ARMA SPECTRAL ESTIMATION: AN EFFICIENT CLOSED FORM PROCEDURE. (U)
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ARMA Spectral Estimation: An Efficient Closed Form Procedure

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ARMA SPECTRAL ESTIMATION:
AN EFFICIENT CLOSED FORM PROCEDURE

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ABSTRACT

In this report, an effective procedure for effecting an ARMA spectral model of a time series is described. This procedure is predicated on minimizing as set of "basic error" terms as generated from an ARMA model that is hypothesized as characterizing the time series under analysis. The spectral estimation performance achieved in using this approach has been empirically found to be generally superior to that obtained using such contemporary methods as maximum entropy and the periodogram.
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I. INTRODUCTION

In this report, a description of the high performance ARMA spectral estimator, as in-part developed under the AFOSR contract 80-0069, shall be given. To begin, the main objective of the project was that of developing an effective method for estimating the $a_k$ and $b_k$ coefficients governing the ARMA model

$$x(n) + \sum_{k=1}^{p} a_k x(n - k) = \sum_{k=0}^{q} b_k \epsilon(n - k)$$

(1)

in which the excitation time series $\{\epsilon(n)\}$ is taken to be white. These coefficients are to be selected so that this ARMA model is "most consistent" with a set of time series observations.

$$x(1), x(2), \ldots, x(N).$$

(2)

which are available. The term "most consistent" is here being used in the sense that the hypothesized ARMA model is most compatable with the task of minimizing a set of "basic error terms". A brief description of this procedure shall be now given while a more detailed description is to be found in the appendix.
II. DESCRIPTION OF ARMA MODELING METHOD

The basis for the ARMA method is dependent on the so-called basic error terms. These terms are generated by first multiplying each side of relationship (1) by the delayed entity \(x^*(n-M)\) to obtain

\[
e(m,n) = x(n)x^*(n-m) + \sum_{k=1}^{p} a_k x(n-k)x^*(n-m)
\]

\(\sum_{k=1}^{q+1} a_k\) \(\leq m < N\)

\(\max (m,p) \leq n \leq N\)

It can be shown that these basic error terms \(\varepsilon(m,n)\) are each zero mean when the underlying time series in an ARMA process of order less than or equal to \((q,p)\) and the \(a_k\) coefficients in expression (3) correspond to the exact process' autoregressive coefficients.

With this in mind, a method for selecting the ARMA models \(a_k\) coefficients is suggested. Namely, they are chosen so as to cause the basic error terms (3) to be as close to their mean value of zero as possible. This objective may be realized by introducing the following quadratic functional

\[
f(a) = e^\dagger We
\]

in which \(e\) is a column vector whose elements are appropriate arrangements of the ensemble of basic error terms (3), \(W\) is a nonegative matrix, \(a\) is the autoregressive coefficient vector with elements \(a_k\), and, \(+\) denotes the operation of complex conjugate transposition. It is readily shown that the minimization of functional (4) will result in a set of consistent
linear equations for the optimum ARMA model autoregressive coefficients.

Once the $a_k$ coefficients have been obtained, the ARMA model's moving average coefficients are estimated by first generating the so-called residual sequence $\{e(n)\}$ according to

$$e(n) = x(n) + \sum_{k=1}^{p} a_k x(n - k) \quad p + 1 \leq n \leq N \quad (5)$$

The spectrum of this residual time series is theoretically given by

$$| \sum_{k=0}^{q} b_k e^{-jkw} |^2.$$  

A technique for obtaining a moving average estimate of this residual time series is fully described in the appendix and entails the utilization of the smoothed periodogram. Once this smoothed periodogram has been generated, the required ARMA spectral estimate is achieved. Empirically derived results indicate that this report's procedure provides a superior spectral estimation performance than that achieved by such contemporary alternatives as the maximum entropy method, and the periodogram.

III. CONTRACT PUBLICATIONS

The following two refereed publications resulted from the sponsored AFOSR contract.


Autoregressive Moving Average Spectral Estimation: A Model Equation Error Procedure

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Abstract—A procedure is presented for generating an autoregressive moving average (ARMA) spectral model of a stationary time series based upon a finite set of time series' observations. The ARMA model's autoregressive coefficients are estimated by minimizing a quadratic function of a set of basic error terms. In examples treated to date, this method has demonstrated an exceptional ability in resolving closely spaced narrow band signals in a low signal-to-noise environment where other procedures such as the maximum entropy method often fail. Its effectiveness on other classes of time series also shows promise and a more general evaluation is presently being conducted. With this in mind, the new ARMA procedure promises to be an important spectral estimation tool.

