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A Robust Nonparametric Likelihood Ratio Test

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University of Missouri-Columbia

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A ROBUST NONPARAMETRIC LIKELIHOOD RATIO TEST

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The likelihood ratio principle is employed to suggest a nonparametric test for testing equality of two distributions against a stochastic ordering alternative. The test appears to be robust against a wide range of alternatives. Percentage points for sample sizes less than or equal to twenty are provided as well as a comparison of power values for the Kolmogorov-Smirnov and Mann-Whitney-Wilcoxon tests.

A Robust Nonparametric Likelihood Ratio Test

1. Introduction.

The likelihood ratio principle for obtaining test statistics has proven to be a popular and fruitful method in parametric situations. It leads to such appealing tests as the Student t test when testing for equality of means (variances assumed equal) and the F test when testing for equality of variances when underlying normal distributions are assumed.

In this paper, it is proposed that the likelihood ratio principle be used in a nonparametric setting to construct two sample test statistics which will compare favorably with such nonparametric tests as the Mann-Whitney-Wilcoxon and Kolmogorov-Smirnov tests.

Exact small sample tables are provided in Section 3 as well as a comparison of power with standard nonparametric tests. Theorems on symmetry and unbiasedness are proven in the appendix.

2. The Tests.

Let us assume that we have independent random samples from two discrete populations with respective survival functions (1 - cdf's) $P(t)$ and $Q(t)$. (For convenience we assume that $P(0) = Q(0) = 1$.) If we wish to test the hypothesis of stochastic ordering versus all other alternatives, then our hypotheses may be written as

$$H_0: \; P \overset{st}{\geq} Q \quad (P(t) \geq Q(t) \;\forall\; t)$$

$$H_1: \; P \overset{st}{\ngeq} Q \quad (P(t) < Q(t) \text{ for at least one } t).$$

To construct a likelihood ratio test we will need to find mle's (maximum likelihood estimates) of P and Q subject to a stochastic ordering restriction. Thus if our first sample contains δ_1 observations at x_1, δ_2 observations at x_2, etc., and the second sample d_1 observations at y_1, d_2 observations at y_2, etc., we must be able to maximize the expressions

$$L(P,Q) = \prod_{i=1}^{n_1} (P(x_i-) - P(x_i))^{\delta_i} \prod_{j=1}^{m_1} (Q(y_j-) - Q(y_j))^{d_j} \quad (2.1)$$

subject to the restriction $P \overset{st}{\geq} Q$.

This maximization is discussed in Brunk, et. al. (1966) and in more tractable form in Dykstra (1980). To obtain the desired maximal value of (2.1), proceed as follows:

a) Let $s_1 < s_2 < \ldots < s_m$ denote the distinct combined values of the x_i's and y_i's (combined observed values).

b) Now redefine $\delta_i(d_i)$ to be the number of observations from the first (second) sample at the point s_i.

c) Let $n_j = \sum_{i=j}^{m} \delta_i$ $(m_j = \sum_{i=j}^{m} d_i)$ denote the number of observations from the first (second) sample at least as large as s_j.

d) For $1 \leq a \leq b \leq m$, define $k_{a,b}^{+}$ by

$$k_{a,b}^{+} = \max\{(m_a \sum_a^b \delta_i - n_a \sum_a^b d_i)(\sum_a^b (d_i + \delta_i))^{-1}, 0\}.$$

(Note that $k_{a,b}^{+}$ is a weighted average of m_a and $-n_a$ if positive.)

e) For $1 \le i \le m$, define

$$\hat{k}_i = \min_{a\le i}\ \max_{i\le b}\ k_{a,b}^{+},$$

$$\hat{p}_i = \frac{n_i - \delta_i + \hat{k}_i}{n_i + \hat{k}_i}, \text{ and}$$

$$\hat{q}_i = \frac{m_i - d_i - \hat{k}_i}{m_i - \hat{k}_i}.$$

If $\hat{k}_i = \hat{m}_i > 0$, $\hat{q}_i = 0/0$ is indeterminate. In these cases we must define $\hat{q}_i = \hat{p}_i$. If $m_i = 0$, we define $\hat{q}_i$ to also be zero.

