $E$ of unordered pairs of distinct elements of $V$ (set of edges). (We allow no loops neither multiple edges). The vertex connectivity $k(G)$ of an undirected graph $G$ is the minimum number of vertices whose removal results in a disconnected graph or a trivial graph (consisting of just one vertex). Note that we follow here [Matula, 78] in defining $k$-connectivity, which we find to be most natural. [McLane, 37] gives a (somewhat different) definition of triconnectivity so that he can have the theorem that a graph is planar if its triconnected components are. [McLane, 37] shows that his triconnected components are homeomorphic to 3-blocks. Vertex $k$-connectivity seems to be a fundamental property of a graph and has numerous applications to other graph problems (such as planarity testing, routing problems etc). It is relevant to questions concerning vulnerability of a graph to separation. Cluster analysis methods considering the nature and inherent reliability of proximity...
data use the theory of k-connectivity to find groups of likes and dislikes in object pair association graphs ([Matula, 77], [Matula, 78] also [Jardine, Sibson, 71]).

A k-block of an undirected graph G is a maximal k-connected subgraph. A k-block is trivial if it has only one vertex [Matula, 78]. Clearly, each k-block consists of \( \geq k \) vertices or it is trivial.

[Matula, 78] examined certain properties of k-blocks in graphs (number of them, separation lemma) and [Ardos, Renyi, 60] and [Karp, Tarjan, 80] examined the distribution of the size of the biggest 1 and 2-blocks in random graphs \( G_{n,p} \) with \( p > \frac{c}{n} \) and \( G_{n,N} \) with \( N > cn \). They proved that there is a giant k-block for \( k=1,2 \) with exponentially decaying probability of error. For \( p > \frac{1}{2} \frac{\log n}{n} \) [Ardos, Renyi 60] showed that \( G_{n,p} \) becomes almost surely 2-connected.

In our paper we examine k-connectivity in the model \( G_{n,p} \) defined precisely as follows: For \( 0 < p < 1 \) and \( n \geq 0 \) let \( G_{n,p} \) be a random variable whose values are graphs on the vertex set \( \{1, 2, ..., n\} \). If \( e = \{u,v\} \) and \( u,v \in \{1, 2, ..., n\} \) then \( \text{Prob}(e \text{ is an edge}) = p \) and these probabilities are independent for different \( e \).

We prove that for each constant \( k > 0 \) and for each \( \epsilon (0 < \epsilon < 1) \) and \( \alpha > 1 \), there is a k-block of cardinality \( \geq \epsilon n \) in \( G_{n,p} \) with \( p > \frac{c(k, \epsilon, \alpha)}{n} \) with probability \( \geq 1 - e^{-\alpha n} \). We furthermore prove that for any \( k > 0 \) and \( 0 < m < \frac{n}{2k} \) there are constants \( c(k), d(k) > 0 \) such that the size of the biggest k-block of \( G_{n,p} \) where \( p \geq c(k) \frac{\log n}{n} \) is equal to \( n-m \) with probability \( e^{-m^k d(k)} \). From that we get as corollaries, that there are \( c(k), d(k) > 0 \) and \( d'(k) > 1 \) such that the size of the biggest k-block of \( G_{n,p} \) is \( \geq n-\log n \) with prob \( \geq 1-2n^{1-d(k)\log n} \) and that \( G_{n,p} \) is k-connected with prob \( \geq 1-2n^{-d'(k)} \).
Finally, we prove that for any \( m = o(n) \) \( 3c_1(k) > k+2 \) and a function \( c_1(k) \log n \)
\[ t(n) = \frac{c_1(k) \log n}{m} \]
such that, if \( p > \frac{t(n)}{n} \) then the biggest \( k \)-block of \( G_{n,p} \) has size \( > n-m \) with probability \( > 1 - n/e^{t(n)m} \) as \( n \to \infty \). A corollary is that if \( p > \frac{c_1(k)}{n} \) then the biggest \( k \)-block of \( G_{n,p} \) has cardinality \( > n - \log n \) with probability \( > 1-n^{-2} \). These results were known by [Erdős, Rényi, 60] only for \( k=1 \) and \( c(1) > \frac{1}{2} \).

