AD-A107 665 ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND ABERD--ETC F/G 20/11
CRITICAL ANGLES AND GRAZING INCIDENCE: THE BREAKDOWN OF REGULAR--ETC(U)
NOV 81 T W WRIGHT
UNCLASSIFIED ARBRL-TR-02381 NL

1 OF 1
AD A
107665

END
DATE
FILMED
1-82
DTIC

5L

(12) LEVEL II

AD

TECHNICAL REPORT ARBRL-TR-02381

AD A1

CRITICAL ANGLES AND GRAZING INCIDENCE: THE BREAKDOWN OF REGULAR SHOCK REFLECTION IN SOLIDS

Thomas W. Wright

DTIC ELECTE NOV 2 5 1981 S D B

November 1981

US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
BALLISTIC RESEARCH LABORATORY
ABERDEEN PROVING GROUND, MARYLAND

Approved for public release; distribution unlimited.

DTIC FILE COPY

81 11 23009

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER TECHNICAL REPORT ARBRL-TR-02381	2. GOVT ACCESSION NO. AD-A107 665	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Critical Angles and Grazing Incidence: The Breakdown of Regular Shock Reflection in Solids		5. TYPE OF REPORT & PERIOD COVERED 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Thomas W. Wright		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Director USA ARRADCOM (DRDAR-BLT) Ballistic Research Laboratory Aberdeen Proving Ground, MD 21005		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 1L161102AH43
11. CONTROLLING OFFICE NAME AND ADDRESS US Army Armament Research & Development Command US Army Ballistic Research Laboratory (DRDAR-BL) Aberdeen Proving Ground, MD 21005		12. REPORT DATE November 1981 13. NUMBER OF PAGES 56
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release, distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Shock waves
Solid mechanics
Shock reflections
Finite amplitudes

20. ABSTRACT (Continue on reverse side if necessary and identify by block number) bet 2972

The regular pattern for shock reflection in a nonlinear hyperelastic solid is a centered array of shocks and simple wave fans. As the angle of incidence approaches grazing incidence or a critical angle, the reflection pattern overtakes the incident wave until finally the regular pattern can no longer be sustained. By expanding the reflection solution in powers of amplitude about the linear reflection solution, it is possible to develop a procedure to solve the reflection problem for weak but finite shocks for any material symmetry. (OVER)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Explicit solutions can be exhibited for isotropic materials, including relationships between amplitude and limiting angle for various boundary conditions and incident waves. Some problems require consideration of a nonlinear boundary condition even in the first approximation. Typically, these cases lead to considerable amplification in the leading reflected wave.

TABLE OF CONTENTS

		Page
	TABLE OF CONTENTS	3
	LIST OF FIGURES	5
I.	INTRODUCTION	7
II.	STATEMENT OF THE PROBLEM AND WAVE ANALYSIS	9
III.	REFLECTION CALCULATIONS: GENERAL CONSIDERATIONS	24
IV.	EXAMPLES: REFLECTIONS IN ISOTROPIC MATERIALS	26
	REFERENCES	42
	APPENDIX A: Nearly Equal Wave Speeds	43
	APPENDIX B: Elastic Moduli	47
	LIST OF SYMBOLS	49
	DISTRIBUTION LIST	51

LIST OF FIGURES

Figure		Page
1	Reflection Geometry for the General Case	10
2	Weak Shock Pattern as Perturbation from the Linear Pattern	13
3	Reflected Shock Wave	15
4	Relationship Between Wave Curve and Reflection Point	18
5	Reflected Simple Wave	20
6	Reflected Shock Amplitude at a Rigid/Lubricated Boundary for an Incident Longitudinal Shock	32
7	Limiting Angle - Amplitude Behavior for Longitudinal Shocks. Upper Curve Corresponds to a Reflected Simple Wave. Lower Curve Corresponds to a Reflected Shock from a Rigid/Lubricated Boundary	33
8	Limiting Angle - Amplitude Behavior for Incident Shear Waves, Points to the Right of the Limiting Lines are Allowed	37
9	Plot of the Cubic $f(\xi_1) = A\xi_1^3 - B\xi_1^2 + R^2$ for Several Values of B. The Curves are Normalized to $A = R^2 = 1$	40

I. INTRODUCTION

If a plane shock wave in a nonlinear hyperelastic material is obliquely incident upon a plane material boundary, then, provided that neither the angle of incidence nor the amplitude is too great, a unique reflection pattern will be created and will be composed of shocks and simple waves all centered on the point of reflection. Such a pattern may be called regular reflection. Two previous papers by the author describe the basic situation[1,2], and a third[3] clarifies a point in the linear theory that simultaneously guarantees unique solutions in the corresponding nonlinear theory. Other authors have dealt with aspects of the reflection problem in isotropic solids[4,5,6].

It happens, however, that as the amplitude of the incident wave increases or as the angle of incidence increases, the leading reflected or transmitted wave may move so rapidly that it must overtake the incident wave at the boundary no matter what its direction of propagation, and thus the pattern of regular reflection will be destroyed. In linear elasticity the limiting angle of incidence for regular reflection to occur is called the angle of grazing incidence if, for that angle, the incident wave is the fastest that can occur, and it is called a critical angle if the incident wave is one of the slower waves. In nonlinear elasticity the distinction seems artificial so the phrase "limiting angle" will be used to cover all cases.

The essentials for shock reflection may be simply stated. For a fixed angle of incidence the incident shock wave may be specified completely by a single parameter, thus fixing the deformation gradient and particle velocity immediately behind the wave. The problem now is to fit reflected waves so as to connect the state just fixed with some state at the boundary that is compatible with the boundary conditions. Evidently, the situation is mathematically similar to the Riemann initial value problem[7] or to the problem of waves

---

[1] T. W. Wright, *Reflection of Oblique Shock Waves in Elastic Solids, Int. J. Solids Structures,* 7, *161-181 (1971).*

[2] T. W. Wright, *Uniqueness of Shock Reflection Patterns in Elastic Solids, Arch. Rat. Mech. Anal.,* 42, *115-127 (1971).*

[3] T. W. Wright, *A Note on Oblique Reflections in Elastic Crystals, Quart. J. Mech. Appl. Math.,* 29, *15-24 (1976).*

[4] G. Duvaut, *Phénomènes De Réflexion, Réfraction, Intersection d Ondes Planes Uniformes dans des Matériaux Élastiques Non Linéaires, C. R. Acad. Sc. Paris, Série A,* 264, *883-886 (1967).*

[5] G. Duvaut, *Ondes dans des Matêriaux de Type Harmonique. Rêflexion Oblique d une Onde de Choc Plane Longitudinale sur une Paroi Fixe, C. R. Acad. Sci. Paris, Série A,* 266, *246-249 (1968).*

[6] S. R. Reid, *The Influence of Nonlinearity Upon the Reflection of Finite Amplitude Shock Waves in Elastodynamics, Quart. J. Mech. Appl. Math.,* 25, *185-206 (1972).*

[7] P. Lax, *Hyperbolic Systems of Conservation Laws II, Comm. Pure Appl. Math.,* 10, *537-566 (1957).*

initiated by uniform impulsive loads on the boundary of a semi-infinite half space[8,9]. Although there are differences in detail, most notably in the criteria for admissibility of reflected waves and in the condition that reflected waves all have a common reflection point rather than a fixed direction of propagation, the methods for constructing a solution to the reflection problem and for establishing existence and uniqueness for small but finite amplitudes follow closely those used by Lax[7], and are described in detail elsewhere[2].

In summary, in Reference 2 it was shown that each reflected wave connects a fixed state ahead of the wave with a one parameter family of states behind the wave. Furthermore, parameterization may be so arranged that positive values correspond to simple waves and negative values correspond to shock waves. In anisotropic solids there are three possible families of reflected waves so that a sequence of such waves connects the state behind the incident shock with a three parameter family of states adjacent to the boundary. In general, there will be three independent boundary conditions from which to find the parameters of the reflected waves (γ_1, γ_2, γ_3) in terms of the parameter of the incident wave γ_o. Symbolically, we have

$$B_m(\gamma_1, \gamma_2, \gamma_3; \gamma_o) = 0; \; m = 1, 2, 3 \tag{1.1}$$

The situation for a reflection/transmission problem is exactly analogous, but now there will be three transmitted waves as well as three reflected waves so there will be six parameters to be found from six boundary conditions. In special cases of material symmetry or degeneracy there may actually be less than three waves in a reflection problem or less than six waves in a reflection/transmission problem. Some cases of this kind have been discussed in Reference 1.

A simple application of the implicit function theorem shows that (1.1) can be inverted to obtain three functions $\gamma_m = \Gamma_m(\gamma_o)$; m = 1, 2, 3 provided that γ_o is suitably small and provided that the corresponding linear problem can be solved[1,2], for the Jacobian determinant of (1.1) is identically the same as the one that occurs in the equivalent linear problem.

Here the reflection problem diverges sharply from the Riemann or the impulsive loading problems. In the latter two cases, if the material is hyperelastic with nonvanishing wave speeds, the determinant is always nonzero. However, in reflection problems under the same conditions, it was shown in Reference 3 that the linear determinant is guaranteed to be nonzero only if the incident angle is neither critical nor of grazing incidence. In these cases of limiting angle no information can be given in general. Each case must be examined separately since examples show that sometimes the determinant vanishes and sometimes not[3]. In any event there is always some loss of differentiability in the mathematical

---

[8]*L. Davison, Propagation of Plane Waves of Finite Amplitude in Elastic Solids, J. Mech. Phys. Sol.,* 14, *249-270)1966).*

[9]*A. S. Abou-Sayed and R. J. Clifton, Analysis of Combined Pressure-Shear Waves in an Elastic/Viscoplastic Material, J. Appl. Mech.,* 44, *79-84 (1977).*

structure of the reflected waves as the limiting angle is approached. Nothing comparable occurs in either the Riemann or impulsive loading problem.

This loss in uniformity of the solution may be clearly seen in the structure of reflected waves as follows. In the construction of a composite family of simple waves and shock waves (see Reference 2) the two halves of the family are joined together at zero amplitude where the following equation holds.

$$4\rho\, V(0)\, \Theta'(0) \left\{ \underset{\sim}{b}(0) \cdot \underset{\sim}{t}(0) - V_h(0) \cos \Theta(0) \right\} = \underset{\sim}{C}_3(0) \left\{ \underset{\sim}{r}(0) \otimes \underset{\sim}{N}(0) \right\}^3 \tag{1.2}$$

The angle of a reflected shock is Θ, and the prime represents differentiation with respect to the parameter of the wave family. The right hand side is a contraction of third order elasticities and is generally nonzero. The term in brackets on the left hand side has a geometric interpretation. Terminology and symbols are explained precisely in the sequel, but for now it is enough to comment that as the limiting angle is approached, this left hand factor tends to zero for the leading reflected wave so that $\Theta'(0)$ becomes infinite.

In this paper the reflection of weak but finite amplitude shocks is examined, with particular care being taken to obtain solutions that remain valid all the way to the limiting angle. In the process, the limiting angle itself as a function of incident amplitude is obtained. The procedure to obtain these results is one of expansions in small parameters for a reflection pattern that is close to the pattern appropriate for the corresponding linearized problem. In the next section the structure of weak shocks and simple waves, taken individually either as incident or as reflected waves, is developed. In Section III general procedures for solving reflection problems in materials of any symmetry are laid out in a step by step fashion. In Section IV the general procedures are used to obtain explicit solutions in isotropic materials for incident shock waves that are either longitudinal or quasi-transverse and for several boundary conditions in each case.

II. STATEMENT OF THE PROBLEM AND WAVE ANALYSIS

Suppose a plane shock wave with weak but finite amplitude is incident upon a plane boundary in a nonlinear, simple, hyperelastic solid, and suppose that a regular reflection pattern of shocks and simple waves is established. The situation is shown schematically in Figure 1. The deformation is given by a function

$$\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X}, t) \tag{2.1}$$

where $\underset{\sim}{x}$ is the spatial position of a material particle, $\underset{\sim}{X}$ is the position of the same particle in a fixed reference configuration, and t is time. Both $\underset{\sim}{x}$ and $\underset{\sim}{X}$ are referred to the same fixed Cartesian coordinate system. The Piola-Kirchhoff stress tensor, $\underset{\sim}{T}$, is derivable from a stored energy function W, which in turn is assumed to be a function only of the deformation gradient, $\underset{\sim}{F}$. If the material does not conduct heat, no conclusions are changed by letting W depend on specific entropy as well since entropy change is at most third order in the amplitude of a weak shock.

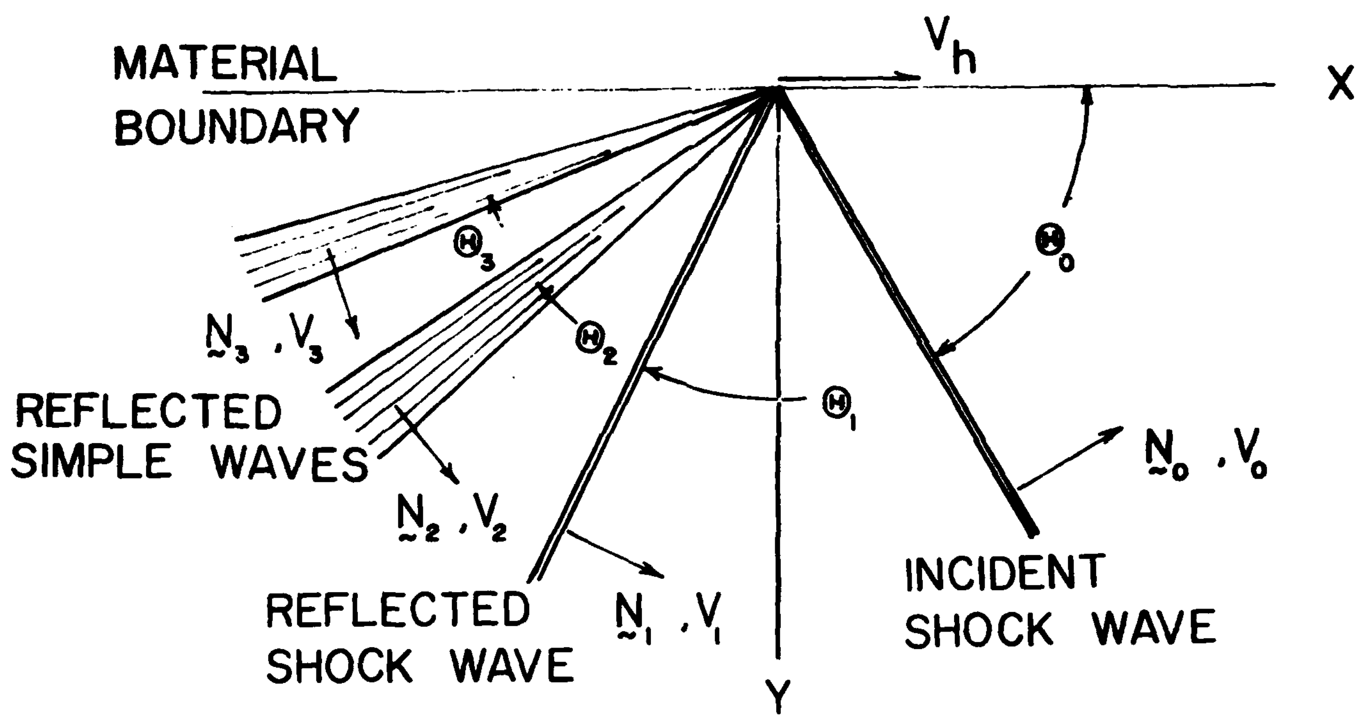

Figure 1. Reflection Geometry for the General Case

$$W = W(\underset{\sim}{F})$$

$$\underset{\sim}{T} = \frac{\partial W}{\partial \underset{\sim}{F}} = \underset{\sim}{T}(\underset{\sim}{F}) \ ; \text{ or } T_{i\alpha} = T_{i\alpha}(F_{j\beta}) \tag{2.2}$$

$$\underset{\sim}{F} = \frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{X}} \ ; \text{ or } F_{j\beta} = \frac{\partial x_j}{\partial X_\beta} = x_{j,\beta}$$

When subscript notation is used, Latin indices will refer to spatial coordinates and Greek indices will refer to material coordinates.

The relations to be satisfied across a shock wave are as follows[10].

$$[\underset{\sim}{T}]\underset{\sim}{N} + \rho V[\underset{\sim}{u}] = \underset{\sim}{0}$$
$$[\underset{\sim}{F}] = -\underset{\sim}{a} \otimes \underset{\sim}{N} \tag{2.3}$$
$$[\underset{\sim}{u}] = V\, \underset{\sim}{a}$$

The square brackets indicate the jump of a quantity across a shock; for example, $[\underset{\sim}{T}] = \underset{\sim}{T}^- - \underset{\sim}{T}^+$ where the (+) and (-) signs refer to the front and the rear of the shock, respectively. The unit normal and normal speed of the shock in the reference configuration are $\underset{\sim}{N}$ and V. The density in the reference configuration is ρ, the particle velocity is $\underset{\sim}{u} = \frac{\partial \underset{\sim}{x}}{\partial t}$, and $\underset{\sim}{a}$, defined by $(2.3)_3$, is the amplitude.

In a simple wave, all field quantities are functions of a single parameter, say γ, and all quantities are constant on propagating planes, called wavelets, which are also parameterized by γ. Throughout a simple wave the following equations must be satisfied[11].

$$(\underset{\sim}{Q} - \rho V^2 \underset{\sim}{1})\underset{\sim}{m} = \underset{\sim}{0}$$
$$\underset{\sim}{F}' = -\, \underset{\sim}{m} \otimes \underset{\sim}{N} \tag{2.4}$$
$$\underset{\sim}{u}' = V\underset{\sim}{m}$$

Here Q is the acoustic tensor with components $Q_{ij} = C_{i\alpha j\beta}N_\alpha N_\beta$ where $C_{i\alpha j\beta} = \partial^2 W/\partial F_{i\alpha}\partial F_{j\beta}$. The prime indicates differentiation with respect to γ, $\underset{\sim}{N}$ and V are the unit normal and normal speed of a wavelet in the reference configuration, and $\underset{\sim}{m}$ is a unit vector. As in the shock case ρ is the reference density. Naturally, it is required that ρV^2 be a proper number and $\underset{\sim}{m}$ the corresponding proper vector of Q.