f) The maximal value of (2.1) subject to $P \overset{st}{\ge} Q$ is then given by

$$L(\hat{P},\hat{Q}) = \prod_{i=1}^{m} (1 - \hat{p}_i)^{\delta_i}\, \hat{p}_i^{\,n_i-\delta_i} (1 - \hat{q}_i)^{d_i}\, \hat{q}_i^{\,m_i-d_i} \tag{2.2}$$

where we adopt the convention that $0^0 = 1$.

It is well known that the unrestricted maximum of (2.1) occurs by putting equal probability at the observations from each sample. This is equivalent to requiring the $\hat{k}_i$'s in e) to be zero. Thus if

$$p_i^* = \frac{n_i - \delta_i}{n_i} \text{ and } q_i^* = \frac{m_i - d_i}{m_i},\ i = 1,\dots,m,$$

we have

$$\sup_{P,Q} L(P,Q) = L(P^*,Q^*) = \prod_{i=1}^{m} (1 - p_i^*)^{\delta_i}\, p_i^{*\,n_i-\delta_i} (1 - q_i^*)^{d_i}\, q_i^{*\,m_i-d_i}.$$

The likelihood ratio principle asserts that H_0 should be rejected if

$$\lambda = \frac{\sup\limits_{\substack{st \\ P \geq Q}} L(P,Q)}{\sup\limits_{P,Q} L(P,Q)} = \frac{L(\hat{P},\hat{Q})}{L(P^*,Q^*)}$$

$$= \prod_{i=1}^{m} \left(\frac{1 - \hat{p}_i}{1 - p_i^*}\right)^{\delta_i} \left(\frac{\hat{p}_i}{p_i^*}\right)^{n_i-\delta_i} \left(\frac{1 - \hat{q}_i}{1 - q_i^*}\right)^{d_i} \left(\frac{\hat{q}_i}{q_i^*}\right)^{m_i-d_i} \tag{2.3}$$

is sufficiently small. Equivalently, since λ may become very small with increasing sample size, we use as our test statistic

$$T = - 2\ell n\lambda$$

and reject H_0 if T becomes too large. Since the size of such a test with critical point t_0 is given by

$$\sup_{\substack{st \\ P \geq Q}} \Pr(T \geq t_0),$$

it is fortunate that if this supremum is attained, it is attained in the class of distributions where P = Q (assuming continuous distributions).

Theorem 2.1. Assuming P and Q are continuous,

$$\sup_{\substack{st \\ P \geq Q}} \Pr(T \geq t_0) = \sup_{P = Q} \Pr(T \geq t_0)$$

for all t_0.

Proof. (See Appendix.)

If we sample from continuous distributions, then with probability one, all the δ_i's and d_i's will be zero or one. Since the n_i's, m_i's, δ_i's and d_i's can all be determined from the ranks of the combined samples, the test statistic T is a function of the combined ranks, and hence the proposed test is nonparametric in nature. While we realize that the test was derived under the assumption of discrete distributions, since a continuous distribution may be approximated arbitrarily closely by a discrete distribution, we feel the test is still reasonable in the continuous setting.

Note that if we consider only continuous P and Q, the supremum in Theorem 2.1 is attained for any P = Q. Because of this, it seems inherently reasonable to restrict the class of alternatives to a particular subset of $P \overset{st}{\not\geq} Q$, namely $P \overset{st}{<} Q$, and use the same test statistic for testing

$$H_0': \; P = Q$$

$$\text{vs } H_1': \; P \overset{st}{<} Q \;\; (P(t) \le Q(t) \text{ for all } t \text{ and } P(t) < Q(t) \text{ for some } t).$$

Similar arguments to those used in Theorem 2.1 will suffice to show that the proposed test is unbiased.