I. INTRODUCTION

In this paper, we shall be concerned with the task of estimating the statistical characteristics of a stationary random time series \( x(n) \) from a finite set of observations. For many applications, knowledge of the time series' underlying autocorrelation sequence as formally defined by

\[
\rho_x(n) = E \{ x(n + k) x^*(k) \} \tag{1}
\]

conveys all the information required. In this expression, \( E \) and \( * \) denote the expected value and complex conjugation operations, respectively. It is often advantageous to equivalently characterize stationary time series in the frequency domain where their intrinsic properties may be more discernible. This is particularly true for so-called narrow-band processes. The vehicle for this characterization is the associated power spec-

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This approach often provides good estimates and does not be effective [2] and [3]. A more desirable closed-form convergence, and typically requires a relatively long data length performance is not always as good as that provided by AR methods.

A useful method for estimating the ARMA model's AR coefficients from the set of given observations will be now presented. This first entails multiplying each side of expression (4) by the entity \( x(n - m) \) to yield the "basic error terms" where the range on the \( m \) and \( n \) variables is dictated by the given time series range of 1 \( \leq n \leq N \) and the desire to have the random variables \( e(n - k) \) and \( x^*(n - m) \) be uncorrelated. If the time series is in fact an ARMA process of order less than or equal to \( (p, q) \) it follows that the basic error terms \( e(m, n) \) are each zero mean random variables. 1 This is a consequence of the causality of model (4) and the whiteness of the excitation which results in the random variables \( e(n - k) \) and \( x^*(n - m) \) being uncorrelated.

With these thoughts in mind, relationship (6) provides an ideal vehicle for determining a set of AR coefficients which are consistent with the given time series observations. Specifically, the \( a_k \) coefficients will be selected so as to cause each of the basic error terms (6a) to be as close to their mean value of zero as possible. This goal is achieved by minimizing a squared error criterion of the form

\[
f(a) = e^T We
\]

in which \( e \) is a column vector whose elements are appropriate arrangements of the ensemble of basic error terms (6a). \( W \) is a

\[ r_x(0) a^2 \sum_{k=0}^{q} |b_k|^2. \]
nonnegative definite square matrix, and, \( \dagger \) denotes the operation of complex conjugate transposition. This criterion is readily shown to be a function of the AR coefficients upon substitution of expression (6a) into criterion (7).

The weighting matrix \( W \) is to be selected based on statistical reasoning which must be tempered by an appreciation that little is generally known about the time series' statistics. To illustrate this point, let us consider the usually hypothesized condition that the excitation \( \{ e(n) \} \) is a Gaussian white process. In this case, if the matrix \( W \) is set equal to \( E \{ ee' \}^{-1} \), the minimization of criterion (7) will result in a maximum likelihood estimate of the AR coefficients. Unfortunately, the generation of this particular matrix requires knowledge of the time series' second-order statistics. This statistical information, however, is precisely what we are seeking to estimate via the ARMA model (4).\(^2\) This dilemma has arisen due to the unrealistic desire to seek a maximum likelihood ARMA model. More modest objectives must be sought if a tractable procedure is to be evolved.

A reasonable implementation of the above autoregressive coefficient selection philosophy then depends on a prudent choice for the weighting matrix. In this paper, we will be concerned exclusively with a choice that results in the following mean-squared error criterion:

\[
f(a) = \sum_{m=q+1}^{N-1} w(m) \left( \sum_{n=1}^{N} e(m,n) \right)^2
\]

(8)

although other choices are suggested in Section V. In this expression, the \( w(m) \) are nonnegative parameters which are logically selected to be inversely proportional to the variance of the term which they multiply. A general expression for these variance entities, unfortunately, also reveals a dependency on the time series' unknown second-order statistics and the ARMA model coefficients. Empirical evidence suggests, however, that a choice of the weighting parameters as given by

\[
w(m) = [N - m]^4 \quad q + 1 \leq m \leq N - 1
\]

(9)

results in satisfactory performance. An examination of other weighting parameter selections is currently being conducted.

Using standard calculus, it is readily shown that the set of AR coefficients which render criterion (8) a minimum must satisfy the following linear system of equations:

\[
C \hat{a} = c
\]

(10)

where \( \hat{a} \) is the \( p \times 1 \) AR coefficient vector estimate with elements \( a_k \), \( C \) is a \( p \times p \) nonnegative definite Hermitian matrix with elements

\[
c_{ik} = \sum_{n=1}^{N} \sum_{m=q+1}^{N} \sum_{s=1}^{N} w(m) x(n-k) x(s-m) \cdot x^*(n-m) x^*(s-i), \quad \text{for} \ i, k = 1, 2, \ldots, p
\]

(11a)

where \( t = 1 + \max(m,p) \) and \( c \) is a \( p \times 1 \) vector with elements

\[
c_i = \alpha_i, \quad \text{for} \ 1 \leq i \leq p.
\]

(11b)

Thus the optimum set of ARMA autoregressive coefficient estimates are obtained by solving the linear system of equations (10). It is possible to utilize the Hermitian structure of matrix \( C \) to yield an efficient solution procedure.

