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KINEMATICS, DYNAMICS, AND ESTIMATION OF RIGID-BODY MOTION USING EULER PARAMETERS (QUATERNIONS)

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This report explores the algebraic and measure-theoretic properties of the Euler parameter (quaternion) representation of the rotation group. A family of probability densities on this group is examined, and schemes for numerical integration and estimation of satellite attitude are presented with numerical examples. An effort has been made wherever possible to deal with nonlinear problems directly rather than linearizing them. Several questions for further research are raised regarding convolutions of densities on the group, Brownian motion, and the statistics of spin and torque vectors considered as stochastic variables.

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ABSTRACT

This report explores the algebraic and measure-theoretic properties of the Euler parameter (quaternion) representation of the rotation group. A family of probability densities on this group is examined, and schemes for numerical integration and estimation of satellite attitude are presented with numerical examples. An effort has been made wherever possible to deal with nonlinear problems directly rather than linearizing them. Several questions for further research are raised regarding convolutions of densities on the group, Brownian motion, and the statistics of spin and torque vectors considered as stochastic variables.

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1. INTRODUCTION

Euler's Theorem states that the most general rotational (or angular) displacement of a rigid body can be accomplished by a single rotation through an angle, θ, about a line. Letting the direction of this line be designated by a unit vector

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} , \tag{1}$$

we have a numerical description of this rotational displacement (henceforth, just "displacement"). It is not unique, however, since both u and -u define the same line and θ has not been restricted to an interval of length 2π. This representation is not particularly economical, using four variables and one constraint, $|u| = 1$, to describe a mathematical object that has three degrees of freedom.

There are many choices for parameterizing the rotation group with three variables. One of the most common is the z-x-z Euler angles, three plane rotations specified in "running coordinates;" another is the roll-pitch-yaw system, or x-y-z Euler angles, used in models of ship motion. These, too, have their drawbacks stemming from the differences between the topology of this group and that of R^3, the real 3-dimensional space of the three angles. Such representations cannot be uniformly continuous — "small" rotations are not always described by small parameter values. This presents numerical difficulties when values of these parameters are being estimated from observations of the orientation of a rigid body. Furthermore, the propagation equations for rigid-body motion are not as simple as one would like and suffer from the same topological singularities.

The elements of the direction cosine matrix,

$$C = [c_{ij}] \quad i,j = 1,2,3, \tag{2}$$

are nicely behaved in this sense, but they are nine numbers representing three degrees of freedom and hence subject to six constraints. They do, however, possess one useful property relating to the structure of the rotations as a group under the operation of "composition" or sequential rotation. Composition of two rotations is represented by the matrix product. The rotation

$$C_3 = C_2 \, C_1 \tag{3}$$

is the result of first rotating by C_1 and then by C_2. The direction cosine matrix is, of course, similarly useful for mapping vectors in cartesian coordinates directly (for further discussion see Appendix A).

The Euler parameters have all the advantages of the matrix product representation of the group except this point mapping property. They are uniformly continuous; that is, they do not have any singularities to be contended with in their calculations or statistics. Their primary disadvantages are the following:

1. They are obscure,

2. They are four parameters with one constraint, and

3. The representation is bivalued.

The constraint is a problem to be dealt with whenever inexact calculations of Euler parameters arise: solutions to differential equations, estimations based on linear approximations, etc. The two representations of a given rotational displacement differ only in sign, but this is a serious problem in numerically comparing two rotations, and it must constantly be kept in mind while developing numerical algorithms.

2. DEFINITIONS AND FUNDAMENTAL PROPERTIES

While the motivation may not be clear, it is convenient to list here some fundamental terminology. In terms of the direction, u, and magnitude, θ, of a rotational displacement, the Euler parameters are given by

$$\begin{aligned} \beta_0 &= \cos\frac{\theta}{2} \\ \beta_i &= u_i \sin\frac{\theta}{2}, \quad i = 1,2,3. \end{aligned} \tag{4}$$

For simplicity, when β appears without a subscript it refers to the column vector $(\beta_0, \beta_1, \beta_2, \beta_3)$. The constraint is

$$\sum \beta_i^2 = 1, \tag{5}$$

which is obvious from Eq. 4. Thus, the β_i are bounded by 1, and the bivalued correspondence comes from the entrance of the angle as θ/2 so that a 2π rotation added to θ changes the sign of all the β_i.

The direction cosine matrix is given by

$$C(\beta) = \begin{bmatrix} \beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2 & 2(\beta_1\beta_2+\beta_0\beta_3) & 2(\beta_1\beta_3-\beta_0\beta_2) \\ 2(\beta_1\beta_2-\beta_0\beta_3) & \beta_0^2-\beta_1^2+\beta_2^2-\beta_3^2 & 2(\beta_2\beta_3+\beta_0\beta_1) \\ 2(\beta_1\beta_3+\beta_0\beta_2) & 2(\beta_2\beta_3-\beta_0\beta_1) & \beta_0^2-\beta_1^2-\beta_2^2+\beta_3^2 \end{bmatrix} \tag{6}$$

and it is clear that $C(\beta) = C(-\beta)$. It is easy to show that $(\beta_1, \beta_2, \beta_3)$ is an eigenvector of $C(\beta)$ (with eigenvalue 1). This is the matrix having the successive transformation property described by Eq. 3.

A similar property holds for the following matrix:

$$S(\beta) = \begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & \beta_3 & -\beta_2 \\ \beta_2 & -\beta_3 & \beta_0 & \beta_1 \\ \beta_3 & \beta_2 & -\beta_1 & \beta_0 \end{bmatrix} . \qquad (7)$$

This matrix is introduced in Ref. 1 as having the property (established in Ref. 2) that successive rotational displacements β and β' result in a rotational displacement β'' given by

$$\beta'' = S(\beta')\beta \qquad (8)$$

in the same sense that, as in Eq. 3,

$$C(\beta'') = C(\beta')\ C(\beta). \qquad (9)$$

Directly, one can show that Eq. 8 expands to

$$S(\beta'') = S(\beta')\ S(\beta), \qquad (10)$$

which contains Eq. 8 as its first column. Thus it is clear that the Euler parameters provide a two-valued representation of the rotation group as a group of linear transformations, one that happens to be a subgroup of the rotation group in four dimensions (as represented by 4 × 4 real orthogonal matrices of determinant 1). Not all orthogonal 4 × 4 matrices are of the form of Eq. 7.

The group identity is $S(b_0) = I_4$, where $b_0 = (1,0,0,0)$. And for each β, the inverse of $S(\beta)$ is its transpose

$$S(\beta)^T = S(\bar{\beta}) \quad ,$$

Ref. 1. H. S. Morton, Jr., J. L. Junkins, and J. N. Blanton, "Analytic Solutions for Euler Parameters," Celestial Mech. Vol. 10 (1974).

Ref. 2. E. T. Whittaker, A Treatise on the Analytic Dynamics of Particles, 4th ed., Dover (1944).

where

$$\bar{\beta} = (\beta_0, -\beta_1, -\beta_2, -\beta_3) \quad . \tag{11}$$

Consider next the matrix

$$K = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and define, for any $\alpha \varepsilon R^4$,

$$R(\alpha) = KS(\alpha)^T K. \tag{12}$$

Then

$$R(\alpha) = \begin{bmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix} . \tag{13}$$

This matrix is also given in Ref. 1. It has a property similar to that of S, Eq. 10: from

$$S(\beta'') = S(\beta')\ S(\beta), \tag{14}$$

we have

$$R(\beta'') = KS(\beta)^T\ S(\beta')^T K$$

$$= R(\beta)R(\beta'), \tag{15}$$

and from column one, $\beta'' = R(\beta)\beta'$.

By direct evaluation we can establish the very interesting, even amazing, and useful property

$$R(\alpha)^T\ S(\beta) = S(\beta)\ R(\alpha)^T \tag{16}$$

for any β and any α in R^4. (This also implies that $R(\alpha)$ and $S(\beta)$ commute.) Next let us observe that when $\alpha = \beta$, this product takes on the form

$$\Sigma(\beta) = R(\beta)^T S(\beta) = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & C(\beta) & \\ 0 & & & \end{array}\right] , \tag{17}$$

where $C(\beta)$ is the direction cosine matrix defined in Eq. 6.

This definition is convenient to establish the law of composition stated in Eq. 9.

$$\begin{aligned} \Sigma(\beta')\Sigma(\beta) &= [R(\beta')^T S(\beta')]\ [R(\beta)^T S(\beta)] \\ &= R(\beta')^T\ \ [R(\beta)^T S(\beta')]\ \ S(\beta) && (18) \\ &= R(\beta'')^T S(\beta'') && (19) \\ &= \Sigma(\beta'') \ , \end{aligned}$$

where $\beta'' = S(\beta')\beta$, and Eq. 18 follows from Eq. 16, and Eq. 19 follows from Eq. 15 (for R) and Eq. 10 (for S).

The matrix

$$T(\alpha,\beta) = R(\alpha)S(\beta), \tag{20}$$

where α and β are unit vectors in R^4, has six degrees of freedom, the same as the 4×4 orthogonal matrices. Indeed, we can show that any orthogonal 4×4 matrix having determinant 1 is of this form. We proceed as follows: the first column of such a matrix, X, is a unit vector. Calling it ζ, we form

$$S(\zeta)^T X \ ,$$

which then has the form

$$\left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & C & \\ 0 & & & \end{array}\right] ,$$

where C is orthogonal and has determinant 1. Thus $S(\zeta)^T X$ is of the form of Eq. 17:

$$S(\zeta)^T X = \Sigma(\eta) \text{ for some } \eta. \tag{21}$$

Multiplying by $S(\zeta)$ and substituting from Eq. 17,

$$X = S(\zeta)R(\eta)^T S(\eta). \tag{22}$$

We have shown that $S(\zeta)$ and $R(\eta)^T$ commute, so

$$X = R(\eta)^T S[S(\zeta)\eta], \tag{23}$$

and letting $\alpha=\bar{\eta}$ (as defined in Eq. 11) and $\beta = S(\zeta)\eta$, the result follows. In summary, we have shown that the four-dimensional real orthogonal unimodular (determinant +1) group, the rotations in R^4, modulo $(I_4, -I_4)$, is isomorphic to the direct product of the groups of S and R matrices, modulo $(I_4, -I_4)$.

3. QUATERNIONS AND OTHER REPRESENTATIONS OF THE ROTATION GROUP

We have dealt up to this point with several representations of the rotation group: the direction-cosine matrices, the S and R matrix representations that we associated with the term "Euler parameters," and others were mentioned. Some authors prefer to call the Euler parameters quaternions, and indeed the representations are isomorphic. Historically, the parameters were first developed by Euler and Rodrigues and were published in 1776. They have variously been called "the homogeneous parameters of Euler," "Euler's symmetric parameters," and "Euler-Rodrigues parameters." It was in 1843 that Hamilton introduced quaternions and the properties of this division algebra, not subject to the length constraint.