Corollary 2.1. For all t_0, and continuous P and Q,

$$\inf_{P \overset{st}{\le} Q} \Pr(T \ge t_0) = \inf_{P = Q} \Pr(T \ge t_0).$$

There exists some symmetry properties between $n_1 = \sum_{i=1}^{m} \delta_i$ and $m_1 = \sum_{i=1}^{m} d_i$ which are not readily apparent. In particular, we have the following theorem.

Theorem 2.2. There exists a 1 - 1 mapping between those strings (of x's and y's) containing n_1x's and m_1y's and those strings containing m_1x's and n_1y's such that corresponding strings have the same value of λ (and hence T).

Proof. (See Appendix.)

Since the number of permutations n_1x's and m_1y's is the same as for m_1x's and n_1y's, and since if $P = Q$ (continuous), all permutations are equally likely, we have the following corollary.

Corollary 2.1. If $P = Q$ (continuous), the distribution of T is symmetric in n_1 and m_1.

This result halves the work of finding critical points for the null distribution of T.

3. Distribution of the Test.

Robertson and Wright (1980) have derived the asymptotic distribution of T under H_o when $P = Q$ are of the discrete type with only a finite number of points of positive probability. They have shown that asymptotically the survival function of T (under H_o) is a weighted average of chi-square survival functions.

Wolfe and Lee (1976) have discussed a Mann-Whitney-Wilcoxon type test based upon the restricted mle's $\hat{P}$ and $\hat{Q}$ obtained under the stochastic ordering restriction, but were unable to obtain the asymptotic distribution.

To obtain the null distribution of T for the case $P = Q$ (continuous) for small sample sizes, we have used the computer to enumerate all possible rankings of the samples along with the corresponding values of the statistic T. From these we have computed the .90, .95 and .99 percentage points of the null distribution of T. For values of n and m too large to consider all rankings, we have simulated the percentage points based upon 4000 independent observations for each value of n and m through 20. The percentage points are displayed in Table 1.

We have been unable to determine the asymptotic distribution of T.

4. Power of the Test.

To obtain some feeling for the power of our proposed test compared to the power of the Mann-Whitney and the Kolmogorov-Smirnov tests, we simulated observations from various distributions. We considered sample sizes of (5,5), (5,10) and (5,15) and generated 1000 observations to obtain each empirical estimate of the power curve. For each test procedure, we chose our critical points to make the size of the test as close to .05 as possible without exceeding it. Thus if we were unable to construct a test whose size was near .05, we would expect the power function of that test to be lower. This happened sometimes with the Kolmogorov-Smirnov test.

We constructed our alternatives to be basically of three types. The first type consisted of standard families of distribution which naturally gave rise to a stochastic ordering. The families of distributions which we considered were the normal, the exponential and the uniform distributions.

For testing normality against a one-sided shift alternative, we know the Mann-Whitney test is very good. However, the new test performs nearly as well. Values of the power function for the normal distribution, as well as stochastically ordered alternatives for the exponential and uniform distributions, are given in Table 2. In each of these situations, the new test is nearly as powerful as the Mann-Whitney test, and usually more powerful than the Kolmogorov-Smirnov test.

Of course stochastically ordered alternatives may occur in many ways. One way is to have Lehmann type alternatives, i.e., the alternative cdf is a fixed power of the null hypothesis cdf. We considered this kind of alternative for cases where the null hypothesis was an exponential ($\mu = 1$) distribution and the case where it was a uniform [0,1] distribution. A Monte Carlo power study obtained from 1000 independent samples is presented in Table 3. Under these alternatives the new test performs better than the Mann-Whitney for the sample sizes of (5,5), and nearly as well for the sample sizes of (5,10) and (5,15). It performs uniformly better than the Kolmogorov-Smirnov test for these alternatives.

