SIGNIFICANT STRESSES OF ARRESTED SALINE WEDGES (U)

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SIGNIFICANT STRESSES OF ARRESTED SALINE WEDGES

by

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<th>The hydrodynamical relations for the fresh water above and the arrested saline wedge below the interface are developed with a view toward evaluating the average interfacial stress and the average bottom stress along the entire length of the wedge. The relations are also verified by applying the principle of momentum separately to the fresh water and next to the saline wedge. Stresses are evaluated for arrested wedges observed in a laboratory channel, the densimetric Froude number falling close to 0.40. It is (Continued)</th>
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found that the interfacial stresses are of viscous origin and are not modified in the cases when unilateral mixing is present.
PREFACE

The study reported herein was completed in the Hydraulics Laboratory, U. S. Army Engineer Waterways Experiment Station (WES), Vicksburg, Mississippi, as a portion of the Navigation Hydraulics Program being conducted for the Office, Chief of Engineers, U. S. Army. The experimental work was completed at the National Bureau of Standards.

This report was prepared under the direction of Mr. H. B. Simmons, Chief of the Hydraulics Laboratory, by Dr. G. H. Keulegan, Special Assistant to the Chief, Hydraulics Laboratory.

Commanders and Directors of WES during the preparation and publication of this report were COL Nelson P. Conover, CE, and COL Tilford C. Creel, CE. Technical Director was Mr. F. R. Brown.
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SIGNIFICANT STRESSES OF ARRESTED SALINE WEDGES

PART I: INTRODUCTION

1. In 1946, an experimental program on the model laws of density currents was initiated at the National Bureau of Standards for the U. S. Army Engineer Waterways Experiment Station. The experimental work continued until 1961. Results of various studies have been reported with the exception of two studies—one relating to the interfacial stresses of the arrested saline wedges and the other, the effect of the Richardson's number on mixing. The present paper takes up the question of stresses.

2. Laboratory studies are needed on stresses of arrested saline wedges to provide guidance in the interpretation of physical models for the motion of saline waters into water courses free of large tidal influences. Constructing the model on the Froudian scale, it is necessary that the relative densities be retained. In small models the associated interfacial stresses are laminar in origin and are dependent on Reynolds number of flow (Keulegan 1955b). Since in all probability the corresponding stresses in nature would be turbulent in character, transference from the model to the prototype must be made with caution.

3. The present study reexamines further the results discussed in the report cited. Additional information is provided regarding the distribution of velocity in a cross section of the two layers, fresh and saline waters. Here, as in the previous report, attention is focused on the mean values of the stresses at the interface and at the bottom of the wedge. Flow equations are depth-integrated in both layers, the fresh water and the saline wedge. Next, resulting equations are integrated lengthwise, thereby yielding expressions for the mean stresses. Here, the derivations are repeated with more detail for clarity. Further, to examine the validity of the results involving the average stresses, the principle of momentum is applied separately once to the entire volume of fresh water lying over the wedge and next to the volume of saline water in the wedge.
PART II: APPARATUS AND PROCEDURES

4. The apparatus consists of a river forebay, a sea, and a channel connecting the two. The horizontal channel, 400 ft* long, 2 ft high, and 9 in. wide, is rectangular in cross section. The bottom consists of planed and painted wood. The sides are of glass or lucite plates, inserted into rectangular steel frames which in turn with the wooden bottom between them are attached to a steel base. The frames are 4 ft long and 2 ft high and the entire channel assembly rests on 4-ft-high concrete columns.

5. Over the forebay a metering tank is mounted to control the discharge into the channel representing the river. The bottom of the tank consists of a 1/8-in.-thick brass plate provided with numerous circular orifices of different sizes. Discharge into the channel is controlled by a chosen combination of orifices, their size and number, and the water-surface elevation in the tank. The orifices are calibrated in place by measuring the volume of water collected in the channel during a known time period with the sea end of the channel closed. The forebay is square in shape, each side is 4 ft long, and the bottom is aligned with the river channel bottom. The entrance into channel is rounded.

6. The sea is the most critical part of the entire apparatus. The 10-ft-long, 8-ft-wide, and 2-ft-deep rectangular tank representing the sea is elevated with its horizontal bottom aligned with the channel bottom. Despite the limited dimensions, the sea is required to behave as an unbounded sea. It is necessary to resort to a separate saline water reservoir placed directly below to establish a steady exchange of saline water. Return of saline water from the sea is governed by elevated crests of considerable length which also maintain a selected sea water-surface elevation. During a test run there is considerable mixing between fresh and saline waters at the sea surface and to maintain a

* Multiply feet by 0.3048 to convert to metres; multiply inches by 25.4 to convert to millimetres.
constant salinity of the sea, saline water from the reservoir below is constantly introduced into the sea. Previous observations have indicated that if \( \rho + \Delta \rho \) be the density of seawater and \( \rho + \Delta \rho' \) the density of fresh water at the locale of efflux from the sea area, as the result of mixing, the ratio \( \Delta \rho' / \Delta \rho \) equals 0.34 (Keulegan 1957b). This surprisingly high value of mixing requires that the density of the saline water in the reservoir be maintained at a constant value through the continued addition of salt (Figure 1).

![Figure 1. Plane view of experimental sea](image)

7. Saline water densities are determined electrically by a probe, the exploring end consisting of two parallel copper or platinum wires 2 cm in length. The salinity of an electrolyte is related indirectly to the resistance between the two parallel wires immersed in a solution. If the ratio of the diameter of the wires to the spacing between them is small, the resistance is given by (Keulegan 1949):

\[
R = \frac{1}{\pi L} \log_e \left[ \frac{2s}{d} \left(1 - \frac{d^2}{4s^2}\right) \right]
\]

\( (1) \)
where
\[ R = \text{effective resistance, ohms} \]
\[ \tau = \text{specific resistance of the electrolyte, ohms-cm} \]
\[ L = \text{length of the individual wires, cm} \]
\[ s = \text{spacing between wires, cm} \]
\[ d = \text{diameter of the wires, cm} \]

Since the relation applies accurately only for wires of great length, it is used for the purpose of designing the probe and selecting the ammeter. The probe is calibrated against a standard solution of relative density \( \Delta \rho / \rho = 0.10 \). The standard solution is prepared on the assumption that if \( V_s \) is the volume of salt of density \( \rho_s \) to be dissolved in water of volume \( V_w \), the volume of the solution, all at room temperature, is \( V_w + V_s \). On this basis, the density of the solution would be

\[ \rho + \Delta \rho = \left( \rho_s V_s + \rho_w V_w \right) / (V_s + V_w) \]  \hspace{1cm} (2)

Lower density solutions are obtained by adding water to standard solution. The calibration consists of noting the ampere versus density when the a-c voltage applied across the wires in 6 volts.

8. The experimental procedure for the eventual formation of an arrested saline wedge is rather simple. In the beginning of a test the sea is isolated from the river by means of a gate. The side gate at one of the channel walls about 1 ft away from the sea is left open to permit the lateral efflux of the river water. The gate is manipulated to establish the desired depth of water for the channel. With this selection, the waters in the channel and the saline water of the sea are at the same level. With this condition realized, the side gate is pushed down and the sea gate is pulled up simultaneously. This allows the entering of the river current into the sea area.

9. In the initial stages of the movement the velocity of the advancing saline front is large. With distance, the velocity decreases gradually. In the initial stages the front is rounded and also elevated. At some critical distance, the roundness of the front disappears and the front assumes the appearance of a slowly moving wedge. This form of the wedge is maintained even after the saline wave is arrested.
PART III: BASIC DATA OF THE EXPERIMENTS

10. Experiments were restricted to two river depths, \( H_0 = 45.5 \text{ cm} \) and \( H_0 = 23.0 \text{ cm} \) (Figure 2); these are the depths

![Figure 2. Notation diagram, arrested saline wedge](image)

of water immediately upstream of the tip of the arrested saline wedge. Since \( B \), the channel width, is 22.9 cm, the corresponding depth-width waters are 2 and 1, respectively. In these tests the excess density of the sea saline waters over the density of fresh water, \( \Delta \rho \), falls in the range 0.01 to 0.08. The river velocity \( U_0 \) just upstream of the wedge is selected such that \( n \), the ratio of the depth of saline waters of wedge at the river mouth \( h_{sl} \) to water depth \( H_0 \), is close to 1/2. The data of observations regarding the river velocity \( U_0 \), the density difference \( \Delta \rho \), the length of arrested saline wedge \( x_0 \), the depth of saline waters at the river mouth \( h_{sl} \), the fall of surface waters at the river mouth \( \Delta H \), and water temperature \( \Theta \) are shown in Tables 1 and 2 for the two test series. Determination of the average interfacial stress \( T_s \), stress averaged along the entire length of the interface, and the average bottom stress \( T_0 \) will be evaluated on the basis of the entries shown in these two tables. The values of \( \Delta \rho \) in Table 1 are 10 percent smaller than the original measurements and the values of \( h_{sl} \) shown are 1-1/2 cm less than the original observations. The reason for these adjustments will be given subsequently.
11. Expressions for the length of the arrested saline wedges and the depth of saline waters at the river mouth in laboratory channels were established previously (Keulegan 1957b). Thus, a comparison with these previous results may indicate if the value of some particular observation of this study requires adjustment.