3. Properties of \( k \)-blocks

**Proposition 1** [Matula, 78] For each \( k \geq 0 \), any two \( k \)-blocks have no more than \( k-1 \) vertices in common.

**Definition** [Matula, 78] A **separating set** \( S \) of \( G \) is a vertex subset \( S \subseteq V(G) \) such that \( G - S \) is disconnected. A **minimum separating set** \( S \subseteq V(G) \) has \( |S| = k(G) \).

**Definition** Let \( G \) be a graph \( (V,E) \) and let \( S \subseteq V \) be a set of vertices. Then by \( \langle S \rangle \) we denote the subgraph induced by \( S \) on \( G \).

**Lemma 1** [Matula, 78] (Block separation lemma) Let \( S \subseteq V(G) \) be a minimum separating set of the noncomplete graph \( G \) with \( \langle A_1 \rangle, \langle A_2 \rangle, \ldots, \langle A_m \rangle, m \geq 2 \) the components of \( G - \langle S \rangle \) and let \( k \geq k(G) + 1 \). Then each \( k \)-block of \( G \) is a \( k \)-block of \( \langle A_i \cup S \rangle \) for precisely one value of \( i \), and each \( k \)-block of \( \langle A_i \cup S \rangle \) for every \( i \) is a \( k \)-block of \( G \).

For a proof, see [Matula, 78].

**Remark** [Matula, 78] shows that for each \( k \geq 1 \) the total number of nontrivial \( k' \)-blocks for \( 1 \leq k' \leq k \), is \( \leq \left\lfloor \frac{2n-1}{3} \right\rfloor \) for any graph \( G \) with \( n \) vertices.
4. Giant $k$-blocks in Random Graphs

In the following we introduce special notation for very large subgraphs.

For each $\epsilon$, $0 \leq \epsilon \leq 1$, a subgraph $H$ of a graph $G$ of $n$ vertices is called an $\epsilon$-giant of $G$ if the cardinality of the vertex set of $H$ is $\geq \epsilon n$.

**DEFINITION:** Given a vertex set $S \subseteq V$ in the graph $G = (V,E)$, the boundary vertices of $S$ is the set $B(S) = \{u \in S | \exists v \in V - S \text{ such that } \{u,v\} \in E\}$.

**DEFINITION:** Let $X$ be a random variable whose values are the cardinality of the maximum $k$-block of instances of $G_{n,p}$. Let $F_{n,p,k}(a) = \text{Prob}\{X < a\}$ be the distribution function of $X$.

**THEOREM 1:** For every $\epsilon$ on $(0,1)$, $\alpha > 1$ and $k > 0$ there is a $c = c(k,\epsilon,\alpha) > 0$ such that, for $p \geq \frac{c}{n}$, $F_{n,p,k}(\epsilon n) \leq e^{-\alpha n}$. In other words, the random graph $G_{n,p}$ with $p \geq \frac{c}{n}$ has an $\epsilon$-giant $k$-block with probability at least $1 - e^{-\alpha n}$. To prove this theorem, we shall need the following definition and lemma.

**DEFINITION:** If $G = (V,E)$ and $A,B$ are subsets of $V$, then $E(A,B) = \{e = \{u,v\} \in E | u \in A \text{ and } v \in B\}$.

**LEMMA 2:** For any $\alpha_1, \epsilon_1, \epsilon_2 > 0$ where $\epsilon_1 + \epsilon_2 < 1$ and $\alpha_1 \geq 1$ there are constants $c, \epsilon_3, \epsilon_4 > 0$ such that a random graph $G_{n,p}$ with $p \geq \frac{c}{n}$ has the property (*) with probability $\geq 1 - e^{-\alpha_1 n}$.