A. Forward and Backward Data Approach

One can more effectively use the observed data (3) to achieve an ARMA spectral estimate by using that data’s backward version as specified by

\[
\tilde{x}(1), \tilde{x}(2), \ldots, \tilde{x}(N)
\]

(12)

in which \( \tilde{x}(n) = x^*(N+1-n) \). In particular, one formulates a mean-squared error criterion as given by

\[
f(\hat{a}) = \sum_{m=q+1}^{N-1} w(m) \left\{ \sum_{n=1}^{N} e(m,n)^2 \right\}
\]

\[
+ \left\{ \sum_{n=1}^{N} \tilde{e}(m,n)^2 \right\}
\]

(13)

where \( \tilde{e}(m,n) \) denotes the basic error term associated with the backward sequence (12). It is a simple matter to show that the AR coefficient vector which minimizes this forward-backward criterion must satisfy the linear system of equations

\[
[C + \tilde{C}] \hat{a} = c + \tilde{c}
\]

(14)

where \( C \) and \( \tilde{C} \) are each \( p \times p \) nonnegative definite Hermitian matrices given by relationship (11a) with the forward (3) and backward (12) data used, respectively, and, the \( p \times 1 \) vectors \( c \) and \( \tilde{c} \) are given similar interpretations.

It is important to note that the rank of matrix \( [C + \tilde{C}] \) at least equals the rank of either of its constituent nonnegative definite matrices \( C \) or \( \tilde{C} \). Thus, in using this forward and backward approach to estimate the AR coefficients, one is typically able to use a higher order model (i.e., larger value of \( p \)) than would be the case for the forward or backward models only. Moreover, it has been empirically found that the forward and backward approach usually results in better spectral estimates.

III. Numerator Dynamics

A variety of procedures exist for determining the numerator dynamics of an ARMA time series once the AR coefficients have been estimated [4]-[9]. In this section, a procedure which has been found to produce improved results will be presented. It makes use of the characteristic equation (4) which can be interpreted as generating the auxiliary "residual" sequence \( \{ e(n) \} \) according to

\[
e(n) = x(n) + \sum_{k=1}^{p} a_k x(n-k)
\]

(15a)

\[
= \sum_{k=0}^{q} b_k e(n-k).
\]

(15b)

One may straightforwardly show that the spectrum of the moving average residual time series \( S_e(\omega) \) is given by \( |B(\omega)|^2 \sigma^2 \).
This observation in conjunction with relationship (5) provides the
vehicle for obtaining the underlying time series spectral es-
timate, that is
\[ S_x(\omega) = S_e(\omega) / |A(\omega)|^2. \]  
(16)

With this in mind, a method will be now given for obtaining a
qth-order moving average spectral estimate of the residual se-
quence. This estimate will be based on the specific residual
elements
\[ e(p+1), e(p+2), \ldots, e(N) \]  
(17)
which are calculated using relationship (15a) and the given
time series observations (3). The AR coefficient used in (15a)
will correspond to the \( a_k \) estimate obtained upon solving ex-
pression (14).

The approach to be now presented is an adaption of the
well-known method of Welch for obtaining smoothed peri-
odogram estimates \[10]. In essence, one first segments the
calculated residual elements (17) into \( L \) segments each of
length \( q + 1 \) as specified by
\[ e_k(n) = a(n) e(n + 1 + p + kd), \quad 0 \leq n \leq q \]
\[ 0 \leq k \leq L - 1 \]  
(18)
where \( a(n) \) is a data window and "\( d \)" a positive integer which
specifies the time shift between adjacent segments. These in-
dividual segments will overlap for a shift selection of \( d < q \).
Furthermore, to include only observed data, the relevant pa-
rameters must be selected so that \( p + q + 1 + (L - 1)d < N \).
Finally, the periodogram of each of the \( L \) segments is taken
and these are averaged to obtain the desired smoothed qth-
order periodogram, that is
\[ \tilde{S}_e(\omega) = \frac{1}{L} \sum_{k=0}^{L-1} \left\{ \frac{1}{q+1} \left| \sum_{n=0}^{q} a(n) e(n + p + kd)^{-j\omega n} \right|^2 \right\}. \]  
(19)
The data window is to be normalized according to \( \sum a^2(n) = 1 \).
In using this segmentation, the variance of the estimate \( S_e(\omega) \)
is reduced. The price paid for this reduction, however, is a loss
in frequency resolution, an increased bias of the estimate, and,
a possible deterioration in power level estimates. It must be
noted, however, that the overall resolution capability of this
ARMA procedure is predominately influenced by the AR co-
efficient selection. If one is basically interested in resolution
performance, an examination of the ARMA model's pole loca-
tions then need only be considered.