12. The length of an arrested saline wedge on the basis of dimensional analysis would be

\[ \frac{x_o}{H_o} = f \left( \frac{V_H}{v}, \frac{2V_r}{V_H}, \frac{B}{H_o} \right) \]  

(3)

where \( V_r \) is the river velocity and

\[ V_\Delta = \sqrt{\frac{\Delta \rho}{\rho g H_o}} \]  

(4)

The ratio \( V_r/V_\Delta \) is referred to as the densimetric Froude number and \( V_H/V_\Delta \), as the densimetric Reynolds number. When \( B/H_o \) equals unity, the wedge length is

\[ \frac{x_o}{H_o} = 0.19 \left( \frac{V_H}{v} \right)^{1/2} \left( \frac{2V_r}{V_\Delta} \right)^{-5/2}, \frac{B}{H_o} = 1 \]  

(5)

in the densimetric Reynolds number of range \( 1 \times 10^4 \) to \( 10.0 \times 10^4 \). In the present study the determination of length for the case \( H_o = 23.0 \) cm and \( B = 22.9 \) cm on the basis of values shown in Table 2 shows agreement with the above as the mean value of the coefficient is also 0.19. When \( B/H_o \) equals 2 in the densimetric Reynolds number of range \( 4.0 \times 10^4 \) to \( 40.0 \times 10^4 \) the length is given

\[ \frac{x_o}{H_o} = 0.12 \left( \frac{V_H}{\Delta \rho} \right)^{1/2} \left( \frac{2V_r}{V_\Delta} \right)^{-5/2}, \frac{B}{H_o} = 1 \]  

(6)
When the determinations of length for the case $H_0 = 45.5$ cm and $B = 22.9$ cm were made on the basis of the recorded values of $\Delta p/\rho$, which were 10 percent higher than those shown in Table 1, the results were not in agreement with Equation 6; the mean value of the coefficient was 0.10. Repeating the determination with $\Delta p/\rho$ values as shown in Table 1, the mean value of the coefficient is 0.12 in agreement with Equation 6. Due to error of probe calibration, sea densities are not correctly determined. The error does not affect the evaluation of the interfacial mean stress $T_s$, but it would affect the bottom mean stress $T_b$, using the formulae subsequently established.

13. Previous studies (Keulegan 1957b) have indicated that the depth of seawater at the river mouth expressed as ratio $h_{sl}/H_0$ is practically independent of the density of seawater, the width-depth ratio of the channel, and the densimetric Reynolds number; it is merely a function of the densimetric Froude number $V_f / V_\Delta$ as shown in Table 3. The entries represent averages from tests with different channels and different width-depth ratios (Keulegan 1957b).

14. As first shown by Shijf and Schoenfeld (1953), the flow of fresh water at the river mouth is critical,

$$\frac{U_1^2}{\Delta \rho \rho gh_{w1}} = 1$$  (7)

where $h_{w1}$ is the depth of fresh water at the river mouth and $U_1$ is the velocity of freshwater current (Keulegan 1955a). Assuming that the fall of surface water, $AH$, is negligible one has $h_{w1} = H_0 - h_{sl}$ and thus

$$\frac{h_{sl}}{H_0} = 1 - \left(\frac{V_f}{V_\Delta}\right)^{2/3}$$  (8)

The entries in Table 3 are in good agreement with this expression for $V_f / V_\Delta$ close to 0.5 only. For smaller values of $V_f / V_\Delta$, theory yields depth values greater than those given in this table.
15. In the derivation of the theoretical expression, the theory assumes tacitly that the distribution of pressure is hydrostatic. On the other hand, when the densimetric Froude number is small the interface in the area of river mouth would be curved, a condition wherein the pressure would be no longer hydrostatic. Thus, it is desirable to obtain the depth $h_{sl}$ from a relation based on observation.

16. In this study the position of the interface was obtained visually. To enhance a stronger contrast between the fresh and saline waters, the water of the sea was colored by introducing a chrome compound. This is ideal if there is no mixing at the interface. Unfortunately, if agitation is present small traces of this material may affect a deep coloring and the true position of the interface may not be inferred correctly. This was the case with the tests carried with water depth $H_o = 45.5$ cm and thus the inferred values of $h_{sl}$ are not reliable. Indeed, the ratio $H_{sl}/H$ when computed from the inferred values are at variance with the entries of Table 3. However, agreement would be obtained only after subtracting 1.5 cm from the inferred $h_{sl}$. These adjusted values of $h_{sl}$ are shown in Table 1. A similar adjustment is not necessary for the tests carried out with water depth $H_o = 23.0$ cm.
PART V: SHAPE OF WEDGES AND FALL OF SURFACE WATERS

17. Information on the shape of the arrested saline wedge and the fall of surface waters over the wedge plays an important part in determining the average boundary stress at the interface and at the bottom from formulae.

18. Previous studies have shown that within a definite range of densimetric Froude number the shapes are affine to each other irrespective of river velocities, the density of seawater, and channel width (Keulegan 1957b). Denoting by \( h_s \) the depth of saline waters in the wedge at a point of distance \( x \) from the tip of the wedge

\[
\frac{h_s}{h_{sl}} = f \left( \frac{x}{x_0} \right)
\]

where \( x_0 \) denotes the length of the wedge.

Introducing the relative distance

\[
\zeta = \frac{x}{x_0}
\]

the affine shape is

\[
\frac{h}{h_{sl}} = f(\zeta)
\]

This functional relation is shown in Table 4.

19. For the present study the observation of shape of arrested saline wedge, selecting the river water depth of ratio of \( H_o = 23 \) cm, is restricted to tests with a common densimetric Froude number \( V_r/V_\Delta = 0.40 \) and five different seawater densities. Tests with a given seawater density are repeated and the averages of \( h_s/h_{sl} \) are sought; these are shown in Table 5. Comparison between the entries of different columns indicates that for a constant densimetric Froude number the affine shape is independent of the river velocity or the
density of the seawater. The mean values from the columns are shown in the last column and these differ but little from the entries of Table 5, which apply to arrested saline wedges of varying densimetric Froude number previously studied (Keulegan 1957b). The affine depth data from Tables 4 and 5 are shown in Figure 3. The curve drawn through the points yields the relation

\[
\frac{h_s}{h_{s1}} = \zeta + a \sin 2\pi \zeta, \quad a = 0.09
\] (12)

Figure 3. Affine shape of arrested saline wedge
20. To measure the fall of surface waters, along the 60-m length of the experimental channel, 12 glass manometer tubes were mounted vertically against the glass sidewalls. Next to the tubes, paper scales, (reading in centimetres) were glued to the glass. The meniscus positions were read by a telescope. Displacement of the menisci was readily estimated within a tenth of a millimetre. To establish a common zero for the scales, the exit gates of the channel were closed and the channel was filled with fresh water to a designed depth $H_o$, 23.0 cm or 45.5 cm. The readings of the menisci in all the manometer tubes were taken after all the surface oscillations had ceased. An example of the measurement of the fall of the freshwater free surface is given in Figure 4. The

![Figure 4](image)

Figure 4. An example of observed values of fall of water surface

part of the curve corresponding to $x$ negative represents the fall upstream of the tip of the arrested wedge. The slope here is in accord with the relation
\[
\frac{\Delta h}{\Delta x} = -\lambda \frac{U_0^2}{2gR_h}
\]

where \( R_h \) is the hydraulic radius and \( \lambda \) the coefficient of resistance

\[
\lambda = 0.0559 \left( \frac{U_0 R_h}{V} \right)^{-1/4}
\]

The part of the curve corresponding to \( x \) positive represents the surface fall of fresh water lying over the arrested saline wedge. Denoting the fall of this surface at the river mouth by \( \Delta H \) and that at a point \( \xi = x/x_0 \) by \( \Delta h \), the ratio \( \Delta h/\Delta H \) will be referred to as the relative fall of surface waters. This quantity as a function of \( \xi \) is, determined from curves similar to the one shown in Figure 4. Determinations made with varying seawater densities and for a depth of water \( H_0 = 23.0 \, \text{cm} \) are shown in Table 6 and those corresponding to \( H_0 = 45.5 \, \text{cm} \) in Table 7. It appears that within the range of the densimetric Froude number used the ratio is independent of this number. The average values of relative fall from Tables 6 and 7 are transferred to Figure 5 and the curve through the points yields the relation

\[
\frac{\Delta h}{\Delta H} = \xi - b \sin \pi \xi + c \sin 2\pi \xi
\]

where \( b = 0.17 \) and \( c = 0.03 \).

21. In the case of an arrested saline wedge with the fresh water above it in motion, it would be easier to evaluate the average interfacial stress by directly considering the hydrodynamics of the freshwater layer. For this determination it is important that both \( \Delta H \) and \( \Delta h/\Delta H \) are correctly observed and evaluated.
Figure 5. Relative fall of water surface over arrested wedge
PART VI: VELOCITY PATTERNS

22. The determination of the mean stress $T_S$ of the interface and the mean bottom stress $T_o$ on the basis of the dynamics of the arrested saline wedge requires that the pattern of the velocities in this area is accurately known. The problem becomes more complicated if there is a flow due to mixing across the interface.