(*): If $A, B$ are any two vertex subsets of $V$ such that $|A| \geq \lfloor \epsilon_1 n \rfloor$, $|B| \geq \lfloor \epsilon_2 n \rfloor$ and $A \cap B = \emptyset$ then $|E(A,B)| > 0$.

**PROOF OF LEMMA:** The complement of (*) is: "There are two vertex subsets $A, B$ such that $|A| \geq \lfloor \epsilon_1 n \rfloor$, $|B| \geq \lfloor \epsilon_2 n \rfloor$, $A \cap B = \emptyset$ and
E(A,B) = \emptyset. Clearly

\[ \operatorname{Prob}(E(A,B) = \emptyset) \leq (1 - p)^{\varepsilon_1 n^2} \leq \left(1 - \frac{c_2}{n}\right)^n \leq e^{-c_2 n^2} \]

Since there are at most \( \frac{1}{2} \cdot 4^n \) ways to select these A,B, and upper bound on the probability of the complement of (*) is

\[ \sum_{\text{all } A, B} \operatorname{Prob}(E(A,B) = \emptyset) \leq \frac{1}{2} \left(\frac{-c_1}{4 \varepsilon_1^2}\right)^n \leq \frac{\varepsilon_1 n}{e} \]

for

\[ \varepsilon_1 \geq \frac{\varepsilon_1 + \log_e 4}{\varepsilon_1 \varepsilon_2} \]

Now we return to the proof of the Theorem 1. Let \( G = (V,E) \) be an instance of the random graph \( G_{n,p} \). Let \( \mathcal{E}_1 \) be the event "G has no \( \varepsilon \)-giant k-block". Assume event \( \mathcal{E}_1 \) be true in the instance \( G \) of \( G_{n,p} \). Let initially the set \( A = \emptyset \). Do the following construction just until \( A \) has cardinality \( \geq \varepsilon' \cdot n/2 \), where \( \varepsilon' = \min(\varepsilon,1-\varepsilon) \).

(a) Find a minimum separating set \( S \) of \( G \). Let \( \langle A_1 \rangle, \ldots, \langle A_m \rangle \) \( m \geq 2 \) be the components of \( G-S \). Let \( \langle A_1 \rangle \) be the smallest of them. Let \( A + (A_1 \cup S) \cup A \). Let \( B \) be the union of the rest of the components and let \( G + \) the graph induced by \( B \cup S \). If \( |A| < \varepsilon' \cdot n/2 \), then go to (a).

By the above method of constructing \( A \), each addition of a component in \( A \) adds at most \( k-1 \) vertices to \( B(A) \) (i.e. the vertices of the
cut) and at least one vertex to $A - B(A)$ (by the block separation lemma and by the fact that $k$-blocks have at least $k$ vertices if they are non-trivial) or causes the transformation of a boundary to a nonboundary vertex. Thus, at least $1/k$ of the vertices of $A$ are not in $B(A)$.

By this construction, finally the $k$-blocks of $G$ are going to be separated. Because all $k$-blocks have been assumed to have cardinality $\leq \varepsilon n$, we will finally have

$$\varepsilon' \frac{n}{2} \leq |A| \leq \min\left[\varepsilon', \frac{n}{2} + \varepsilon n, \varepsilon' \frac{n}{2} \frac{3}{2}\right]$$

So

$$|A - B(A)| \geq \frac{\min(\varepsilon, 1 - \varepsilon')}{2k} \cdot n$$

and

$$|V - A| \geq n\left(1 - \min\left[\varepsilon + \frac{\varepsilon'}{2}, 3\varepsilon'/4\right]\right)$$

(obviously $|V - A| > 0$ for any $\varepsilon$ on $(0,1)$). Let $Y = A - B(A)$ and $Z = V - A$ then $|Y| \geq \varepsilon_1 n$ and $|Z| \geq \varepsilon_2 n$ where $\varepsilon_1 = \frac{\varepsilon'}{2k}$, $\varepsilon_2 = 1 - \min\left[\varepsilon + \frac{\varepsilon}{2}, 3\varepsilon'/4\right]$ and $E(Y, Z) = \emptyset$ by construction.