Figure 1 illustrates the diversity of terminology in the standard texts and technical literature that has evolved over the last 200 years. Keep in mind that all of this addresses one primary mathematical object: the group of rotations in 3-space. Representations linked by double arrows are identical or nearly identical. Numbers identify references at the end of this report. Arrows without numbers can mostly be found in Chapter 4 of Ref. 5. Other direct transformations beyond those shown by the arrows certainly exist or can be derived, for example, the calculation of z-x-z Euler angles from direction cosines. To make matters worse or at least more confusing, there is no universal standard notation within individual representations. Euler angles, for example, may be defined in many different ways.

By the term "principal axis form" is meant a specification of the axis of a rotation and an angle. This can be a unit vector and a scalar for the angle, or the product of the angle and the unit vector. The Rodrigues parameters are not discussed in any of the references, but they are essentially like the principal axis form: the unit vector is multiplied by the tangent of one-half of the rotation.

A representation of a group must model the group operation. The ease with which this is done provides one distinction between representations. The direction cosines and Euler parameters take the form of real matrices and combine according to the ordinary

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Ref. 3. G. Birkhoff and S. MacLane, A Survey of Modern Algebra, MacMillan (1953).

Ref. 4. H. C. Corben and P. Stehle, Classical Mechanics, 2nd ed., Wiley (1950).

Ref. 5. H. Goldstein, Classical Mechanics, Addison-Wesley (1950).

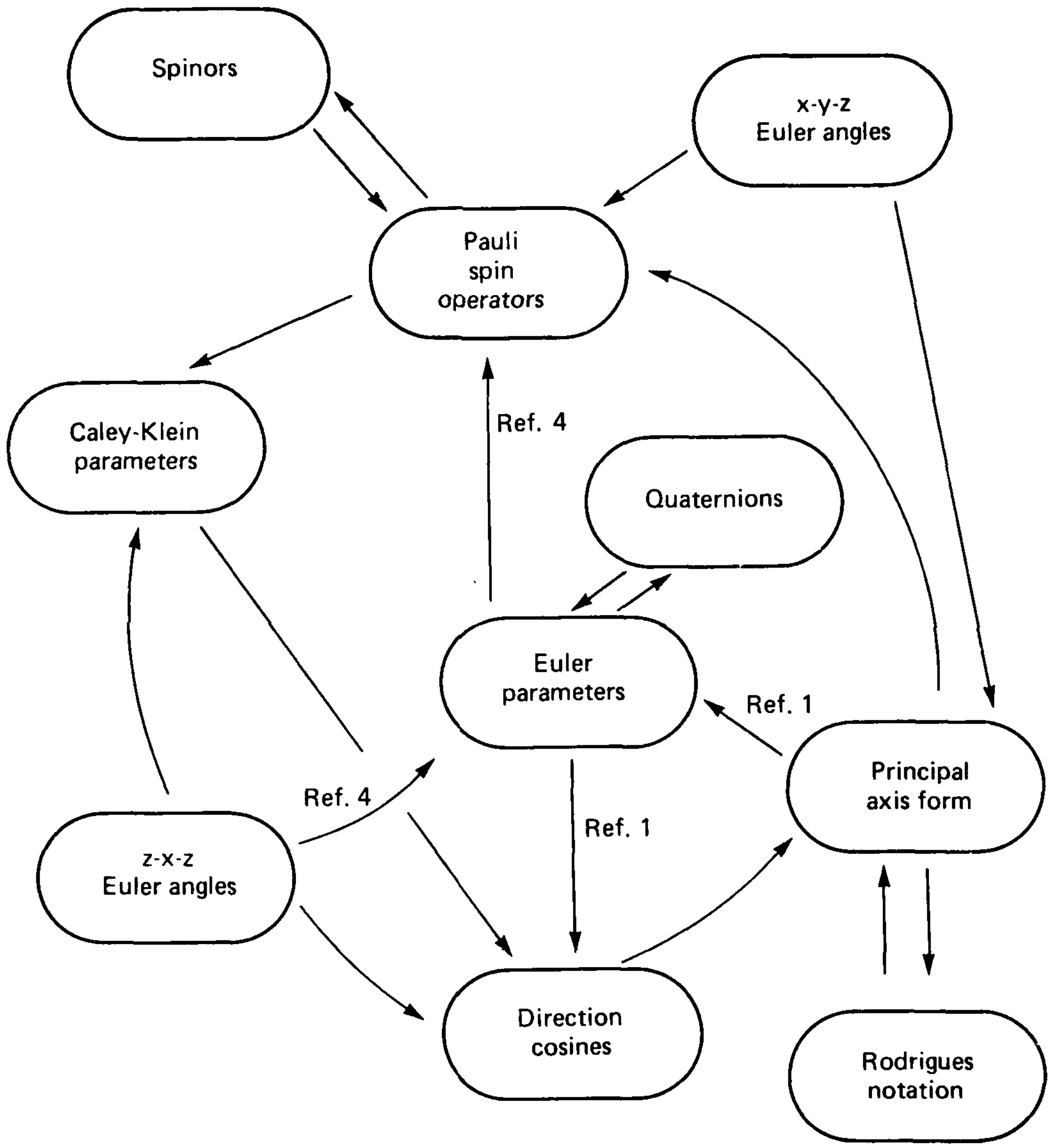

Fig. 1 Representations of the group of rotations.

matrix multiplication. The Pauli spin and Caley-Klein representations use compiex matrices of lower order and are basically equivalent in computational complexity to quaternions and Euler parameters. Principal axis and Rodrigues notation use vector cross and dot products, and Euler angles require complicated trigonometric calculations plus logical branching to avoid problems at singularities.

4. KINEMATICS AND DYNAMICS OF ROTATIONAL MOTION

From Eq. 10 come the interesting properties of the Euler parameters. Differentiating Eq. 4, we obtain

$$\dot{\beta} = \cos\theta/2 \begin{pmatrix} 0 \\ \dot{\theta}\mu_1/2 \\ \dot{\theta}\mu_2/2 \\ \dot{\theta}\mu_3/2 \end{pmatrix} + \sin\theta/2 \begin{pmatrix} -\dot{\theta}/2 \\ \dot{\mu}_1 \\ \dot{\mu}_2 \\ \dot{\mu}_3 \end{pmatrix} . \tag{24}$$

This form can be simplified a bit by noting that when $\theta = 0$ we have

$$\dot{\beta} = \frac{1}{2} \begin{pmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} , \tag{25}$$

where $\omega_i = \dot{\theta}\mu_i$, which are the three components of the instantaneous spin vector. Next we can differentiate Eq. 10 to obtain:

$$S(\dot{\beta}'') = S(\dot{\beta}')S(\beta) + S(\beta')S(\dot{\beta}). \tag{26}$$

Now let β be a constant, selected to be equal to β'' at some time, t_0. Then $\dot{\beta} = 0$ and at t_0, $\beta'' = \beta$ so that at t_0 we have $\beta' = b_0$, the identity. Thus the above form applies to $\dot{\beta}'$ and Eq. 26 becomes

$$S[\dot{\beta}''(t_0)] = S(\omega/2)S[\beta''(t_0)] = \frac{1}{2} S(\omega)S[\beta''(t_0)]. \tag{27}$$

The first column of Eq. 27 becomes

$$\dot{\beta}''(t_0) = \frac{1}{2}S[\omega(t_0)]\beta''(t_0),$$

which holds for any choice of t_0; thus at t:

$$\dot{\beta}(t) = \frac{1}{2}S[\omega(t)]\beta(t), \tag{28}$$

where $\omega(t) = \begin{bmatrix} 0 \\ \omega_1(t) \\ \omega_2(t) \\ \omega_3(t) \end{bmatrix}$

and the last three components are the instantaneous spin vector in body-fixed coordinates. Thus

$$S[\dot{\beta}(t)] = \frac{1}{2}S[\omega(t)]S[\beta(t)] \ . \tag{29}$$

The reason ω appears in body-fixed coordinates is that the spin motion follows the displacement β. This is due to the choice in Eq. 27 to hold β constant. Had we held β' constant and let β vary, the spin in inertial coordinates would appear.

Also we see that the form of Eq. 10 is that of the state transition matrix:

$$\beta(t) = \Phi(t,t_0)\beta(t_0) \tag{30}$$

in Ref. 1. Let us define, for any $\beta(t)$, a matrix

$$\Phi(t,t_0) = S[\beta(t)]S[\beta(t_0)]^T \ , \tag{31}$$

which is a rotational motion, the inverse of $\beta(t_0)$ followed by $\beta(t)$. Call its first column $\alpha(t)$, so that

$$\Phi(t,t_0) = S[\alpha(t)] \tag{32}$$

is defined properly, and observe from

$$S[\beta(t)] = S[\alpha(t)]S[\beta(t_0)] \tag{33}$$

that Eq. 30 does indeed hold for this choice (Eq. 31) of $\Phi(t,t_0)$. Differentiating Eq. 33 and applying Eq. 28 we obtain

$$\frac{1}{2}S[\omega(t)]S[\beta(t)] = \dot{\Phi}(t,t_0)S[\beta(t_0)] \; ,$$

or

$$\dot{\Phi}(t,t_0)S[\beta(t_0)] = S[\omega(t)]\Phi(t,t_0)S[\beta(t_0)] \; ,$$

and thus

$$\dot{\Phi}(t,t_0) = \frac{1}{2}S[\omega(t)]\Phi(t,t_0) \; , \tag{34}$$

as the differential equation for the transition matrix. All of the preceding is kinematic and is true for any history of $\omega(t)$.

We can use Eq. 17 in developing the dynamics of rigid body motion. We need the derivative of $C(\beta)$ with respect to time in terms of the body-fixed spin components, ω.

$$\frac{d}{dt}\Sigma(\beta) = \dot{\Sigma}(\beta) = \dot{R}(\beta)^T S(\beta) + R(\beta)^T \dot{S}(\beta) \; . \tag{35}$$

Because R and S are linear in β (Σ is quadratic),

$$\dot{\Sigma}(\beta) = R(\dot{\beta})^T S(\beta) + R(\beta)^T S(\dot{\beta}) \; . \tag{36}$$

The form of interest is

$$\dot{\Sigma}(\beta)\Sigma(\beta)^T = R(\dot{\beta})^T S(\beta)S(\beta)^T R(\beta) + R(\beta)^T S(\dot{\beta})S(\beta)^T R(\beta) \tag{37}$$

$$= R(\dot{\beta})^T R(\beta) + S(\dot{\beta})R(\beta)^T R(\beta)S(\beta)^T$$

$$= R(\dot{\beta})^T R(\beta) + S(\dot{\beta}) \; S(\beta)^T \; . \tag{38}$$

We know $S(\dot{\beta})$ from Eq. 29 and using Eq. 12 we find $R(\dot{\beta})^T$. Evaluation of these yields $\frac{1}{2}S(\omega)$ for each term in Eq. 38, so

$$\dot{\Sigma}(\beta)\Sigma(\beta)^T = S(\omega) \; , \tag{39}$$

where $\omega = (0, \omega_1, \omega_2, \omega_3)$ in this notation.