23. Examination of velocities in the arrested wedge and in the part of fresh water lying over the wedge was restricted to tests with a river depth of 23.0 cm. Interfacial velocities, surface water velocities, and velocities of fresh water and saline water in a cross section were measured separately.

24. For the observation of the interface velocities a small globule of butyl phthalate dissolved in xylene, with a density slightly less than the density of the seawater, is injected by means of an eye-dropper into the fresh water a few centimetres below the surface. After the descent of the drop is completed and it reaches the interface, subsequent motion of the drop from one locality to another is timed. Let $U_i$ be this interface velocity observed at the point $x$ and $U$ be the mean velocity of fresh water in the cross section through $x$.

Forming the ratio $U_i/U$, observed values from the tests corresponding to various seawater densities are shown in Table 8 as function of $\xi$. For a given $x/x_o$ the ratios $U_i/U$ reveal small differences from one column to the other. These differences may be ignored. An interface with a sharp discontinuity in the densities does not exist. Actually, there is a gradation of densities between the two liquids and one takes the surface where the density equals $1 + 1/2 \Delta \rho$ as the effective interface. The density of the butyl phthalate globules cannot be accurately controlled and in different tests the globules are differently placed with respect to the interface. It is thus sufficient to consider the average values entered in the last column of Table 8. The plotting in Figure 6 is on this basis. Starting from zero, the value of $U_i/U$ increases very rapidly and then slowly reaches asymptotically the limit $U_i/U = 0.53$. The surface velocity $U_s$ is determined by timing the
Figure 6. Interface and surface velocities, $F_0 = 0.40$

motion of small paraffin particles riding on the surface waters. Also, in this case the ratio $U_s/U$ is formed and the values as a function of $\zeta$ from the tests using different seawater densities are entered in Table 9. Here also $U_s/U$ values reveal small differences from one column to the other. The average values are entered in the last column and the plotting in Figure 6 is on this basis.

25. The device to determine the distribution of the velocities in a flow cross section consisted of a thin phosphor bronze strip about 1/4 in. wide and 8-1/2 in. long with one of the ends of the strip soldered to a rectangular rod. Touching one of the sidewalls of the
experimental channel, the rod was held vertical with the strip facing
the current normally in a horizontal plane. The deflection $\delta$ of the
free end of the strip is an indication of the velocities in the plane of
the strip. An argument of dimensional analysis yields

$$\frac{\delta}{\ell} = f\left(\frac{\rho u^2}{E}, \frac{t}{\ell}\right)$$

(16)

where

$\delta = \text{deflection of the free end of the strip}$

$\ell$ and $t = \text{length and thickness, respectively, of the strip}$

$\rho = \text{density of liquid}$

$u = \text{root mean square of the velocities in the strip plane}$

$E = \text{Young's modulus of the strip natural}$

As the deformation of the strip is flexural in the main

$$\frac{\delta}{\ell} = f\left[\frac{\rho u^2}{E} \left(\frac{t}{\ell}\right)^3\right]$$

(17)

For a given strip $E$, $t$ and $\ell$ are constants. Then,

$$\rho^{1/2}u = f(\delta)$$

(18)

where the quantities are measured in standard units, cgs. Although
the relation may be established using Kirchhoff's theory of slender
wires, it is more practical to resort to calibration. For this purpose
the strip is held at middepth of a current in the experimental channel
of discharge $Q$ and the end deflection $\delta$ is noted. The average
velocity $u$ is deduced from the discharge and the strip elevation from
the channel bottom using Blasius' velocity relation. It is assumed
that the velocities of all the points in a cross section having the
same common elevation are equal. This introduces a small calibration
error since the velocities at the points of strip level are not constant.
This, however, will not severely affect the value of $u/U$, where $u$ is
the root-mean-square value of the velocities at points of a common elevation from the bottom and $U$ is the average velocity of the freshwater
current in the same cross section, all the velocities being based on a chosen calibration.

26. Using five different seawater densities the examination of velocities was confined to a central area of \( x/x_0 = 0.75 \) where the depth of saline water was 7.0 cm and that of the fresh water above, 16.0 cm. All the arrested wedges realized, corresponding to a densimetric Froude number \( F_0 = 0.40 \), were nearly of a common length \( x_o = 2000 \) cm.

27. Relative velocities \( u/U \) of the saline wedge area are given in Table 10 and those of fresh water over the wedge in Table 11. The entered values were read from individual curves drawn smoothly through points of observation. Actually, there was some scatter in the observed values. The small variation in the entry values from column to column suggests that the pattern of velocities in a cross section is hardly affected by the seawater densities, all the arrested wedges corresponding to a common densimetric Froude number. Figure 7 was prepared using the mean values shown in the last column of Tables 10 and 11.

28. Flow in the tip area of the arrested saline wedge with \( \zeta \) less than 0.1 may be supposed to be irrotational, the same as in the case where the boundary is composed of two rigid walls inclined at an angle less than 90 deg. In the remaining part of the wedge if there is mixing at the interface, this will influence the velocities in the lower areas of the arrested wedge where the motion is away from the sea. Using the methods of an earlier investigation on mixing in arrested saline wedges (Beta 1957), it may be estimated that the total quantity of saline water traversing the interface into fresh water due to mixing is about 8 percent of the saline water inflow into the wedge from the sea. The effect of this may be ignored. One may then assume that the velocity patterns from one cross section to another are similar to each other. It was already seen that the interfacial velocity \( U_i/U \) has practically the same value along almost the entire length of the interface except in the area near the tip.
Figure 7. Fresh and saline water velocities in arrested saline wedges, $F_o = 0.40$
PART VII: HYDRODYNAMICS OF FRESHWATER AND INTERFACIAL STRESS

29. Let the origin of the rectangular axis system be placed at the tip of the arrested saline wedge and at the midpoint of the channel bottom width. Let the axis of \( x \) be drawn horizontally in the direction of motion of the fresh water and \( z \) vertically upward. Components of the velocities along the axes \( x \), \( y \), and \( z \) are \( u \), \( v \), and \( w \), respectively. Elevation of the water surface measured from the horizontal channel bottom may be denoted by \( h \) and that of the interface by \( h_s \). Let the width of the channel be \( B \), \( B = 2b \) (Figure 8).

\[ \text{Figure 8. Notation diagram, analysis of wedge stresses} \]

30. As the length of the arrested saline wedge is large in comparison with the maximum depth \( h_s \) of the wedge at the river mouth, the vertical component of the fluid acceleration will be neglected and the pressure will be evaluated hydrostatically. Under this assumption, using the notation of Lamb, the equations of motion are

\[
\frac{u}{\partial x} \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{\rho} \left( \frac{\partial p_{zx}}{\partial x} + \frac{\partial p_{yu}}{\partial y} \right)
\]

(19)
Here $p_{zx}$ denotes the tangential stress which the liquid of increasing $z$ exerts on a plane surface normal to $z$ axis and in the direction of $x$. Similarly $p_{yx}$ is the tangential stress which the liquid of the increasing $y$ exerts on a plane surface normal to $y$ axis and in the direction of $x$. Again, $\rho$ is the density of water, $p$ is the pressure, and $g$ is the gravitational contact.

31. The hydrodynamics of the freshwater flow over the arrested wedge will be considered first. Multiply the terms in Equation 19 by $dz\,dy$ and integrate over a cross section between the channel walls extending from the interface to the free surface. Hence,

$$0 = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g \quad (20)$$

$$\int_{-b}^{b} \int_{h_s}^{h} \left( u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) dz\,dy + \frac{1}{\rho} \int_{-b}^{b} \int_{h_s}^{h} \frac{\partial p_{zx}}{\partial z} dz\,dy + \frac{1}{\rho} \int_{-b}^{b} \int_{h_s}^{h} \frac{\partial p_{yx}}{\partial y} dz\,dy \quad (21)$$

Since the liquid is incompressible

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

and one now has

$$w \frac{\partial u}{\partial z} = \frac{\partial}{\partial z} (uw) + u^2 \frac{\partial u}{\partial x}$$

Substituting this in the left-hand member of Equation 19 this now becomes...
\[ \int_{-b}^{+b} \int_{h}^{h} \left[ \frac{\partial}{\partial x} (u^2) + \frac{\partial}{\partial z} (uw) \right] dz \, dy \]

or

\[ \int_{-b}^{+b} \int_{h}^{h} \frac{\partial u^2}{\partial x} dz \, dy + \left[ (uw)_s - (uw)_i \right] B \]

where the subscript "s" refers to the surface and "i" to the interface.

As a particle on the water surface stays on this surface and a particle on the interface stays on the interface in the absence of mixing,

\[ \omega_s = u_s \frac{dh}{dx}, \quad \omega_i = u_i \frac{dh}{dx} \]

Then, the final form of the left-hand member of Equation 21 is

\[ \int_{-b}^{+b} \int_{h}^{h} \frac{\partial u^2}{\partial x} dz \, dy + \left( u_s^2 \frac{dh}{dx} - u_i^2 \frac{dh}{dx} \right) B \quad (22) \]

One now introduces \( \alpha \), the Boussinesq coefficient of velocity distribution, defined by

\[ \alpha U^2 A = \int_{-b}^{+b} \int_{h}^{h} u^2 dx \, dz \quad (23) \]

where \( U \) is the average value of \( u \) in a cross section \( A = B(h - h_s) \).