Hence, there are disjoint sets $Y' \subseteq Y$ and $Z' \subseteq Z$ such that $|Y'| = \varepsilon_1 n$, $|Z'| = \varepsilon_2 n$ and $E(Y', Z') = \emptyset$. Call $\mathcal{E}_2$ the above event. We have just shown $\mathcal{E}_1$ implies $\mathcal{E}_2$. So,

$$\text{prob}\{\mathcal{E}_1\} \leq \text{prob}\{\mathcal{E}_2\} \leq \varepsilon^{*n}$$

by Lemma 2.
NOTE: According to Lemma 2, any $\alpha \geq 1$ and $c = \frac{\alpha + \log 4}{e^2}$ satisfy the theorem. Replacing $c_1, e_2$ with the expressions found, we get

$$c \geq 2k \left[ \frac{\alpha + \log 4}{e' \cdot (1 - \min(e + \frac{1}{2}e', \frac{3}{4}e'))} \right]$$

5. $k$-blocks of dense random graphs.

This section considers edge density $p > c \frac{\log n}{n}$.

**THEOREM 2.** For any constant integer $k > 0$ and any $n$ and $m < \frac{n}{2k}$ there are constants $c(k), d(k) > 0$ such that the cardinality $X$ of the biggest $k$-block of the graph $G_{n,p}$ with $p > c(k) \frac{\log n}{n}$ satisfies the property

$$\text{Prob}\{X = n-m\} \leq n^{-md(k)}$$

**PROOF:** Let $G$ be an instance of $G_{n,p}$ and let the event $X = n-m$ be true in that instance. Let $A$ be a $k$-block with $|A| = X$. For every $u \in V-A$, we have that

$$|\{v \in E(G) : v \notin A\}| \leq k - 1$$

(since, otherwise $u$ would belong to $A$). Let

$$A_1 = \{v \in A : \exists u \in V-A : \{u, v\} \in E(G)\}$$

then

$$|A_1| \leq (k-1) |V-A| = (k-1)m$$
Let $A_2 = A - A_1$. We get

$$|A_2| \geq n - m - (k - 1)m = n - km$$

Furthermore, there is no edge from $V - A$ to $A_2$.

Let $\mathcal{E}$ be the above event. The probability of $\mathcal{E}$ is bounded above by

$$u(m, n) = \binom{n}{m} \binom{n - m}{n - km} (1 - p)^{(n - km)} (n - km)$$

But

$$(1 - p) \leq \left(1 - \frac{c \log n}{n}\right) \leq e^{-c \frac{\log n}{n}}$$

since

$$p \geq \frac{c \log n}{n}$$

Also

$$\binom{n - m}{n - km} \leq \binom{n - m}{(k - 1)m} \leq e^{(k - 1)m \log(n - m)}$$

since

$$(k - 1)m < \frac{n - m}{2}$$

and

$$\binom{n}{m} \leq e^{m \log n}$$

since

$$m < \frac{n}{2}$$

Thus $u(n, m) \leq n^{-d(n, m)}$ where $d(n, m) = c m \left(1 - \frac{km}{n}\right) - m - (k - 1)m \log(n - m) \log n$

$$> c m \left(1 - \frac{km}{n}\right) - m - (k - 1)m$$

$$> \frac{c m}{2} - km$$ (by our assumption).

So, $d(n, m) > m d(k)$ where $d(k) = \frac{c}{2} - k$. Note that $d(k) > 0$ iff $c(k) > 2k$.

So, $\text{Prob}(\mathcal{E}) \leq n^{-m d(k)}$.  

\[\square\]
THEOREM 3: For any constant integer $k > 0$ and any $n >> k$ there is a constant $c(k) > 0$ and a $d(k) > 0$ such that the cardinality $X$ of the biggest $k$-block of the graph $G_{n, p}$ with $p > c(k) \frac{\log n}{n}$ satisfies the property

\[ \text{Prob}(X \leq n - \log n) < 2n^{(1-d(k)\log n)} \]

PROOF: By using theorem 2, we get

\[ \text{Prob}\left\{ \log n \leq n - X < \frac{n}{2k}\right\} = \sum_{m=\log n}^{n/2k} \frac{n}{n} \cdot m^{-d(k)} \]

with $d(k) = \frac{c(k)}{2} - k > 0$ for $c(k) > 2k$.