---

[10] *C. Truesdell and R. Toupin, The Classical Field Theories, Flügge's Handbuch der Physik III/1, Springer, Berlin-Gottingen-Heidelberg (1960).*

[11] *E. Varley, Simple Waves in General Elastic Materials, Arch. Rat. Mech. Anal., 20, 309-328 (1965).*

The response to an incident shock of weak but finite amplitude will be near to the response for the corresponding linearized problem. Accordingly, all calculations may be based on variables that measure small variations from the pattern and response from infinitesimal elasticity. In Figure 2 the reflection pattern for the linear problem is shown by solid lines, the actual pattern is indicated by the dashed lines, and the angles α_1, α_2, α_3 are small. Since all waves are weak, the general constitutive equation $(2.2)_1$ need only be approximated to second order as follows.

$$\underset{\sim}{T} = \underset{\sim}{C}_2\,(\underset{\sim}{F}-1) + \tfrac{1}{2}\,\underset{\sim}{C}_3\,(\underset{\sim}{F}-\underset{\sim}{1})^2 \tag{2.5}$$

$\underset{\sim}{C}_2$ and $\underset{\sim}{C}_3$ are tensors of second and third order elastic constants, $\underset{\sim}{C}_2 = \partial^2 W/\partial \underset{\sim}{F}^2$, $\underset{\sim}{C}_3 = \dfrac{\partial^3 W}{\partial \underset{\sim}{F}^3}$ where both tensors now refer to an unstressed reference state wherein $\underset{\sim}{F} = \underset{\sim}{1}$. In component form, the symbol $\underset{\sim}{C}_2(\underset{\sim}{F}-\underset{\sim}{1})$ is the contraction $C_{i\alpha j\beta}(F_{j\beta}-\delta_{j\beta})$ and similarly $\underset{\sim}{C}_3(\underset{\sim}{F}-\underset{\sim}{1})^2$ is the contraction $C_{i\alpha j\beta k\gamma}(F_{j\beta}-\delta_{j\beta})(F_{k\gamma}-\delta_{k\gamma})$. The more compact notation will usually be used hereafter.

Three types of wave must now be examined; the incident shock, a reflected shock, and a reflected simple wave.

(i) Incident Shock.

This wave propagates at a fixed angle of incidence Θ_o, but its speed depends on the incident amplitude and will be slightly greater than the corresponding speed in linear elasticity. For the fixed direction of propagation there exists in the linear theory a tensor Q with unit proper vectors $\underset{\sim}{p}$, $\underset{\sim}{q}$, $\underset{\sim}{r}$ and corresponding proper numbers ρV_p^2, ρV_q^2, ρV_r^2. We have

$$(Q - \rho V_p^2\,\underset{\sim}{1})\underset{\sim}{p} = \underset{\sim}{0}, \text{ etc.} \tag{2.6}$$

where Q is computed as shown following (2.4) but using $\underset{\sim}{C}_2$. It may be assumed that the linear wave speeds are ordered

$$V_p \geq V_q \geq V_r$$

so that they may be referred to as fast, intermediate, or slow. Since Q is symmetric, $\underset{\sim}{p}$, $\underset{\sim}{q}$, $\underset{\sim}{r}$ are orthogonal, and therefore the amplitude of the incident shock has the representation

$$\underset{\sim}{a} = \xi\underset{\sim}{p} + \eta\underset{\sim}{q} + \zeta\underset{\sim}{r} \tag{2.7}$$

where ξ, η, ζ are small quantities. The shock speed may be written

$$V_o = V(1+v) \tag{2.8}$$

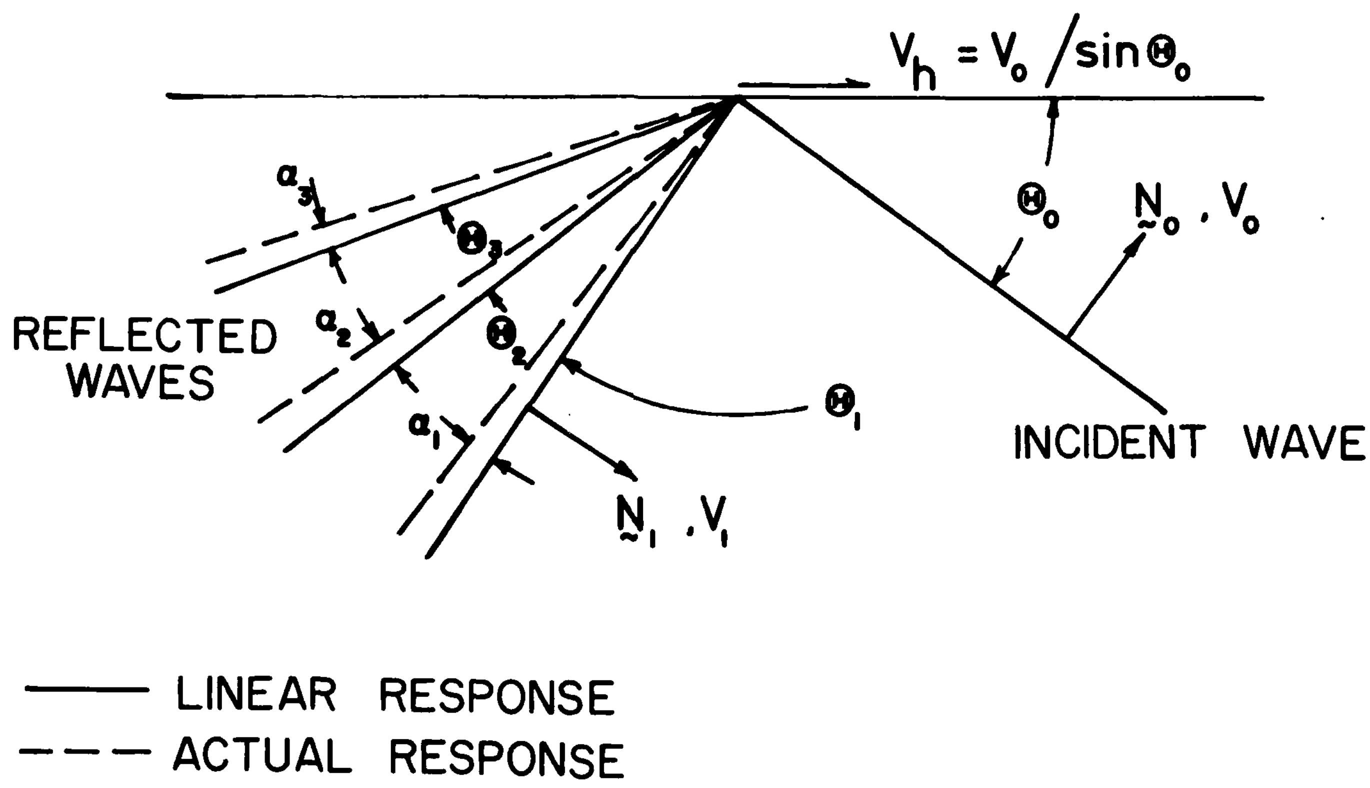

Figure 2. Weak Shock Pattern as Perturbation from the Linear Pattern

where V is one of the speeds from (2.6) and v is a small correction. Finally it may be assumed that the deformation gradient ahead of the incident shock, $\underset{\sim}{F}^+$, differs little from the identity tensor $\underset{\sim}{1}$, that is, all components of $\underset{\sim}{F}^+-\underset{\sim}{1}$ are small. The relationships among the various small quantities are found as follows. With $(2.3)_2$ and $(2.3)_3$ substituted in $(2.3)_1$ we have

$$\left\{\underset{\sim}{T}(\underset{\sim}{F}^+ - \underset{\sim}{a} \otimes \underset{\sim}{N}) - \underset{\sim}{T}(\underset{\sim}{F}^+)\right\} \underset{\sim}{N} + \rho V^2 \underset{\sim}{a} = \underset{\sim}{0} \qquad (2.9)$$

Now put (2.5), (2.7) and (2.8) in (2.9) with $\underset{\sim}{N} = \underset{\sim}{N}_0$ and resolve the resulting equation along $\underset{\sim}{p}$, $\underset{\sim}{q}$, and $\underset{\sim}{r}$. To first order the result is

$$\begin{aligned} \rho(V^2 - V_p^2)\xi &= 0 \\ \rho(V^2 - V_q^2)\eta &= 0 \\ \rho(V^2 - V_r^2)\zeta &= 0 \end{aligned} \qquad (2.10)$$

where use has been made of (2.6). These three equations require that V be equal to one of the linear elastic wave speeds (as already assumed), say $V = V_p$. Then the amplitude ξ is an arbitrary quantity of first order, but η and ζ are zero to first order provided $V_p \neq V_q$ and $V_p \neq V_r$. To second order the components along $\underset{\sim}{p}$, $\underset{\sim}{q}$, and $\underset{\sim}{r}$ yield

$$\begin{aligned} &2\rho V_p^2\, v - \underset{\sim}{C}_3(\underset{\sim}{p} \otimes \underset{\sim}{N})^2(\underset{\sim}{F}^+-\underset{\sim}{1}) \\ &\qquad + \tfrac{1}{2}\, \underset{\sim}{C}_3(\underset{\sim}{p} \otimes \underset{\sim}{N})^3\xi = 0 \\ &\rho(V_p^2 - V_q^2)\eta - \underset{\sim}{C}_3(\underset{\sim}{p} \otimes \underset{\sim}{N})(\underset{\sim}{q} \otimes \underset{\sim}{N})(\underset{\sim}{F}^+-\underset{\sim}{1})\xi \\ &\qquad + \tfrac{1}{2}\, \underset{\sim}{C}_3(\underset{\sim}{p} \otimes \underset{\sim}{N})^2(\underset{\sim}{q} \otimes \underset{\sim}{N})\xi^2 = 0 \end{aligned} \qquad (2.11)$$

There is a third equation, similar to $(2.11)_2$ but with the permutations $q \to r$, $\underset{\sim}{q} \to \underset{\sim}{r}$ and $\eta \to \zeta$. Equation $(2.11)_1$ shows that v is a first order term, and $(2.11)_2$ shows that η (and ζ) is a second order term provided $|V_p-V_q|/V_p > O(\xi)$, i.e. provided V_p and V_q are not too near to each other. (See Appendix A.)

(ii) Reflected Shock.

A wave of this type propagates at an angle $\theta + \alpha$, but its speed along the boundary is determined by the incident wave at the point of reflection. Refer to Figure 3. The amplitude of the reflected wave is again given by (2.7), repeated here

$$\underset{\sim}{a} = \xi \underset{\sim}{p} + \eta \underset{\sim}{q} + \zeta \underset{\sim}{r} \qquad (2.7)$$

but now $\underset{\sim}{p}$, $\underset{\sim}{q}$, $\underset{\sim}{r}$ are proper vectors of $\underset{\sim}{Q}(\underset{\sim}{N}^o)$. That is, they are proper vectors of the acoustic tensor $C_{i\alpha j\beta}N_\alpha^o N_\beta^o$, where $\underset{\sim}{N}^o$ is the normal to the reflected wave in the linear problem. Furthermore, the speed is given by

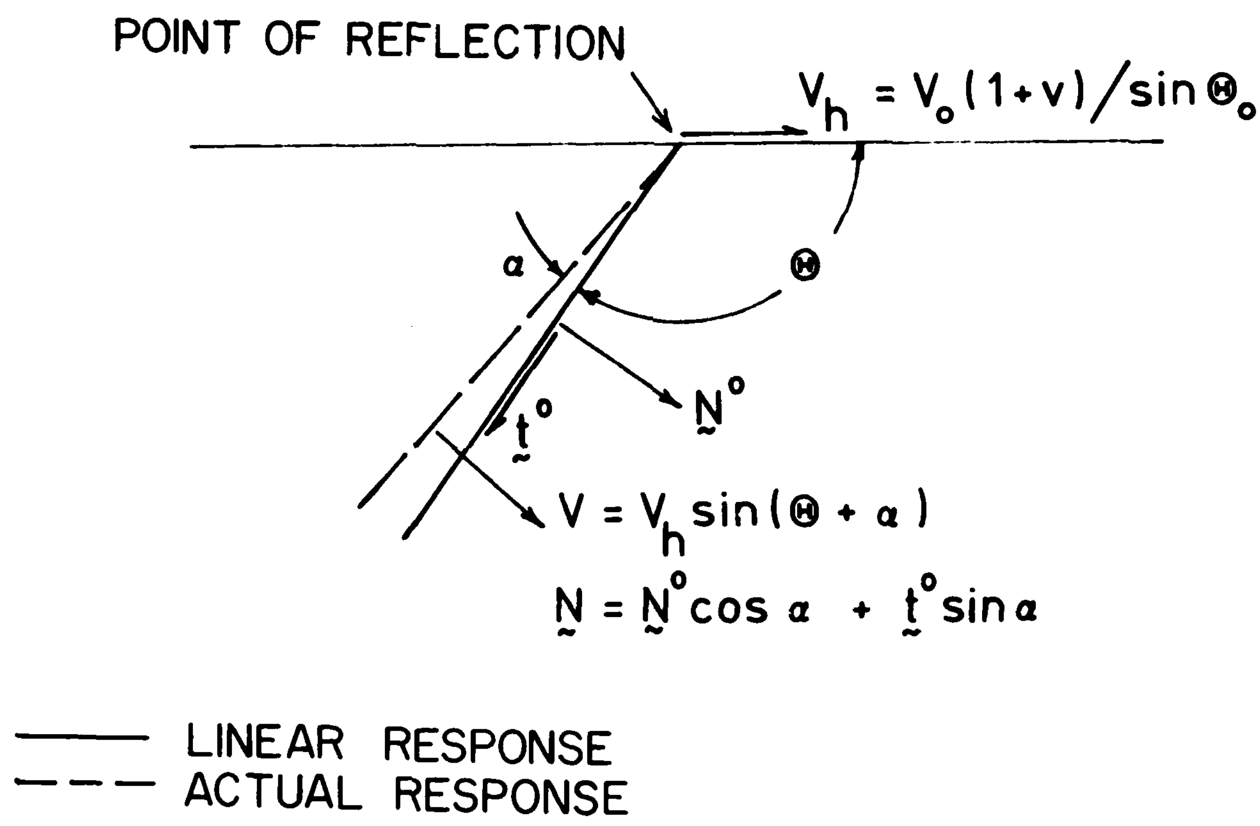

Figure 3. Reflected Shock Wave

$$V = V_h \sin(\theta+\alpha) = \frac{V_o(1+v)}{\sin\theta_o} \sin(\theta+\alpha) \tag{2.12}$$

and the normal to the shock is given by

$$\underset{\sim}{N} = \underset{\sim}{N}^o \cos\alpha + \underset{\sim}{t}^o \sin\alpha \tag{2.13}$$

where $\underset{\sim}{N}^o$, $\underset{\sim}{t}^o$ are the normal and the tangent to the reflected shock in the linear problem.

Previously, (2.9) was used to determine the relationships among v and other small quantities. Here we are interested in the relationships among α and other small quantities. Note first, however, that the deformation gradient ahead of the wave is determined by that ahead of the incident shock plus the sum of the variations across all preceding waves. For example, if we consider the first reflected wave, the deformation gradient ahead of it is

$$\underset{\sim}{F}^+ - \underset{\sim}{a}_o \otimes \underset{\sim}{N}_o$$

where the subscript zero refers to the incident shock. It turns out, as in (2.11), that the deformation gradient is only required to first order so that only first order terms need be retained in $\underset{\sim}{a}_o$. It is convenient to write simply $\underset{\sim}{F}^+$ in this section, meaning that the sum of first order variations are to be included. Now with (2.5), (2.7), (2.12) and (2.13) in (2.9) we arrive at the desired equation, which may be resolved along the proper vectors of $Q(\underset{\sim}{N}^o)$. The first order results are as follows.

$$\begin{aligned} \rho\,\xi\left(V_p{}^2 - V_{h_o}{}^2 \sin^2\theta\right) &= 0 \\ \rho\,\eta\left(V_q{}^2 - V_{h_o}{}^2 \sin^2\theta\right) &= 0 \\ \rho\,\zeta\left(V_r{}^2 - V_{h_o}{}^2 \sin^2\theta\right) &= 0 \end{aligned} \tag{2.14}$$

where $V_{h_o} = V_o/\sin\theta_o$. The interpretation of these equations is similar to that of (2.10). Only one of the linear wave speeds, say V_p, can equal $V_{h_o} \sin\theta$. In that case ξ is an arbitrary quantity of first order, and η and ζ are of higher order provided $V_p \neq V_q$, $V_p \neq V_r$.

To second order, the components along $\underset{\sim}{p}$, $\underset{\sim}{q}$, and $\underset{\sim}{r}$ are as follows.

$$2\rho V_p \left\{ \underset{\sim}{b}\cdot\underset{\sim}{t} - V_{h_o} \cos\theta \right\} \alpha - \tfrac{1}{2} \underset{\sim}{C}_3 \,(\underset{\sim}{p} \otimes \underset{\sim}{N})^3 \xi$$

$$-2\rho V_p^{\,2}\, v + \underset{\sim}{C}_3\, (\underset{\sim}{p} \otimes \underset{\sim}{N})^2 (\underset{\sim}{F}^{+} - \underset{\sim}{1}) = 0 \qquad (2.15)$$

$$\rho \left(V_p^{\,2} - V_q^{\,2}\right) \eta - \underset{\sim}{C}_2 \left\{ (\underset{\sim}{p} \otimes \underset{\sim}{N})(\underset{\sim}{q} \otimes \underset{\sim}{t}) + (\underset{\sim}{p} \otimes \underset{\sim}{t})(\underset{\sim}{q} \otimes \underset{\sim}{N}) \right\} \alpha\xi$$

$$-\underset{\sim}{C}_3 (\underset{\sim}{p} \otimes \underset{\sim}{N})(\underset{\sim}{q} \otimes \underset{\sim}{N})(\underset{\sim}{F}^{+} - \underset{\sim}{1})\xi + \tfrac{1}{2} \underset{\sim}{C}_3 (\underset{\sim}{p} \otimes \underset{\sim}{N})^2 (\underset{\sim}{q} \otimes \underset{\sim}{N}) \xi^2 = 0$$

There is again a third equation, similar to $(2.15)_2$ but with the permutations $q \leftrightarrow r$, $\underset{\sim}{q} \leftrightarrow \underset{\sim}{r}$, $\eta \leftrightarrow \zeta$ as before. The term in braces in $(2.15)_1$ may be interpreted graphically as shown in Figure 4. The vector $\underset{\sim}{b}$ is the radius of the wave surface at the point of tangency of the reflected wave. It may be computed from the formula

$$\rho V_p b_\alpha = C_{i\alpha j\beta}\, p_i p_j N_\beta$$

The projection of $\underset{\sim}{b}$ in the X-Y plane is shown in the figure.

Equations (2.15) will fail if either $|V_p - V_q|/V_p$ or $(\underset{\sim}{b}\cdot\underset{\sim}{t} - V_{h_o} \cos\theta)/V_p$ is small. The first case is treated in Appendix A. The second case occurs as the incident shock approaches a limiting angle and must be treated separately. It is reasonable to assume that the two cases do not occur simultaneously since the fastest acoustic speed is an isolated speed for most solids and limiting angles involve only the fastest reflected wave. Thus, if nearly equal speeds occur, they will be less than the fastest speed for that direction and cannot occur at a limiting angle.