Differentiating the two sides of this last equation with respect to \( x \),
in accordance with the Leibnitz rule,

\[
\frac{d}{dx} (\alpha U^2 A) = \int_{-b}^{+b} \int_{h_s}^{h} \frac{\partial}{\partial x} u^2 \, dz \, dy + \left( u_s^2 \frac{dh}{dx} - u_i^2 \frac{dh}{dx} \right) B
\]

Hence after comparing with Equation 22, the left-hand member of Equation 21 is

\[
\int_{h_s}^{h} \int_{-b}^{+b} \left( u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) \, dz \, dy = \frac{d}{dx} \left( \alpha U^2 A \right) \tag{24}
\]

A slight variation of \( \alpha \) with respect to \( x \) may be ignored and as \( UA \) is constant

\[
\frac{d}{dx} (\alpha U^2 A) = UA \frac{d}{dx} (\alpha U) = \frac{\alpha}{2} A \frac{dU^2}{dx} = \frac{\alpha}{2} \frac{dU^2}{dx} (h - h_s)B \tag{25}
\]

Thus finally

\[
\int_{-b}^{+b} \int_{h_s}^{h} \left( u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) \, dz \, dy = \frac{\alpha}{2} \frac{dU^2}{dx} (h - h_s)B \tag{26}
\]

This is an evaluation of the integral on the left-hand side of Equation 21. From Equation 20

\[ p = p_a + pg(h - z) \quad h > h_s \]
where \( p_a \) is the atmospheric pressure. Accordingly, the first integral on the right-hand side of Equation 21 reduces to

\[
\frac{1}{\rho} \int_{-b}^{+b} \int_{h_s}^{h} \frac{\partial p_{zx}}{\partial z} \, dz \, dy = -\frac{1}{\rho} \int_{-b}^{+b} \frac{dh}{dx} (h - h_s)B \tag{27}
\]

In the absence of air currents, the stress at the water surface vanishes. Let the traction of the fresh water on the saline water be denoted by \( \tau_s \). Thus

\[
p_{zx} = 0 \quad z = h
\]

\[
p_{zx} = \tau_s \quad z = h_s
\]

and the second integral on the right-hand side of Equation 21 reduces to

\[
\frac{1}{\rho} \int_{-b}^{+b} \int_{h_s}^{h} \frac{\partial p_{zx}}{\partial z} \, dz \, dy = -\frac{1}{\rho} \int_{-b}^{+b} \tau_s \, dy = -\frac{1}{\rho} \bar{\tau}_s B \tag{28}
\]

where \( \bar{\tau}_s \) is the average value of the interfacial stress along the channel width. It is a positive quantity. Denoting the resistive force of the vertical wall by \( \tau_w \), a positive quantity,

\[
p_{yx} = -\tau_w \quad y = b
\]

\[
p_{yx} = \tau_w \quad y = -b
\]
the third integral on the right-hand side of Equation 21 reduces to

\[
\frac{1}{\rho} \int_{-b}^{b} \int_{h_s}^{h} \frac{\partial p}{\partial y} \, dz \, dy = \frac{2}{\rho} \int_{h_s}^{h} \tau \, dz = \frac{2}{\rho} \tau_w (h - h_s) \tag{29}
\]

where \( \tau_w \) is the average value of the wall shear across the span \( h - h_s \). In view of Equations 26, 27, 28, and 29, the original Equation 21 reduces to

\[
\frac{\tau_s}{\rho} = -g \frac{dh}{dx} (h - h_s) - 2 \frac{\tau_w (h - h_s)}{\rho B} - \frac{\alpha}{2} \frac{dU^2}{dx} (h - h_s) \tag{30}
\]

This is the expression for the local stress in terms of the surface slope, the surface elevation, the saline wedge height, and the current mean velocity. All these refer to the situation at the point \( x \). However, owing to the difficulties in measuring the surface fall rate \( dh/dx \) accurately, it will prove to be more serviceable to consider the average \( \tau_s \) of the interface stress, averaged along the entire length of the saline wedge.

\[
\tau_s \bigg|^{x_0}_0 = \int_{0}^{x_0} \tau_s \, dx \tag{31}
\]

and in view of Equation 30
\[ T_s = -g \int_0^{x_0} \frac{dh}{dx} \left( h - h_s \right) \frac{dx}{x_0} \]

\[ - \frac{2}{B} \int_0^{x_0} \frac{\tau_w}{\rho} \left( h - h_s \right) \frac{dx}{x_0} - \frac{\alpha}{2} \int_0^{x_0} \frac{dU^2}{dx} \left( h - h_s \right) \frac{dx}{x_0} \quad (32) \]

Since

\[ U_{OH} = U_1 (H_0 - \Delta H - h_{s1}) = U(h - h_s) \quad (33) \]

where \( U_1 \) is the velocity of fresh water at the river mouth and \( \Delta H \) the fall of surface, the last term in Equation 32 changes to

\[ \frac{U^2}{\alpha} \left( \frac{H_o - \Delta H - h_{s1}}{x_0} \right) - \frac{U^2}{x_0} \quad (34) \]

Also, neglecting \( \Delta H^2 \)

\[ \frac{1}{x_0} \int_0^{x_0} h \frac{dh}{dx} \, dx = \frac{1}{x_0} \int_0^{x_0} h \, dh = \frac{H_0 \Delta H}{x_0} \quad (35) \]

and

\[ \int_0^{x_0} h_s \, \frac{dh}{dx} \, dx = - \int_0^{x_0} h_s \, \frac{d\Delta h}{dx} \, dx \quad (36) \]
For brevity introduce \( T_{w1} \) the total frictional force of the walls on the fresh water

\[
T_{w1} = 2 \int_{0}^{x_0} \tau_w (h - h_s) \, dx \quad (37)
\]

In view of the last four relations, Equation 32 changes to

\[
\frac{T_s}{\rho} = gH_0 \left( \frac{\Delta H}{x_0} - \int_{0}^{x_0} \frac{h_s}{H_0} \frac{d\Delta h}{dx} \, dx \right) - \frac{T_{w1}}{\rho B x_0} - \alpha \left[ \frac{2}{U_1} \left( \frac{H_0}{\rho} - \frac{\Delta H}{x_0} \right) - \frac{2}{U_1^2} \right] \quad (38)
\]

32. One may also obtain this relation by applying the principle of momentum to the body of water found at the instant \( t \) over the wedge with its boundary ABDE as shown in Figure 9. Let \( \frac{dM}{dt} \) denote the rate of change of momentum of this water in the direction \( x \) positive.

Figure 9. Notation diagram relating to momentum of fresh water
The forces producing this change are the pressure forces $P_1$ and $P_2$ acting on the upstream and the downstream forces $AB$ and $ED$, and the pressure force $P_1$ on the interface. The tangential traction forces taking part in this change of momentum are $T_{w1}$, the retraction force from the two sidewalls, and $T_{sBx}$, the retraction force due to the saline wedge. Thus

$$\frac{\delta M}{\delta t} = P_1 - P_2 - P_1 - T_{w1} - T_{sBx}$$  \hspace{1cm} (39)$$

The particles of water which at time $t$ are in the planes $AB$ and $DE$ move into the surfaces $A'B$ and $D'E'$ at time $t + \delta t$, respectively. Let $M_0$ be the momentum of the liquid contained in $A'BDE$; $M_1$ of that contained in $ABA'$; and $M_2$ of that contained in $ED'D'E'$. On this basis the momentum of liquid under consideration is $M_0 + M_1$ at the instant $t$ and $M_0 + M_2$ at the instant $t + \delta t$. Thus the rate of change of momentum is

$$\frac{\delta M}{\delta t} = (M_2 - M_1)\delta t$$  \hspace{1cm} (40)$$

Now,

$$M_1 = \rho \int_{-b}^{b} \int_{H_0}^{H_0 + b} u_0^2 \, dz \, dy \, \delta t = \alpha_1 \rho u_0^2 H_0 B \delta t$$

and

$$M_2 = \rho \int_{-b}^{b} \int_{h_{s1}}^{H_0 - \Delta H} u_1^2 \, dz \, dy \, \delta t = \alpha_2 \rho u_1^2 (H_0 - \Delta H - h_{s1}) \delta t$$

where $U_0$ is the river current average velocity at the tip of the saline wedge and $U_1$ the average velocity of fresh water at the river mouth.
Introducing these in Equation 40 and assuming $\alpha_2 = \alpha_2 = \alpha$

$$\frac{\delta M}{\delta t} = \rho \alpha \left[ u_1^2 (H_o - \Delta H - h_{s1}) - u_0^2 H_0 B \right]$$

(41)