So, \[ \text{Prob}\left\{ \log n \leq n - X < \frac{n}{2k}\right\} < n \cdot -\log n \cdot d(k) < \frac{1}{n} \cdot d(k) \log n. \]

Also, by theorem 1 and using $\varepsilon = \frac{1}{2k}$ we get

\[ \text{Prob}\left\{ n - X > \frac{n}{2k}\right\} < e^{-\varepsilon n} \]

for any $\alpha > 1$ and $c(k) > \frac{\alpha + \log 4}{\varepsilon_1 \varepsilon_2}$ and $\varepsilon_1 \varepsilon_2 = \frac{1}{2k} \left(1 - \frac{3}{8k}\right)$.

So, for $c(k) > \max\left(2k, \frac{\alpha + \log 4}{\varepsilon_1 \varepsilon_2}\right)$

or $c(k) > (\alpha + \log 4)16k^2$

we get

\[ \text{Prob}\left\{ \log n \leq n - X\right\} < e^{-\alpha n} + \frac{1}{n} \cdot \log n \cdot d(k) \]

or

\[ \text{Prob}\{X \leq n - \log n\} < 2n^{1-d(k) \cdot \log n} \]

for sufficiently large $n$. \qed
NOTE: Theorem 3 says that for \( p > c(k) \frac{\log n}{n} \) the graph \( G_{n,p} \) has a k-block of size \( \geq n - \log n \) with probability limiting to 1 as \( n \to \infty \).

**THEOREM 4:** For any constant integer \( k > 0 \) and \( n >> k \) there are constants \( c(k) > 0, d'(k) > 1 \) such that the random graph \( G_{n,p} \) with \( p > c(k) \frac{\log n}{n} \) is k-connected with probability

\[
\geq 1 - 2n^{-d'(k)}.
\]

**PROOF:** Let \( R = n - X \) where \( X = \) cardinality of the biggest k-block of \( G_{n,p} \). By using theorems 2, 3 and \( c(k) > 2 + \max \left( 2k, \frac{\alpha + \log 4}{\varepsilon_1 \varepsilon_2} \right) \) with

\[
\varepsilon_1 \varepsilon_2 = \frac{1}{2k} \left( 1 - \frac{3}{8k} \right)
\]

we get that

\[
\text{Prob}\{1 \leq R\} < e^{\alpha n} + n^{1/(\alpha^2 - k)}.
\]

Let

\[
d'(k) = \frac{c(k)}{2} - k - 1.
\]

Then \( d'(k) > 1 \) for \( c(k) > 2 + \left( \max 2k, \frac{\alpha + \log 4}{\varepsilon_1 \varepsilon_2} \right) \)

and

\[
\text{Prob}\{1 \leq R\} < e^{\alpha n} + n^{-d'(k)} < 2n^{-d'(k)}
\]

for large \( n \).

Hence

\[
\text{Prob}\{R = 0\} > 1 - 2n^{-d'(k)}
\]

\[\square\]
6. \textit{k-blocks for intermediate edge densities.}

Let \( \frac{c}{n} < p < c' \cdot \frac{\log n}{n} \). We wish to study the \(k\)-connectivity of this class of random graphs.