Finally we should note that since T was derived as a test against the alternatives $P \overset{st}{\ngtr} Q$, it will still have some power in detecting differences between distributions whose cdf's cross. The Mann-Whitney and Kolmogorov-Smirnov tests usually have little power in these situations. Table 4 gives power values for uniform and normal distributions with mean 0 when the variance is allowed to vary. Finally, Table 5 shows power values when the coefficient of variation (μ/σ) if fixed at .5, and then μ is allowed to change. We see in Tables 4 and 5 that T outperforms the other tests in this situation.

An extensive power study of numerous situations indicates that the test statistic T performs well under a wide range of circumstances. For small nearly equal sample sizes, it does especially well.

5. Appendix.

Proof (of Theorem 2.1). Since $L(P^*,Q^*) = (\frac{1}{n})^n (\frac{1}{m})^m$ is free of $(X_1,\ldots,X_n\ Y_1,\ldots,Y_m)$ a.s., T is a nonincreasing function of $\sup_{P \overset{st}{\geq} Q} L(P,Q)$ a.s. However, as discussed in Dykstra (1980),

$$\sup_{P \overset{st}{\geq} Q} L(P,Q) = \sup_{P \overset{st}{\geq} R^* \overset{st}{\geq} Q} L_1(P)L_2(Q) \tag{A.1}$$

where R^* is the empirical survival function of the combined sample and $L_1(L_2)$ is the likelihood of the first (second) sample.

Now, for fixed $y_1, y_2,\ldots,y_m$

$$\sup_{R^* \overset{st}{\geq} Q} L_2(Q) \tag{A.2}$$

is nondecreasing in x_k since R^* is nondecreasing in x_k. Consider now the jump in $\hat{P}$ at the point $x_k = s_j$ say. From earlier considerations this is

$$\prod_{i<j} \frac{n_i - 1 + \hat{k}_i}{n_i + \hat{k}_i} \frac{1}{n_j + \hat{k}_j} .$$

Now $\hat{P} \geq R^*$ implies that

$$\prod_{i<j} \frac{n_i - 1 + \hat{k}_i}{n_i + \hat{k}_i} \geq \frac{n_j + m_j}{n_1 + m_1} . \tag{A.3}$$

Moreover,

$$\hat{k}_j = \min_{a \le j} \max_{j \le b} k_{a,b}^{+} \le \max_{j \le b} k_{j,b}^{+} \le m_j, \qquad \text{(A.4)}$$

so that

$$\frac{1}{n_j + \hat{k}_j} \ge \frac{1}{n_j + m_j} . \qquad \text{(A.5)}$$

Combining (A.4) and (A.5), we see that the probability which $\hat{P}$ places at x_k is never less than the probability R^* places at x_k.

Suppose now that $\tilde{P}$ denotes the survival function which is identical to $\hat{P}$ except it places the probability which $\hat{P}$ puts at x_k on the value $x_k + \Delta(\Delta > 0,\ x_k + \Delta \ne x_j$ for all j).

Thus if $\tilde{R}$ denotes the combined empirical survival function with x_k replaced by $x_k + \Delta$, we must have

$$\tilde{P}(t) \ge \tilde{R}(t) \text{ for all } t$$

(since $\tilde{P}$ is increased by at least as much as $\tilde{R}$ over the interval $[x_k,\ x_k + \Delta)$ and $\tilde{P}$ and $\tilde{R}$ are unchanged elsewhere).

However, the likelihood under $\hat{P}$ and $\tilde{P}$ are equivalent. Thus

$$\sup_{P \ge R^*} L_1(P) \le \sup_{P \ge \tilde{R}} L_1(P) \qquad \text{(A.6)}$$

where the first likelihood is evaluated at $x_1,\ldots,x_k,\ldots,x_n$ and the second at $y_1,\ldots,x_{k+\Delta},\ldots,x_n$.