Ignoring the pressure of the atmosphere, the end pressure forces are

$$P_1 = \rho g \frac{H_o^2}{2} B$$

and

$$P_2 = \rho g \frac{(H_o - \Delta H - h_{s1})^2}{2} B$$

Thus, ignoring $\Delta H^2$

$$P_1 - P_2 = \rho g B \left( H_o \Delta H + H_o h_{s1} - \Delta H h_{s1} - \frac{h_{s1}^2}{2} \right)$$

(42)

The total pressure force from the interface is

$$P_i = \rho g B \int_0^{h_{s1}} (H_o - \Delta h_o - h_s) \, dh_s$$

or

$$P_i = \rho g B \left( H_o h_{s1} - \frac{h_{s1}^2}{2} - \int_0^{x_0} \frac{dh_s}{dx} \, dx \right)$$

Now

$$\frac{d}{dx} (\Delta h_s) = \frac{dh_s}{dx} + h_s \frac{d\Delta h}{dx}$$
and as

\[ \int_{0}^{x_0} \frac{d}{dx} (\Delta h_s) \, dx = \Delta h_{s1} \]

finally

\[ P_1 = \rho g B \left( h_0 \Delta h_s - \frac{h_{s1}^2}{2} - \Delta h_{s1} + \int_{0}^{x_0} h_s \frac{d\Delta h}{dx} \, dx \right) \quad (43) \]

From Equations 42 and 43

\[ P_1 - P_2 - P_i = \rho g B \left( h_0 \Delta H - \int_{0}^{x_0} h_s \frac{d\Delta h}{dx} \, dx \right) \quad (44) \]

Substituting from Equations 41 and 44 in Equation 39 and dividing the resulting equation by \( \rho B x_0 \)

\[ \frac{T_s}{\rho} = g h_0 \left( \Delta H - \int_{0}^{x_0} \frac{h_s \, d\Delta h}{dx} \, dx \right) \frac{T_w}{\rho B x_0} - \alpha \left[ \frac{U_1^2 (H_0 - \Delta H - h_{s1}) - U_0^2 H_0}{x_0} \right] \quad (45) \]

This last equation is identical with Equation 38, the alternate form of Equation 32 which was derived by integrating vertically the Eulerian flow, Equation 19.
PART VIII: HYDRODYNAMICS OF ARRESTED SALINE WEDGE AND AVERAGE BOTTOM STRESS

33. Multiplying the terms in Equation 19 by $dz \, dy$ and integrating over the cross section of the wedge at $x$,

\[
\int_{-b}^{+b} \int_{o}^{h} \left( u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) dz \, dy = -\frac{1}{\rho'} \int_{-b}^{+b} \int_{o}^{h} \frac{\partial p}{\partial x} dz \, dy
\]

\[
+ \frac{1}{\rho'} \int_{-b}^{+b} \int_{o}^{h} \frac{p_{zx}}{\partial z} dz \, dy + \frac{1}{\rho'} \int_{-b}^{+b} \int_{o}^{h} \frac{\partial p_{yx}}{\partial y} dz \, dy
\]

(46)

where $\rho'$ is the density of saline waters of the edge. Since in the normal cross section of the arrested saline wedge the mean velocity $U$ practically vanishes, one now introduces, in establishing an expression similar to Equation 23, the interfacial velocity $U_i$ and defines a new coefficient of velocity $\beta$ as

\[
\beta U_i^2 A = \int_{-b}^{+b} \int_{o}^{h} u^2 dz \, dy
\]

(47)

Using the same argument that was applied to the freshwater part it may be shown that

\[
\int_{o}^{h} \int_{-b}^{+b} u \left( \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) dz \, dy = B \frac{d}{dx} \left( \beta U_i^2 h_s \right)
\]

(48)
Assuming that the pressure is hydrostatic also in the wedge

\[ p = p_a + pg(h - h_s) + g(\rho + \Delta\rho)(h_s - z), \quad z \leq h_s \]

where \( \Delta\rho \) is the excess of density of the saline water of the wedge over that of fresh water; \( \rho' = \rho + \Delta\rho \). Differentiating with respect to \( x \)

\[ \frac{\partial p}{\partial x} = g \left( \rho \frac{\partial h}{\partial x} + \Delta\rho \frac{\partial h_s}{\partial x} \right) \]

and setting in the first integral of the right-hand side of Equation 46

\[ - \int_{-b}^{+b} \int_{-b}^{+b} \frac{1}{\rho' \rho \partial x} \frac{\partial p}{\partial x} \, dz \, dy = -g \left( \frac{\partial h}{\partial x} + \frac{\Delta\rho \partial h_s}{\rho \partial x} \right) h_s B \quad (49) \]

since \( \rho' / \rho \) is practically unity. Denoting the frictional stress of the bottom by \( \tau_0 \)

\[ p_{zX} = \tau_s \quad \text{at} \quad z = h_s \]

\[ p_{zX} = -\tau_0 \quad \text{at} \quad z = 0 \]

the quantities \( \tau_s \) and \( \tau_0 \) being regarded as positive. The second integral on the right-hand side of Equation 42 now becomes, neglecting again the difference between \( \rho \) and \( \rho' \),

\[ \frac{1}{\rho' \rho} \int_{-b}^{+b} \int_{-b}^{+b} \frac{\partial p_{zX}}{\partial z} \, dz \, dy = \frac{1}{\rho} \int_{-b}^{+b} (\tau_s + \tau_0) \, dy = \frac{1}{\rho} \left( \tau_s + \tau_0 \right) B \quad (50) \]

In regard to the sidewall effect
\[ p_{yx} = -\tau_w \quad y = h \]
\[ p_{yx} = +\tau_w \quad y = -b \]

Here \( \tau_w \) is positive when the flow in the neighborhood of the wall is in the direction of the main flow of fresh water above and negative when in the opposite direction. The sign of \( \tau_w \) changes from the positive to the negative in moving from the interface downward. Now,

\[
\frac{1}{\rho} \int_{-b}^{+b} \int_0^h s \frac{\partial p_{yx}}{\partial y} \, dz \, dy = \frac{2}{\rho} \int_0^h \tau_w \, dz = -\frac{2}{\rho} \tau_w h_s \quad (51)
\]

Substituting the expression from Equations 48, 49, 50, and 51 in Equation 46, dividing by \( B \), and transferring terms

\[
\frac{\pi}{\rho} + \frac{T_o}{\rho} = g \left( \frac{\partial h}{\partial x} + \frac{\Delta \rho}{\rho} \frac{\partial h}{\partial x} \right) h_s + \frac{2}{\rho} \tau_w \frac{h_s}{B} + \frac{d}{dx} \left( \beta U_1^2 h_s \right) \quad (52)
\]

Multiplying the terms in this equation by \( dx \), integrating between the limits \( x = 0 \) and \( x = x_o \), and writing

\[
T_{o\cdot o} = \int_0^{x_o} \pi \, dx \quad (53)
\]

\[
\frac{\pi}{\rho} + \frac{T_o}{\rho} = g \int_0^{x_o} \frac{dh}{dx} h_s \, dx + \frac{1}{2} \frac{\Delta \rho}{\rho} \frac{h_{s1}^2}{x_o} + \frac{2}{\rho} \int_0^{x_o} \tau_w \frac{h_s}{B} \, dx + \beta U_1^2 \frac{h_{s1}}{x_o} \quad (54)
\]

This last relation can be obtained also by applying the momentum principle to the entire body of the salt wedge at the instant \( t \) and contained in the boundary ADCB as shown in Figure 10. Let the rate of change of
momentum of this body of liquid in the direction of x positive be denoted by

\[
\frac{\delta M}{\delta t}
\]

The forces producing this change are the total pressure forces \( P_1 \) acting on the interface and \( P_2 \) acting on the limiting face of the saline wedge with the depth \( h_s \). The tractive forces also taking part in this are the interfacial total stress \( T_s Bx_0 \) and the retarding force of the sidewalls \( T_w2 \).

Thus

\[
\frac{\delta M}{\delta t} = P_1 - P_2 + T_s Bx_0 + T_0 Bx_0 - T_w2 \tag{55}
\]

In the notation previously adopted

\[
T_w2 = 2 \int_0^x \tau_{ws} h_s \, dx \tag{56}
\]
The particles of saline water which at time \( t \) are in the planes \( AO \) and \( OC \), move into the surfaces \( ADO \) and \( OC' \), respectively, at time \( t + \delta t \). Let \( M_0 \) be the momentum of liquid in \( ADOC \), \( M_1 \) of that contained in \( ADO \), and \( M_2 \) of that contained in \( OC' \). On this basis the momentum of the saline body of water under consideration is \( M_0 + M_1 \) at time \( t \) and \( M_0 + M_2 \) at the later instant \( t + \delta t \). Thus the rate of change of momentum is

\[
\frac{\delta M}{\delta t} = \frac{(M_2 - M_1)}{\delta t}
\]  

(57)

Now,

\[
M_1 = -\rho B \int_{-b}^{b} \int_{h_{sl}}^{+b} u_1^2 \, dz \, dy \, \delta t
\]

and

\[
M_2 = \rho B \int_{-b}^{b} \int_{h_{sl}}^{+b} u_1^2 \, dz \, dy \, \delta t
\]

as \( u_1 \) is negative for \( z \) smaller than \( \delta \), and positive for \( z \) greater than \( \delta \). Substituting in Equation 53

\[
\frac{\delta M}{\delta t} = B \rho \int_{-b}^{b} \int_{h_{sl}}^{+b} u_1^2 \, dx \, dy
\]