The fundamental physical law of rotation of rigid bodies is

$$\frac{d}{dt} L_i = \tau_i \ , \tag{40}$$

where τ is the torque applied to the body and L is the angular momentum. The subscript "i" reminds us that this is true in inertial reference frames. We can define the inertia tensor

$$J = \int_{body} (r^T rI - rr^T) \ \rho dV \tag{41}$$

in which dV is differential volume, ρ is density, I is the 3 × 3 identity, and

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{42}$$

is a point in the body. The matrix $(r^T rI-rr^T)$ is then

$$\begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} , \tag{43}$$

which is valid only in the specific coordinate frame in which r is defined. Thus a rotating body will have an inertia tensor that may be time-varying in any coordinates other than body-fixed.

Angular momentum is conveniently separated as the product

$$L = J\omega, \tag{44}$$

and

$$\dot{L} = \dot{J}\omega + J\dot{\omega}. \tag{45}$$

As an alternative to writing $\dot{J}$, we can transform Eq. 40 into body-fixed coordinates using

$$J_i = C(\beta)^T \ J_b \ C(\beta) \quad ; \tag{46}$$

$$\omega_i = C(\beta)^T \omega_b \quad ; \tag{47}$$

$$\tau_i = C(\beta)^T \tau_b \quad . \tag{48}$$

We must substitute for $\dot{J}$ and $\dot{\omega}$ in Eq. 45

$$\dot{J}_i = \dot{C}(\beta)^T J_b C(\beta) + C(\beta)^T J_b \dot{C}(\beta) \quad ; \tag{49}$$

$$\dot{\omega}_i = \dot{C}(\beta)^T \omega_b + C(\beta)^T \dot{\omega}_b \quad . \tag{50}$$

This produces, in place of Eq. 45

$$\dot{L}_i = \dot{C}(\beta)^T J_b \omega_b + C(\beta)^T J_b \dot{C}(\beta) C(\beta)^T \omega_b + C(\beta)^T J_b C(\beta) \dot{C}(\beta)^T \omega_b + C(\beta)^T J_b \dot{\omega}_b \quad . \tag{51}$$

Equating $\dot{L}_i$ to torque (Eq. 40) and substituting with Eq. 48 to involve body-fixed representation exclusively, we obtain

$$\dot{L}_i = \tau_i = C(\beta)^T \tau_b \tag{52}$$

and

$$C(\beta) \dot{L}_i = \tau_b \quad . \tag{53}$$

Multiplying Eq. 51 on the left by $C(\beta)$ simplifies it somewhat and and makes all occurrences of $\dot{C}$ be of the form $\dot{C}C^T$ or the transpose of this. From Eq. 39 we can extract the lower right 3 × 3 to leave C in place of Σ, and write (calling this submatrix of $S(\omega)$ by the same name),

$$\tau_b = S(\omega_b)^T J_b \omega_b + J_b S(\omega_b)\omega_b + J_b S(\omega_b)^T \omega_b + J_b \dot{\omega}_b \quad . \tag{54}$$

Note that ω_b is used in S because we have chosen in Eq. 46 through Eq. 48 the transformation $C(\beta)$ to be from inertial to body-fixed.

Next, observe that Eq. 54 simplifies by skew-symmetry of $S(\omega_b)$ to the familiar

$$J_b \dot{\omega}_b = S(\omega_b)\ J_b \omega_b + \tau_b \quad . \tag{55}$$

This is the body-fixed form of Eq. 40 with $\dot{J}_i$ terms in more tractable form.

Equations 55 and 28 or 29 form a set of differential equations for the motion of a rigid body. In this context we need to indicate the dependence of τ on the other variables, specifically β, ω, and time:

$$\tau_b = \tau_b\ (\beta,\omega,t) \quad . \tag{56}$$

Nowhere else in Eq. 55 does β appear, and thus a distinction can be made between two classes of problems: those in which τ depends on β and those in which it does not. For numerical purposes, it is $\dot{\omega}$ that is of interest, and it always depends on ω except in trivial problems.

Torques that do not depend on β are those due to on-board rockets, momentum wheels, fluids, nonrigidity, or other mass-movement effects. Those torques derived from interaction with external masses or fields (drag, radiation pressure, magnetic, gravity gradient) do depend on β. The simplification in the former case allows solution numerically to the differential equation (Eq. 34) for the entire state transition matrix without knowledge of the initial attitude, $\beta(t_0)$. This problem is a first-order differential equation (nonlinear) in three variables; Eq. 34 for $\beta(t)$ is reduced a quadrature.

There are important implications here for satellite attitude determination in the sense that the transition matrix, Φ, for the β variables is itself an S-matrix (Eq. 32) and can be represented as $S[\alpha(t)]$ for some time-varying Euler parameters, $\alpha(t)$. Then the matrix $C[\beta(t_0)]$ can be propagated as well:

$$C[\beta(t)] = C[\alpha(t)]\ C[\beta(t_0)] \quad , \tag{57}$$

and this inverted to

$$C[\beta(t_0)] = C[\alpha(t)]^T\ C[\beta(t)] \tag{58}$$

to translate back to t_0 any future information about the satellite attitude in the form of observation of directions in space. Most attitude sensors (magnetometers, solar detectors, star cameras) that are not rate sensors are of this type.

The implication of Eq. 58 specifically is this: in the absence of β-dependent torques, a knowledge of the initial spin, $\omega(t_0)$, reduces the attitude determination problem to:

1. A first-order differential equation in ω only,
2. A quadrature to yield $C[\alpha(t)]$, and
3. A least-squares estimation of $\beta(t_0)$.

That is, the time dependence is removed from the estimation process; it is as if all measurements are taken simultaneously at t_0. One can even propagate the covariance effects of errors in the knowledge of ω back from the time of each measurement to derive appropriate weights for the estimation of β at t_0.

Numerical solutions of Eqs. 28 and 56

$$\dot{\beta} = \frac{1}{2} S(\omega) \beta \quad ,$$
$$\dot{\omega} = J^{-1} S(\omega) J\omega + J^{-1} \tau(\beta,\omega,t) \quad , \tag{59}$$

may be improved considerably in computation time by the following method in cases when ω varies much more slowly than β. In general this will include only problems in the class having torques independent of β. In this case there is much to be gained by replacing the approximation

$$\Delta\beta = \dot{\beta}\, \Delta t \tag{60}$$

by

$$\beta(t+\Delta t) = e^{\frac{1}{2}S(\omega)\Delta t} \beta(t) \quad , \tag{61}$$

where β and ω denote values at time t, which we fix at the beginning of the interval of length Δt. The matrix exponential serves the role of the transition matrix when ω is constant:

$$\Phi(t+\Delta t,t) = e^{\frac{1}{2} S(\omega)\Delta t} \tag{62}$$

as in Eq. 30 and satisfies the differential equation (Eq. 34) when the derivative is taken with respect to Δt holding t (and ω) constant.

It is not necessary to use the series expansion for the matrix exponential or even to retain the exponential notation, for we may refer back to the fundamental definitions (Eqs. 4 and 7) and write

$$\Phi(t+\Delta t,t) = S\left(\cos\frac{|\omega|\Delta t}{2},\ u \sin\frac{|\omega|\Delta t}{2}\right), \tag{63}$$

where u is the unit vector as given in Eq. 4 with an appropriately chosen sign. Appendix B shows a different approach to the development of Eqs. 62 and 63.

A Runge-Kutta fourth-order algorithm based on Eq. 63 was developed by straightforward replacement of all $\beta(t + \Delta t) = \beta(t) + \dot{\beta}(t)\Delta t$ or similar steps by the rotational equivalent. Some attention is also required, however, to the order of matrix products, since these are no longer commutative operations. Let h be the time step and define

$$\Theta(\omega,\Delta t) = S\begin{bmatrix} \cos\frac{|\omega|\Delta t}{2} \\ \frac{\omega}{|\omega|}\sin\frac{|\omega|\Delta t}{2} \end{bmatrix}. \tag{64}$$

Thus $\Theta(\omega,\Delta t)$ represents the rotation generated over an interval Δt by the constant spin vector, ω. The Runge-Kutta algorithm is as follows:

$$\lambda_1 = \Theta[\omega(t_i),\ h/2] \tag{65}$$

$$k_1 = \frac{h}{2}\ \tau[\beta(t_i),\ \omega(t_i),\ t_i] \tag{66}$$

$$\beta' = \lambda_1\ \beta(t_i) \tag{67}$$

$$\omega' = k_1 + \omega(t_i) \tag{68}$$

$$\lambda_2 + \Theta(\omega',\ h/2) \tag{69}$$

$$k_2 = \frac{h}{2}\ \tau(\beta',\ \omega',\ t_i + h/2) \tag{70}$$

$$\beta^* = \lambda_2\ \beta(t_i) \tag{71}$$

$$\omega^* = k_2 + \omega(t_i) \tag{72}$$

$$\lambda_3 = \Theta(\omega^*,\ h) \tag{73}$$

$$k_3 = h\ \tau(\beta^*,\ \omega^*,\ t_i + h/2) \tag{74}$$

$$\beta^+ = \lambda_3\ \beta(t_i) \tag{75}$$

$$\omega^+ = k_3 + \omega(t_i) \tag{76}$$

$$k_4 = \frac{h}{2}\ \tau(\beta^+,\ \omega^+,\ t_i + h) \tag{77}$$

$$\beta(t_{i+1}) = \Theta(\omega^+,\ h/6)\ \Theta(\frac{\omega^*+\omega'}{2},\ h/3)\ \Theta[\omega(t_i), h/6]\ \beta(t_i) \tag{78}$$

$$\omega(t_{i+1}) = \omega(t_i) + (k_1 + 2k_2 + k_3 + k_4)/3 \quad . \tag{79}$$

When the torque, τ, does not depend on β, it is possible to to modify this algorithm to integrate $\Phi(t,t_0)$ from the differential equation (Eq. 34) and the initial condition $\Phi(t_0,t_0) = I_4$, the 4×4 identity matrix. The intermediate calculations of β would be unnecessary and $\Phi(t_{i+1},t_0)$ would be produced in Eq. 78 by replacing $\beta(t_i)$ by $\Phi(t_i,\ t_0)$.