Thus, combining (A.2) and (A.6), we see, in light of (A.1) that $T(x_1,\ldots,x_n,\ y_1,\ldots,y_m)$ is a nonincreasing function of each x_i (all other variables held constant), and hence

$$I_{[T \ge t_0]}(x_1,\ldots,x_n,\ y_1,\ldots,y_m)$$

is nonincreasing in each x_i. Now write $P(T \ge t_0)$ as an iterated integral

$$P(T \geq t_0) = \underset{X_1}{E} \cdots \underset{X_n}{E} \underset{\underset{\sim}{Y}\ (T \geq t_0)}{E} I(X_1, \ldots, X_n, \underset{\sim}{Y}) \tag{A.7}$$

and note that $\underset{\underset{\sim}{Y}\ (T \geq t_0)}{E} I\ (X_1, \ldots, X_n, \underset{\sim}{Y})$ is nonincreasing in each X_i,

Since it is well known that for nonincreasing g and stochastically ordered survival functions $P \overset{st}{\geq} Q$

$$\int_{-\infty}^{\infty} g(t)\, d\,P(t) \leq \int_{-\infty}^{\infty} g(t)\, dQ(t)$$

it clearly follows that $\sup\limits_{P \overset{st}{\geq} Q} P(T \geq t_0)$

must occur when $P = Q$.

<u>Proof</u> (of Theorem 2.2). Since the value of λ (and also T) depends only upon the ranks, we may assume that the combined sample consists of the points $\frac{1}{n_1 + m_1}, \frac{2}{n_1 + m_1}, \ldots, 1$. Then, letting $\eta = n_1 + m_1$, we can write

$$\lambda = C \sup_{P(x) \geq 1 - x \geq Q} \prod_{i=1}^{\eta} \left[P\left(\frac{i-1}{\eta}\right) - P\left(\frac{i}{\eta}\right)\right]^{\delta_i} \left[Q\left(\frac{i-1}{\eta}\right) - Q\left(\frac{i}{\eta}\right)\right]^{d_i} \tag{A.8}$$

$$= C \sup_{Q^* \geq 1 - x \geq P^*} \prod_{i=1}^{\eta} \left[P^*\left(\frac{\eta - i}{\eta}\right) - P^*\left(\frac{\eta + 1 - i}{\eta}\right)\right]^{\delta_i} \left[Q^*\left(\frac{\eta - i}{\eta}\right) - Q^*\left(\frac{\eta + 1 - 1}{\eta}\right)\right]^{d_i} \tag{A.9}$$

for a suitable constant C where

$$P^*(x) = 1 - P(1 - x), \text{ and}$$

$$Q^*(x) = 1 - Q(1 - x).$$

Noting that $P^*(Q^*)$ is a survival function iff $P(Q)$ is, it is apparent that (A.9) is of the exact same form as (A.8).

Thus if two strings are such that $\delta_i^{(1)} = d_{\eta-i}^{(2)}$ and $d_i^{(1)} = \delta_{\eta-i}^{(2)}$, they must give the same value of λ. This of course happens if a particular string of x's and y's is written in reverse order, and then the x's replaced by y's and vice versa. This then defines the 1 - 1 mapping mentioned in Theorem 2.2.

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TABLE 1.

90, 95 and 99^{th} percentiles for T under the assumption P = Q (continuous)