In the case of a limiting angle, $(2.15)_1$ must be replaced by an equation with terms up to α^2 since the coefficient of α will be a first order term. Since it is assumed that V_p and V_q are not nearly equal, equation $(2.15)_2$ is sufficiently complete for η and neither $(A.2)_2$ nor $(A.2)_3$ are needed. From $(2.15)_2$ it can be seen that η is $O(\alpha\xi)$ at least, and when inserted into $(A.2)_1$, will give rise to terms in α^2. Considerable care must be taken so that all terms that contribute equally to each power of α will be included. The equation to replace $(2.15)_1$ has the form

$$A\alpha^2 - B\alpha + C = 0 \qquad (2.16)$$

where A includes only terms of order zero and B and C include first order terms. We have

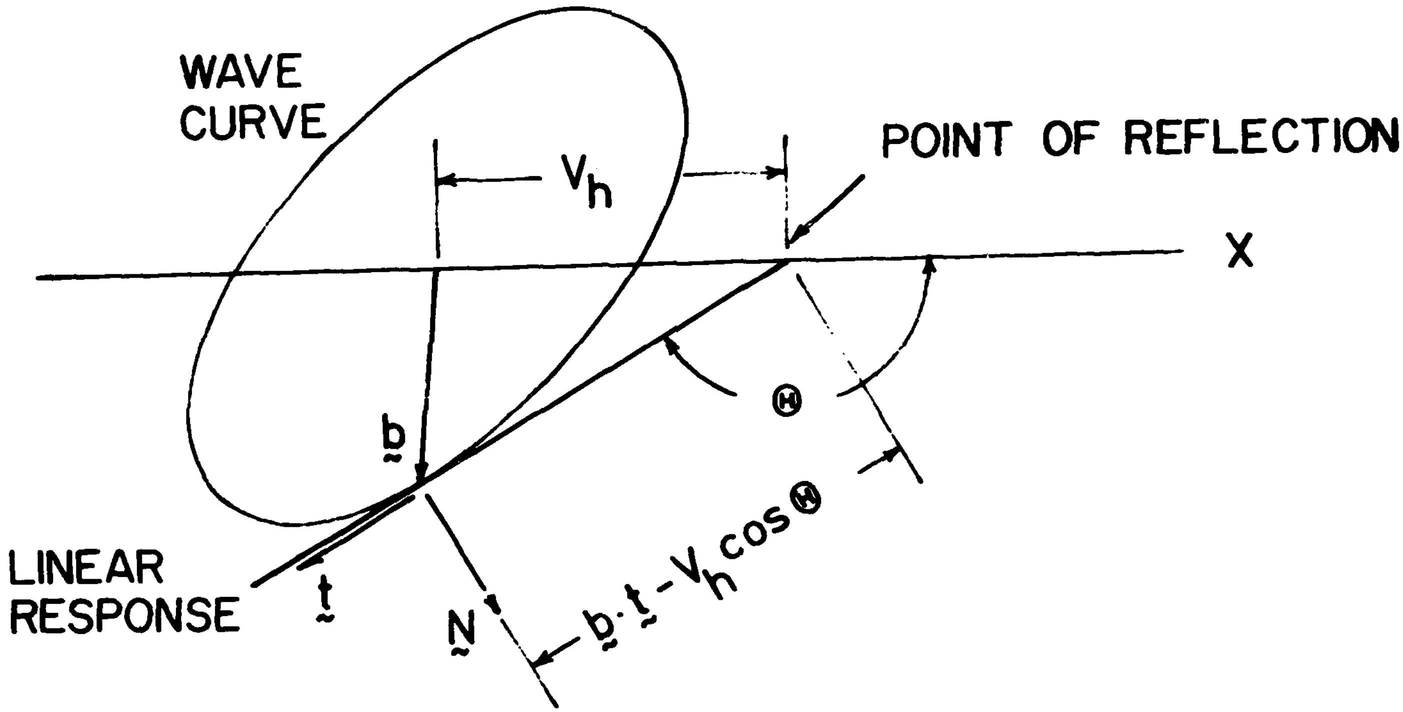

Figure 4. Relationship Between Wave Curve and Reflection Point

$$
\begin{aligned}
A &= \rho V_{h_o}^{\,2}\cos^2\theta - \mathbf{C}_2(\mathbf{p}\otimes\mathbf{t})^2 \\
&\quad - \left\{\frac{\mathbf{C}_2\left[(\mathbf{p}\otimes\mathbf{N})(\mathbf{q}\otimes\mathbf{t}) + (\mathbf{p}\otimes\mathbf{t})(\mathbf{q}\otimes\mathbf{N})\right]}{\rho(V_p^{\,2} - V_q^{\,2})}\right\}^2 \\
&\quad - \left\{\frac{\mathbf{C}_2\left[(\mathbf{p}\otimes\mathbf{N})(\mathbf{r}\otimes\mathbf{t}) + (\mathbf{p}\otimes\mathbf{t})\ (\mathbf{r}\otimes\mathbf{N})\right]}{\rho(V_p^{\,2} - V_r^{\,2})}\right\}^2 \\
B &= 2\rho V_p(\mathbf{b}\cdot\mathbf{t} - V_{h_o}\cos\theta) - \frac{3}{2}\ \mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})^2(\mathbf{p}\otimes\mathbf{t})\xi \\
&\quad + 2\ \mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})(\mathbf{p}\otimes\mathbf{t})(\mathbf{F}^+-\mathbf{1}) - 4\rho V_p V_{h_o}\cos\theta\ v \\
&\quad + \mathbf{C}_2\left[(\mathbf{p}\otimes\mathbf{N})(\mathbf{q}\otimes\mathbf{t}) + (\mathbf{p}\otimes\mathbf{t})(\mathbf{q}\otimes\mathbf{N})\right]\cdot \\
&\qquad \cdot\left\{\frac{2\ \mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})(\mathbf{q}\otimes\mathbf{N})(\mathbf{F}^+-\mathbf{1})}{\rho(V_p^{\,2} - V_q^{\,2})} - \frac{3}{2}\ \frac{\mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})^2(\mathbf{q}\otimes\mathbf{N})\xi}{\rho(V_p^{\,2} - V_q^{\,2})}\right\} \\
&\quad + \mathbf{C}_2\left[(\mathbf{p}\otimes\mathbf{N})(\mathbf{r}\otimes\mathbf{t}) + (\mathbf{p}\otimes\mathbf{t})(\mathbf{r}\otimes\mathbf{N})\right]\cdot \\
&\qquad \cdot\left\{\frac{2\ \mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})(\mathbf{r}\otimes\mathbf{N})(\mathbf{F}^+-\mathbf{1})}{\rho(V_p^{\,2} - V_r^{\,2})} - \frac{3}{2}\ \frac{\mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})^2(\mathbf{r}\otimes\mathbf{N})\xi}{\rho(V_p^{\,2} - V_r^{\,2})}\right\} \\
C &= \tfrac{1}{2}\ \mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})^3 + 2\rho V_p^{\,2}\ v - \mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})^2(\mathbf{F}^+-\mathbf{1})
\end{aligned}
\tag{2.17}
$$

Equation $(2.15)_1$ is a specialization of (2.16) when α is a first order quantity.

(iii) Reflected Simple Wave.

The analysis for a reflected simple wave is slightly more complicated than that for a reflected shock since the differential equations (2.4) must be satisfied throughout the wave rather than the difference equations (2.3) across the wave. Again it is convenient to base the analysis on the proper vectors from (2.6). Refer to Figure 5. Within the simple wave the acoustic tensor Q depends on a variable deformation gradient and a variable normal since

$$
\begin{aligned}
\hat{\mathbf{C}}_2 &= \mathbf{C}_2 + \mathbf{C}_3(\mathbf{F}-\mathbf{1}) \\
\mathbf{N} &= \mathbf{N}^o\cos\alpha + \mathbf{t}^o\sin\alpha
\end{aligned}
\tag{2.18}
$$

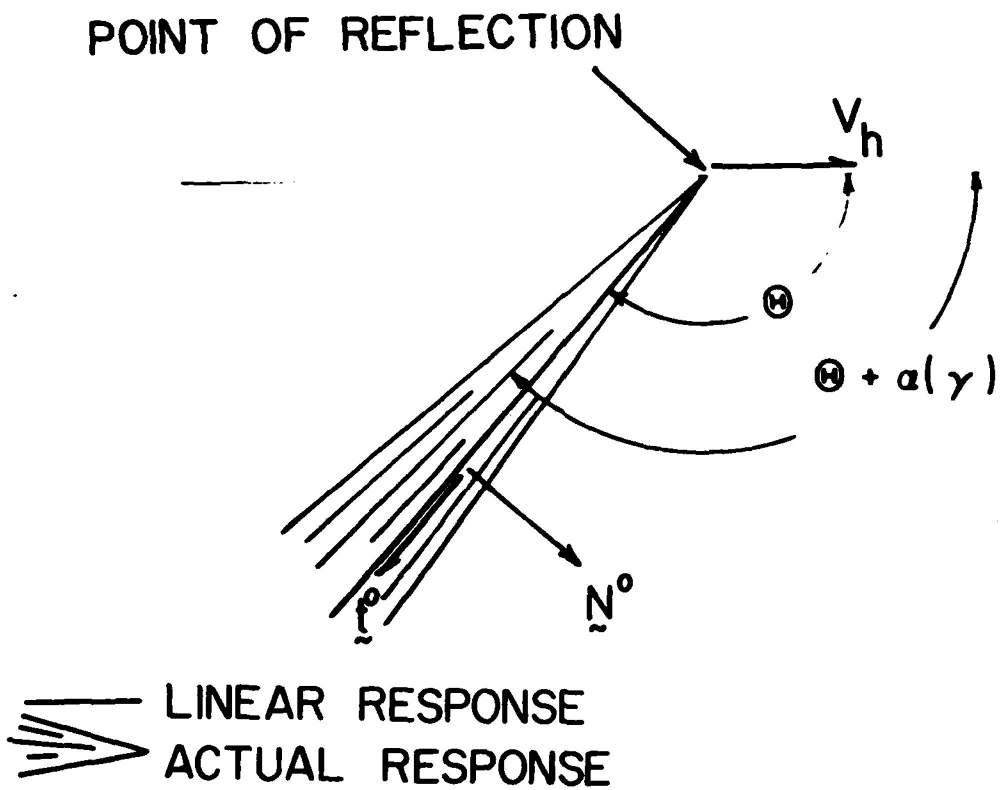

Figure 5. Reflected Simple Wave

where both $\underset{\sim}{F}$ and α depend on the wave parameter γ. The unit vector $\underset{\sim}{m}$, which is a proper vector of Q, may be written

$$\underset{\sim}{m} = \xi\underset{\sim}{p} + \eta\underset{\sim}{q} + \zeta\underset{\sim}{r} \tag{2.19}$$

where $\underset{\sim}{p}$, $\underset{\sim}{q}$, $\underset{\sim}{r}$, as before, are the proper vectors of $Q(\underset{\sim}{N}^o)$. It is assumed that $\underset{\sim}{m}$ is close to one of the other vectors, say $\underset{\sim}{m} \doteq \underset{\sim}{p}$, so that $\xi = \sqrt{1-\eta^2-\zeta^2} \doteq 1$. From $(2.4)_2$ we may write

$$\underset{\sim}{F}(\gamma) = \underset{\sim}{F}^+ - \int_o^\gamma \underset{\sim}{m}(\gamma) \otimes \underset{\sim}{N}(\gamma)\, d\gamma \tag{2.20}$$

With only the lowest order terms retained, this becomes

$$\begin{aligned} \underset{\sim}{F}(\gamma) = \underset{\sim}{F}^+ &- \underset{\sim}{p} \otimes \underset{\sim}{N}^o\xi\gamma - \underset{\sim}{p} \otimes \underset{\sim}{t}^o \int_o^\gamma \alpha\, d\gamma \\ &- \underset{\sim}{q} \otimes \underset{\sim}{N}^o \int_o^\gamma \eta\, d\gamma - \underset{\sim}{r} \otimes \underset{\sim}{N}^o \int_o^\gamma \zeta\, d\gamma \end{aligned} \tag{2.21}$$

Now with (2.18), (2.19), and (2.21) inserted in $(2.4)_1$, and with $V = V_h \sin(\theta+\alpha)$ a vector equation relating α, η, ζ and γ is obtained. At lowest order we obtain (2.15) again. Since $\xi = O(1)$, we have $V_p = V_{h_o} \sin\theta$, as required by linear theory, and η, ζ are higher order terms. At first order the following equations are obtained.

$$\begin{aligned} &2\rho V_p(\underset{\sim}{b}\cdot\underset{\sim}{t} - V_{h_o}\cos\theta)\alpha \\ &\quad -\underset{\sim}{C}_3(\underset{\sim}{p} \otimes \underset{\sim}{N}^o)^3\,\gamma - 2\rho V_p^{\,2}\, v \\ &\qquad +\underset{\sim}{C}_3(\underset{\sim}{p} \otimes \underset{\sim}{N}^o)^2(\underset{\sim}{F}^+ - \underset{\sim}{1}) = 0 \\ &\rho\left(V_p^{\,2} - V_q^{\,2}\right)\eta - C_2\left[(\underset{\sim}{p} \otimes \underset{\sim}{N}^o)(\underset{\sim}{q} \otimes \underset{\sim}{t}^o) + (\underset{\sim}{p} \otimes \underset{\sim}{t}^o)(\underset{\sim}{q} \otimes \underset{\sim}{N}^o)\right]\alpha \\ &\quad -\underset{\sim}{C}_3(\underset{\sim}{p} \otimes \underset{\sim}{N}^o)(\underset{\sim}{q} \otimes \underset{\sim}{N}^o)(\underset{\sim}{F}^+ - \underset{\sim}{1}) \\ &\qquad +\underset{\sim}{C}_3(\underset{\sim}{p} \otimes \underset{\sim}{N}^o)^2(\underset{\sim}{q} \otimes \underset{\sim}{N}^o)\gamma = 0 \end{aligned} \tag{2.22}$$

There is a third equation for ζ, similar to $(2.22)_2$. Provided that the coefficients of α and η are not small, these equations, upon comparison with (2.15), show that to second order a simple wave may be replaced by a shock of equal amplitude and located so as to bisect the wedge of the simple wave. To see that this is true, note that the wedge is defined by $\alpha(\gamma)$ as γ varies in the range $[0, \gamma_m]$ for some extreme value γ_m. With γ replaced by $\frac{1}{2}\gamma_m$, $(2.22)_1$

is identical in form to $(2.15)_1$. Insertion of $(2.22)_2$ into (2.21) followed by integration shows that $\mathbf{F}(\gamma_m)$ is the same as $\mathbf{F}^-$ calculated from $(2.3)_2$ with (2.7) and $(2.15)_2$.

If $\left(\mathbf{b}\cdot\mathbf{t} - V_{h_o}\cos\theta\right)/V_p$ is small, then equation $(2.22)_1$ must be replaced by one with terms in α^2, but $(2.22)_2$ remains unchanged. By the same process used to obtain (2.22), but with care taken to retain all terms quadratic in small quantities, we again obtain an equation of the form of (2.16) with coefficients A, B, C as follows.

$$
\begin{aligned}
A = {} & \rho V_{h_o}^2\cos^2\theta - \mathbf{C}_2(\mathbf{p}\otimes\mathbf{t})^2 \\
& - \frac{\left\{\mathbf{C}_2\left[(\mathbf{p}\otimes\mathbf{N})(\mathbf{q}\otimes\mathbf{t}) + (\mathbf{p}\otimes\mathbf{t})(\mathbf{q}\otimes\mathbf{N})\right]\right\}^2}{\rho(V_p^2 - V_q^2)} \\
& - \frac{\left\{\mathbf{C}_2\left[(\mathbf{p}\otimes\mathbf{N})(\mathbf{r}\otimes\mathbf{t}) + (\mathbf{p}\otimes\mathbf{t})(\mathbf{r}\otimes\mathbf{N})\right]\right\}^2}{\rho(V_p^2 - V_r^2)}
\end{aligned}
$$

$$
\begin{aligned}
B = {} & 2\rho V_p\left(\mathbf{b}\cdot\mathbf{t} - V_{h_o}\cos\theta\right) - 2\,\mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})^2(\mathbf{p}\otimes\mathbf{t})\gamma \\
& + 2\,\mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})(\mathbf{p}\otimes\mathbf{t})(\mathbf{F}^+ - \mathbf{1}) - 4\rho V_p V_{h_o}\cos\theta\, v \\
& + \frac{2\,\mathbf{C}_2\left[(\mathbf{p}\otimes\mathbf{N})(\mathbf{q}\otimes\mathbf{t}) + (\mathbf{p}\otimes\mathbf{t})(\mathbf{q}\otimes\mathbf{N})\right]}{\rho(V_p^2 - V_q^2)}\,\cdot \\
& \quad \cdot\left\{\mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})(\mathbf{q}\otimes\mathbf{N})(\mathbf{F}^+ - \mathbf{1}) - \mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})^2(\mathbf{q}\otimes\mathbf{N})\gamma\right\} \\
& + \frac{2\,\mathbf{C}_2\left[(\mathbf{p}\otimes\mathbf{N})(\mathbf{r}\otimes\mathbf{t}) + (\mathbf{p}\otimes\mathbf{t})(\mathbf{r}\otimes\mathbf{N})\right]}{\rho(V_p^2 - V_r^2)}\,\cdot \\
& \quad \cdot\left\{\mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})(\mathbf{r}\otimes\mathbf{N})(\mathbf{F}^+ - \mathbf{1}) - \mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})^2(\mathbf{r}\otimes\mathbf{N})\gamma\right\}
\end{aligned}
\tag{2.23}
$$

$$
C = \mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})^3\gamma + 2\rho V_p^2\, v - \mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})^2(\mathbf{F}^+ - \mathbf{1})
$$

Equations (2.23) should be compared with (2.17), and $(2.22)_2$ with $(2.15)_2$. In obvious shortened notation, the equations for α in the case of a shock or a simple wave are as follows.