As a numerical example, the torque-free motion of a body whose inertia moments were $J_{11} = 43.1$, $J_{22} = 40.6$, and $J_{33} = 44.3$ was integrated from an initial $\beta(t_0) = (1, 0, 0, 0)$, and $\omega(t_0) = 10$ rpm, offset by 0.1 radian (about 6°) from the z-axis in the +x direction.

Figure 2 shows the spin history for 500 sec, about 83 revolutions. Precession has carried the spin axis through four trips around the z-axis. Using the Runge-Kutta scheme presented here, one can select a step size h based on this lower rate of ω variation rather than the high rate of β periodicity. Variations of β are shown over 50 sec in Fig. 3. Table 1 shows roughly the order of magnitude of accumulated errors in β and in ω at the end of 500 sec. These values were obtained by comparison of results with values obtained by integrating at 2 sec steps.

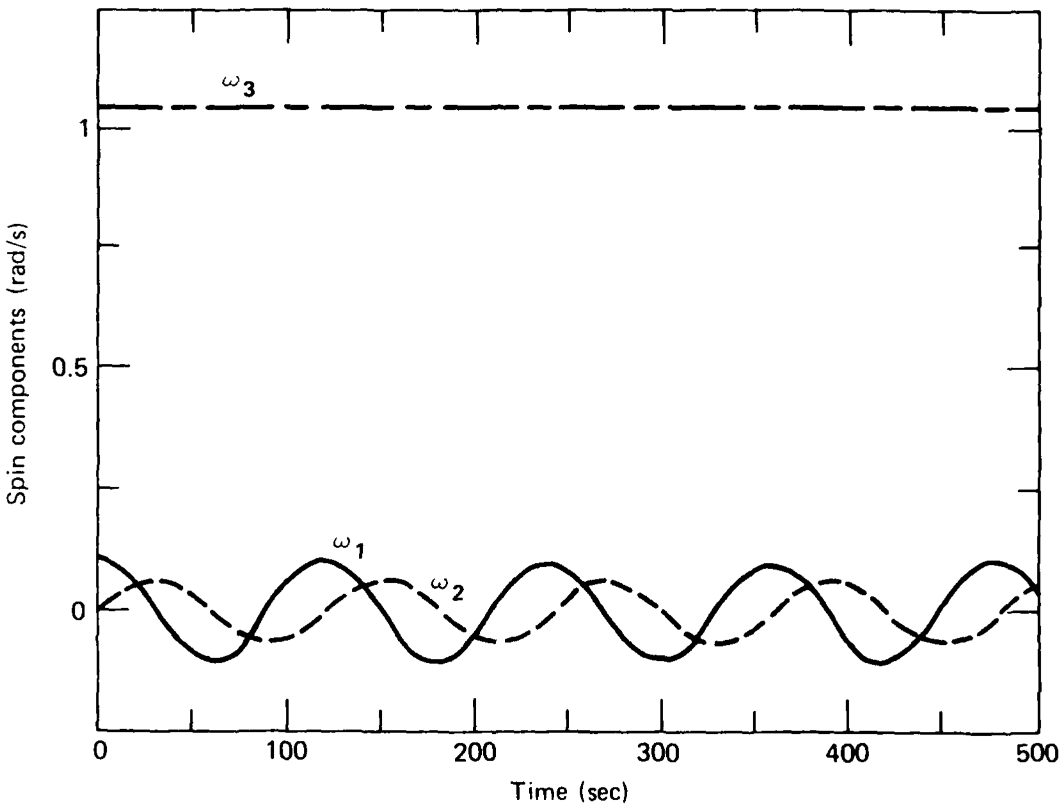

Fig. 2 Spin rate history, 500 sec.

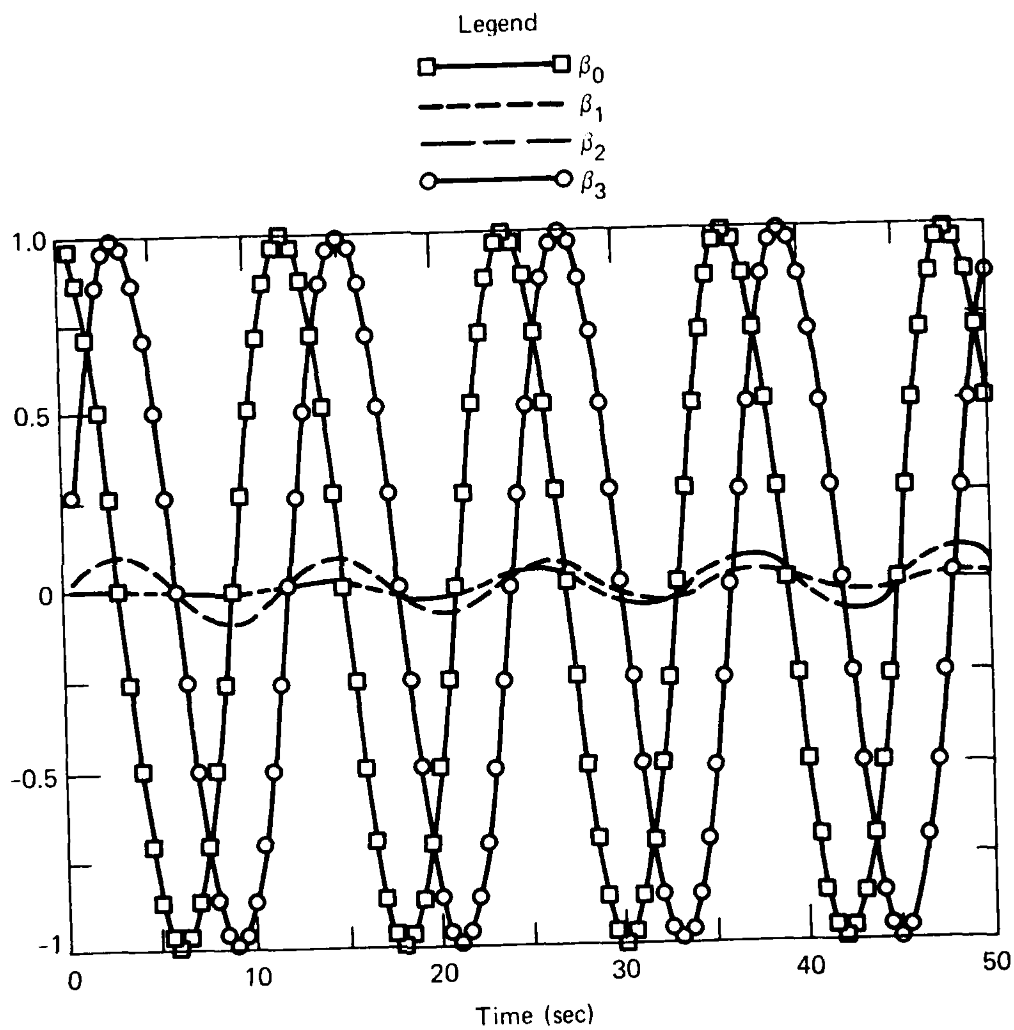

Fig. 3 Orientation history.

Table 1

Errors in ω and β from numerical sources.

Step size (sec)

	5	10	20
$\Delta\omega$	0.0001	0.001	0.015
$\Delta\beta$	0.008	0.02	0.10

The errors in β fall within the range from $\frac{\Delta\omega}{2\omega}$ to $\frac{t\Delta\omega}{4}$, where they are expected to be if they are due to errors in ω. Thus, one would conclude in this case that the β integration is "more linear" than the ω integration. The opposite would be true in the case of large but constant (in body coordinates) torques.

5. A PROBABILITY FUNCTION ON THE GROUP

The size or "amount" of rotation indicated by a vector, β, of Euler parameters can be measured directly by calculating the angle θ in the definition given by Eq. 4. A comparison of the "difference" between two orientations, β' and β, can be made by using the group operation and inverse as a means of "subtraction." On writing

$$\beta'' = S(\beta')^T \beta \quad , \tag{80}$$

we could calculate the angle θ'' that corresponds to β''; however, there are equivalent measures of the amount of rotation that are simpler to calculate. For example, we can define a metric on the rotation group as follows:

$$\rho(\beta,\beta') = |\sin \theta''/2| \quad , \tag{81}$$

where θ'' is as defined above. The reason this is simpler is that it eliminates trigonometric calculations by simplifying to

$$\rho(\beta,\beta') = \sqrt{1 - (\beta^T\beta')^2} \quad . \tag{82}$$

This can be seen by observing that the first element of Eq. 80, β_0'', is $\beta^T\beta'$, which is also $\cos \frac{\theta''}{2}$. Thus

$$(\cos \frac{\theta''}{2})^2 = (\beta^T\beta')^2 \quad , \tag{83}$$

and Eq. 82 follows directly. Note that

$$\rho(\beta,-\beta) = 0 \quad , \tag{84}$$

which means that ρ is a metric on the rotation group and not on the unit sphere in R^4 because it does indeed identify antipodal points. We have not shown that ρ satisfies the definition of a metric, in particular the triangle inequality. This calculation is tedious but not difficult.

An important observation is the invariance of ρ under rotation,

$$\rho[S(\alpha)\beta,\ S(\alpha)\beta'] = \rho(\beta,\beta') \quad , \tag{85}$$

which follows directly from the invariance of $\beta^T\beta'$ under such an operation. It is also interesting to note that the function ρ can be written as

$$\sqrt{1 - \frac{(\beta^T\beta')^2}{|\beta|^2\ |\beta'|^2}}$$

and extended to all of R^4, where the projection back to the unit sphere is incorporated in ρ directly. Thus, one should beware of numerical errors in the length of β when applying ρ in the form Eq. 82, which is not corrected for length.

This metric notion of the amount of rotation leads directly to the concept of the scalar magnitude of a rotation defined as

$$\|\beta\| = \rho(\beta, b_0) \quad , \tag{86}$$

so that $\|b_0\| = 0$, $\|\beta\| \leq 1$ for any β and, from Eqs. 81 and 4,

$$\|\beta\| = |\sin\theta/2| \quad . \tag{87}$$

We can speak of a differential rotation

$$\delta\beta = \begin{cases} \cos\frac{d\theta}{2} \\ u_i \sin\frac{d\theta}{2} \quad , \quad i = 1,2,3 \end{cases} \tag{88}$$

as in Eq. 4, which has a differential scalar magnitude although it is a unit vector in R^4. The first component is large and is not $d\beta_0$ in the usual linear sense; for this reason we choose the Greek δ rather than d, emphasizing this distinction.