n

m	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	5.545 5.545 5.545	3.958 6.730 6.730	4.866 7.638 7.638	4.552 5.603 8.376	5.178 6.225 8.997	5.716 6.762 9.535	5.509 6.189 10.008	5.427 6.612 10.431	5.407 6.994 10.813	5.617 7.343 11.162	5.742 6.985 11.483	5.752 6.776 9.008	5.555 6.650 9.284	5.723 6.908 9.543	5.828 7.013 9.785	5.903 7.242 10.014	5.798 7.262 10.231	5.861 7.180 9.390	5.798 7.125 9.585
3		4.499 5.545 8.318	5.742 6.788 9.561	6.086 6.766 10.585	5.912 6.959 11.457	5.626 7.213 9.445	6.161 7.346 10.118	5.858 7.468 9.677	5.728 7.454 10.226	6.172 6.818 10.729	6.015 7.961 10.513	5.940 7.804 10.944	6.283 7.526 10.840	6.212 7.844 10.675	6.163 7.577 11.029	6.323 7.544 10.880	6.273 7.690 10.723	6.297 7.702 10.934	6.355 7.768 11.082
4			6.086 7.271 11.090	5.774 7.361 9.593	5.822 7.915 9.641	6.103 7.690 10.602	6.279 8.206 10.778	6.488 7.731 11.044	6.341 7.754 11.206	6.584 7.895 11.118	6.612 7.987 11.265	6.503 8.023 11.499	6.704 8.069 11.431	6.666 8.270 11.239	6.583 8.246 11.640	6.623 8.233 11.453	6.613 8.390 11.359	6.705 8.363 11.693	6.616 8.305 11.583
5				6.225 7.271 10.044	6.335 7.520 10.660	6.740 7.983 11.297	6.738 7.821 11.582	6.653 8.205 11.519	6.730 8.350 11.457	6.553 8.418 11.557	6.735 8.322 11.600	6.850 8.380 11.710	6.875 8.425 11.660	6.938 8.447 11.403	6.983 8.504 11.962	7.010 8.501 12.033	6.990 8.582 11.948	7.132 8.549 12.198	7.209 8.772 12.204
6					6.225 8.260 11.229	6.716 8.384 11.215	6.788 8.089 11.483	6.810 8.054 11.815	6.940 8.354 11.667	6.936 8.614 11.664	6.876 8.547 11.882	7.062 8.651 12.216	7.110 8.705 12.159	7.137 8.829 12.119	7.053 8.976 11.737	7.333 9.127 12.034	7.333 9.048 12.762	7.161 8.647 12.434	7.073 8.715 11.826
7						7.043 8.260 11.229	6.866 8.511 11.730	6.882 8.550 11.922	6.940 8.631 11.944	7.054 8.899 11.840	7.129 8.824 12.537	7.169 8.722 12.401	7.464 8.877 12.365	7.263 8.619 12.245	7.367 9.030 11.529	7.153 8.855 12.236	7.409 9.135 13.070	7.375 8.927 11.928	7.560 9.377 12.622
8							7.022 8.720 11.940	7.210 8.540 12.051	7.049 8.683 11.914	6.999 8.723 12.037	7.373 9.010 12.861	7.243 8.815 12.666	7.144 8.753 13.252	7.481 9.350 12.[illegible]6	7.638 9.283 12.608	7.585 8.975 12.248	7.662 9.182 12.514	7.688 9.315 13.041	7.607 9.390 13.167
9								7.009 8.652 11.770	7.348 8.918 12.057	7.406 9.168 12.420	7.512 9.100 12.381	7.471 8.9[illegible]3 12.[illegible]44	7.562 9.458 12.845	7.481 9.378 12.692	7.754 9.444 12.758	7.734 9.442 12.774	7.480 9.363 12.578	7.513 8.983 12.434	7.670 9.351 12.558
10									7.271 8.997 12.759	7.309 8.874 12.737	7.456 9.218 12.580	7.589 9.312 12.397	7.648 9.232 12.945	7.847 9.523 13.058	7.904 9.693 12.882	7.548 9.331 13.665	7.736 9.506 13.008	7.930 9.562 12.840	8.020 9.763 13.530
11										7.446 9.194 12.554	7.686 9.411 13.377	7.552 9.197 12.650	7.528 9.115 12.518	7.784 9.368 12.932	7.718 9.591 13.559	7.840 9.602 13.247	7.895 9.589 13.610	7.858 9.781 13.142	8.045 9.962 13.495
12											7.584 9.368 13.101	7.734 9.526 12.910	7.404 9.210 12.867	8.031 9.570 12.764	8.009 9.873 13.524	7.918 9.666 13.475	8.231 9.946 13.693	8.206 9.895 13.878	8.116 9.935 13.634
13												7.778 9.459 12.832	7.712 9.386 13.293	7.862 9.257 12.780	7.615 9.575 13.158	7.987 9.647 13.549	7.889 9.636 12.749	8.065 9.942 13.407	8.208 9.955 13.855
14													7.986 9.677 13.023	7.707 9.595 13.211	8.104 9.878 13.575	8.080 9.648 13.001	8.200 9.909 13.662	8.117 9.825 13.616	8.107 9.875 13.749
15														8.045 9.873 13.396	7.974 9.842 13.704	8.330 10.057 13.930	8.130 10.114 13.780	7.846 9.596 13.992	8.178 10.116 13.743
16															8.224 10.113 14.001	8.212 9.811 13.631	8.221 9.975 13.477	8.095 9.942 13.677	8.240 9.840 13.281
17																8.202 10.022 13.209	8.192 9.872 13.875	8.297 9.905 14.233	8.273 10.023 14.099
18																	8.438 9.955 13.999	8.350 10.196 14.147	8.212 10.065 14.495
19																		8.108 9.964 14.172	8.317 10.180 13.365
20																			8.469 10.187 13.778