Shock Wave:

$$A\,\bar{\alpha}^2 - \bar{\alpha}\left(B_1 - \frac{3}{4}B_2\xi + b\right) + \frac{1}{2}C_1\xi + c = 0 \tag{2.24}$$

Simple Wave:

$$A\,\alpha^2 - \alpha\left(B_1 - B_2\gamma + b\right) + C_1\gamma + c = 0$$

If B_1 is large, then the terms with A, B_2 and b may be dropped, but near a limiting angle B_1 will be small so that all terms must then be retained. Strictly speaking near a limiting angle a simple wave can no longer be replaced by an equivalent shock for computational purposes. To do so would require the equivalence of (2.21) (with $\gamma = \xi$) and $(2.3)_2$, written here to second order.

$$\begin{aligned} \underset{\sim}{F}^- = \underset{\sim}{F}^+ - \underset{\sim}{p} \otimes \underset{\sim}{N}\xi - \underset{\sim}{p} \otimes \underset{\sim}{t}\,\xi\bar{\alpha} \\ - \underset{\sim}{q} \otimes \underset{\sim}{N}\eta - \underset{\sim}{r} \otimes \underset{\sim}{N}\zeta \end{aligned} \tag{2.25}$$

If it could be shown that a good second order approximation could be obtained by setting

$$\xi\bar{\alpha} = \int_0^\xi \alpha(\gamma)\, d\gamma \tag{2.26}$$

then comparison of $(2.22)_2$ with $(2.15)_2$ would complete the required equivalence. Even if (2.26) does not hold to second order, for practical purposes it is still useful and will introduce only a small error. To justify this last statement, first solve $(2.24)_2$ for γ.

$$\gamma = -\frac{A\alpha^2 - (B_1 + b)\alpha + c}{C_1 + B_2\alpha} \tag{2.27}$$

The curve for γ as a function of α is nearly parabolic for small α. To represent a simple wave there must be a branch that intersects the axis on which $\gamma = 0$ and that has positive and increasing values of γ for increasing α. If this condition has been met, there are two extreme cases to consider. First, if the roots of the numerator of (2.27) are widely separated, then the branches may be approximated locally near $\gamma = 0$ by linear functions. This occurs only when B_1 is large and is the case already considered. In the other extreme condition the roots of the numerator of (2.27) are equal, say $\alpha = \hat{\alpha}$, and we have the following approximate expressions.

$$\alpha - \hat{\alpha} = \sqrt{-\frac{C_1}{A}}\;\gamma^{1/2}$$

$$\int_0^\xi \alpha d\gamma = \hat{\alpha}\,\xi + \frac{2}{3}\sqrt{-\frac{C_1}{A}}\;\xi^{3/2}$$

On the other hand from $(2.24)_1$ in the same circumstances we have approximately

$$\bar{\alpha} - \hat{\alpha} = \sqrt{-\frac{C_1}{2A}}\,\xi^{\frac{1}{2}}$$

and

$$\bar{\alpha}\,\xi = \hat{\alpha}\,\xi + \frac{1}{\sqrt{2}}\sqrt{-\frac{C_1}{A}}\;\xi^{3/2}$$

Since 2/3 and $1/\sqrt{2}$ differ by less than 6%, the approximation (2.26) should prove useful even in extreme circumstances.

III. REFLECTION CALCULATIONS: GENERAL CONSIDERATIONS

All the usual boundary conditions require the calculation of stresses or velocities or both behind the reflection pattern. With reference to Figure 2, the deformation gradient and velocity behind the pattern may be written:

$$\begin{aligned} \underset{\sim}{F}_B &= \underset{\sim}{F}_I + \sum_{i=0}^{3} \Delta_i\, \underset{\sim}{F} \\ \underset{\sim}{u}_B &= \underset{\sim}{u}_I + \sum_{i=0}^{3} \Delta_i\, \underset{\sim}{u} \end{aligned} \tag{3.1}$$

where the subscript B indicates the region behind the wave pattern and adjacent to the boundary, the subscript I indicates the region ahead of the incident wave, and the symbol Δ_i indicates the variation across the i^{th} wave. In the treatment of individual waves in the last section, it was shown that to second order all waves may be treated as if they were shock waves. (The only possible exception is the last case treated, i.e., a simple wave near the limiting angle.) Accordingly, the amplitudes of the waves may be represented as follows.

$$\begin{aligned} \underset{\sim}{a}_o &= \xi_o\, \underset{\sim}{p}_o + \eta_o\, \underset{\sim}{q}_o + \zeta_o\, \underset{\sim}{r}_o \\ \underset{\sim}{a}_1 &= \xi_1(1+\phi)\underset{\sim}{p}_1 + \eta_2\, \underset{\sim}{q}_1 + \zeta_1\, \underset{\sim}{r}_1 \\ \underset{\sim}{a}_2 &= \xi_2\, \underset{\sim}{p}_2 + \eta_2(1+\psi)\underset{\sim}{q}_2 + \zeta_2\, \underset{\sim}{r}_2 \\ \underset{\sim}{a}_3 &= \xi_3\, \underset{\sim}{p}_3 + \eta_3\, \underset{\sim}{q}_3 + \zeta_3(1+\omega)\underset{\sim}{r}_3 \end{aligned} \tag{3.2}$$

In (3.2) the subscripts refer to the various waves as shown in Figure 2. The polarizations $\underset{\sim}{p}_n$, $\underset{\sim}{q}_n$, $\underset{\sim}{r}_n$ correspond to the directions of waves in the linear pattern. Thus

$$
\underset{\sim}{Q}_n \begin{Bmatrix} \underset{\sim}{p}_n \\ \underset{\sim}{q}_n \\ \underset{\sim}{r}_n \end{Bmatrix} = \rho \begin{Bmatrix} V_{p_n}^{2}\, \underset{\sim}{p}_n \\ V_{q_n}^{2}\, \underset{\sim}{q}_n \\ V_{r_n}^{2}\, \underset{\sim}{r}_n \end{Bmatrix} \tag{3.3}
$$

where $\left\{\underset{\sim}{Q}_n\right\}_{ij} = C_{i\alpha j\beta} N^n_\alpha N^n_\beta$. The normal to the incident wave is given as well as the primary amplitude and polarization of the incident wave, i.e., one of ξ_o, η_o or ζ_o is given. The primary amplitudes of the reflected waves are ξ_1, η_2 and ζ_3. All other terms, including ϕ, ψ and ω represent higher order corrections. Transmitted waves could be represented analogously.

The calculation of the reflection/transmission pattern and of the wave amplitudes now proceeds in a stepwise manner once the boundary condition, angle of incidence, and the polarization and amplitude of the incident shock are all specified.

Step (i) The angles of reflection/transmission for the linear problem are computed. This corresponds to the fact that the reflection/transmission pattern for an incident shock of vanishing amplitude is the same as the pattern in linear elasticity. The pattern is computed by the standard methods, e.g., see Musgrave[12].

Step (ii) Corrections to the speed of the incident shock wave and to the polarization may be computed from (2.11) or (A1) as appropriate.

Step (iii) The amplitudes of all waves, given by (3.2), are inserted in the expressions for stresses and velocities, and these in turn are inserted in the boundary conditions. The lowest order terms, considered separately, give exactly the same expressions as would be obtained from linear elasticity. This may be written as follows.

$$
B \begin{Bmatrix} \xi_1 \\ \eta_2 \\ \zeta_3 \end{Bmatrix} = M(\xi_o) \tag{3.4}
$$

Here B is the 3 x 3 matrix of boundary coefficients from linear elasticity, which operates on the column matrix of primary reflection amplitudes, and M, which is a linear function of ξ_o, is a column matrix that originates from the lowest order terms of the incident wave. For illustrative purposes, only reflected waves have been considered in (3.4), and the incident wave has been assumed to have primary polarization $\underset{\sim}{p}_o$. Equation (3.4) may be solved provided

---

[12] *M. J. P. Musgrave, Crystal Acoustics, Holden-Day, San Francisco (1970).*

that det B $\neq$ 0. In Reference 13 it is shown that, generally, det B $\neq$ 0 except possibly at a limiting angle. For the moment let us assume that det B is non-vanishing for all angles. The case when det B = 0 at the limiting angle requires special treatment and is discussed separately in Step (vii).

Step (iv) Angle corrections α_1, α_2, α_3 and amplitude corrections η_1, ζ_1, ξ_2, ζ_2, ξ_3, η_3 may be computed from (2.15) or (A.2) in all cases that do not involve a limiting angle. If reflection occurs near a limiting angle, corrections may be found from (2.16) with (2.17) (or with (2.23) if greater accuracy is desired in the case of a simple wave).

Step (v) The results of (ii), (iii), and (iv) are used to express the second order terms of the boundary conditions. These may be written as follows

$$B \begin{Bmatrix} \phi\ \xi_1 \\ \psi\ \eta_2 \\ \omega\ \zeta_3 \end{Bmatrix} = N\left(\xi_0,\ \xi_i,\ \eta_i,\ \zeta_i,\ \alpha_i\right) \tag{3.5}$$

Here B is the same boundary matrix for the linear problem as before, ϕ, ψ, ω are correction factors for the primary amplitudes, and the column vector N is quadratic in first order terms and linear in second order terms.

Step (vi) If the angle of incidence is near the linear limiting angle, then (2.16) must be used to compute the nonlinear limiting angle. For shock waves a limiting angle occurs when α becomes complex with (2.17) used in (2.16). For simple waves a limiting angle occurs when α becomes complex with (2.23) used in (2.16). For this last case it is necessary to set γ = 0 in (2.23) since the limit occurs when the leading edge of the simple wave overtakes the incident wave. Note that only Step (iii), the linear solution, is required to compute the limiting angle.

Step (vii) If the angle of incidence is near the limiting angle and det B = 0 at the linear limiting angle, then it is necessary to combine steps (ii) - (vi) into one step. The boundary equations have then become essentially nonlinear, and the simplest approximate solution is obtained from a set of quadratic equations rather than a sequence of linear equations.

IV. EXAMPLES: REFLECTIONS IN ISOTROPIC MATERIALS

The procedure given at the end of the last section is perfectly general for anisotropic materials. No doubt numerical results could be produced by computer for arbitrary symmetries, but only isotropic materials seem sufficiently simple that explicit formulas may be derived.

In Reference 13 calculations were made for three different boundary conditions with an incident longitudinal shock wave. The results of those

---

[13] *T. W. Wright, Oblique Reflections, in Propagation of Shock Waves in Solids, ed. E. Varley, AMD Vol. 17, ASME, New York (1976).*

calculations are summarized below since they are illustrative of the kinds of behavior possible. In addition, new results for incident shear waves are derived. All waves will be approximated by shocks as described in Section II. Notation for elastic moduli is explained in detail in Appendix B.

Longitudinal Waves

If the incident wave in Figure 1 is a longitudinal wave progressing into an unstressed region, the linear wave speed is $c_1 = \sqrt{c_{11}/\rho}$, also $\underset{\sim}{p}_o = \underset{\sim}{N}_o$, $\eta_o = \zeta_o = 0$, and the first order correction from $(2.11)_1$ is

$$v = - \frac{C}{4c_{11}} \xi_o \tag{4.1}$$

where $C = C_{11\ 11\ 11}$ and $c_{11} = \lambda + 2\mu$, λ, μ being the usual Lamé constants. For most materials C is negative so that compressive waves (i.e., $\xi_o > 0$) correspond to shock waves, as is well known.

There are only two reflected waves in (3.2), so $\underset{\sim}{a}_3 = \underset{\sim}{0}$. Also $\zeta_1 = \zeta_2 = 0$, and $\underset{\sim}{p}_i = \underset{\sim}{N}_i$, $\underset{\sim}{q}_i = \underset{\sim}{t}_i$ for i = 0, 1, 2. The reflection angles for the linear problem are given by $\theta_1 = \pi - \theta_o$ and $\theta_2 = \pi - \bar{\theta}$ where $\bar{\theta} = \sin^{-1}\left(\frac{c_2}{c_1} \sin \theta_o\right)$ and c_2 is the linear elastic speed for shear waves. Angular corrections α_1 and α_2 are found from $(2.15)_1$

$$\begin{aligned} &-2\, c_{11} \cot\theta_o\, \alpha_1 + \tfrac{1}{2} C(\xi_1 + \xi_o) \\ &\qquad - (C-c) \sin^2 2\theta_o\, \xi_o = 0 \\ &2 \frac{c_2}{c_1} c_{11} \frac{\cos\bar{\theta}}{\sin\theta_o} \alpha_2 + \tfrac{1}{2} \frac{c_{44}}{c_{11}} C\, \xi_o \\ &\qquad - \tfrac{1}{2}\left[\tfrac{1}{2}(C-c) + c_{12} + c_{44}\right](\xi_o + \xi_1) = 0 \end{aligned} \tag{4.2}$$

Amplitude corrections η_1 and ξ_2 are found from $(2.15)_2$

$$\begin{aligned} &(c_{12} + c_{44})(\eta_1 - \alpha_1\xi_1) + \tfrac{1}{4}(C-c) \sin 4\theta_o \xi_o \xi_1 = 0 \\ &-(c_{12} + c_{44})(\xi_2 + \alpha_2\eta_2) + \tfrac{1}{4}\left[\tfrac{1}{2}(C-c) + c_{12} + c_{44}\right]\eta_2^2 \\ &\qquad + \tfrac{1}{4}(C-c)\left[\xi_o \sin 2(\theta_o + \bar{\theta}) - \xi_1 \sin 2(\theta_o - \bar{\theta})\right]\eta_2 = 0 \end{aligned} \tag{4.3}$$

In (4.2) and (4.3) primary amplitudes ξ_1, η_2 are to be found from the linear boundary value problem, and $c = C_{112} = C_{11\ 11\ 22}$.

Note that $(4.2)_1$ clearly fails as θ_o approaches 90°, the limiting angle in this case, since the coefficient of α_1 tends to zero. With $\theta_o = \pi/2 - \alpha_o$, where α_o is small, $(4.2)_1$ must be replaced by a quadratic equation, obtained from (2.16).

$$c_{11} \alpha_1{}^2 + 2 c_{11} \alpha_o \alpha_1 - \tfrac{1}{2} C(\xi_o + \xi_1) = 0 \qquad (4.4)$$

Since α_1 must be real, the discriminant of (4.4) must be positive. It is this requirement that determines the limiting angle as a function of amplitude. The discriminant vanishes when

$$\xi_o = - \frac{2\alpha_o{}^2}{\frac{C}{c_{11}} \left(1 + \frac{\xi_1}{\xi_o}\right)} \qquad (4.5)$$

and then $\alpha_1 = - \alpha_o$ so that $\theta_1 + \alpha_1 = \frac{\pi}{2}$. In (4.5) the ratio ξ_1/ξ_o is a function of α_o and comes from the reflection problem for linear elasticity.

Equation (4.5) was obtained with the approximation, made for computational purposes, that amplitudes may be calculated correctly to second order from formulas for shock waves. If the first reflected wave is actually a simple wave, then the regular reflection pattern fails when the leading edge of the simple wave overrides the incident wave, not when the approximating shock does, since it is embedded within the real simple wave. These remarks should be made clear by reference to Fig. 5. The net result is that for simple wave fans it is necessary to set $\xi_1/\xi_o = 0$ in (4.5) to obtain the correct limiting relationship.

Formulas (4.1) through (4.4) are perfectly general for any incident longitudinal wave and any boundary condition, but (4.5) gives the correct limiting condition only if the linear boundary determinant does not vanish at the limiting angle.

The formulas for velocities and surface tractions behind the reflection pattern, correct to second order in amplitudes, are found by combining (2.5), (3.1) and (3.2) for isotropic materials.

$$
\begin{aligned}
\frac{\underset{\sim}{u}_B \cdot \underset{\sim}{N}_B}{(1+\nu)V_h} &= \tfrac{1}{2}(\xi_o - \xi_1)\sin 2\theta_o + \alpha_1\xi_1\cos^2\theta_o - \eta_2\frac{c_{44}}{c_{11}}\sin^2\theta_o \\
&\quad -\eta_1\sin^2\theta_o - (\xi_2 - \alpha_2\eta_2)\frac{c_2}{c_1}\cos\bar{\theta}\,\sin\theta_o \\[2ex]
\frac{\underset{\sim}{u}_B \cdot \underset{\sim}{t}_B}{(1+\nu)V_h} &= (\xi_o + \xi_1)\sin^2\theta_o - \tfrac{1}{2}\eta_2\sin 2\bar{\theta} - \tfrac{1}{2}(\eta_1 + \alpha_1\xi_1)\sin 2\theta_o \\
&\quad + \alpha_2\eta_2\cos^2\bar{\theta} + \xi_2\sin^2\bar{\theta} \\[2ex]
\frac{1}{c_{44}}\underset{\sim}{N}_B \cdot (\underset{\sim}{T}\underset{\sim}{N}_B) &= -(\xi_o + \xi_1)\left(\frac{c_{12}}{c_{44}} + 2\cos^2\theta_o\right) - \eta_2\sin 2\bar{\theta} \\
&\quad - (\eta_1 + \alpha_1\xi_1)\sin 2\theta_o - \frac{c_{12}}{c_{44}}(\xi_2 + \alpha_2\eta_2) \\
&\quad - 2\left(\xi_2\cos^2\bar{\theta} + \alpha_2\eta_2\sin^2\bar{\theta}\right) \\
&\quad + \tfrac{1}{2}\left(\xi_o^{\,2} + \xi_1^{\,2}\right)\left(\frac{c}{c_{44}}\sin^2\theta_o + \frac{C}{c_{44}}\cos^2\theta_o\right) + \tfrac{1}{4}\eta_2^{\,2}\left\{\tfrac{1}{2}\,\frac{C-c}{c_{44}} + \left(\frac{c_{12}}{c_{44}} + 1\right)\right\} \\
&\quad + \xi_o\xi_1\left\{\frac{c}{c_{44}}(1 + 2\cos^2\theta_o)\sin^2\theta_o + \frac{C}{c_{44}}\cos 2\theta_o\cos^2\theta_o\right\} \\
&\quad + \tfrac{1}{2}\,\frac{C-c}{c_{44}}\eta_2\cos\theta_o\left\{\xi_o\sin(\theta_o + 2\bar{\theta}) + \xi_1\sin(\theta_o - 2\bar{\theta})\right\} \\[2ex]
\frac{1}{c_{44}}\underset{\sim}{t}_B \cdot (\underset{\sim}{T}\underset{\sim}{N}_B) &= (\xi_o - \xi_1)\sin 2\theta_o - \eta_2\cos 2\bar{\theta} + (\eta_1 + \alpha_1\xi_1)\cos 2\theta_o \\
&\quad + (\xi_2 - \alpha_2\eta_2)\sin 2\bar{\theta} + \tfrac{1}{4}\left(\xi_o^{\,2} - \xi_1^{\,2}\right)(C-c)\sin 2\theta_o \\
&\quad + \tfrac{1}{2}\eta_2(\xi_o + \xi_1)\left\{\left(\frac{c_{12}}{c_{44}} + 1\right) + \tfrac{1}{2}\,\frac{C-c}{c_{44}}\cos 2\bar{\theta}\right\}
\end{aligned}
\tag{4.6}
$$

For a clamped boundary, the first two of (4.6) must be identically zero, and similarly for the last two in the case of a free boundary. The linear boundary determinants in these two cases are as follows.

Clamped:

$$
\begin{aligned}
\det B &= \cos(\theta_o - \bar{\theta}) \\
&\to c_2/c_1 \text{ as } \theta_o \to \frac{\pi}{2}
\end{aligned}
\tag{4.7$_1$}
$$

Free:

$$\det B = \left(\frac{c_{12}}{c_{44}} + 2\cos^2\theta_o\right)\cos 2\bar{\theta} + \sin 2\theta_o \sin 2\bar{\theta}$$

$$\rightarrow \frac{c_{12}^{\;2}}{c_{44}\, c_{11}} \quad \text{as } \theta_o \rightarrow \frac{\pi}{2} \tag{4.7$_2$}$$

Since the determinants remain positive all the way to the limiting angles (except for a Poisson's ratio of zero in the second case), the first five steps at the end of Section III are all that are needed to determine solutions to second order. These two cases were discussed further in Reference 13.