The calculus of these differential rotations and of distributions or functions defined on the rotation group can be extracted from the more general and abstract methods presented in Refs. 6 through 10; however, in an effort to gain clarity and insight, we choose to develop methods in the specific context not only of the rotation group but of the particular representations of it that are most familiar, such as Euler angles, Euler parameters, etc.

Let us denote by G the rotations as an abstract group, independent of any representation by parameters. Suppose f is a real-valued function defined on G. We will find it useful to be able to integrate f over all or part of G, such as one integrates probability densities to obtain probabilities. But the symbols

$$\int_G f(g)dg \tag{89}$$

are not well defined without some definition of what is meant by dg.

We need a "measure" on G, and the symmetry or uniformity of G ought to imply that for any fixed rotation $g_0 \epsilon G$,

$$\int_G f(gg_0)dg = \int_G f(g)dg \quad , \tag{90}$$

that is, that this measure is uniform over G. Take, now, as an example a representation of G, the z-x-z Euler angles, which we denote by (ϕ,θ,ψ). Write g as a function of these so that $f(g) = f(\phi,\theta,\psi)$ and

$$\int_G f(g)dg = \int_{0,0,0}^{2\pi,\pi,2\pi} f(\phi,\theta,\psi)I(\phi,\theta,\psi)d\phi d\theta d\psi \ . \tag{91}$$

Ref. 6. I. M. Gel'fand and Z. Ya. Šapiro, "Representations of the Group of Rotations in Three-Dimensional Space and Their Applications," AMS Trans., Series 2, Vol. 2 (1956).

Ref. 7. F. D. Murnaghan, The Theory of Group Representations, Dover (1963) (also pub. by Johns Hopkins Press, 1938).

Ref. 8. U. Grenander, Probabilities on Algebraic Structures, Wiley (1963).

Ref. 9. C. Chevalley, Theory of Lie Groups, Vol. I, Princeton (1946).

Ref. 10. H. Flanders, Differential Forms, Academic Press (1968).

If we can find a "weight function," I, with the property, Eq. 90, and normalize it so that

$$\int_G dg = 8\pi^2 \quad , \tag{92}$$

where

$$dg = I(\phi,\theta,\psi)d\phi d\theta d\psi \quad , \tag{93}$$

then Eqs. 91 and 93 can become a definition of dg that makes Eq. 89 well defined. The choice of $8\pi^2$ will be clear later.

Intuitively, we see that for the differential rotations $(d\phi,d\theta,d\psi)$ to be orthogonal, we must have $\theta = \frac{\pi}{2}$, for which $d\phi$ is about z, $d\theta$ about x, and $d\psi$ about y. The alignment of the ϕ and ψ angles when θ is small means that the $I(\phi,\theta,\psi)$ value ought to be small too. As it turns out, $I(\phi,\theta,\psi) = \sin\theta$ works. In this example, the key feature of this weighting function is that it represents the nonorthogonality of the three differential rotations. The invariant (under rotation) quantities of interest are the "differential volume," represented by differential rotations in three degrees of freedom, and the "relative orthogonality" of these axes or "angles" between them. The function $I(\phi,\theta,\psi) = \sin\theta$ represents the volume of a parallelepiped whose edges are unit vectors in the directions of these three Euler rotations as viewed in a fixed (not the running z-x-z) coordinate frame. It corrects the quantity $d\phi d\theta d\psi$ to the differential volume, dg, that is needed as an invariant under rotation so that the integral satisfies Eq. 90.

We can do the same thing using the Euler parameters. Since there are four of these and only three degrees of freedom in the rotation group, the differential volume will be a "three-form" in the exterior algebra over R^4, an elaborate way of saying that the previous

$$dg = I(\phi,\theta,\psi)d\phi d\theta d\psi$$

will be replaced by

$$\begin{aligned} dg = {} & I_0(\beta)d\beta_1 d\beta_2 d\beta_3 - I_1(\beta)d\beta_0 d\beta_2 d\beta_3 \\ & + I_2(\beta)d\beta_0 d\beta_1 d\beta_3 - I_3(\beta)d\beta_0 d\beta_1 d\beta_2 \end{aligned} \tag{94}$$

and $I(\beta)$ (a vector) is to be chosen to make dg an invariant under rotations. For notational simplicity Eq. 94 will be written

$$dg = I(\beta)^T *d\beta \quad , \tag{95}$$

where $*d\beta$ denotes a vector (* is the Hodge star operator, Ref. 10),

$$*d\beta = \begin{bmatrix} d\beta_1 d\beta_2 d\beta_3 \\ -d\beta_0 d\beta_2 d\beta_3 \\ d\beta_0 d\beta_1 d\beta_3 \\ -d\beta_0 d\beta_1 d\beta_2 \end{bmatrix} \quad . \tag{96}$$

Next, from Eq. 88 we see that at $\beta = b_0 = (1,0,0,0)$ we ought to have $dg = 8\ d\beta_1 d\beta_2 d\beta_3$. Thus we conclude we want $I(1,0,0,0) = (8,0,0,0)$. The requirement of invariance is, in our current notation,

$$I(\beta')^T *d\beta' = I(\beta)^T *d\beta \tag{97}$$

for any choice of β and β'. Fortunately, we know how Euler parameters are related; there exists an α such that

$$\beta' = S(\alpha)\beta \quad . \tag{98}$$

From this it follows that, considering α a constant,

$$d\beta' = S(\alpha)d\beta \tag{99}$$

and

$$*d\beta' = S(\alpha)*d\beta \quad , \tag{100}$$

which can be argued from the framework of exterior algebra in which "*" and "d" are linear operators, or by accepting Eq. 99 on the basis of "d" being linear and calculating Eq. 100 from the definition of $*d\beta$.

In any event, multiplying Eq. 100 on the left by the 1×4 matrix $I(\beta')^T$,

$$I(\beta')^T *d\beta' = I(\beta')^T S(\alpha) *d\beta \quad , \tag{101}$$

and thus

$$I(\beta)^T = I(\beta')^T S(\alpha) \quad , \tag{102}$$

$$I(\beta') = S(\alpha) \quad I(\beta) \quad , \tag{103}$$

which holds for any β, specifically $\beta = (1,0,0,0)$. So we obtain by substituting this β, and $\beta' = \alpha$,

$$I(\alpha) = 8\alpha \quad . \tag{104}$$

Thus the invariant differential volume element is

$$\begin{aligned} dg = 8(&\beta_0 d\beta_1 d\beta_2 d\beta_3 - \beta_1 d\beta_0 d\beta_2 d\beta_3 \\ &+ \beta_2 d\beta_0 d\beta_1 d\beta_3 - \beta_3 d\beta_0 d\beta_1 d\beta_2) \quad . \end{aligned} \tag{105}$$

This differential form is defined on the surface of the unit ball in R^4, so we can apply Stokes' Theorem and actually evaluate the integral to obtain

$$\int_{|\beta|=1} dg = 32 \int_{|\beta| < 1} d\beta_0 d\beta_1 d\beta_2 d\beta_3 = 16\pi^2 \quad . \tag{106}$$

This integral is twice as large as Eq. 92 because of the bivalued representation of the rotation group; that is

$$\int_G dg = \frac{1}{2}\int_{|\beta|=1} dg = 8\pi^2 \tag{107}$$

as in Eq. 92. Note that the topological and geometric properties result in expressions symmetric in the four components of β (Eqs. 105 and 82); the uniqueness of β_0 as in Eq. 4 appears in expressions relating to the group structure and in particular the group identity.

Having thus demonstrated an invariant volume element, dg, we can think about probability densities on G, those functions, f, normalized to have their integral Eq. 89 over the group be unity. For example, the constant

$$f(g) = \frac{1}{8\pi^2}$$

is one such and represents the "equal likelihood" density function, or the "uniform density function." The finite volume of the group and the existence of a uniform density function do not parallel the structure of the real line and the theory of probability distributions of real variables. Another major distinction is that the properties of linear spaces are enjoyed by the real line and not by the rotation group. Both, however, have a group structure that allows the fundamental operation of convolution to be defined. A theory of Fourier transforms can be developed for G as it can for the real line (Refs. 7 and 8).

We are going to investigate an analog of the multivariate normal density on Euclidean space, the probability density function

$$p_D(\beta) = k(D)\, e^{-\frac{1}{2}\beta^T D\beta} \quad , \tag{108}$$

where D is a real 4 × 4 symmetric matrix, and k(D) is a normalization constant defined by

$$1 = \int_G p_D(\beta)dg = \frac{1}{2}\, k(D) \int_{|\beta|=1} e^{-\frac{1}{2}\beta D\beta} dg \quad . \tag{109}$$

There are two primary motivations for dealing with this choice: first, given two such functions, the product is another such after normalization and admits a maximum-likelihood estimation analogous to that for the multivariate normal probability functions; and second, densities of this form represent well the information obtained from single measurements of commonly used attitude determination instruments on spacecraft.

A number of facts are immediately obvious. The choice of D to represent the density function is not unique. Let $D' = D + \lambda I$, λ a constant, then

$$\beta^T D'\beta = \lambda + \beta^T D\beta \quad , \tag{110}$$

and

$$p_{D'}(\beta) = k(D')\, e^{-\lambda/2}\, e^{-\frac{1}{2}\beta^T D\beta} \quad . \tag{111}$$

Therefore, $p_D(\beta)$ and $p_{D'}(\beta)$ differ at most by a constant, but by Eq. 109 they must be equal:

$$p_D(\beta) = p_{D'}(\beta)$$

and

$$k(D') = k(D)\, e^{\lambda/2} \quad . \tag{112}$$

We also can observe that the uniform distribution is a member of this class of functions. Letting $D = \lambda I$, the dependence on β is eliminated, for example, with $\lambda = 1$:

$$p_I(\beta) = k(I)\ e^{-\frac{1}{2}\beta^T\beta} = k(I)\ e^{-\frac{1}{2}} , \tag{113}$$

and thus

$$k(I) = \frac{e^{1/2}}{8\pi^2} . \tag{114}$$

What we have is a one-to-one correspondence between these density functions and the equivalence classes of real symmetric 4 × 4 matrices defined by the equivalence relation that two matrices are equivalent if their difference is a constant times the identity matrix. This is the reason we have not mentioned, for example, positive definiteness; it is not a class property, although each class clearly has a positive definite representative.

We cannot tie estimation to the "expectation" of β (nor even define it) because G is not a linear space, so we must appeal to maximum likelihood for an estimation of β. If we were dealing with a probability over the full four-dimensional space, we could solve for the peak value of p_D (the most likely β) by differentiating with respect to β and setting this equal to (0,0,0,0):

$$\frac{d\ p_D(\beta)}{d\beta} = -k(D)\ e^{-\frac{1}{2}\beta^T D\beta}\ (\beta^T D) . \tag{115}$$

But this leads only to $\beta^T D = 0$ and $\beta = 0$, which is not very useful. Incorporating the constraint $|\beta| = 1$, however, we can find the desired solution by determining points for which the gradient of p_D is parallel to β. From among these we select the one for which p_D is largest. When

$$\frac{d\ p_D(\beta)}{d\beta} = c\ \beta^T , \tag{116}$$

with c constant, the directional derivatives perpendicular to β are 0, these being the only directions that preserve $|\beta| = 1$.