(58)

and in agreement with Equation 47, \( A = h_{sl} B \)

\[
\frac{\delta M}{\delta t} = B u_1^2 h_{sl} B
\]

(59)
From Equation 39, rewriting

\[ P_1 = \rho g B \left( H_0 h_{s1} - \frac{h_{s1}^2}{2} - \Delta h h_{s1} + \int_{0}^{x_0} h_s \frac{d\Delta h}{dx} \, dx \right) \]  

(60)

Now

\[ P_2 = B \int_{0}^{h_{s1}} p_2 z \, dz \]

where

\[ P_2 = \rho g \left( H_0 - \Delta h - h_{s1} \right) + g(\rho + \Delta \rho) h_{s1} - g(\rho + \Delta \rho) z \]

The integration yields

\[ P_2 = \rho g B \left( H_0 h_{s1} - \Delta h h_{s1} - \frac{h_{s1}^2}{2} + \frac{\Delta \rho}{\rho} \frac{h_{s1}^2}{2} \right) \]  

(61)

Substituting in Equation 51 from Equations 59, 60, and 61 and dividing by \( \rho B x_o \), results in

\[ \frac{T_s}{\rho} + \frac{T_o}{\rho} = -g \int_{0}^{x_0} h_s \frac{d\Delta h}{dx} \, dx + \frac{g}{2} \frac{\Delta \rho}{\rho} \frac{h_{s1}^2}{x_o} + \frac{2}{\rho B} \int_{0}^{x_0} h_s \frac{d\Delta h}{dx} \, dx + \frac{\beta U_{11}^2}{\rho} \frac{h_{s1}}{x_o} \]  

(62)

which agrees with Equation 54, since \( h = H_0 - \Delta h \).
34. In evaluating the average interface and bottom stresses on the basis of Equations 32 and 50 using experimental data, it will be useful first to express these equations in a dimensionless form. It is further required that the average resistive stress \(2\tau\) on the channel walls, from fresh water and saline water, be ascertained analytically since these stresses are not amenable to direct measurements.

35. In laboratory experiments the Reynolds number of flow accounted is generally small and hence the resistance of the sidewalls may be determined by the Blasius relation

\[
\frac{-\bar{u}}{u} = a \left( \frac{u_X}{v} \right)^{1/7}, \quad a = 7.64
\]

where \(u_X = (\tau/\rho)^{1/2}\) and \(\bar{u}\) is the mean velocity at points having a constant distance \(z'\) from the interface. From this

\[
\frac{\tau}{\rho} = a_1 \left( \frac{u}{v} \right)^{-1/4} u^{-2}, \quad a_1 = 0.0284
\]

and

\[
\frac{\tau}{\rho \bar{u}^2} = a_1 \left( \frac{u}{v} \right)^{-1/4} \frac{\bar{u}^{7/4}}{\bar{U}}
\]

where \(\bar{U}\) is the average velocity of the fresh water in a section. Then for the average shear on the wall

\[
\frac{-\bar{t}}{\rho \bar{u}^2} = a_1 \left( \frac{u}{v} \right)^{-1/4} \int_0^1 \left( \frac{\bar{u}}{\bar{U}} \right)^{7/4} ds, \quad s = \frac{z'}{h_w}
\]
$z'$ is distance measured from the interface and $h_w$ is the depth of fresh water. First affecting the integration on the basis of data shown in Table 11, one then has, after introducing $U_o$, the river mean velocity,

$$\frac{2\tau}{\rho U_o^2} = \lambda_o \left( \frac{U}{U_o} \right)^{7/4}$$  \hspace{1cm} (64)

where

$$\lambda_o = 0.051 \left( \frac{U b}{v} \right)^{-1/4}$$  \hspace{1cm} (65)

This estimate is only provisional as it assigns a larger value to the resistance. The derivation ignores the effects of turbulence of the free surface and of the interface. At such points one expects lower stresses than assumed. In the saline water area the frictional effect of the wall is not very critical. Since in one part motion is directed toward the sea and in the other part away from the sea, resistance effects are somewhat neutralized. As the application of the above method to this case is less likely to be valid, it is better to ignore it for the present.

36. Dividing Equation 32 by $U_o^2$, the result is

$$\frac{T_s}{\rho U_o^2} = \frac{g H_o}{U_o^2} \frac{\Delta H}{x_o} I_1 - \sigma \frac{H_o}{x_o} I_2 - \lambda_o \frac{H_o}{R} I_3$$  \hspace{1cm} (66)

where

$$I_1 = 1 - n \int_0^1 \frac{h_s}{h_{s1}} \frac{d(h/\Delta H)}{d\xi} d\xi, \hspace{0.5cm} n = \frac{h_{s1}}{H_o}$$

$$I_2 = \frac{h_{s1} + \Delta H}{(H_o - \Delta H - h_{s1})} = \frac{h_{s1}}{H_o - h_{s1}}$$
\begin{align*}
I_3 &= \int_0^1 \left( \frac{U}{U_o} \right)^{-7/4} \left( \frac{h - h_s}{H_o} \right) \, d\xi = \int_0^1 \left( 1 - \frac{n}{h_s} \right)^{-3/4} \, d\xi \quad (67) \\

\text{Utilizing the experimental values of } h_s/h_{s1} \text{ and } \Delta h/\Delta H \text{ expressed in terms of } \xi, \text{ see Equations 12 and 15, and affecting the integration and remembering that } n = h_{s1}/H_o, \text{ the above given multipliers are} \\
I_1 &= 1 - 0.59n \quad (68) \\
I_2 &= n/(1 - n) \quad (69) \\
I_3 &= 1 + 0.37n + 0.20n^2 \quad (70)
\end{align*}

37. Dividing Equation 54 by $U_o^2$, ignoring the small value term from the sidewall fraction, the result is

\begin{align*}
\frac{T_o}{\rho U_o^2} + \frac{T_s}{\rho U_s^2} &= -n \frac{gH_o}{U_o^2} \frac{\Delta h}{\Delta H} I_4 + n \frac{gH_o}{U_o^2} \frac{\Delta p}{\rho} \frac{h_{s1}}{x_o} + I_5 \frac{H_o}{x_o} \quad (71)
\end{align*}

where

\begin{align*}
I_4 &= \int_0^1 \frac{d(\Delta h/\Delta H)}{d\xi} \frac{h_s}{h_{s1}} \, d\xi = 0.59 \quad (72) \\
I_5 &= \beta \left( \frac{U_{11}}{U_{11}} \right)^2 \frac{n}{(1 - n)^2}
\end{align*}

The definition of the Boussinesq velocity coefficients $\alpha$ is given by
Equation 23 and that of $\beta$ by Equation 47. These coefficients may be evaluated through the entries of the last columns in Tables 10 and 11, respectively. It will be remembered that these tables give the root-mean-square values of velocity $u$ of points of as common distance $z'$ from the interface in a given cross section. This fact is due to the manner in which the determinations are made. The hydrodynamic forces acting on the ribbon and causing it to deflect are proportional in the square of the velocity of the particles striking the ribbon. With this interpretation

$$a = 1.012$$  \hspace{1cm} (73)

and

$$\beta = 0.14$$  \hspace{1cm} (74)

since $U_{11}/U_1 = 0.53$ and $\beta = 0.14$ the last multiplier reduces to

$$I_5 = \frac{0.033n}{(1 - n)^2}$$  \hspace{1cm} (75)

Accordingly, after neglecting the term involving $\lambda$, representing the effect of wall friction the final form of the equation to evaluate the average bottom shear is

$$\frac{T_0}{\rho U_o^2} + \frac{T_s}{\rho U_o^2} = \frac{n}{U_o^2} \frac{g H_o}{U_o} \left( -0.59 \frac{\Delta H}{x_o} + \frac{1}{2} \frac{\Delta \rho}{\rho} \frac{h}{x_o} \right) + \frac{0.033n}{(1 - n)^2}$$  \hspace{1cm} (76)

Subtracting Equation 64 from this last equation and recalling that $I_1 = 1 - 0.59n$, the result is

$$\frac{T_o}{\rho U_o^2} = \frac{g H_o}{U_o^2} \left( - \frac{\Delta H}{x_o} + \frac{n}{2} \frac{\Delta \rho}{\rho} \frac{h}{x_o} \right) + a \frac{H_o}{x_o} I_2 + \lambda \frac{H_o}{B} I_3 + \frac{H_o}{x_o} 15$$  \hspace{1cm} (77)

It is preferable to use this equation to evaluate the average bottom stress under the arrested saline wedge.
PART X: EXPERIMENTAL VALUES OF INTERFACIAL STRESS