Percentiles above underscore are exact

Percentiles below underscore are simulated

(4000 observations)

The table is symmetric in n and m.

Table 2

Estimated Power of Two-Sample One-Sided Tests

P: N(0, 1)
Q: N(θ, 1)

n	m	θ	K - S	M - W	T
5	5	0.0	.0360	.0410	.0480
		.6140	.1810	.2100	.2180
		1.0400	.3530	.4280	.4090
		1.4670	.5640	.6480	.6270
		2.0810	.8070	.8830	.8630
5	10	0.0	.0450	.0490	.0370
		.5320	.2010	.2050	.1890
		.9010	.4000	.4680	.4270
		1.2700	.6070	.6880	.6520
		1.8020	.8430	.9040	.8780
5	15	0.0	.0300	.0470	.0520
		.5010	.1160	.2260	.2020
		.8490	.2580	.4210	.4090
		1.1980	.5170	.7080	.7060
		1.6990	.7890	.9160	.9050

P: exp(1)
Q: exp(θ)

n	m	θ	K - S	M - W	T
5	5	1.000	.0460	.0530	.0530
		.600	.1160	.1430	.1380
		.400	.2780	.3180	.3070
		.250	.4130	.4690	.4640
		.100	.7860	.8020	.8130
5	10	1.000	.0640	.0560	.0470
		.600	.1870	.1870	.1560
		.400	.3660	.3820	.3460
		.250	.6410	.6300	.5710
		.150	.8540	.8580	.8210
5	15	1.000	.0290	.0530	.0420
		.600	.1000	.1650	.1590
		.400	.2660	.4190	.3790
		.250	.5050	.7130	.6540
		.150	.8110	.9120	.8840

P: U[0, 1]
Q: U[0, θ]

n	m	θ	K - S	M - W	T
5	5	1.00	.0450	.0550	.0540
		.80	.0910	.1170	.1140
		.60	.2590	.2950	.3020
		.40	.4880	.4720	.5220
		.20	.8120	.7520	.8340
5	10	1.00	.0490	.0500	.0470
		.80	.1480	.1420	.1250
		.60	.4150	.3540	.3010
		.40	.7930	.6430	.6080
		.20	.9820	.9290	.9180
5	15	1.00	.0310	.0510	.0540
		.80	.0720	.1300	.1260
		.60	.2320	.3500	.3340
		.40	.6340	.7020	.7280
		.20	.9700	.9680	.9830

Table 3

Estimated Power of Two-Sample One-Sided Tests

P: U[0, 1]