This equation is written, from Eqs. 115 and 116, as

$$- p_D(\beta)\ (\beta^T D) = c\beta^T \quad . \tag{117}$$

Solutions of this equation are the eigenvectors of D, for which there are four real eigenvalues, $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. Let a set of orthogonal eigenvectors corresponding to these be $\alpha_1, \ldots, \alpha_4$ and suppose the λ_i are denumerated in ascending order. We now have

$$p_D(\alpha_i) = k(D)\ e^{-\frac{1}{2}\lambda_i} \quad , \tag{118}$$

so p_D is largest for the smallest of the λ_i, namely λ_1, although α_1 may not be unique, as is the case for p_I, for example.

Another important property is that a probability density function may be "rotated." If we form

$$p_D[S(\alpha)\beta] = k(D)\ e^{-\frac{1}{2}\beta^T S(\alpha)^T D S(\alpha)\beta} \quad , \tag{119}$$

we obtain $p_{D'}(\alpha)$, where

$$D' = S(\alpha)^T\ DS(\alpha) \tag{120}$$

and by the invariance of the normalizing integral,

$$k(D') = k(D) \quad . \tag{121}$$

That D' is symmetric is obvious; it clearly has the same eigenvalues as D and eigenvectors $S(\alpha)^T\ \alpha_i$ corresponding to them. For example, setting $\alpha = \alpha_1$ makes b_0 [b_0 =(1,0,0,0)] the most likely orientation.

As was previously mentioned, the product of p_D and $p_{D'}$ is a function, $p_{D''}$, of the same type:

$$p_{D'}(\beta)\ p_D(\beta) = k(D')\ k(D)\ e^{-\frac{1}{2}\beta^T (D+D')\beta} \ . \tag{122}$$

On writing

$$D'' = D+D' \quad , \tag{123}$$

we find $p_{D''}$ is normalized according to

$$p_{D''}(\beta) = \frac{k(D+D')}{k(D)\,k(D')}\, p_{D'}(\beta)\, p_D(\beta) \quad . \tag{124}$$

There is a similarity here to the process of combining two linear least-squares estimates based on independent sets of observations. These "D" matrices play the role of the "information matrix" or the inverse of the "error covariance estimate." It is the differences rather than the similarities that are of interest. As we have shown, the "zero information" matrix is not only the zero matrix but any constant times the identity. Addition of any of these "zero information" matrices leaves the corresponding density function, p_D, unchanged (after normalizing).

In the case of very good information about the most likely orientation, or small error covariances, these probability density functions ought to behave like the multivariate normal. To see that this is the case, let us look at a D matrix whose density function, p_D, is concentrated in the vicinity of b_0, the group identity. First, b_0 being a eigenvector implies

$$Db_0 = \lambda b_0 = \begin{pmatrix} \lambda \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad , \tag{125}$$

so, by symmetry of D,

$$D = \left[\begin{array}{c|ccc} \lambda & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & X & \\ 0 & & & \end{array}\right] \quad , \tag{126}$$

where X is a symmetric 3 × 3 matrix. We want the eigenvalues of X to be large in comparison to λ; as with all such D matrices, we are free at this point to subtract λI from D, the effect being absorbed in the normalization function, $k(D')$, where $D' = D-\lambda I$. For small variations around b_0, the three degrees of freedom reside in the last three components of $\delta\beta$, as we have seen in Eqs. 4 or 88. Thus

$$D' = D-\lambda I = \left[\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & X' & \\ 0 & & & \end{array}\right] \tag{127}$$

does indeed look like a multivariate normal density in the three independent differential rotations about the x, y, and z axes under the correspondence

$$X' = \frac{Y^{-1}}{4}, \tag{128}$$

where $Y = E(\theta\theta^T)$, $\theta = (\theta_x, \theta_y, \theta_z)$, and E is the expectation operator.

Noting that the correspondence in Eq. 128 arises from the approximation $\sin \frac{\theta}{2} \approx \frac{\theta}{2}$ for small θ in the definition of β_1, β_2, and β_3, and that β_0 involves $\cos \frac{\theta}{2}$, it is natural to ask whether D^{-1} might be obtainable as $E(\beta\beta^T)$. Certainly the matrix $E(\beta\beta^T)$ exists and is symmetric for any probability density, p_D. For the uniform density function Eq. 113, it is not difficult to evaluate $E(\beta\beta^T)$,

$$\frac{1}{2} \int_{|\beta|=1} p_I(\beta)\ \beta\beta^T\ dg = \begin{bmatrix} 1/4 & & & 0 \\ & 1/4 & & \\ & & 1/4 & \\ 0 & & & 1/4 \end{bmatrix}, \tag{129}$$

where dg is given in Eq. 105. This gives $D = 4I$ and shows that in this case

$$D^{-1} = E(\beta\beta^T) \tag{130}$$

for an appropriately selected D from the class (of scalar multiples of I) corresponding to the uniform density function. It is not difficult to show that each class $\{D+\lambda I, \lambda \text{ real}\}$, where D is symmetric, has one choice of λ that makes

$$\text{Trace } (D+\lambda I)^{-1} = 1 \quad . \tag{131}$$

The trace of $(D+\lambda I)^{-1}$ is the sum of the reciprocals of the eigenvalues of $D+\lambda I$. For large λ these reciprocals are all positive, but small; let μ_i be the eigenvalues of D and let λ decrease toward the largest quantities $-\mu_i$, and the trace of $(D+\lambda I)^{-1}$ will increase, reaching 1 at some value of λ.

The significance of this is in the fact that $E(\beta\beta^T)$ must have

$$\text{Trace } E(\beta\beta^T) = 1 \tag{132}$$

because $\beta^T\beta = 1$. Is it the case, we may ask, that given any real symmetric (4×4) matrix D

$$E(\beta\beta^T) = \frac{k(D)}{2} \int_{|\beta|=1} \beta\beta^T \quad e^{-\frac{1}{2}\beta^T D\beta} dg = (D+\lambda I)^{-1} \text{ ?} \tag{133}$$

Here λ would be that number for which Eq. 131 is satisfied. No attempt here will be made to resolve this speculation.

Consider the diagonalization of a real symmetric (4×4) matrix D by an orthogonal matrix of determinant 1:

$$D = T(\alpha,\gamma)^T \Lambda T(\alpha,\gamma) \tag{134}$$

where $T(\alpha,\beta) = R(\alpha)S(\beta)$ as in Eq. 20, and

$$\Lambda = \begin{bmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} . \tag{135}$$

This clearly generalizes Eqs. 119 and 120 to two types of "rotations" that arise because of the noncommutativity of composition of rotations. The same argument used to establish Eq. 121 shows that

$$k(D) = k(\Lambda) \quad . \tag{136}$$

The density function Eq. 108 can thus be reduced by a change of variable that diagonalizes the quadratic form, $\beta^T D\beta$, to the function

$$p_\Lambda(\delta) = k(\Lambda)\ e^{-\frac{1}{2}\delta^T\Lambda\delta} \tag{137}$$

where Λ is obtained from D by the change of variables

$$\delta = R(\alpha)\ S(\gamma)\beta \quad . \tag{138}$$

To accomplish this reduction to standard form Eq. 137, we required not one but two transformations of the stochastic variable, δ, in defining the new stochastic variable, β. By commutativity of R and S matrices (Eq. 16) and relationship, Eq. 15, this transformation can also be written

$$S(\delta) = S(\gamma)\ S(\beta)\ S(\alpha) \quad , \tag{139}$$

where the nature of α as a "precedent" rotation and γ as a "subsequent" rotation is more clearly shown.

6. ESTIMATION OF ORIENTATION

The fundamental result of the previous section that we will draw on is the means of calculating the most likely orientation, β^*, from a probability density function, $p_D(\beta)$. It was shown that to find β^* from D it was necessary to look for the eigenvector corresponding to the smallest eigenvalue. This answer may not be unique; two eigenvalues may be equal or nearly so. The other feature of the family of functions chosen is that the product of two is also in the same class and the matrix resulting from such an operation is the sum of the two matrices for the two factors (Eq. 123).

So far we have seen only one specific example of a D matrix interpreted physically: D = 4I, which represents "no information" and generates the uniform probability density. Figures 4 and 5 show two examples of the shape of the function

$$f(\beta) = e^{-\frac{1}{2}\beta^T D \beta}$$

plotted for the two-dimensional analog of the Euler parameters. The inner curve is the unit circle, $|\beta| = 1$, and the outer curve has its radius increased to $1 + f(\beta)$. In Fig. 4, the eigenvectors of D are aligned with the coordinate axes; in Fig. 5, they have been rotated as indicated. Following the discussion of the matrix given in Eq. 126, we see that if

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 \\ 0 & 0 & d_2 & 0 \\ 0 & 0 & 0 & d_3 \end{bmatrix} , \qquad (140)$$

where d_1, d_2, d_3 denote any positive real values very much larger than 1, the function $p_D(\beta)$ is a maximum at $\beta^* = (1,0,0,0)$. A matrix D' having an equally well defined eigenvector at an arbitrary point β' is given by