The rigid-lubricated boundary by definition requires the first and the fourth of (4.6) to vanish. The linear terms give the solution $\xi_1/\xi_o = 1$, $\eta_2/\xi_o = 0$, but the boundary determinant in that case is given by

$$\det B = \sin\theta_o \cos\theta_o$$

$$\rightarrow 0 \text{ as } \theta_o \rightarrow \frac{\pi}{2} \tag{4.8}$$

Since $\sin\theta_o$ appears in (4.8) only because of the normalization by V_h in $(4.6)_1$, there is no difficulty at $\theta_o = 0$, but as θ_o approaches 90° the linear solution no longer gives a good first approximation. It is then necessary to use non-linear boundary conditions right from the start.

With $\theta_o = \frac{\pi}{2} - \alpha_o$, where α_o is a small angle, and with a small angle approximation used throughout, it is readily found from $(4.2)_2$ and $(4.3)_2$ that

$$\alpha_2 = O\,(\xi_o)$$

$$\xi_2 = O\,(\alpha_2\eta_2). \tag{4.9}$$

Furthermore, from $(4.3)_1$ and (4.4) we have to lowest order

$$\eta_1 = \alpha_1\xi_1$$

$$\alpha_1 = -\alpha_o + \left[\alpha_o^{\;2} + \tfrac{1}{2}\frac{C}{c_{11}}\,(\xi_o + \xi_1)\right]^{\frac{1}{2}} \tag{4.10}$$

Boundary condition $(4.6)_1$ reduces to

$$\frac{c_{44}}{c_{11}}\,\eta_2 = (\xi_o - \xi_1)\,\alpha_o - \alpha_1\xi_1 \tag{4.11}$$

where $(4.10)_1$ has been used. Finally, boundary condition $(4.6)_4$, with the use of (4.9), (4.10) and (4.11) becomes

$$-1 + \frac{\xi_1}{\xi_o}\left[1 + \frac{C\,\xi_o}{2c_{11}\alpha_o^2}\left(1 + \frac{\xi_1}{\xi_o}\right)\right]^{\frac{1}{2}} = 0 \tag{4.12}$$

This equation has only one acceptable real solution.

$$\frac{\xi_1}{\xi_o} = \tfrac{1}{2}\,D - \left\{\tfrac{1}{4}\,D^2 - D\right\}^{\frac{1}{2}}$$

$$\text{where } \frac{1}{D} = -\tfrac{1}{2}\,\frac{C}{c_{11}}\,\frac{\xi_o}{\alpha_o^2} \tag{4.13}$$

$$\text{and } 4 \leq D < \infty$$

The extreme case of D large leads to the linear elastic result, but the limit D = 4 gives

$$\frac{\xi_1}{\xi_o} = 2 \tag{4.14}$$

and the limiting amplitude-angle relation

$$\xi_o = -\frac{\alpha_o^2}{2C/c_{11}} \tag{4.15}$$

The results in (4.14) and (4.15) were previously given by Reid[6], but not (4.13). Note that (4.15) is similar to (4.5), but much stronger.

Equation (4.13) is plotted in Fig. 6. The limiting angle/amplitude relations, given by equation (4.5) for a simple wave and by (4.15) in the present case, are plotted in Fig. 7 for elastic constants $C/c_{11} = -8.5$. Fig. 6 shows the rapid change in reflection amplitude as the limiting angle is approached. Fig. 7 shows how large the limiting angle α_o may be even for relatively small strains. This point is discussed further in Reference 13.

Shear Waves

If the incident wave in Fig. 1 is a shear wave progressing into an unstressed region, the linear elastic wave speed is $c_2 = \sqrt{c_{44}/\rho}$, with $\underline{p}_o = \underline{t}_o$, and first order corrections v and ξ_o from (2.11) are given by

$$v = 0$$

$$-(c_{12} + c_{44})\,\xi_o + \frac{1}{8}\left\{C-c + 2\,(c_{12} + c_{44})\right\}\eta_o^2 = 0 \tag{4.16}$$

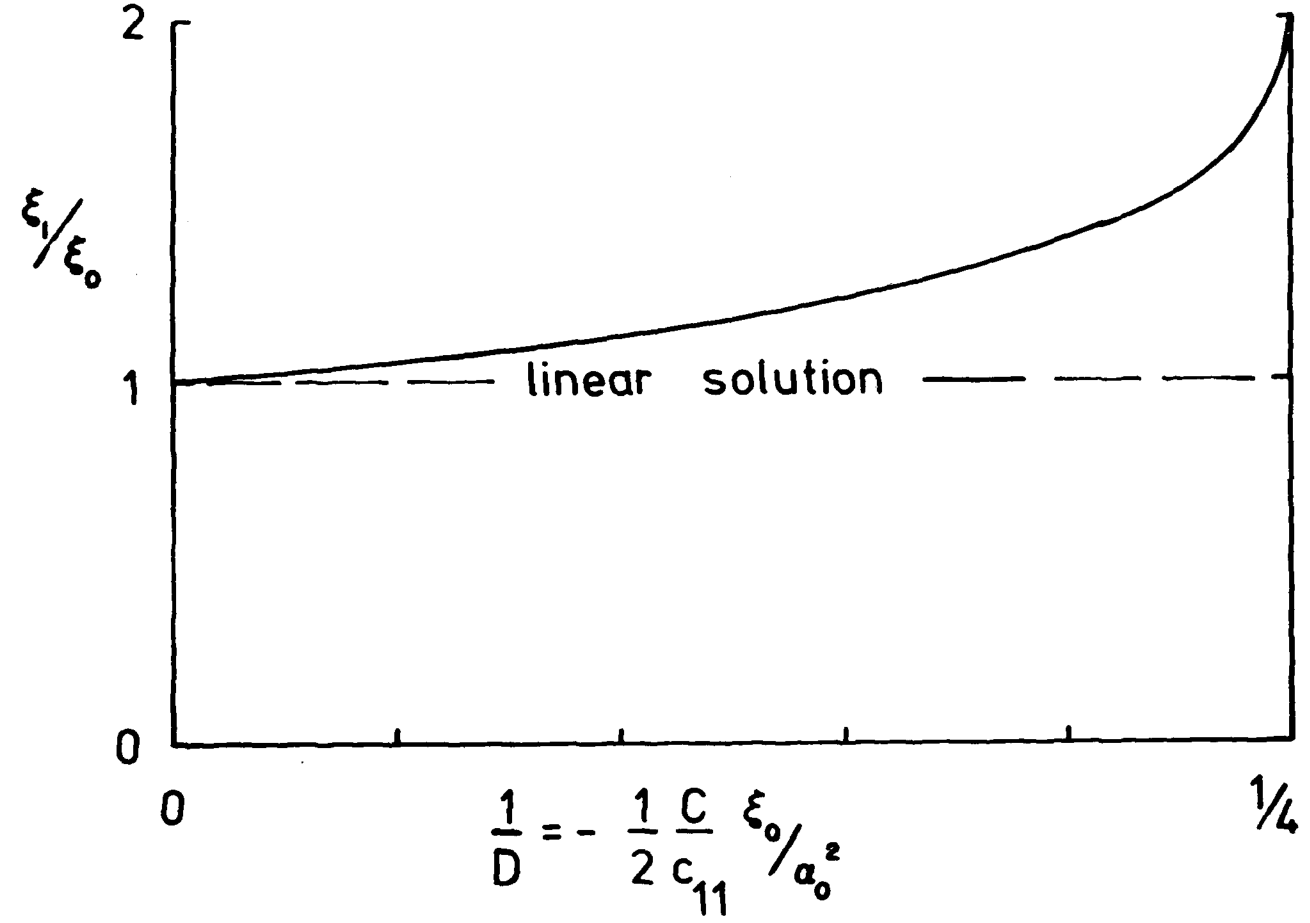

Figure 6. Reflected Shock Amplitude at a Rigid/Lubricated Boundary for an Incident Longitudinal Shock. (Symbols are explained in the text)

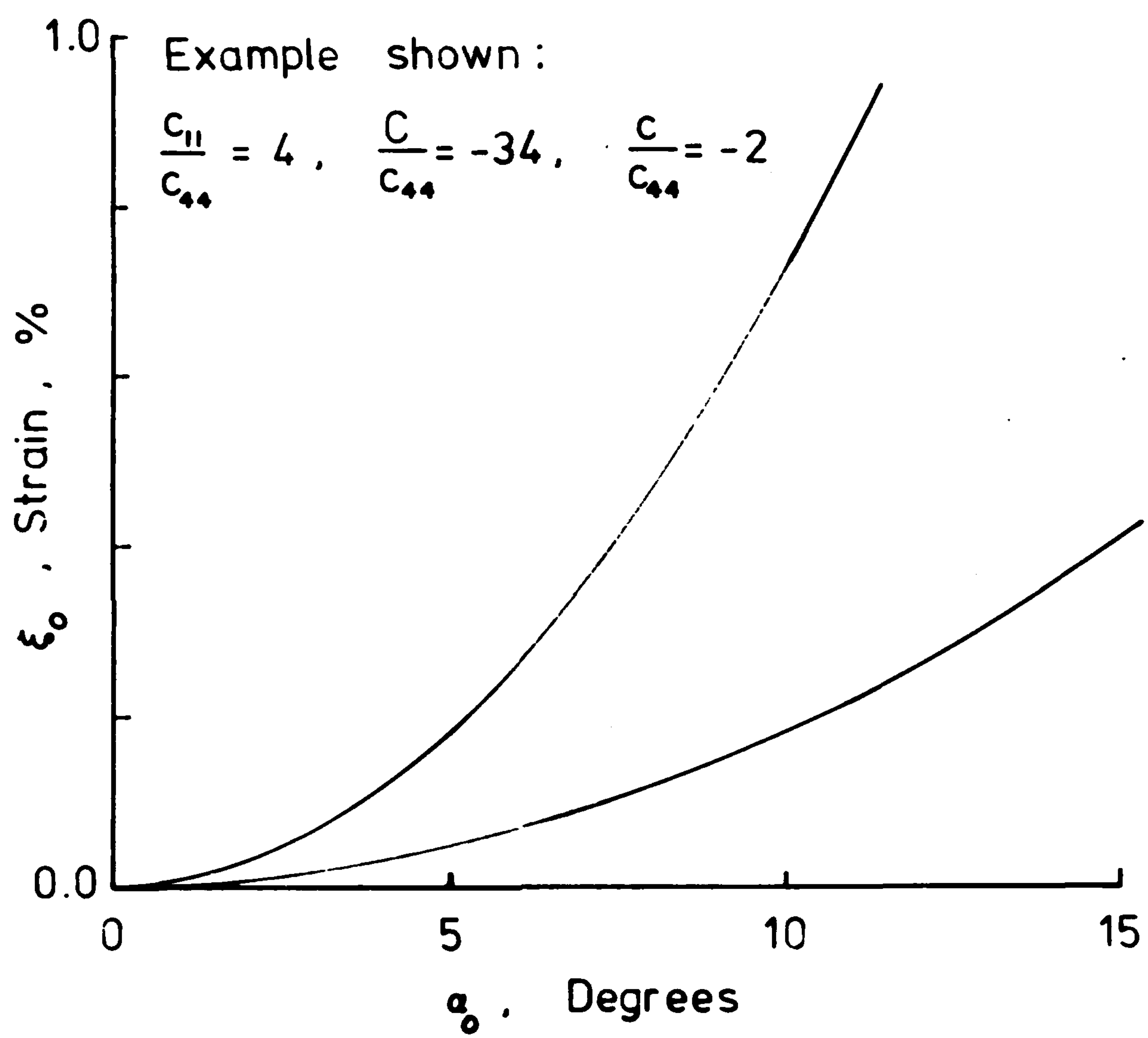

Figure 7. Limiting Angle - Amplitude Behavior for Longitudinal Shocks. Upper Curve Corresponds to a Reflected Simple Wave. Lower Curve Corresponds to a Reflected Shock from a Rigid/Lubricated Boundary. (Symbols are explained in the text)

Again, there are only two reflected waves with angles $\theta_2 = \pi - \theta_o$ and $\theta_1 = \pi - \bar{\theta}$ where $\bar{\theta} = \sin^{-1}\left(\frac{c_1}{c_2}\sin\theta_o\right)$. The linear critical angle occurs for $\bar{\theta} = \frac{\pi}{2}$ or $\theta_o = \sin^{-1}\frac{c_1}{c_2}$. Angular corrections α_1 and α_2 are found from $(2.15)_1$.

$$
\begin{aligned}
&2\, c_{11}\frac{c_2}{c_1}\frac{\cos\bar{\theta}}{\sin\theta_o}\alpha_1 - \tfrac{1}{2} C\,\xi_1 - \tfrac{1}{2}(C-c)\sin 2(\theta_1 - \theta_o)\eta_o = 0 \\
&2\, c_{44}\frac{\cos\theta_o}{\sin\theta_o}\alpha_2 - \tfrac{1}{4}\left[C-c + 2(c_{12} + c_{44})\right]\xi_1 = 0
\end{aligned}
\tag{4.17}
$$

Amplitude corrections η_1 and ξ_2 are found from $(2.15)_2$.

$$
\begin{aligned}
&(c_{12} + c_{44})(\eta_1 - \alpha_1\xi_1) \\
&\quad + \tfrac{1}{4}\left[(C-c)\cos 2(\theta_o + \bar{\theta}) + 2\,(c_{12} + c_{44})\right]\eta_o\xi_1 = 0 \\
&(c_{12} + c_{44})(\xi_2 + \alpha_2\eta_2) \\
&\quad - \tfrac{1}{4}\left[(C-c)\cos 4\,\theta_o + 2\,(c_{12} + c_{44})\right]\eta_o\eta_2 \\
&\quad + \tfrac{1}{4}\,(C-c)\sin 2(\bar{\theta} - \theta_o)\xi_1\eta_2 - \frac{1}{8}\left[C-c + 2(c_{12} + c_{44})\right]\eta_2^{\,2} = 0
\end{aligned}
\tag{4.18}
$$

As before ξ_1 and η_2 are to be found from the linear boundary value problem.

The equation for α_1 again fails near the limiting angle since the coefficient of α_1 in $(4.17)_1$ tends to zero. With $\theta_o = \sin^{-1}\frac{c_2}{c_1} - \alpha_o$ and $\theta_1 + \alpha_1 = \frac{\pi}{2} + \hat{\alpha}_1$, where α_o and $\hat{\alpha}_1$ are small angles, eqn. $(4.17)_1$ must be replaced by a quadratic equation, obtained from (2.16) as before. It is more convenient to work with $\hat{\alpha}_1$ rather than α_1. To be consistent the basis vectors for the first reflected wave now are $\underset{\sim}{p}_1 = \underset{\sim}{e}_x$ and $\underset{\sim}{q}_1 = \underset{\sim}{e}_y$. Eqns. (4.18) remain correct with $\bar{\theta} = \frac{\pi}{2}$ and ξ_1 and η_1 interpreted as components along the redefined $\underset{\sim}{p}_1$ and $\underset{\sim}{q}_1$.

$$
\begin{aligned}
&c_{11}\,\hat{\alpha}_1^{\,2} + (C-c)\,\frac{c_{12}}{c_{11}}\,\eta_o\hat{\alpha}_1 \\
&\quad -\tfrac{1}{2}\, C\,\xi_1 - (C-c)\,\frac{c_2}{c_1}\left(1-\frac{c_2^{\,2}}{c_1^{\,2}}\right)^{1/2}\eta_o - 2\,c_{11}\left(\frac{c_1^{\,2}}{c_2^{\,2}} - 1\right)^{1/2}\alpha_o = 0
\end{aligned}
\tag{4.19}
$$

The last term in (4.19) does not come from $(2.17)_3$, but rather from $(2.14)_1$ since the use of $\hat{\alpha}_1$ instead of α_1 leaves a second order term left over from $(2.14)_1$ that must be accounted for. Solution of (4.19) gives

$$\hat{\alpha}_1 = -\frac{C-c}{2c_{11}}\frac{c_{12}}{c_{11}}\eta_o + \left\{\left(\frac{C-c}{2c_{11}}\;\frac{c_{12}}{c_{11}}\right)^2 \eta_o^{\,2} + \left[\frac{C-c}{c_{11}}\frac{c_2}{c_1}\left(1-\frac{c_2^{\,2}}{c_1^{\,2}}\right)^{1/2} + \tfrac{1}{2}\frac{C}{c_{11}}\frac{\xi_1}{\eta_o}\right]\eta_o + 2\left(\frac{c_1^{\,2}}{c_2^{\,2}} - 1\right)^{1/2}\alpha_o\right\}^{1/2} \tag{4.20}$$

where the positive radical has been chosen so that $\hat{\alpha}_1$ has the correct value for fixed α_o with $\eta_o = 0$. To ensure that $\hat{\alpha}_1$ is real, the terms under the radical must be positive. As before, it is this condition that determines the limiting relationship between η_o and α_o. The ratio ξ_1/η_o must be obtained from the linear problem (or nonlinear problem) if ξ_1 denotes a shock wave or be set equal to zero if ξ_1 denotes a simple wave fan. It should be noted that in the development given here α_o is required to be positive since only then can ξ_1/η_o be found from the linear problem. If $\alpha_o < 0$, perhaps an estimate for a lower bound of η_o could be found in some cases by appealing directly to the nonlinear problem. Of the two values of η_o that make the radical be zero only the smaller one is physically reasonable.

$$\eta_o = -\frac{2\left(\dfrac{c_1^{\,2}}{c_2^{\,2}} - 1\right)^{1/2}\alpha_o}{\dfrac{C-c}{c_{11}}\;\dfrac{c_2}{c_1}\left(1-\dfrac{c_2^{\,2}}{c_1^{\,2}}\right)^{1/2} + \tfrac{1}{2}\dfrac{C}{c_{11}}\;\dfrac{\xi_1}{\eta_o}} \tag{4.21}$$

It is worth considering (4.20) and (4.21) further for clamped or free boundary conditions. In these cases it is readily found that the linear solution gives

$$\text{Clamped:}\qquad \frac{\xi_1}{\eta_o} = -\frac{\sin 2\theta_o}{\dfrac{c_1}{c_2}\cos(\bar{\theta} - \theta_o)} \tag{4.22}$$

$$\rightarrow -2\frac{c_2}{c_1}\left(1 - \frac{c_2^{\,2}}{c_1^{\,2}}\right)^{1/2} \quad \text{as } \theta_o \rightarrow \sin^{-1}\frac{c_2}{c_1}$$

$$\text{Free:} \quad \frac{\xi_1}{\eta_o} = \frac{\sin 4\Theta_o}{\dfrac{c_1^2}{c_2^2}\cos^2 2\Theta_o + \sin 2\Theta_o \sin 2\bar{\Theta}}$$

$$\rightarrow \frac{4\dfrac{c_2}{c_1}\left(1 - \dfrac{c_2^2}{c_1^2}\right)^{\frac{1}{2}}}{\dfrac{c_1^2}{c_2^2} - 2} \quad \text{as } \Theta_o \rightarrow \sin^{-1}\frac{c_2}{c_1} \tag{4.23}$$

Since shocks are compressive in most materials, for the present discussion it will be assumed that $C < c < 0$. For the clamped case if $\eta_o > 0$ near the limiting angle, then $\xi_1 < 0$ which indicates a simple wave. Then ξ_1/η_o must be set equal to zero in (4.21) and η_o has the limits

$$0 \leq \eta_o < -2\frac{c_{11}}{C-c}\frac{c_1^2}{c_2^2}\alpha_o \tag{4.24}$$

On the other hand, if $\eta_o < 0$, then $\xi_1 > 0$, which indicates a shock wave, and η_o has the limits

$$0 \geq \eta_o \cdot 2\frac{c_{11}}{c}\frac{c_1^2}{c_2^2}\alpha_o \tag{4.25}$$

If c is actually positive, rather than negative as assumed, then (4.25), or more properly (4.20), seems to indicate that α_o may be negative. For the free boundary case similar considerations apply. If $\eta_o > 0$, then $\xi_1 > 0$, indicating a shock, and η_o is limited by (4.21). But if $\eta_o < 0$, then $\xi_1 < 0$, indicating a simple wave, and (4.21) or (4.20) seem to indicate that α_o may be negative, i.e., that the limiting angle is greater than $\sin^{-1}\frac{c_2}{c_1}$. Fig. 8 shows an example of the limiting relations between η_o and α_o for these two boundary conditions.