38. Using Equations 66 and 77 the experimental values of average interfacial stress $T_s$ and average bottom stress under the wedge $T_0$ are evaluated on the basis of data shown in Tables 1 and 2. The evaluation as function of $U_0 H_0 / v$ are collected in Tables 12 and 13 for water depth $H_0 = 23.0$ cm and $45.5$ cm, respectively. Since the values thus arrived should be independent of the ratio $H/B$, the values of $T_s$ and $T_0$ for the two depths are shown in Figures 11 and 12. Here, $2T_s / \rho U_0^2$ is plotted logarithmically against $U_0 H_0 / v$ and also $2T_0 / \rho U_0^2$ against $U_0 H_0 / v$. The scatter of the points in both figures is very large. This is hardly surprising and was expected since the values of $T_s$ and $T_0$ are obtained in both cases by subtracting quantities of nearly equal order of magnitude, each of these quantities separately being open to errors as large as 5 percent. In particular, the fall of water surface $\Delta H$ cannot be determined with certainty. In this study $\Delta H$ is not measured directly; it is deduced from a curve of observed points $\Delta H$ versus $x$ and there is an element of subjectivity. Another cause for the scatter is the fact that the points of the figures as regards to $n$ are not comparable. By dimensional analysis it may be shown that

$$\frac{2T_s}{\rho U_0^2} = f_1 \left( \frac{U_0 H_0}{v}, n \right)$$

(78)

and

$$\frac{2T_0}{\rho U_0^2} = f_2 \left( \frac{U_0 H_0}{v}, n \right)$$

(79)

For the data shown in the figures $n$ is not constant but varies from 0.35 to 0.55. A good average value of $n$ for all the points is $n = 0.45$. In Figure 11 a straight line is drawn through the points, with a slope equaling 1. Half of the observed points are above the line and the other half below the line; and judging by the eye, the
Figure 11. Average interfacial stress of arrested saline wedge

squares of the deviations are at their least value. In Figure 12, the
slope of the straight line drawn through the points equals 1/4. Ac-
cordingly the average stresses are

\[
\frac{2T_s}{\rho U_o^2} = 54 \left( \frac{U o H}{U o V} \right)^{-1}, \quad n = 0.45
\]  

(80)
Figure 12. Average bottom stress of arrested saline wedges

and

\[
\frac{2T_0}{\rho U_0^2} = 0.03 \left( \frac{U_0 H_0}{\nu} \right)^{-1/4}, \quad n = 0.45
\]

(81)

The examined data cover the range of Reynolds number \( R = 4,000 \) to \( R = 110,000 \). These are in agreement with the previous investigation (Keulegan 1957b) excepting a slight decrease in \( T_0 \).
PART XI: THEORETICAL EVALUATION OF AVERAGE STRESSES

39. The evaluation of the stresses may be attempted on the basis of the measured velocity distribution given in Tables 10 and 11. It is expected that this velocity pattern will apply almost to the whole length of the arrested wedge except a stretch adjacent to the tip.

40. Consider first the evaluation of the average interfacial stress $T_s$. The stress at the interface is

$$\tau_s = \mu \frac{du}{dz} \quad z = 0$$  \hspace{1cm} (82)

or

$$\tau_s = \mu \frac{U_o}{h_s} \frac{du}{dz} \frac{1}{h_s}$$ \hspace{1cm} (83)

Putting

$$m = \frac{du}{U_o} \frac{dz}{h_s}$$  \hspace{1cm} (84)

and dividing by $\rho U_o^2$

$$\frac{\tau_s}{\rho U_o^2} = m \left( \frac{U_o H_o}{V} \right)^{-1} \times \frac{U_o H_o}{U_o h_s}$$

Remembering that $n = h_{s1}/H_o$

$$\frac{\tau_s}{\rho U_o^2} = m \left( \frac{U_o H_o}{V} \right)^{-1} \frac{1}{n} \frac{U_o h_{s1}}{U_o h_s}$$

Introducing the average shear $T_s$

45
\[
\frac{T_s}{\rho U_o^2} = 2m \left( \frac{U H_o}{\alpha_o v} \right)^{-1} \frac{1}{n} \int_0^1 \frac{d\zeta}{h_s h_{s1} \left( 1 - n \frac{h_s}{h_{s1}} \right)}
\]  

(85)

Placing

\[
F_1 = \frac{1}{n} \int_0^1 \frac{d\zeta}{h_s h_{s1} \left( 1 - n \frac{h_s}{h_{s1}} \right)}
\]  

(86)

one now has

\[
\frac{2T_s}{\rho U_o^2} = 2m F_1 \left( \frac{U H_o}{\alpha_o v} \right)^{-1}
\]  

(87)

Taking \( h_s/h_{s1} \) from Table 4, \( F_1 \) is computed by numerical integration for a few selected values of \( \varepsilon \) and \( n \). These values are given in Table 14.

41. Toward the evaluation of the factor \( m \) one needs to consider the velocity rates in the area of the interface. The appropriate quantities are taken from the entries of Tables 10 and 11 and are shown in graph form in Figure 13. The inclinations of the straight lines, one for the region of the fresh water and the other for the salt water, are somewhat different in value; this is in accordance with the fact that the viscosity of salt water is somewhat greater than that of fresh water. On the basis of these data the factor \( m \) amounts to 3.0. As the average value of \( n \) of all the tests is 0.45, the corresponding \( F_1 \) is read from Table 14 and for an assumed \( \varepsilon \), equal to 0.05, is 6.53. Substitution of these in Equation 87 yields for the average interfacial stress the result
The plot of this relation is shown in Figure 11. Agreement between the theory and the observation is quite fair.

Although a slight mixing is present between fresh and saline waters, this fact fails to destroy the effective laminar regime of the interface. However, one expects the initiation of a turbulent regime for high Reynolds number, but the present tests fail to give any indication of the exact value of \( \frac{U_H}{V} \) needed. The maximum \( U_H \) noted is \( 1.2 \times 10^4 \).

Flow in the saline water close to the bottom is turbulent. As the Reynolds number of flow is small, according to Blasius' hypothesis the velocities would vary as the one-seventh power of distance from bottom. Thus, one expects that

\[
\frac{2T_S}{\rho U_o^2} = 39 \left( \frac{U_H}{V} \right)^{-1}
\]

(88)
\[
\frac{u}{U} = A_0 \left( \frac{2}{h_s} \right)^{1/7}
\]  

(89)

where \( h_s \) is the depth of saline water and \( U \) freshwater current mean velocity in a cross section. The channel bottom consists of planed and painted wood. From the experiments of Bazin with planed wooden surfaces one may deduce the velocity law

\[
\frac{u}{u_\infty} = a \left( \frac{u_\infty z}{v} \right)^{1/7}, \quad a = 5.9
\]  

(90)

where \( u_\infty = \sqrt{\frac{1}{\rho}} \). Dividing Equation 89 by Equation 90

\[
\frac{U}{u_\infty} = \frac{a}{A_0} \left( \frac{u_\infty h_s}{v} \right)^{1/7}
\]  

(91)

Solving for \( \tau_0 \),

\[
\tau_0 = \left( \frac{a}{A_0} \right)^{-7/4} \left( \frac{h_s}{v} \right)^{-1/4} U^{7/4}
\]  

(92)

or

\[
\frac{\tau_0}{\rho U_0^2} = \left( \frac{a}{A_0} \right)^{-7/4} \left( \frac{U H_0}{V} \right)^{-1/4} \left( \frac{U}{U_0} \right)^{-7/4}
\]

Introducing \( U H_0 = U(H_0 - h_s) \) and putting \( h_{s1} = nH_0 \) one obtains

\[
\frac{\tau_0}{\rho U_0^2} = \left( \frac{a}{A} \right)^{-7/4} \left( \frac{U H_0}{V} \right)^{-1/4} \left( \frac{h_s}{n H_{s1}} \right)^{-1/4} \left( 1 - n \frac{h_s}{H_{s1}} \right)^{-7/4}
\]
Multiply the two sides of this equation by \(2d\xi\), integrating between \(\xi = \varepsilon\) and \(\xi = 1\), and introducing the average bottom shear for the entire length of the wedge \(T_o\), we now have

\[
\frac{2T_o}{\rho U_o^2} = 2 \left( \frac{A_o}{a} \right)^{7/4} \cdot F_2 \cdot \left( \frac{UH_o}{v} \right)^{-1/4}
\]

(93)

where

\[
F_2 = \int_{\varepsilon}^{1} \left( \frac{1 - n \frac{h_s}{h_s1}}{1 - n \frac{h_s}{h_s1}} \right)^{-7/4} \left( \frac{n \frac{h_s}{h_s1}}{n \frac{h_s}{h_s1}} \right)^{-1/4} d\xi
\]

Again taking the values of \(h_s/h_s1\) from Table 4, the definite integral may be computed by numerical integration for selected values of \(n\) and \(\varepsilon\). These are given in Table 15.