$$D' = S(\beta')\, DS(\beta')^T \quad , \qquad (141)$$

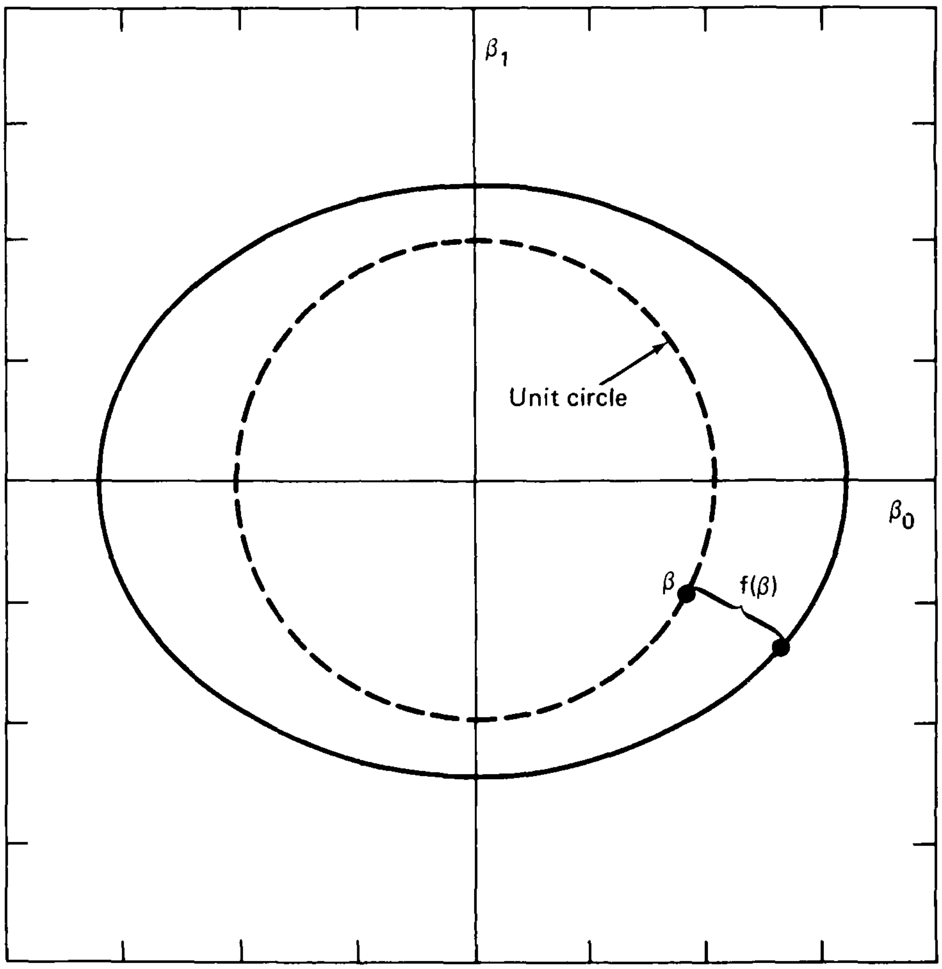

Fig. 4 Showing the shape of f (β) for β on the unit circle.
$$f(\beta) = e^{-\frac{1}{2}\beta^T D \beta},\ D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

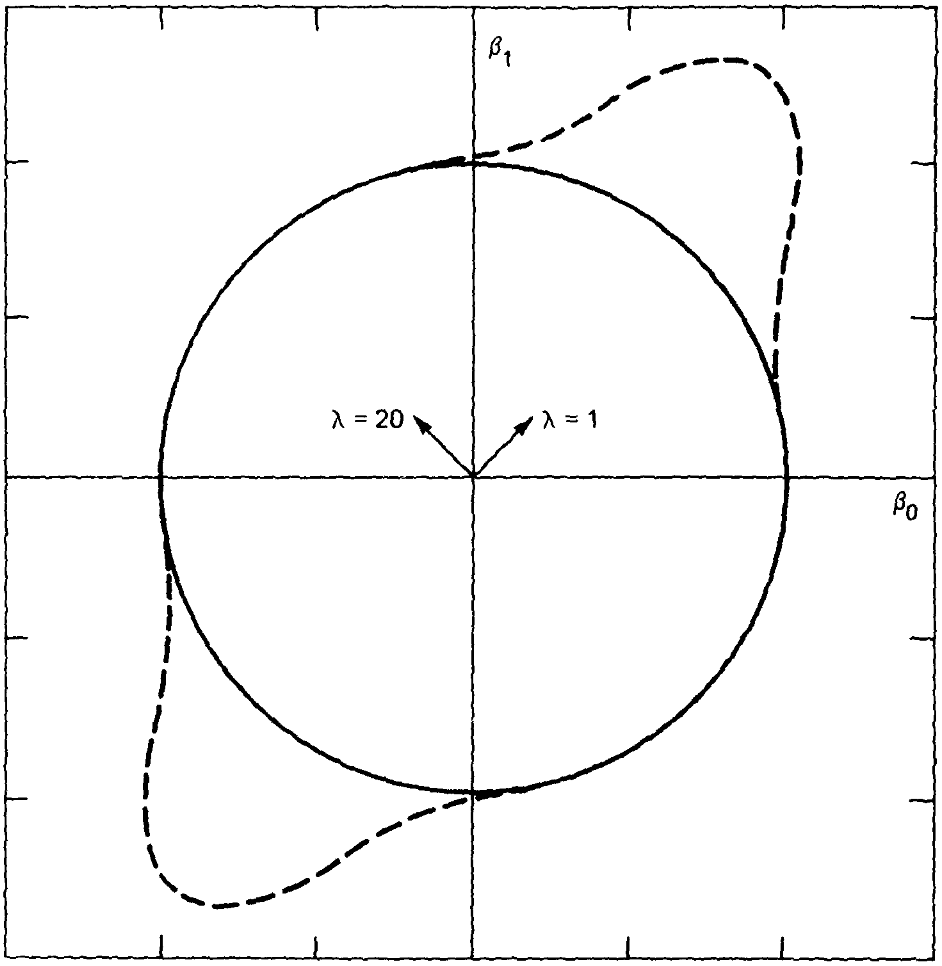

Fig. 5 Showing the shape of $f(\beta)$ for β on the unit circle.

$$f(\beta) = e^{-\frac{1}{2}\beta^T D\beta},\ D = Q^T \Lambda Q,\ \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 20 \end{bmatrix},\ Q = \begin{bmatrix} \cos 0.3\pi & \sin 0.3\pi \\ -\sin 0.3\pi & \cos 0.3\pi \end{bmatrix}.$$

since $S(\beta')^T$ carries β' into (1,0,0,0). There is another choice at hand that has this same eigenvector:

$$D'' = R(\beta')\, DR(\beta')^T \quad . \tag{142}$$

The distinction between D' and D" when $\beta' \neq (1\ 0\ 0\ 0)$ is that of the order in which β' and the stochastic variable are applied as was discussed at the end of the previous section. An easy way to visualize this is to let β' become a function of time with $\beta'(0) = (1\ 0\ 0\ 0)$. Now

$$S(\beta')^T\, D'\, S(\beta')$$

equals D and is constant in time, whereas

$$D'''(t) = S(\beta')^T\, D'' S(\beta') = \Sigma(\beta')^T\, D\, \Sigma(\beta') \quad , \tag{143}$$

where $\Sigma(\beta')$ is defined in Eq. 17 and $D'''(t)$ is clearly the representation of the constant matrix D in a coordinate frame that rotates according to $\beta'(t)$; by this we mean that the choice of the three coordinate axes in Eqs. 1 and 4 is a rotating frame, as is clear from Eq. 17. Note then that the distinction between Eqs. 141 and 142 must disappear in the special case where $d_1 = d_2 = d_3$.

Next, consider the case of a satellite attitude measuring system such as a vector magnetometer or solar attitude detector that provides at some instant of time the information that a body-fixed unit vector, u, is aligned with an inertial unit vector, U. No information is provided about the orientation of the spacecraft around that axis. Given only these data, all orientations around this axis are equally likely. Starting with any α such that

$$C(\alpha)\, u = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} , \tag{144}$$

let

$$D' = \Sigma(\alpha)^T \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \Sigma(\alpha) \quad , \tag{145}$$

where again $d \gg 1$ (depending on the precision of the measuring instrument). The probability density function $p_{D'}(\beta)$ now takes on its maximum at $\beta = (1,0,0,0)$, but this point is not unique. Since any β of the form

$$\beta' = \begin{pmatrix} \cos\frac{\theta}{2} \\ u \sin\frac{\theta}{2} \end{pmatrix} \tag{146}$$

is carried by $\Sigma(\alpha)$ into

$$\begin{pmatrix} \cos\frac{\theta}{2} \\ 0 \\ 0 \\ \sin\frac{\theta}{2} \end{pmatrix} ,$$

which is also an eigenvector of the diagonal matrix in Eq. 145, it follows that β' is an eigenvector of D' having an eigenvalue of 2. Such a set of β' describes all possible orientations obtained by rotations about the vector u. Since the vector u is fixed in the body, we next wish to precede such rotations by one carrying the inertial vector U into u; call this vector δ.

$$u = C(\delta)\, U \quad . \tag{147}$$

Writing

$$D = R(\delta)\, D' R(\delta)^T \quad , \tag{148}$$

we can show that δ is an eigenvector of D. Since δ is carried by $R(\delta)^T$ into (1,0,0,0), which we have shown to be an eigenvector of D' with eigenvalue 2 (the smaller of 2 and d), δ is an eigenvector of D and $p_D(\beta)$ takes on an extremum (maximum) at δ. Each of the other equally likely orientations arises from a δ' such that

$$\beta' = R(\delta)^T \delta' \quad . \tag{149}$$

From this equation, we find

$$\delta' = S(\beta')\delta \quad , \tag{150}$$

which is interpreted as δ' being the rotation δ followed by the rotation β' (Eq. 146).

Next we shall give a numerical example to show how two such observations may be combined by the rule of adding D-matrices (multiplying density functions) to yield a D-matrix whose eigenvector having the smallest eigenvalue provides a uinque solution of the satellite attitude. Consider first the simplest case in which the body-fixed and inertial frames happen to be aligned; the Euler parameters describing this relationship are (1,0,0,0), the identity. Suppose we have two observations, one of the body z-axis and one of the body x-axis. To describe the first by a D-matrix, D_1, we need to find α and δ according to Eqs. 144 and 147. In this case we may choose (neither choice is unique) $\alpha = \delta = (1,0,0,0)$ because both u and U are the respective z-axis unit vectors, (0,0,1), of their coordinate frames. We obtain

$$D_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \tag{151}$$

and will reserve the choice of d.

Next, the second observation is defined by u = (1,0,0) and U = (1,0,0), and we assume this is made simultaneously with the first. Here again, δ may be selected as (1,0,0,0). The choice of α must provide the point transformation (see Appendix A) of (1,0,0) into (0,0,1), so let

$$C(\alpha) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \tag{152}$$

which we obtain from

$$\alpha = \begin{pmatrix} \cos 45^\circ \\ 0 \\ \sin 45^\circ \\ 0 \end{pmatrix} \tag{153}$$

in Eq. 6. This yields

$$D_2 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & d \end{bmatrix} . \tag{154}$$

Now we can write the matrix of combined information:

$$D = D_1 + D_2 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & d+2 & 0 & 0 \\ 0 & 0 & 2d & 0 \\ 0 & 0 & 0 & d+2 \end{bmatrix} , \tag{155}$$

which, for large d has a unique eigenvector, (1,0,0,0), corresponding to its smallest eigenvalue (4).