The boundary determinants for the clamped and free boundary have the same limits as $(4.7)_1$ and $(4.7)_2$ so that second order solutions may be found by following the first five steps at the end of Section III.

The rigid-lubricated boundary again provides the most interesting case since the linear boundary determinant vanishes as in (4.8), and again it is necessary to use nonlinear boundary conditions right from the start. The formulas for shear traction and normal velocity behind the reflection pattern,

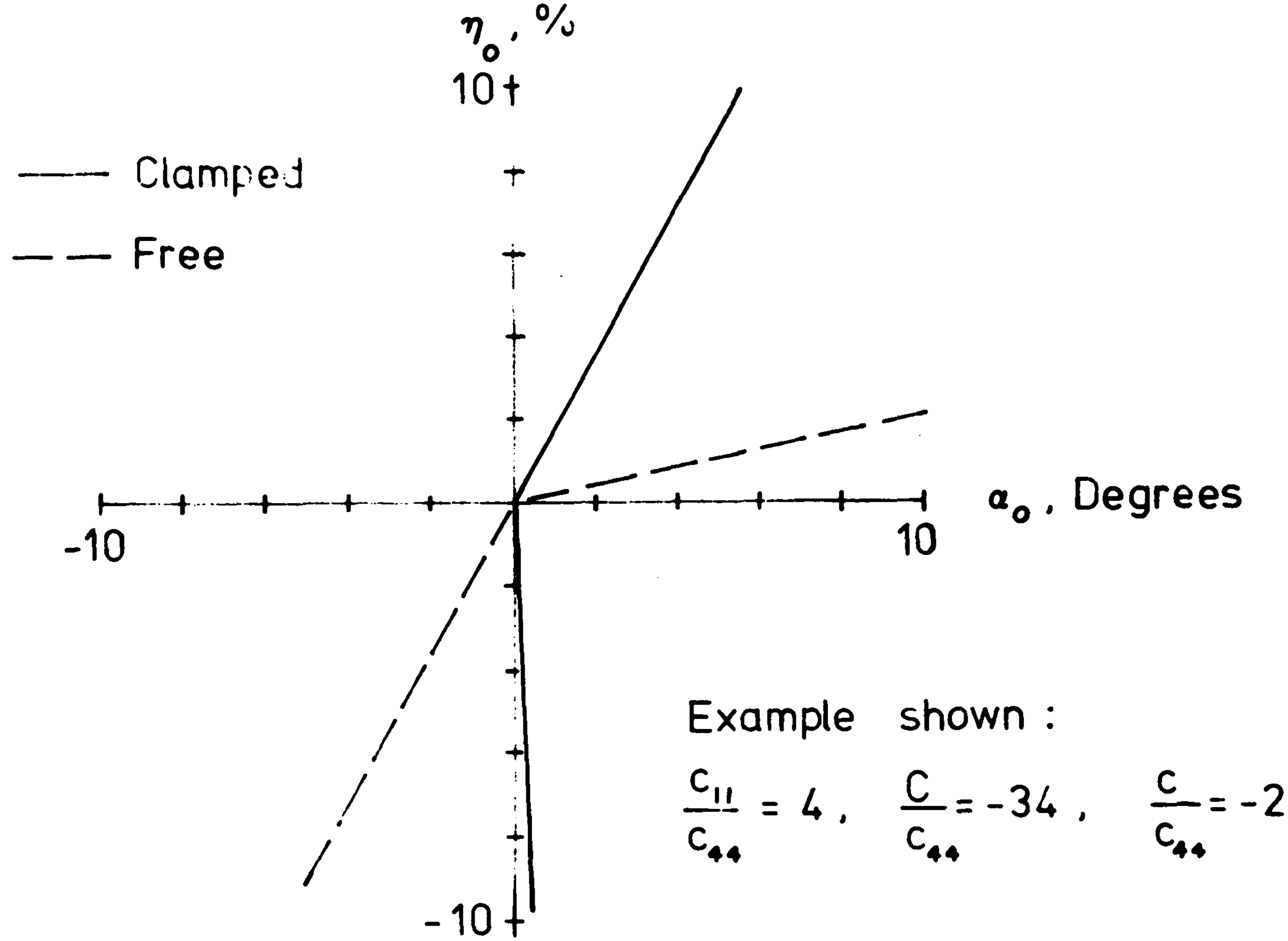

Figure 8. Limiting Angle - Amplitude Behavior for Incident Shear Waves, Points to the Right of the Limiting Lines are Allowed. (Symbols are explained in the text)

correct to second order in amplitudes are found, as before, by combining (2.5), (3.1) and (3.2) for isotropic materials.

$$\frac{1}{c_{44}} \underset{\sim}{t}_B \cdot \underset{\sim\sim}{TN}_B = 0$$

$$= -(\eta_o + \eta_2)\cos 2\theta_o - (\xi_o - \xi_2)\sin 2\theta_o + \eta_1 + \hat{\alpha}_1\,\xi_1$$

$$\frac{1}{V_h} \underset{\sim}{N}_B \cdot \underset{\sim}{u}_B = 0 \tag{4.26}$$

$$= -(\eta_o + \eta_2)\sin^2\theta_o + (\xi_o - \xi_2)\sin\theta_o\cos\theta_o$$

$$- \eta_1 + \alpha_2\,\eta_2\sin\theta_o\cos\theta_o$$

Since the linear problem has the solution $\xi_1/\eta_o = 0$, $\eta_2/\eta_o = -1$ for every incident angle, it seems reasonable to assume here that $\xi_1 = o(\eta_o)$ and $\eta_2 = O(\eta_o)$, i.e., $\lim_{\eta_o\to 0} \xi_1/\eta_o = 0$ and $\lim_{\eta_o\to 0} \eta_2/\eta_o$ = constant. With these estimates of magnitudes, it is easily seen from $(4.16)_2$, $(4.17)_2$, $(4.18)_1$ and $(4.18)_2$, respectively, that

$$\begin{aligned} \xi_o &= O(\eta_o^{\,2}) \\ \alpha_2 &= o(\eta_o) \\ \eta_1 &= \hat{\alpha}_1\,\xi_1 + o(\eta_o^{\,2}) \\ \xi_2 &= O(\eta_o^{\,2}) \end{aligned} \tag{4.27}$$

Since terms higher than quadratic have already been eliminated, (4.26) may be written

$$\begin{aligned} -(\eta_o + \eta_2)\cos 2\theta_o - (\xi_o - \xi_2)\sin 2\theta_o + 2\hat{\alpha}_1\xi_1 &= 0 \\ -(\eta_o + \eta_2)\sin^2\theta_o + \tfrac{1}{2}(\xi_o - \xi_2)\sin 2\theta_o - \hat{\alpha}_1\xi_1 &= 0 \end{aligned} \tag{4.28}$$

To second order we have

$$\begin{aligned} \eta_2 &= -\eta_o + o(\eta_o^{\,2}) \\ \hat{\alpha}_1\xi_1 &= (\xi_o - \xi_2)\sin\theta_o\cos\theta_o + o(\eta_o^{\,2}) \end{aligned} \tag{4.29}$$

Equation $(4.29)_1$ shows that the reflected shear wave has nearly the same amplitude as that calculated in the linear problem. However, $(4.29)_2$ leads to rather different results for ξ_1,

$$\left\{ \tfrac{1}{2} \frac{C}{c_{11}} \xi_1 + \frac{C-c}{c_{11}} \frac{c_2}{c_1} \left(1 - \frac{c_2^{\,2}}{c_1^{\,2}}\right)^{\frac{1}{2}} \eta_o + 2\left(\frac{c_1^{\,2}}{c_2^{\,2}} - 1\right)^{\frac{1}{2}} \alpha_o \right\}^{\frac{1}{2}} \xi_1$$

$$= \tfrac{1}{4} \eta_o^{\,2} \left[\frac{C-c}{c_{12} + c_{22}} \cos 4\Theta_c + 2\right] \frac{c_2}{c_1} \left(1 - \frac{c_2^{\,2}}{c_1^{\,2}}\right)^{\frac{1}{2}} \tag{4.30}$$

where $\Theta_c = \sin^{-1} \dfrac{c_2}{c_1}$. For purposes of graphical interpretation, (4.30) will be rewritten as

$$\left\{- A\, \xi_1 + B\right\}^{\frac{1}{2}} \xi_1 = R$$

Since the positive square root is always intended, ξ_1 has the same sign as R, which depends both on the magnitude of Θ_c and the sign of C-c. Thus ξ_1 must be a root of

$$f(\xi_1) = A\, \xi_1^{\,3} - B\, \xi_1^{\,2} + R^2$$

which is plotted in Fig. 9 for $A > 0$. In this case if $R > 0$, the reflected wave is a shock, so the limiting angle occurs when the minimum in $f(\xi_1)$ just touches the axis. The minimum lies at $\xi_1 = 2B/3A$ and the function vanishes there if $B = 3\, (\tfrac{1}{2}\, AR)^{2/3}$. Returning to the coefficients of (4.30), the limiting conditions are

$$\alpha_o = - \tfrac{1}{2} \frac{C-c}{c_{11}} \frac{c_2^{\,2}}{c_1^{\,2}} \eta_o + O\left(\eta_o^{\,4/3}\right)$$

$$\xi_1 = \frac{2\, \eta_o^{\,4/3}}{\left(- \tfrac{1}{2} \dfrac{C}{c_{11}}\right)^{1/3}} \left(\frac{c_2}{c_1}\right)^{2/3} \left(1 - \frac{c_2^{\,2}}{c_1^{\,2}}\right)^{1/3} \left[\frac{C-c}{c_{12} + c_{44}} \cos 4\Theta_c + 2\right]^{2/3} \tag{4.31}$$

On the other hand, for $A > 0$ but $R < 0$, the reflected wave is a simple wave. With $\xi_1 = 0$ for the leading edge of the wave in (4.20), the radical vanishes at

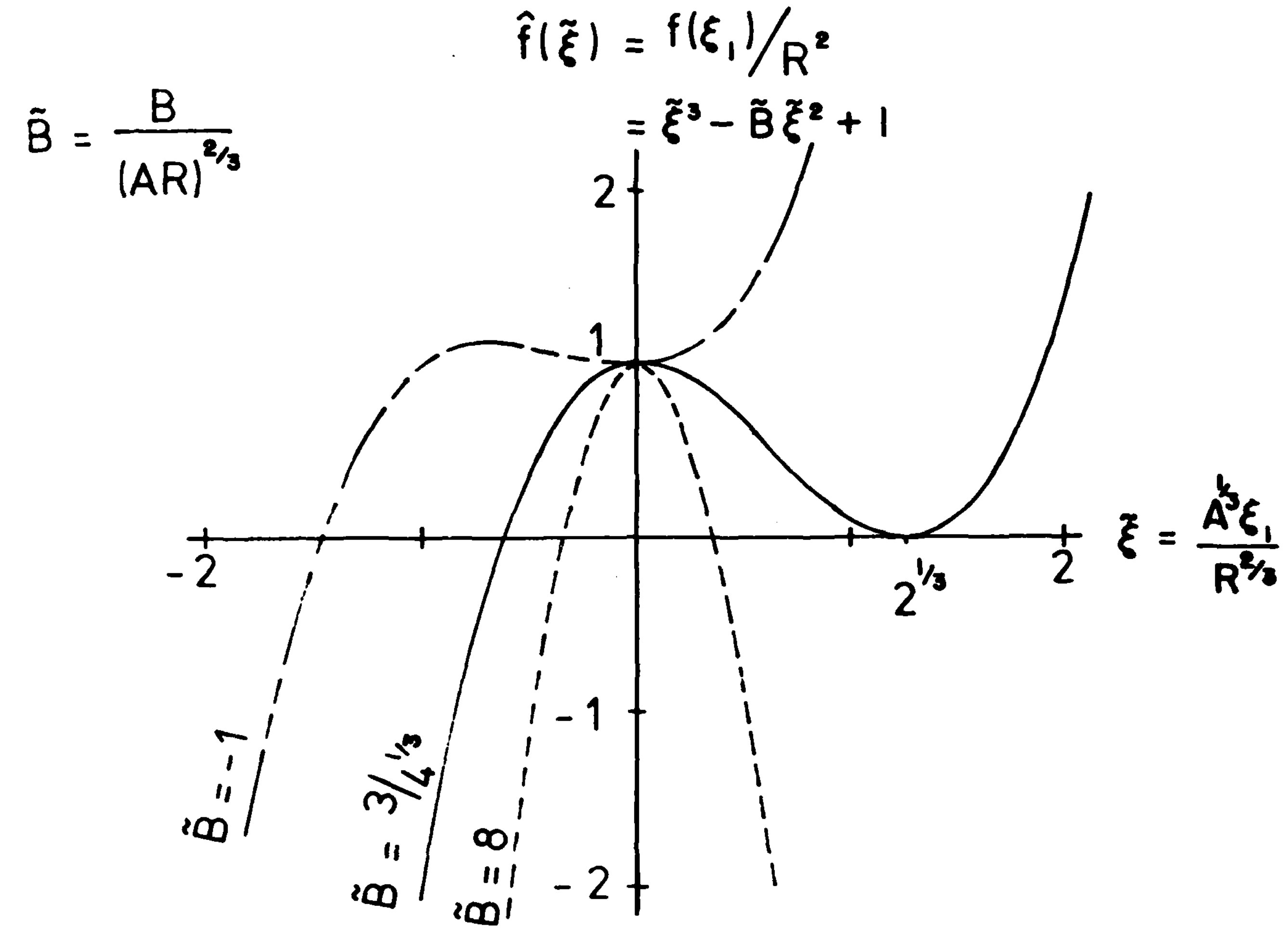

Figure 9. Plot of the Cubic $f(\xi_1) = A\xi_1^{\ 3} - B\xi_1^{\ 2} + R^2$ for Several Values of B. The Curves are Normalized to $A = R^2 = 1$. (Symbols are explained in the text)

$$\alpha_o = -\tfrac{1}{2}\,\frac{C-c}{c_{11}}\,\frac{c_2^{\,2}}{c_1^{\,2}}\,\eta_o + O\left(\eta_o^{\,2}\right) \qquad (4.32)$$

The amplitude, as calculated by the shock wave approximation from (4.30), is given by

$$\xi_1 = -\frac{\eta_o^{\,4/3}}{2^{4/3}\left(-\tfrac{1}{2}\,\dfrac{C}{c_{11}}\right)^{1/3}}\left(\frac{c_2}{c_1}\right)^{2/3}\left(1-\frac{c_2^{\,2}}{c_1^{\,2}}\right)^{1/3}\left[\frac{C-c}{c_{12}+c_{44}}\cos 4\Theta_c + 2\right] \qquad (4.33)$$

which is smaller than $(4.31)_2$ by a factor of $4\cdot 2^{1/3}$.

The argument for the case $A < 0$ follows the same pattern as for $A > 0$, but with $-\xi_1$ replacing ξ_1 at every step, since for $A < 0$, Fig. 9 becomes its own mirror image by reflection across the axis $\xi_1 = 0$.

REFERENCES

1. T. W. Wright, Reflection of Oblique Shock Waves in Elastic Solids, Int. J. Solids Structures, 7, 161-181 (1971).

2. T. W. Wright, Uniqueness of Shock Reflection Patterns in Elastic Solids, Arch. Rat. Mech. Anal., 42, 115-127 (1971).

3. T. W. Wright, A Note on Oblique Reflections in Elastic Crystals, Quart. J. Mech. Appl. Math., 29, 15-24 (1976).

4. G. Duvaut, Phénomenes De Réflexion, Réfraction, Intersection d'Ondes Planes Uniformes dans des Matériaux Élastiques Non Linéaires, C. R. Acad. Sc. Paris, Série A, 264, 883-886 (1967).

5. G. Duvaut, Ondes dans des Matériaux de Type Harmonique. Réflexion Oblique d'une Onde de Choc Plane Longitudinale sur une Paroi Fixe, C. R. Acad. Sci. Paris, Serie A, 266, 246-249 (1968).

6. S. R. Reid, The Influence of Nonlinearity Upon the Reflection of Finite Amplitude Shock Waves in Elastodynamics, Quart. J. Mech. Appl. Math., 25, 185-206 (1972).

7. P. Lax, Hyperbolic Systems of Conservation Laws II, Comm. Pure Appl. Math., 10, 537-566 (1957).

8. L. Davison, Propagation of Plane Waves of Finite Amplitude in Elastic Solids, J. Mech. Phys. Sol., 14, 249-270 (1966).

9. A. S. Abou-Sayed and R. J. Clifton, Analysis of Combined Pressure-Shear Waves in an Elastic/Viscoplastic Material, J. Appl. Mech., 44, 79-84 (1977).

10. C. Truesdell and R. Toupin, The Classical Field Theories, Flügge's Handbuch der Physik III/1, Springer, Berlin-Gottingen-Heidelberg (1960).

11. E. Varley, Simple Waves in General Elastic Materials, Arch. Rat. Mech. Anal., 20 309-328 (1965).

12. M. J. P. Musgrave, Crystal Acoustics, Holden-Day, San Francisco (1970).

13. T. W. Wright, Oblique Reflections, in Propagation of Shock Waves in Solids, ed. E. Varley, AMD Vol. 17, ASME, New York (1976).