IV. NUMERICAL EXAMPLE

In order to compare the effectiveness of the new ARMA
spectral estimator with the maximum entropy method, the
classical problem of resolving two closely spaced (in fre-
quency) sinusoids in white noise will be considered. Specifi-
cally, the time series under study is specified by
\[ x(n) = \sqrt{20} \cos (0.4nm) + \sqrt{2} \cos (0.426mn) + w(n) \]  
(20)
where \( \{w(n)\} \) is a white Gaussian noise process of zero mean
and unity variance. The sinusoids of normalized frequencies
0.4 and 0.426 are readily calculated to have a signal-to-noise
ratio (SNR) of 10 dB and 0 dB, respectively. A sequence of
length 640 defined over $0 < n < 639$ was next generated using this relationship. Furthermore, in order to provide a statistical basis for our comparison, this 640 length sequence was then decomposed into ten disjoint sequences each of length 64 defined on $0 < n < 63, 64 < n < 127, \ldots, 576 < n < 639$. An ensemble consisting of ten subsequences each of length 64 has thereby been generated with each subsequence having a different noise sample and a different initial phase between the two sinu- Fordent. This latter condition is useful in revealing any po- ential noise sample and a different initial phase between the two tral estimation performance. Further investigation of this have given preliminary evidence of providing satisfactory spec- tral estimation performance. Further investigation of this most critical weighting matrix choice is now being conducted and will be subsequently reported upon.

The spectral estimates which resulted when the new ARMA spectral estimator (forward data only) and the maximum en- tropy (covariance) method were applied to these ten random time series samples are displayed in Figs. 1 and 2, respectively. The ordinates were scaled from 0 to 50 dB for each of the ten individual plots with each AR maximum entropy estimate be- ing of order thirty and each ARMA estimate being of order fif- teen (with $d = 16$). The spectral estimates are plotted one above the other (lower one corresponds to $0 < n < 63$) so as to better reveal the consistency of the estimates and to con- serve space. It is clear that the maximum entropy method is able to resolve the sinusoids in only five of the ten sample cases. On the other hand, the new ARMA method successfully resolves the sinusoids in each case and provides excellent fre- quency estimates of $f_1 = 0.3980$ and $f_2 = 0.4295$ obtained by averaging over the ten cases with associated sampled standard deviations of $a_{f_1} = 0.0027$ and $a_{f_2} = 0.0107$.

To illustrate the new ARMA method's effectiveness on long data length sequences, an eighth- and fifteenth-order ARMA model were next obtained using the entire sequence of length 640 as given above. The spectral estimates which resulted are displayed in Fig. 3 where a sharpness in frequency resolution is apparent. In addition, a thirtieth-order AR spectral estimate which arises from this 640-length sequence is shown in Fig. 4. Clearly, the new ARMA method has outperformed the AR maximum entropy method in both the short and long data length examples here considered.

V. CONCLUSIONS

A generalization of a recently developed ARMA spectral estima- tion method has been presented. This has included the intro- duction of an error weighting matrix, the concept of using forward and backward data, and, the utilization of Welch's method for obtaining estimates of the spectrum's numerator dynamics. Empirical evidence suggests that this new proce- dure has an improved spectral resolution performance when compared to popular contemporary methods. Its full poten- tial will be realized, however, only after a number of relevant issues are resolved. These include the task of determining good choices for the weighting matrix $W$, obtaining a data de- pending procedure for estimating the ARMA model order, and, investigating other numerator dynamics methods.

In addition to the specific weight matrix selection consid- ered in this paper, the following two choices of criterion

$$
\sum_{m=q+1}^{N-1} \sum_{m=m+1}^{N} |e(m,n)|^2
$$

and

$$
\sum_{n=q+2}^{N-1} w(n) \sum_{m=q+1}^{N} |e(m,n)|^2
$$

have given preliminary evidence of providing satisfactory spec- tral estimation performance. Further investigation of this most critical weighting matrix choice is now being conducted and will be subsequently reported upon.

REFERENCES

A procedure was devised for generating an autoregressive moving average (ARMA) spectral model of a stationary time series based upon a finite set of time series' observations. The ARMA model's autoregressive coefficients are estimated by minimizing a quadratic function of a set of basic error terms. In examples treated to date, this method has demonstrated an exceptional ability in resolving closely spaced narrow band signals in low signal-to-noise environment where other procedures such as the maximum entropy method often fail.
Its effectiveness on other classes of time series also shows promise and a more general evaluation is presently being conducted. With this in mind, the new ARMA procedure promises to be an important spectral estimation tool.