\textbf{THEOREM 5.} For any constant \( k > 0 \) and any \( m = o(n) \) there is a constant \( c_1(k) > 0 \) and a function \( t(n) = \frac{c_1(k) \log n}{m} \) such that, if \( p > \frac{t(n)}{n} \), then if \( X \) is the cardinality of the biggest \(k\)-block of \( G_{n,p} \) then

\[ \Pr[X < n - m] \leq \frac{n^k}{e^t(n) m} \to 0 \quad \text{as} \quad n \to \infty. \]

\textbf{PROOF:} Assume that in the instance \( G \) of \( G_{n,p} \) the cardinality \( X \) of the biggest \(k\)-block satisfies the inequality \( X < n - m \). Then, we can find two sets \( Y, Z \) (as in proof of theorem 3) such that \( |Y| = m, |Z| = n - km \) and no edge between them. This event is above bounded by the probability \( 1 - q \) where

\[ q = \Pr[\text{for every pair of disjoint sets } Y, Z \text{ of vertices of the above sizes, there is at least one edge between } Y, Z]. \]

We shall show \( q \to 1 \) as \( n \to \infty \). Let us enumerate all possible pairs of sets of vertices of the above sizes. Call them

\[ \{Y_1, Z_1\}, \{Y_2, Z_2\}, \ldots, \{Y_g, Z_g\} \]

where

\[ g = \binom{n}{m} \binom{n - m}{n - km} \binom{n}{m} \binom{n - m}{(k-1)m} \]

We have that \( q = \)
\[ \text{Prob}\{E(Y_1, Z_1) \neq \emptyset \land \ldots \land E(Y_g, Z_g) \neq \emptyset\} \]

where \( E(Y, Z) = \text{set of edges between } Y, Z. \)

So, by Baye's formula, \( q = \)

\[ \text{Prob}\{E(Y_1, Z_1) \neq \emptyset\} \cdot \text{Prob}\left\{ \frac{E(Y_2, Z_2) \neq \emptyset}{E(Y_1, Z_1) \neq \emptyset} \right\} \ldots \text{Prob}\left\{ \frac{E(Y_g, Z_g) \neq \emptyset}{\bigwedge_{i=1}^{g-1} E(Y_i, Z_i) \neq \emptyset} \right\} \]

We need the following enumeration lemma:

**LEMMA 3**: For every two sets \( Y_i, Z_i \) having at least one edge \( e \) between them, there are at least

\[ g_1 = \binom{n-2}{m-1} \binom{n-2-(m-1)}{(k-1)m-1} \]

pairs of sets of sizes \( m, n - km \) which also have this edge between them.

This lemma can be proved easily by taking out the two vertices of \( e \) and enumerating.

**COROLLARY**: There is a suitable enumeration of the sets in the \( q \) product such that for every term \( i \) not equal to 1 the next \( g_1 \) or more terms (conditioned on the existence of an edge from \( A_i \) to \( B_i \)) will be equal to 1.

Hence, the value of \( q \) is

\[ q \geq \left[ \text{Prob}\{E(Y_1, Z_1) \neq \emptyset\} \right]^{g_1/q_1} \]

But

\[ g_1/q_1 \leq \left( \frac{n}{m} \right)^k \text{ as } n \to \infty. \]
Hence,

\[ q \geq [1 - (1-p)^m(n-km)]^{n/m} \]
\[ \geq [1 - ((1-p)^{1/F} pm(n-km))^{n/m}] \]

or

\[ q \geq (1 - e^{pm(n-km)})^{n/m} \]
\[ \geq 1 - \left( \frac{n}{m} \right)^{k} e^{t(n)m} \]

or

\[ q \geq 1 - e^{[t(n)m - k \log n]} > 1 - n^{-2} \]

if

\[ c_{1}(k) > k + 2. \]

(Since \( t(n)m > c_{1}(k) \log n > (k+2) \log n \))

So,

\[ \text{Prob} \{ X < n - m \} < e^{[t(n)m - k \log n]} + 0 \text{ as } n \to \infty \]

for the above values of \( c(k) \)

**COROLLARY:** For \( m = \log n \) and \( t(n) > c_{1}(k) > k + 2 \) we get: For each \( k > 0 \), the graph \( G_{n,p} \) with \( p > \frac{c_{1}(k)}{n} \), has a \( k \)-block of cardinality \( > n - \log n \) with probability \( > 1 - n^{-2} \).
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