14. K. Brugger, Thermodynamic Definition of Higher Order Elastic Coefficients, Phys. Rev., 133, A1611-A1612 (1964).

15. R. Bechman and R. F. S. Hearman, The Third Order Elastic Constants, in Landolt-Börnstein, Numerical Data and Functional Relationships in Science and Technology, New Series, Group III, Vol. 2, ed. K.-H. Hellwege, Springer-Verlag, Berlin, Heidelberg, New York (1969).

APPENDIX A

NEARLY EQUAL WAVE SPEEDS

If $V_p = V_q$ in (2.11) or even if they are near to each other, i.e. $|V_p - V_q|/V_q \ll 1$, then $(2.11)_2$ fails. In that case η must be assumed to be a first order term. With ξ and $\underset{\sim}{F}^+ - \underset{\sim}{1}$ still arbitrary first order terms, it is necessary to find v and η from simultaneous quadratic equations.

$$
\begin{aligned}
&2\rho V_p^{\,2}\, v\xi - \underset{\sim}{C}_3(\underset{\sim}{p}\otimes\underset{\sim}{N})^2(\underset{\sim}{F}^+-\underset{\sim}{1})\xi - \underset{\sim}{C}_3(\underset{\sim}{p}\otimes\underset{\sim}{N})(\underset{\sim}{q}\otimes\underset{\sim}{N})(\underset{\sim}{F}^+-\underset{\sim}{1})\eta \\
&\quad + \tfrac{1}{2}\,\underset{\sim}{C}_3(\underset{\sim}{p}\otimes\underset{\sim}{N})^3\,\xi^2 + \underset{\sim}{C}_3(\underset{\sim}{p}\otimes\underset{\sim}{N})^2(\underset{\sim}{q}\otimes\underset{\sim}{N})\xi\eta \\
&\quad + \tfrac{1}{2}\,\underset{\sim}{C}_3(\underset{\sim}{p}\otimes\underset{\sim}{N})(\underset{\sim}{q}\otimes\underset{\sim}{N})^2\,\eta^2 = 0 \\
&\rho(V_p^{\,2} - V_q^{\,2})\eta + 2\rho V_p^2 v\eta - \underset{\sim}{C}_3(\underset{\sim}{q}\otimes\underset{\sim}{N})^2(\underset{\sim}{F}^+-\underset{\sim}{1})\eta \\
&\quad - \underset{\sim}{C}_3(\underset{\sim}{p}\otimes\underset{\sim}{N})(\underset{\sim}{q}\otimes\underset{\sim}{N})(\underset{\sim}{F}^+-\underset{\sim}{1})\xi + \tfrac{1}{2}\,\underset{\sim}{C}_3(\underset{\sim}{p}\otimes\underset{\sim}{N})^2(\underset{\sim}{q}\otimes\underset{\sim}{N})\xi^2 \\
&\quad + \underset{\sim}{C}_3(\underset{\sim}{p}\otimes\underset{\sim}{N})(\underset{\sim}{q}\otimes\underset{\sim}{N})^2\xi\eta + \tfrac{1}{2}\,\underset{\sim}{C}_3(\underset{\sim}{q}\otimes\underset{\sim}{N})^3\eta^2 = 0
\end{aligned}
\tag{A.1}
$$

The equation for ζ remains unaltered. Equations (2.11) are specializations of (A.1) with η a second order quantity and terms of third or fourth order discarded.

For the case of nearly equal speeds in (2.15) it is necessary to assume that η is of the same order as ξ, and η and α must be found from the following quadratic equations where ξ, v and $\underset{\sim}{F}^+ - \underset{\sim}{1}$ are taken to be known.

$$- 2\rho V_p(\underset{\sim}{b}_p \cdot \underset{\sim}{t} - V_{h_o} \cos\Theta)\alpha\xi + 2\rho V_p^{\,2} v\xi$$

$$- \underset{\sim}{C}_2 \left\{(\underset{\sim}{p}\underset{\sim}{N})(\underset{\sim}{q}\underset{\sim}{t}) + (\underset{\sim}{p}\underset{\sim}{t})(\underset{\sim}{q}\underset{\sim}{N})\right\}\alpha\eta$$

$$- \underset{\sim}{C}_3(\underset{\sim}{p}\underset{\sim}{N})^2(\underset{\sim}{F}^+-\underset{\sim}{1})\xi - \underset{\sim}{C}_3(\underset{\sim}{p}\underset{\sim}{N})(\underset{\sim}{q}\underset{\sim}{N})(\underset{\sim}{F}^+-\underset{\sim}{1})\eta$$

$$+ \tfrac{1}{2} \underset{\sim}{C}_3(\underset{\sim}{p}\underset{\sim}{N})^3\xi^2 + \underset{\sim}{C}_3(\underset{\sim}{p}\underset{\sim}{N})^2(\underset{\sim}{q}\underset{\sim}{N})\xi\eta$$

$$+ \tfrac{1}{2} \underset{\sim}{C}_3(\underset{\sim}{p}\underset{\sim}{N})(\underset{\sim}{q}\underset{\sim}{N})^2\eta^2 = 0$$

$$\rho(V_p^{\,2} - V_q^{\,2})\eta - 2\rho(V_q\underset{\sim}{b}_q \cdot \underset{\sim}{t} - V_pV_h \cos\Theta)\alpha\eta + 2\rho V_p^{\,2} v\eta$$

$$- \underset{\sim}{C}_2 \left\{(\underset{\sim}{p}\underset{\sim}{N})(\underset{\sim}{q}\underset{\sim}{t}) + (\underset{\sim}{p}\underset{\sim}{t})(\underset{\sim}{q}\underset{\sim}{N})\right\}\alpha\xi$$

$$- \underset{\sim}{C}_3(\underset{\sim}{p}\underset{\sim}{N})(\underset{\sim}{q}\underset{\sim}{N})(\underset{\sim}{F}^+-\underset{\sim}{1})\xi - \underset{\sim}{C}_3(\underset{\sim}{q}\underset{\sim}{N})^2(\underset{\sim}{F}^+-\underset{\sim}{1})\eta \qquad \text{(A.2)}$$

$$+ \tfrac{1}{2} \underset{\sim}{C}_3(\underset{\sim}{p}\underset{\sim}{N})^2(\underset{\sim}{q}\underset{\sim}{N})\xi^2 + \underset{\sim}{C}_3(\underset{\sim}{p}\underset{\sim}{N})(\underset{\sim}{q}\underset{\sim}{N})^2\xi\eta$$

$$+ \tfrac{1}{2} \underset{\sim}{C}_3(\underset{\sim}{q}\underset{\sim}{N})^3 \eta^2 = 0$$

$$\rho(V_p^{\,2} - V_r^{\,2})\zeta - \underset{\sim}{C}_2 \left[(\underset{\sim}{p}\underset{\sim}{N})(\underset{\sim}{r}\underset{\sim}{t}) + (\underset{\sim}{p}\underset{\sim}{t})(\underset{\sim}{r}\underset{\sim}{N})\right]\alpha\xi$$

$$- \underset{\sim}{C}_2 \left[(\underset{\sim}{q}\underset{\sim}{N})(\underset{\sim}{r}\underset{\sim}{t}) + (\underset{\sim}{q}\underset{\sim}{t})(\underset{\sim}{r}\underset{\sim}{N})\right]\alpha\eta$$

$$- \underset{\sim}{C}_3(\underset{\sim}{p}\underset{\sim}{N})(\underset{\sim}{r}\underset{\sim}{N})(\underset{\sim}{F}^+-\underset{\sim}{1})\xi - \underset{\sim}{C}_3(\underset{\sim}{q}\underset{\sim}{N})(\underset{\sim}{r}\underset{\sim}{N})\eta$$

$$+ \tfrac{1}{2} C_3(\underset{\sim}{p}\underset{\sim}{N})^2(\underset{\sim}{r}\underset{\sim}{N})\xi^{\,2} + \underset{\sim}{C}_3(\underset{\sim}{p}\underset{\sim}{N})(\underset{\sim}{q}\underset{\sim}{N})(\underset{\sim}{r}\underset{\sim}{N})\xi\eta$$

$$+ \tfrac{1}{2} \underset{\sim}{C}_3(\underset{\sim}{q}\underset{\sim}{N})^2(\underset{\sim}{r}\underset{\sim}{N})\eta^{\,2} = 0$$

The vector $\underset{\sim}{b}_p$ is the same as $\underset{\sim}{b}$ in the formula following (2.15), and $\underset{\sim}{b}_q$ is computed from the same formula with the permutations $p \leftrightarrow q$, $\underset{\sim}{p} \leftrightarrow \underset{\sim}{q}$. Equations (A.2) reduce to (2.15) if η is a second order term and terms higher than second are dropped.

In case $|V_p - V_q|/V_q \ll 1$ in (2.22) then η and ξ are quantities of comparable magnitude, and both (2.21) and (2.22) must be replaced. Since $\underset{\sim}{m}$ is a unit vector, the condition $\xi \doteq 1$ must be replaced by

$$\xi^2 + \eta^2 \doteq 1 \tag{A.3}$$

which is correct to second order. The lowest order terms in (2.20) now are given by $\mathbf{F} = \mathbf{F}^+ - \mathbf{p}\otimes\mathbf{N}\,\xi\gamma - \mathbf{q}\otimes\mathbf{N}\,\eta\gamma$, and by the same process used to obtain (2.24) we have

$$\begin{aligned}
&2\rho V_p(\mathbf{b}\cdot\mathbf{t} - V_h\cos\theta)\alpha\xi - 2\rho V_p^{\,2}\,v\xi \\
&\quad + \mathbf{C}_2\left\{(\mathbf{p}\otimes\mathbf{N})(\mathbf{q}\otimes\mathbf{t}) + (\mathbf{p}\otimes\mathbf{t})(\mathbf{q}\otimes\mathbf{N})\right\}\alpha\eta \\
&\quad + \mathbf{C}_3\,(\mathbf{p}\otimes\mathbf{N})^2(\mathbf{F}^+-\mathbf{1})\xi + \mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})(\mathbf{q}\otimes\mathbf{N})(\mathbf{F}^+-\mathbf{1})\eta \\
&\quad - \mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})^3\,\xi^2\gamma - 2\,\mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})^2(\mathbf{q}\otimes\mathbf{N})\xi\eta\gamma \\
&\quad - \mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})(\mathbf{q}\otimes\mathbf{N})^2\,\eta^2\gamma = 0
\end{aligned}$$

$$\begin{aligned}
&\rho(V_q^{\,2} - V_p^{\,2})\eta + 2\rho(V_q\mathbf{b}_q\cdot\mathbf{t} - V_pV_h\cos\theta)\alpha\eta - 2\rho V_p^{\,2}\,v\eta \\
&\quad + \mathbf{C}_2\left\{(\mathbf{p}\otimes\mathbf{N})(\mathbf{q}\otimes\mathbf{t}) + (\mathbf{p}\otimes\mathbf{t})(\mathbf{q}\otimes\mathbf{N})\right\}\alpha\xi \\
&\quad + \mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})(\mathbf{q}\otimes\mathbf{N})(\mathbf{F}^+-\mathbf{1})\xi + \mathbf{C}_3(\mathbf{q}\otimes\mathbf{N})^2(\mathbf{F}^+-\mathbf{1})\eta \\
&\quad - \mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})^2(\mathbf{q}\otimes\mathbf{N})\,\xi^2\gamma - 2\,\mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})(\mathbf{q}\otimes\mathbf{N})^2\xi\eta\gamma \\
&\quad - \mathbf{C}_3(\mathbf{q}\otimes\mathbf{N})^3\,\eta^2\gamma = 0
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
&\rho(V_r^{\,2} - V_p^{\,2})\zeta + \mathbf{C}_2\left\{(\mathbf{p}\otimes\mathbf{N})(\mathbf{r}\otimes\mathbf{t}) + (\mathbf{p}\otimes\mathbf{t})(\mathbf{r}\otimes\mathbf{N})\right\}\alpha\xi \\
&\quad + \mathbf{C}_2\left\{(\mathbf{q}\otimes\mathbf{N})(\mathbf{r}\otimes\mathbf{t}) + (\mathbf{q}\otimes\mathbf{t})(\mathbf{r}\otimes\mathbf{N})\right\}\alpha\eta \\
&\quad + \mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})(\mathbf{r}\otimes\mathbf{N})(\mathbf{F}^+-\mathbf{1})\xi + \mathbf{C}_3(\mathbf{q}\otimes\mathbf{N})(\mathbf{r}\otimes\mathbf{N})(\mathbf{F}^+-\mathbf{1})\eta \\
&\quad - \mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})^2(\mathbf{r}\otimes\mathbf{N})\xi^2\gamma - 2\,\mathbf{C}_3(\mathbf{p}\otimes\mathbf{N})(\mathbf{q}\otimes\mathbf{N})(\mathbf{r}\otimes\mathbf{N})\xi\eta\gamma \\
&\quad - \mathbf{C}_3(\mathbf{q}\otimes\mathbf{N})^2(\mathbf{r}\otimes\mathbf{N})\eta^2\gamma = 0
\end{aligned}$$

The interpretation of $\underset{\sim}{b}_p$ and $\underset{\sim}{b}_q$ is the same as in (A.2). Clearly equations (A.4) reduce to (2.22) when $|V_p - V_q|/V_p$ is not small, $\xi \doteq 1$, and η is small.

Equations (A.3) and (A.4) are to be solved for α, ξ, η, ζ as functions of γ. Comparison of (A.4) with (A.2) shows that, for purposes of calculation, a simple wave may again be replaced by a shock wave. To see that this is true, first note that ξ, η, ζ in (A.2) are components of $\underset{\sim}{a}$, which is not a unit vector. To make them comparable with the terms of (A.4) we replace ξ, η, ζ in (A.2) by $\xi\gamma_m$, $\eta\gamma_m$, $\zeta\gamma_m$ where $\gamma_m = |\underset{\sim}{a}|$. Next consider a power series solution of (A.4), $\xi = \xi_o + \xi_o'\gamma + \cdots$, $\eta = \eta_o + \eta_o'\gamma + \cdots$, $\alpha = \alpha_o + \alpha_o'\gamma + \cdots$. The terms ξ_o, η_o, α_o are obtained from a quadratic set of equations, and ξ_o', η_o', α_o' from a linear set. These results, when inserted into (2.20), upon integration give $\underset{\sim}{F}(\gamma_m)$ up to terms of second order in small quantities. Similarly, a power series solution of the revised form of (A.2) may be sought. The leading terms are the same as those obtained from (2.20), but each of the second terms is equal to one-half the value previously obtained. Calculation of $\underset{\sim}{F}(\bar{\gamma}_m)$ from $(2.3)_2$, however, gives the same result to second order as that obtained from the assumption of a simple wave. It may be concluded for this case as before, that a simple wave may be replaced by a bisecting shock wave for second order calculations.

APPENDIX B

ELASTIC MODULI

In the literature on ultrasonic wave propagation, it is commonly assumed that the strain energy has the following expansion with respect to an unstressed reference configuration.[14]

$$W\left(x^i_{,\alpha}\right) = \frac{1}{2} c_{\alpha\beta\gamma\delta}\, \eta_{\alpha\beta}\, \eta_{\gamma\delta} + \frac{1}{6} c_{\alpha\beta\gamma\delta\mu\nu}\, \eta_{\alpha\beta}\, \eta_{\gamma\delta}\, \eta_{\mu\nu} + \cdots \qquad (B.1)$$

Where $\eta_{\alpha\beta} \equiv \frac{1}{2}\left(x^i_{,\alpha}\, x^i_{,\beta} - \delta_{\alpha\beta}\right)$ is termed nonlinear strain, and $c_{\alpha\beta\gamma\delta}$ and $c_{\alpha\beta\gamma\delta\mu\nu}$ are termed, respectively, second and third order tensors of elastic moduli. Because of symmetries, the values of the individual moduli are usually reported[15] as entries in the totally symmetric arrays c_{ab} and c_{abc}, where a, b, c = 1, 2, ···, 6. Each Latin subscript corresponds to a pair of Greek subscripts; 1 ↔ 11, 2 ↔ 22, 3 ↔ 33, 4 ↔ 23 or 32, 5 ↔ 13 or 31, 6 ↔ 12 or 21. Thus, $c_{11} = c_{1111}$, $c_{23} = c_{2233}$, $c_{45} = c_{2313}$, $c_{155} = c_{111313}$, etc.

The elasticities for use in this paper may easily be calculated from (B-1).

$$C_{i\alpha j\beta} = x_{i,\sigma}\, x_{j,\rho}\left(c_{\alpha\sigma\beta\rho} + c_{\alpha\sigma\beta\rho\mu\nu}\, \eta_{\mu\nu}\right)$$

$$+ \delta_{ij}\left(c_{\alpha\beta\mu\nu}\, \eta_{\mu\nu} + \frac{1}{2} c_{\alpha\beta\mu\nu\rho\sigma}\, \eta_{\mu\nu}\, \eta_{\rho\sigma}\right) + \cdots$$

$$C_{i\alpha j\beta k\gamma} = x_{i,\sigma}\, x_{j,\rho}\, x_{k,\tau}\, c_{\alpha\sigma\beta\rho\gamma\tau} \qquad (B.2)$$

$$+ \delta_{ik}\, x_{j,\sigma}\, c_{\alpha\gamma\beta\sigma} + \delta_{jk}\, x_{i,\sigma}\, c_{\beta\gamma\alpha\sigma}$$

$$+ \delta_{ij}\, x_{k,\sigma}\, c_{\alpha\beta\gamma\sigma} + \cdots .$$

If the region ahead of the incident shock wave is unstressed, then in (B-2) $x_{i,\alpha} = \delta_{i\alpha}$ and $\eta_{\alpha\beta} = 0$. The elasticities reduce to

---

[14] *K. Brugger, Thermodynamic Definition of Higher Order Elastic Coefficients, Phys. Rev., 133, A1611-A1612 (1964).*

[15] *R. Bechman and R. F. S. Hearman, The Third Order Elastic Constants, in Landolt-Börnstein, Numerical Data and Functional Relationships in Science and Technology, New Series, Group III, Vol. 2, ed. K.-H. Hellwege, Springer-Verlag, Berlin, Heidelberg, New York (1969).*

$$C_{i\alpha j\beta} = \delta_{i\sigma}\,\delta_{j\rho}\,c_{\alpha\sigma\beta\rho}$$

$$\begin{aligned} C_{i\alpha j\beta k\gamma} &= \delta_{i\sigma}\,\delta_{j\rho}\,\delta_{k\tau}\,c_{\alpha\sigma\beta\rho\gamma\tau} \\ &\quad + \delta_{ik}\,\delta_{j\sigma}\,c_{\alpha\gamma\beta\sigma} + \delta_{jk}\,\delta_{i\sigma}\,c_{\beta\gamma\alpha\sigma} \\ &\quad + \delta_{ij}\,\delta_{k\sigma}\,c_{\alpha\beta\gamma\sigma}\,. \end{aligned} \tag{B-3}$$

The reduced notation may also be used for $\underset{\sim}{C}_2$ and $\underset{\sim}{C}_3$. Thus, $C_{111} = C_{11\ 11\ 11}$, $C_{112} = C_{11\ 11\ 22}$, etc.