Q: $U[0, 1]^k$

n	m	k	K - S	M - W	T
5	5	1.00	.0340	.0420	.0450
		1.50	.1060	.1360	.1480
		2.50	.2460	.3020	.3240
		3.50	.3830	.4360	.4730
		5.00	.5410	.5920	.6310
5	10	1.00	.0330	.0570	.0630
		1.50	.0920	.1590	.1470
		2.50	.2640	.3770	.3620
		3.50	.4130	.5410	.5130
		5.00	.6080	.7370	.7090
5	15	1.00	.0160	.0410	.0450
		1.50	.0760	.1320	.1360
		2.50	.2370	.3910	.3700
		3.50	.4470	.6340	.5830
		5.00	.6360	.8030	.7800

P: exp(1)

Q: $[\exp(1)]^k$

n	m	k	K - S	M - W	T
5	5	1.00	.0410	.0420	.0510
		1.50	.1040	.1200	.1380
		2.50	.2130	.2790	.2940
		3.50	.3890	.4420	.4740
		5.00	.5380	.5930	.6350
5	10	1.00	.0320	.0600	.0550
		1.50	.0780	.1370	.1350
		2.50	.2550	.3760	.3370
		3.50	.4180	.5530	.5270
		5.00	.6090	.7450	.7050
5	15	1.00	.0240	.0490	.0480
		1.50	.0560	.1360	.1320
		2.50	.2480	.4060	.3720
		3.50	.4340	.6140	.5870
		5.00	.6750	.8260	.7990

Table 4

Estimated Power of Two-Sample One-Sided Tests

P: U[-1, 1]
Q: U[-θ, θ]

n	m	θ	K - S	M - W	T
5	5	1.00	.0410	.0480	.0490
		.80	.0360	.0440	.0430
		.60	.0580	.0530	.0640
		.40	.0770	.0430	.0820
		.20	.1310	.0470	.1350
		.10	.1280	.0440	.1320
		.05	.1570	.0280	.1570
		.01	.1930	.0290	.1930
5	10	1.00	.0490	.0500	.0470
		.80	.0460	.0510	.0570
		.60	.0590	.0750	.1050
		.40	.0810	.0940	.1300
		.20	.0980	.1020	.1300
		.10	.1700	.1720	.1950
		.05	.1640	.1660	.1730
		.01	.1690	.1690	.1690
5	15	1.00	.0280	.0450	.0500
		.80	.0280	.0640	.0880
		.60	.0460	.0990	.1640
		.40	.0780	.1140	.2360
		.20	.1200	.1420	.3530
		.10	.1580	.1660	.4090
		.05	.1830	.1820	.4740
		.01	.1920	.1960	.5080

P: N(0, 1)
Q: N(0, θ)

n	m	θ	K - S	M - W	T
5	5	1.000	.0440	.0530	.0550
		3.100	.0400	.0430	.0490
		6.390	.0730	.0650	.0760
		13.160	.0850	.0470	.0870
		40.830	.1130	.0430	.1150
5	10	1.000	.0530	.0490	.0470
		2.230	.0420	.0510	.0590
		4.010	.0610	.0750	.0870
		7.550	.0750	.0830	.1090
		21.780	.1100	.1160	.1510
5	15	1.000	.0320	.0560	.0630
		2.050	.0460	.0690	.0790
		3.540	.0380	.0690	.0990
		6.470	.0760	.1130	.1630
		18.260	.1030	.1260	.2550

Table 5

Estimated Power of Two-Sample One-Sided Tests

Constant coefficient of variation = $\frac{\mu}{\sigma}$ = .5

P: $N(.2,(.4)^2)$

Q: $N(\mu, \sigma^2)$

n	m	μ	K - S	M - W	T
5	5	.20	.0450	.0530	.0620
		.50	.2110	.1740	.2470
		1.00	.6770	.5130	.6970
		2.00	.9920	.9140	.9930
5	10	.20	.0260	.0470	.0500
		.50	.2170	.2770	.3440
		1.00	.6940	.7160	.7840
		2.00	.9890	.9890	.9920
5	15	.20	.0280	.0510	.0550
		.50	.2480	.3210	.4380
		1.00	.6960	.7460	.9030
		2.00	.9940	.9940	.9990