As a second example, we let the orientation of the spacecraft be (all calculations carried 15 digits although we show fewer here)

$$\beta^* = \begin{pmatrix} 0.98255 \\ 0.04971 \\ 0.09942 \\ 0.14913 \end{pmatrix}$$

and

$$C(\beta^*) = \begin{pmatrix} 0.93575 & 0.30293 & -0.1804 \\ -0.28316 & 0.95058 & 0.12733 \\ 0.21019 & -0.06803 & 0.97529 \end{pmatrix} .$$

We pick two vectors in inertial coordinates as observations

$$U1 = \begin{pmatrix} -0.26726 \\ -0.53452 \\ 0.80178 \end{pmatrix}$$

and

$$U2 = \begin{pmatrix} 0.96309 \\ -0.24077 \\ 0.12039 \end{pmatrix} .$$

These are mapped, respectively, into

$$u1 = \begin{pmatrix} -0.55677 \\ -0.33033 \\ 0.76216 \end{pmatrix}$$

and

$$u2 = \begin{pmatrix} 0.80654 \\ -0.48626 \\ 0.33622 \end{pmatrix}$$

by $C(\beta^*)$ as in Eq. 147. Now treating these as observations with $d = 10000$, we reconstruct β^* from the sum of the two D-matrices (Eq. 148):

$$D1 = \begin{pmatrix} 319.63 & -712.53 & -1213.32 & -1046.39 \\ -712.53 & 8194.63 & -1929.06 & 3249.87 \\ -1213.32 & -1929.06 & 7917.01 & 3360.56 \\ -1046.39 & 3249.87 & 3360.56 & 3572.71 \end{pmatrix},$$

$$D2 = \begin{pmatrix} 330.32 & -112.05 & -1133.35 & -1370.29 \\ -112.05 & 1905.54 & 3311.83 & -2104.12 \\ -1133.35 & 3311.83 & 8501.14 & 697.31 \\ -1370.29 & -2104.12 & 697.31 & 9266.99 \end{pmatrix},$$

and

$$D1+D2 = \begin{pmatrix} 649.95 & -824.59 & -2346.67 & -2416.69 \\ -824.59 & 10100.17 & 1382.76 & 1154.74 \\ -2346.67 & 1382.76 & 16418.16 & 4057.88 \\ -2416.69 & 1145.74 & 4057.88 & 12839.70 \end{pmatrix}.$$

The eigenvalues and eigenvectors were determined by the Jacobi method and returned β^* correct to at least ten digits.

When observations are not taken simultaneously, but the motion of the satellite is known over the interval between observations, estimation of the orientation is still possible. From knowledge of $\omega(t)$ in body-fixed coordinates we can produce the transition matrix

$$\Phi(t_2,t_1) = S(\beta_2)\ S(\beta_1)^T \tag{156}$$

as in Eq. 31 without actually knowing β_1 or β_2, which represent the spacecraft orientation at t_1 and t_2 when the two observations are taken. The estimate of β_1 can be based on

$$D = \Phi^T\ D_2\Phi + D_1 \quad , \tag{157}$$

where D_1 represents the observation at t_1, and D_2 that at t_2.

The extension of this method to the case in which ω is treated as a stochastic variable is not addressed here. We can speculate that the convolution of two density functions of the form Eq. 108 is another of this type, and further that if ω assumes a normal density that a differential equation for propagation of D-matrices would represent the "diffusion" of these probability densities and a sort of rotational Brownian motion as discussed in Ref. 8 and in other references as far back as 1928.

APPENDIX A

DEFINITION OF ROTATION MATRICES

The confusion that stems from the wide variety of definitions of coordinate frames, Eulerian angles, and rotation matrices cannot be laid aside easily. Reference 5, for example, very carefully discusses the freedom of choice of clockwise versus counterclockwise rotations as positive rotations, the concept of "running coordinates," and the order of matrix multiplication. Consistent with this, Goldstein writes as a rotation about z in a right-handed system with counterclockwise rotations being positive (see Fig. A-1)

$$D = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

He completely neglects to describe this as an "alias" or "coordinate transformation," and the casual reader might be surprised that for $\phi = +90°$ the matrix D carries (1,0,0) into (0,-1,0). The matrix $C(\beta)$ in Eq. 6 is similarly defined as a coordinate transformation.

Reference 3 provides a nice discussion of "alias" versus "alibi." In other works, notably those using tensor notation, the terms "covariant" and "contravariant" appear. The choice of the "alias" or "coordinate transformation" or "contravariant" form above is necessary if the matrix products are to be taken from right to left in "running coordinates." To remember this, consider the succession of two rotations, the first about z as above, and the second about the x-axis after it has been repositioned according to the first rotation. We choose to write the second matrix in "running coordinates;" that is, the axis of x-rotation is to be (1,0,0):

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} .$$

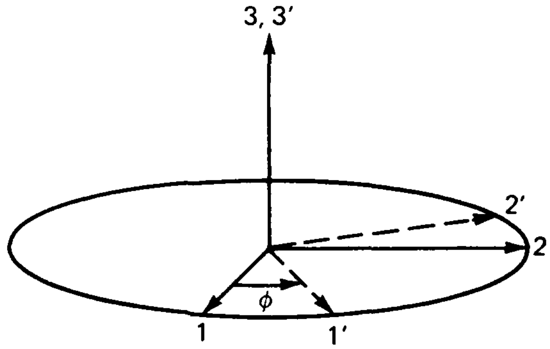

Fig. A-1 The counterclockwise rotation for positive ϕ, about the 3-axis, as in Ref. 5.

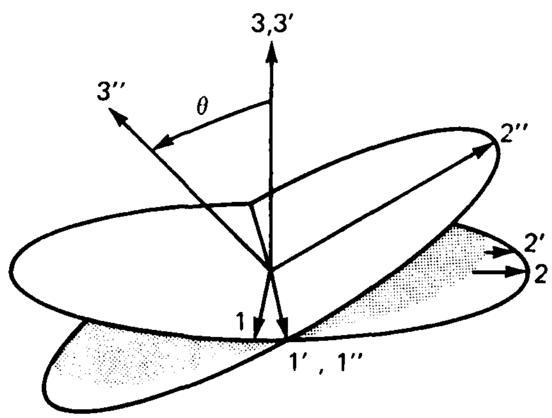

Fig. A-2 The second (subsequent) rotation, again counterclockwise, about the 1′ axis, as in Ref. 5.

See Fig. A-2 for a description of this relative to the first rotation. Now in the "alias" interpretation we fix a point whose coordinates are initially (cos ϕ, sin ϕ, 0). This point will be the final x-axis after repositioning the coordinate frame. Indeed, applying D then C yields (1 0 0).

Using the "point transformation" or "alibi" method, we would piece together this same rotation by first applying a rotation to map

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ,$$

This matrix is D^T. The subsequent rotation must now map

$$\begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} \sin\theta\sin\phi \\ -\sin\theta\cos\phi \\ \cos\theta \end{pmatrix}$$

and is clearly not C^T. It is in fact D^TC^TD, so the product is

$$(D^TC^TD)\, D^T = D^TC^T \quad ,$$

which is reasonably enough the transpose of the alias representation CD. Authors who prefer the alias representation probably feel that it is more natural because the matrices in running coordinates (which are easiest to write) are applied in the same order as are the successive rotations. Authors who prefer the alibi probably feel more comfortable watching a point rotate according to the right-hand rule, e.g., $(1,0,0) \rightarrow (\cos\phi, \sin\phi, 0)$ for "positive" rotation about z. The most common mistake is in selecting the alibi representation and incorrectly composing two rotations as C^TD^T (Ref. 6).

APPENDIX B

DEVELOPMENT OF $e^{S(\omega)t}$ for ω CONSTANT

$$S(\omega) = \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix} , \tag{B.1}$$

by definition, which it is convenient to partition, letting

$$g = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} , \tag{B.2}$$

into

$$S(\omega) = \left[\begin{array}{c|c} 0 & g^T \\ \hline g & G \end{array}\right] . \tag{B.3}$$

Murnaghan (Ref. 7) points out that G satisfies

$$G^3 = - |g|^2 G \tag{B.4}$$

and that therefore e^G is expressible as a quadratic polynomial in G. It is natural to expect the same of $S(\omega)$ and thus of $S(\omega)t$. We observe that

$$Gg = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{B.5}$$

and find then that

$$S(\omega)^2 = \left[\begin{array}{c|c} -g^T g & 0 \\ \hline 0 & G^2 - gg^T \end{array}\right] \qquad (B.6)$$

and

$$S(\omega)^3 = \left[\begin{array}{c|c} -g^T g & 0 \\ \hline 0 & G^2 - gg^T \end{array}\right] \left[\begin{array}{c|c} 0 & -g^T \\ \hline g & G \end{array}\right]$$

$$= \left[\begin{array}{c|c} 0 & g^T g^2 \\ \hline G^2 g - |g^2| g & G^3 - gg^T G \end{array}\right] \quad . \qquad (B.7)$$

On removing G^2g and gg^TG, which are zero by Eq. B.5, and substituting for G^3,

$$S(\omega)^3 = \left[\begin{array}{c|c} 0 & |g|^2 g^T \\ \hline -|g|^2 g & -|g|^2 G \end{array}\right] = - |g|^2 S(\omega) \quad . \qquad (B.8)$$

Thus, the evaluation of $e^{S(\omega)t}$ as a series

$$e^{S(\omega)t} = I_4 + S(\omega)t + \frac{S(\omega)^2 t^2}{2!} + \frac{S(\omega)^3 t^3}{3!} + \ldots , \qquad (B.9)$$

gives

$$e^{S(\omega)t} = I_4 + S(\omega) \left[t - \frac{|g|^2 t^3}{3!} + \frac{|g|^4 t^5}{5!} - \ldots \right]$$

$$+ S(\omega)^2 \left[\frac{t^2}{2!} - \frac{|g|^2 t^4}{4!} + \frac{|g|^4 t^6}{6!} - \ldots \right]$$

$$= I_4 + \frac{S(\omega)}{g} \sin|g|t + \frac{S(\omega)^2}{|g|^2} (1 - \cos|g|t)$$

$$= \left(I_4 + \frac{S(\omega)^2}{|g|^2} \right) + \frac{S(\omega)}{|g|} \sin|g|t - \frac{S(\omega)^2}{|g|^2} \cos|g|t .$$

(B.10)

The constant term is

$$\frac{1}{|g|^2}\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & G^2 \end{array}\right] . \tag{B.11}$$

From the equality

$$\beta(t) = S[\beta(t)]\ b_0, \qquad b_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{B.12}$$

and

$$S[\beta(t)] = e^{\frac{1}{2} S(\omega)t}$$

for constant ω and $\beta(0) = b_0$, we see as expected

$$\beta(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{|g|}\begin{pmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \sin |g|t/2 + \frac{1}{|g|^2}\begin{pmatrix} g^Tg \\ 0 \\ 0 \\ 0 \end{pmatrix} \cos|g|t/2$$

$$= \begin{pmatrix} 0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} \sin|g|t/2 + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cos|g|t/2. \tag{B.13}$$

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