For the isotropic case, there are only two independent second order moduli and three independent third order moduli[15] and then (B-3) becomes

$$C_{i\alpha j\beta} = c_{12}\,\delta_{i\alpha}\,\delta_{j\beta} + c_{44}\left(\delta_{ij}\,\delta_{\alpha\beta} + \delta_{i\beta}\,\delta_{j\alpha}\right)$$

$$\begin{aligned} C_{i\alpha j\beta k\gamma} &= c_{123}\,\delta_{i\alpha}\,\delta_{j\beta}\,\delta_{k\gamma} + c_{144}\left(\delta_{i\alpha}\,\delta_{j\gamma}\,\delta_{k\beta} + \delta_{i\gamma}\,\delta_{j\beta}\,\delta_{k\alpha} + \delta_{i\beta}\,\delta_{j\alpha}\,\delta_{k\gamma}\right) \\ &\quad + \left(c_{12} + c_{144}\right)\left(\delta_{i\alpha}\,\delta_{jk}\,\delta_{\beta\gamma} + \delta_{ik}\,\delta_{j\beta}\,\delta_{\alpha\gamma} + \delta_{ij}\,\delta_{k\gamma}\,\delta_{\alpha\beta}\right) \\ &\quad + \tfrac{1}{2}\left(c_{155} - c_{144} + 2\,c_{44}\right)\left(\delta_{i\beta}\,\delta_{j\gamma}\,\delta_{k\alpha} + \delta_{i\gamma}\,\delta_{j\alpha}\,\delta_{k\beta}\right. \\ &\quad + \delta_{ij}\,\delta_{k\alpha}\,\delta_{\beta\gamma} + \delta_{ij}\,\delta_{k\beta}\,\delta_{\alpha\gamma} + \delta_{ik}\,\delta_{j\alpha}\,\delta_{\beta\gamma} + \delta_{i\beta}\,\delta_{jk}\,\delta_{\alpha\gamma} \\ &\quad \left. + \delta_{ik}\,\delta_{j\gamma}\,\delta_{\alpha\beta} + \delta_{i\gamma}\,\delta_{jk}\,\delta_{\alpha\beta}\right) - c_{44}\left(\delta_{i\beta}\,\delta_{j\gamma}\,\delta_{k\alpha} + \delta_{i\gamma}\,\delta_{j\alpha}\,\delta_{k\beta}\right). \end{aligned} \tag{B-4}$$

For isotropic materials we also have the identities

$$c_{44} = \tfrac{1}{2}\left(c_{11} - c_{12}\right),\ c_{144} = \tfrac{1}{2}\left(c_{112} - c_{123}\right),\ c_{155} = \tfrac{1}{4}\left(c_{111} - c_{112}\right).$$

In this paper the third order moduli most commonly used are

$$\begin{aligned} C &= C_{111} = c_{111} + 3\,c_{11} \\ c &= C_{112} = c_{112} + c_{12} \end{aligned} \tag{B-5}$$

LIST OF SYMBOLS

A, B, C	Polynominal coefficients in eqn. (2.16)
$\underset{\sim}{a}$	Shock amplitude vector, defined by $(2.3)_3$
B_m, B	Boundary equations, boundary matrix
$\underset{\sim}{b}$, $\underset{\sim}{b}_p$, $\underset{\sim}{b}_q$	Propagation vectors, defined by the formula following (2.15)
$\underset{\sim}{C}_2$, $\hat{\underset{\sim}{C}}_2$, $\underset{\sim}{C}_{i\alpha j\beta}$	Second order elastic moduli
$\underset{\sim}{C}_3$, $C_{i\alpha j\beta k\gamma}$	Third order elastic moduli
C, c	Isotropic moduli defined in eqn. (B-5)
$c_{\alpha\beta\gamma\delta}$, $c_{\alpha\beta\gamma\delta\mu\nu}$	Second and third order moduli in Brugger's notation
c_{ab}, c_{abc}	Isotropic moduli in Voight's notation
c_1, c_2	Elastic wave speeds in isotropic materials
D	Defined in eqn. (4.13)
$\underset{\sim}{e}_x$, $\underset{\sim}{e}_y$	Unit vectors along X, Y axes
$\underset{\sim}{F}$, $F_{j\beta}$	Deformation gradient
$\underset{\sim}{F}_B$, $\underset{\sim}{F}_I$	Deformation gradients at the boundary
$\underset{\sim}{m}$	Unit proper vector in simple wave
$\underset{\sim}{N}$, $\underset{\sim}{N}^o$	Unit normal vector
$O(\cdot)$, $o(\cdot)$	Order of magnitude symbols
$\underset{\sim}{p}$, $\underset{\sim}{q}$, $\underset{\sim}{r}$	Unit proper vectors for linear wave
$\underset{\sim}{Q}$	Acoustic tensor
$\underset{\sim}{T}$, $\underset{\sim}{T}_{i\alpha}$	Stress tensor
$\underset{\sim}{t}$, $\underset{\sim}{t}^o$	Unit tangent vector
t	Time
$\underset{\sim}{u}$	Particle velocity
V, V_o	Wave speed
V_p, V_q, V_r	Linear elastic wave speeds, defined as in (2.6)
V_h	Steady speed parallel to the boundary

v	Perturbation of shock speed
W	Strain energy function
$\underset{\sim}{X}$	Material coordinate
$\underset{\sim}{x}$	Spatial coordinate
$x_{i,\alpha}$	Deformation gradient
α, α_m, $\bar{\alpha}$, $\hat{\alpha}$	Small angles
γ, γ_m	Simple wave parameter
Δ_i	Variation across i^{th} wave
$\eta_{\alpha\beta}$	Strain tensor
θ, θ_o	Wave angles
$\bar{\theta}$, θ_c	Limiting angle
λ, μ	Lamé constants
ξ, η, ζ	Components of $\underset{\sim}{a}$
ρ	Density
ϕ, ψ, ω	Second order corrections to amplitudes

DISTRIBUTION LIST

No. of Copies	Organization
12	Commander Defense Technical Info Center ATTN: DDC-DDA Cameron Station Alexandria, VA 22314
1	Deputy Assistant Secretary of the Army (R&D) Department of the Army Washington, DC 20310
1	HQDA (DAMA-ARP-P, Dr. Watson) Washington, DC 20310
1	HQDA (DAMA-MS) Washington, DC 20310
1	Director US Army BMD Advanced Technology Center ATTN: CRDABH-5, W. Loomis P. O. Box 1500, West Station Huntsville, AL 35804
1	Director US Army Ballistic Missile Defense Systems Office 5001 Eisenhower Avenue Alexandria, VA 22333
1	Commander US Army War College ATTN: Lib Carlisle Barracks, PA 17013
1	Commander US Army Command and General Staff College ATTN: Archives Fort Leavenworth, KS 66027
1	Commander US Army Materiel Development and Readiness Command ATTN: DRCDMD-ST 5001 Eisenhower Avenue Alexandria, VA 22333

No. of Copies	Organization
2	Commander US Army Armament Research and Development Command ATTN: DRDAR-TSS Dover, NJ 07801
1	Commander US Army Armament Materiel Readiness Command ATTN: DRSAR-LEP-L, Tech Lib Rock Island, IL 61299
1	Director US Army ARRADCOM Benet Weapons Laboratory ATTN: DRDAR-LCB-TL Watervliet, NY 12189
1	Commander US Army Watervliet Arsenal ATTN: Dr. E. Schneider Watervliet, NY 12189
1	Commander US Army Aviation Research and Development Command ATTN: DRDAV-E 4300 Goodfellow Boulevard St. Louis, MO 63120
1	Director US Army Air Mobility Research and Development Laboratory Ames Research Center Moffett Field, CA 94035
1	Commander US Army Communications Rsch and Development Command ATTN: DRDCO-PPA-SA Fort Monmouth, NJ 07703
1	Commander US Army Electronics Rsch and Development Command Technical Support Activity ATTN: DELSD-L Fort Monmouth, NJ 07703

DISTRIBUTION LIST

No. of Copies	Organization
1	Commander US Army Harry Diamond Labs ATTN: DELHD-TA-L 2800 Powder Mill Road Adelphi, MD 20783
1	Commander US Army Missile Command ATTN: DRSMI-R Redstone Arsenal, AL 35809
1	Commander US Army Missile Command ATTN: DRSMI-YDL Redstone Arsenal, AL 35809
2	Commander US Army Mobility Equipment Research & Development Command ATTN: DRDME-WC DRSME-RZT Fort Belvoir, VA 22060
1	Commander US Army Natick Research and Development Command ATTN: DRXRE, Dr. D. Sieling Natick, MA 01762
1	Commander US Army Tank Automotive Rsch and Development Command ATTN: DRDTA-UL Warren, MI 48090
1	Commander US Army Electronics Proving Ground ATTN: Tech Lib Fort Huachuca, AZ 85613
3	Commander US Army Materials and Mechanics Research Center ATTN: J. Mescall R. Shea S. C. Chou Watertown, MA 02172

No. of Copies	Organization
1	Commander US Army Research Office P. O. Box 12211 Research Triangle Park NC 27709
1	Director US Army TRADOC Systems Analysis Activity ATTN: ATAA-SL, Tech Lib White Sands Missile Range NM 88002
1	Mathematics Research Center US Army University of Wisconsin Madison, WI 53706
1	Office of Naval Research Department of the Navy ATTN: Code 402 Washington, DC 20360
3	Commander Naval Air Systems Command ATTN: AIR-604 Washington, DC 20360
1	Commander Naval Surface Weapons Center ATTN: Code Gr-9, Dr. W. Soper Dahlgren, VA 22448
1	Commander and Director Naval Electronics Laboratory ATTN: Lib San Diego, CA 92152
3	Commander Naval Research Laboratory ATTN: Code 5270, F. MacDonald Code 2020, Tech Lib Code 7786, J. Baker Washington, DC 20375
1	AFATL (DLYW) Eglin AFB, FL 32542

DISTRIBUTION LIST

No. of Copies	Organization
1	AFATL (DLDG) Eglin AFB, FL 32542
1	AFATL (DLDL, MAJ J.E. Morgan) Eglin AFB, FL 32542
1	RADC (EMTLD, Lib) Griffiss AFB, NY 13440
1	AUL (3T-AUL-60-118) Maxwell AFB, AL 36112
1	AFFDL/FB, Dr. J. Halpin Wright-Patterson AFB, OH 45433
1	Director Environmental Science Service Administration US Department of Commerce Boulder, CO 80302
9	Sandia Laboratories ATTN: Mr. L. Davison Div 5163 Dr. C. Harness H. J. Sutherland Code 5133 Code 1721 Dr. P. Chen L. Bertholf W. Herrmann Albuquerque, NM 87115
1	Director Jet Propulsion Laboratory ATTN: Lib (TDS) 4800 Oak Grove Drive Pasadena, CA 91103
1	Director National Aeronautics and Space Administration Lyndon B. Johnson Space Center ATTN: Lib Houston, TX 77058

No. of Copies	Organization
1	IBM Watson Research Center ATTN: R. A. Toupin Poughkeepsie, NY 12601
5	Brown University Division of Engineering ATTN: Prof. R. Clifton Prof. H. Kolsky Prof. A. Pipkin Prof. P. Symonds Prof. J. Martin Providence, RI 02192
4	California Institute of Tech Division of Engineering and Applied Science ATTN: Dr. J. Mikowitz Dr. E. Sternberg Dr. J. Knowles Dr. T. Coguhey Pasadena, CA 91102
4	Carnegie Mellon University Department of Mathematics ATTN: Dr. D. Owen Dr. M. E. Gurtin Dr. B. Coleman Dr. W. Williams Pittsburgh, PA 15213
2	Catholic University of America School of Engineering and Architecture ATTN: Prof. A. Durelli Prof. J. McCoy Washington, DC 20017
1	Harvard University Division of Engineering and Applied Physics ATTN: Dr. G. Carrier Cambridge, MA 02138
2	Iowa State University Engineering Research Lab ATTN: Dr. G. Nariboli Dr. A. Sedov Ames, IA 50010

DISTRIBUTION LIST

No. of Copies	Organization
2	Lehigh University Center for the Application of Mathematics ATTN: Dr. E. Varley Dr. R. Rivlin Bethlehem, PA 18015
1	Massachusetts Institute of Tech ATTN: Dr. R. Probstein 77 Massachusetts Avenue Cambridge, MA 02139
1	Michigan State University College of Engineering ATTN: Prof. W. Sharpe East Lansing, MI 48823
1	New York University Department of Mathematics ATTN: Dr. J. Keller University Heights New York, NY 10053
1	North Carolina State University Dept of Engineering Mathematics ATTN: Dr. W. Bingham P. O. Box 5071 Raleigh, NC 27607
1	Pennsylvania State University Engineering Mechanical Dept. ATTN: Prof. N. Davids University Park, PA 16502
2	Forrestal Research Center Aeronautical Engineering Lab Princeton University ATTN: Dr. S. Lam Dr. A. Eringen Princeton, NJ 08540
1	Purdue University Institute for Mathematical Sciences ATTN: Dr. E. Cumberbatch Lafayette, IN 47907

No. of Copies	Organization
2	Rice University ATTN: Dr. R. Bowen Dr. C. C. Wang P. O. Box 1892 Houston, TX 77001
1	Southern Methodist University Solid Mechanics Division ATTN: Prof. H. Watson Dallas, TX 75221
2	Southwest Research Institute Dept of Mechanical Sciences ATTN: Dr. U. Lindholm Dr. W. Baker 8500 Culebra Road San Antonio, TX 78228
3	SRI International ATTN: D. Curran L. Seaman Y. Gupta Menlo Park, CA 94025
1	Temple University College of Engineering Tech ATTN: Prof. R.M. Haythornthwaite Philadelphia, PA 19122
1	Tulane University Dept of Mechanical Engineering ATTN: Dr. S. Cowin New Orleans, LA 70112
2	University of California ATTN: Dr. M. Carroll Dr. P. Naghdi Berkeley, CA 94704
1	University of California Dept of Aerospace and Mechanical Engineering Science ATTN: Dr. Y. C. Fung P. O. Box 109 La Jolla, CA 90237

DISTRIBUTION LIST

No. of Copies	Organization
1	University of California Department of Mechanics ATTN: Dr. R. Stern 504 Hilgard Avenue Los Angeles, CA 90024
1	University of California at Los Angeles Department of Mechanics ATTN: W. Goldsmith Los Angeles, CA 90024
1	University of Delaware Department of Mechanical Eng ATTN: Prof. J. Vinson Newark, DE 19711
3	University of Florida Dept of Engineering Science and Mechanics ATTN: Dr. C. A. Sciammarilla Dr. L. Malvern Dr. E. Walsh Gainesville, FL 32601
2	University of Houston Dept of Mechanical Engineering ATTN: Dr. T. Wheeler Dr. R. Nachlinger Houston, TX 77004
1	University of Illinois Dept of Theoretical and Applied Mechanics ATTN: Dr. D. Carlson Urbana, IL 61801
2	University of Illinois at Chicago Circle College of Engineering Dept of Materials Engineering ATTN: Prof. A. Schultz Dr. T. C. T. Ting P. O. Box 4348 Chicago, IL 60680
4	University of Kentucky Dept of Engineering Mechanics ATTN: Dr. M. Beatty Prof. O. Dillon, Jr. Prof. P. Gillis Dr. D. Leigh Lexington, KY 40506
2	The University of Maryland Department of Mechanical Eng ATTN: Prof. J. Yang Dr. J. Dally College Park, MD 20742
1	University of Minnesota Dept of Engineering Mechanics ATTN: Dr. R. Fosdick Minneapolis, MN 55455
1	University of Notre Dame Department of Metallurgical Engineering and Materials Sciences ATTN: Dr. N. Fiore Notre Dame, IN 46556
1	University of Pennsylvania Towne School of Civil and Mechanical Engineering ATTN: Prof. Z. Hashin Philadelphia, PA 19105
3	University of Texas Dept of Engineering Mechanics ATTN: Dr. M. Stern Dr. M. Bedford Prof. Ripperger Austin, TX 78712
1	University of Washington Dept of Mechanical Engineering ATTN: Prof. J. Chalupnik Seattle, WA 98105
2	Washington State University Department of Physics ATTN: R. Fowles G. Duvall Pullman, WA 99164

DISTRIBUTION LIST

No. of Copies	Organization
2	Yale University ATTN: Dr. B. Chu Dr. E. Onat 400 Temple Street New Haven, CT 06520

Aberdeen Proving Ground

	Dir, USAMSAA ATTN: DRXSY-D DRXSY-MP, H. Cohen
	Cdr, USATECOM ATTN: DRSTE-TO-F
	Dir, USACSL, Bldg. E3516, EA ATTN: DRDAR-CLB-PA

USER EVALUATION OF REPORT

Please take a few minutes to answer the questions below; tear out this sheet, fold as indicated, staple or tape closed, and place in the mail. Your comments will provide us with information for improving future reports.

1. BRL Report Number ______________________

2. Does this report satisfy a need? (Comment on purpose, related project, or other area of interest for which report will be used.)

____

____

____

3. How, specifically, is the report being used? (Information source, design data or procedure, management procedure, source of ideas, etc.) ______________________

____

____

4. Has the information in this report led to any quantitative savings as far as man-hours/contract dollars saved, operating costs avoided, efficiencies achieved, etc.? If so, please elaborate.

____

____

5. General Comments (Indicate what you think should be changed to make this report and future reports of this type more responsive to your needs, more usable, improve readability, etc.) ______________________

____

____

____

6. If you would like to be contacted by the personnel who prepared this report to raise specific questions or discuss the topic, please fill in the following information.

Name: ______________________

Telephone Number: ______________________

Organization Address: ______________________

____

____

— — — — — — — — — — — FOLD HERE — — — — — — — — — — — —

Director
US Army Ballistic Research Laboratory
Aberdeen Proving Ground, MD 21005

NO POSTAGE NECESSARY IF MAILED IN THE UNITED STATES

OFFICIAL BUSINESS
PENALTY FOR PRIVATE USE, $300

BUSINESS REPLY MAIL
FIRST CLASS PERMIT NO 12062 WASHINGTON, DC

POSTAGE WILL BE PAID BY DEPARTMENT OF THE ARMY

Director
US Army Ballistic Research Laboratory
ATTN: DRDAR-TSB
Aberdeen Proving Ground, MD 21005

— — — — — — — — — — FOLD HERE — — — — — — — — — — —

END

DATE FILMED

1 -82

DTIC