44. The examination of lower velocities of the arrested saline wedge shown in Figure 7 suggests that \(A_o\) equals 0.25. Selecting \(\varepsilon = 0.05\) and \(n = 0.45\), Table 15 gives \(F_2 = 2.3\). As indicated in the above, \(a = 5.9\). Inserting these in Equation 73, the result is

\[
\frac{2T_o}{\rho U_o^2} = 0.018 \left( \frac{UH_o}{v} \right)^{-1/4}
\]

(94)

This is plotted in Figure 12. The estimated values of \(T_o\) are twice as small as the values obtained from Equation 73. Probably the main reason for this large difference was the obvious difficulty of measuring small velocities accurately. In the derivation above it was assumed tacitly that there is similarity in velocity pattern. In the presence of mixing at the interface this assumption would not be valid, and the evaluation should be carried out on a different basis.
A Theory of Affine Shape of Arrested Saline Wedge

45. In a study dealing with the circulation of cooling water between the intake and the outlet of a thermoelectric power plant, Beta (1957) has shown that the affine shape of an arrested wedge computed from the flow equations originally given by Shijf and Schoenfeld (1953), is in agreement with observations. Comparison with theory is made also for observations from Keulegan (1952) covering the range from \( F = 0.06 \) to \( F = 0.40 \). These fall on the theoretical curve corresponding to \( F = 0.55 \). Later, using a similar analysis, Harleman (1961) has indicated that there is agreement between theory and observation as regards the shape of arrested saline wedges. These are significant findings and require further consideration.

46. The equation of motion as relating to the freshwater layer above the saline wedge may be obtained directly from Equation 30. Neglecting the wall friction \( T_w \), taking \( x \) equal to unity, writing \( h_w \) for \( h - h_s \), the desired result is

\[
\frac{dh}{dx} + \frac{U}{g} \frac{dU}{dx} = -\frac{\tau_s}{gh_w}
\]  

From Equation 48 neglecting bottom friction \( T_o \) and placing \( \beta = 0 \) (no motion in the wedge) one has

\[
\frac{dh}{dx} + \frac{\Delta p}{p} \frac{dh_s}{dx} = \frac{\tau_s}{\rho gh_s}
\]  

as relating to a stationary salt wedge. These agree with the equations of Shijf and Schoenfeld (1953). Because of the condition of continuity

\[
U \frac{dh_w}{dx} = -h_w \frac{dU}{dx}
\]
and assuming that the fall of surface water may be neglected

\[
\frac{dh_s}{dx} = \frac{s}{dx}
\]  

(98)

and then

\[
U \frac{dh_s}{dx} = h_w \frac{dU}{dx}
\]

Introducing the latter in Equation 95,

\[
\frac{dh}{dx} + \frac{U^2}{gh_w} \frac{dh_s}{dx} = -\frac{\tau_s}{\rho gh_w}
\]

(99)

Subtracting Equation 99 from Equation 96, the result is

\[
\left(\frac{\Delta \rho}{\rho} - \frac{U^2}{gh_w}\right) \frac{dh_s}{dx} = \frac{\tau_s}{\rho g} \left(\frac{1}{h_w} + \frac{1}{h_s}\right)
\]

(100)

Introducing

\[
\frac{\tau_s}{\rho} = \frac{\lambda_i U^2}{2}
\]

(101)

Equation 100 becomes

\[
\left(\frac{\Delta \rho}{\rho} - \frac{U^2}{gh_w}\right) \frac{dh_s}{dx} = \frac{\lambda_i U^2}{2} \frac{1}{g} \left(\frac{1}{h_w} + \frac{1}{h_s}\right)
\]

which may be written also as

\[
\left[\frac{\Delta \rho}{\rho} - \frac{U_o^2}{gh_o} \left(\frac{U}{U_o}\right)^2 \frac{H_o}{h_w}\right] \frac{dh_s}{dx} = \frac{\lambda_i U_o^2}{2} \frac{1}{gH_o} \left(\frac{U}{U_o}\right)^2 \left(\frac{H_o}{h_w} + \frac{H_o}{h_s}\right)
\]

(102)
Placing
\[ \eta = \frac{h_s}{H_0} \quad ; \quad s = \frac{x}{H_0} \]  \hspace{1cm} (103)

and as, neglecting \( \Delta h \),
\[ \frac{U_0}{U} = 1 - \eta \]

in terms of new variables Equation 98 changes to
\[ \left[ \eta (1 - \eta)^3 - F_o^2 \eta \right] \eta \eta = \frac{\lambda_1}{2} F_o^2 \eta \eta \quad \text{ds} \]  \hspace{1cm} (104)

Here \( F_o \) is the densimetric Froude number
\[ F_o = \frac{V_r}{V_o} = \frac{U_0}{\left( \frac{\Delta \rho}{\rho g H_o} \right)^{1/2}} \]

The solution of Equation 100 subject to the condition \( \eta = 0 \) at \( s = 0 \), is
\[ \left( 1 - F_o^2 \right) \eta^2 \quad - \quad \frac{\eta^3}{2} \quad + \quad \frac{3}{4} \eta^4 \quad - \quad \frac{5}{5} \eta^5 \quad = \quad \frac{\lambda_1}{2} F_o^2 \quad \frac{x}{H_0} \]  \hspace{1cm} (105)

Replacing \( x \) by \( x_o \), the length of the arrested saline wedge, and \( \eta \) by \( \eta_L \),
\[ \eta_L = \frac{h_s 1}{H_0} \]

one also has
\[ \left( 1 - F_o^2 \right) \eta_L^2 \quad - \quad \eta_L^3 \quad + \quad \frac{3}{4} \eta_L^4 \quad - \quad \frac{5}{5} \eta_L^5 \quad = \quad \frac{\lambda_1}{2} F_o^2 \quad \frac{x_o}{H_0} \]  \hspace{1cm} (106)

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This may be written after placing

\[ \phi(\eta_L) = \left(1 - F_o^2\right) \frac{\eta_L^2}{2} - \eta_L^3 + \frac{3}{4} \eta_L^4 - \frac{1}{5} \eta_L^5 \]  

(107)

as

\[ \phi(\eta_L) = \frac{\lambda}{2} \frac{F_o^2}{H_o} \frac{x_0}{H_o} \]  

(108)

Here is an expression which relates the coefficient of resistance of the interface with the length of the arrested wedge. It is equivalent to one already indicated by Harleman (1961). It is valid only in the case where the bottom stress vanishes and the momentum of the wedge can be ignored. Now, dividing Equation 105 by Equation 108 and writing

\[ \zeta = x/x_o \], as before, one has

\[ \left(1 - F_o^2\right) \frac{\eta^2}{2} - \eta^3 + \frac{3}{4} \eta^4 - \frac{1}{5} \eta^5 = \phi(\eta_L) \cdot \zeta \]  

(109)

Assuming that the flow at the river mouth is critical

\[ \eta_L = 1 - F_o^2 \]  

(110)

We note also that

\[ \eta = \frac{h_s}{h_{s1}} \eta_L \]  

(111)

Thus, through Equations 105, 106, and 107, \( \zeta \) may be obtained as a function of \( h_s/h_{s1} \) for a chosen \( F_o \). A few determinations are shown in Figure 14. The circles represent experimental values for an \( F_o \) of 0.40 and are taken from Table 5. Agreement between the theoretical values and the observations is good. In the experimental wedges friction is present at the flume walls and also at the wedge bottom. In the
Figure 14. Theoretical affine shape of arrested wedges

theoretical evaluations these frictions are ignored. The observed close agreement could be hardly expected, unless these two frictions are of like value and may be imagined to be incorporated into the interfacial stress $T_s$. Another point to remember is that the critical flow condition at the river mouth, Equation 110, is valid only for values of $F_o$ close to 0.5. Variation from the expression becomes important as $F_o$ approaches zero. In the usual derivation, it is assumed tacitly that at the river mouth the pressure is hydrostatic. When the depth of salt wedge at the river mouth is increased the interface tends to be more curved, affecting the pressure there, and this is no longer hydrostatic. The
critical conditions are not exactly known and experimental elucidation of the matter is very much desired. This was mentioned before.

47. Equation 108 relates the length of arrested saline wedge to the friction coefficient of the interface $\lambda_i$. The quantity $\Phi(\eta_L)$ is plotted against $F_0$ in Figure 15. For the range between $F_0 = 0.5$ and $F_0 = 0.1$ a good approximation for $\Phi(\eta_L)$ is
\[
\phi(\eta_L)/F_o^2 = 0.015 F_o^{-5/2}
\]

(112)

Thus, a simple formula for the resistance coefficient of the interface \( \lambda_i \) from Equation 108 would be

\[
10 \lambda_i \frac{L}{H_o} = 0.30 F_o^{-5/2}
\]

(113)

As noted before, the length of the arrested wedges observed in laboratory channels with a depth width ratio \( H_o/B \) equal to unity would be

\[
\frac{L_o}{H_o} = 0.19 \left( \frac{\Delta H_o}{v} \right)^{1/2} \left( \frac{2V_r}{V_\Delta^K} \right)^{-5/2}
\]

or, since \( F_o = V_r/V_\Delta \)

\[
\frac{L_o}{H_o} = 0.033 \left( \frac{U_o H_o}{v} \right)^{1/2} F_o^{-3}
\]

Introducing this in Equation 113, practically

\[
10 \lambda_i = 9.1 F_o^{1/2} R^{-1/2}
\]

(114)

Remembering that the mean of the densimetric Froude number of the present tests is \( F_o = 0.45 \), and inserting this

\[
10 \lambda_i = 6.1 (U_o H_o/v)^{-1/2}
\]

(115)

48. This would be the value of the coefficient on the basis of the length of wedges observed in the laboratory channels and the theoretical relation between the coefficient and the length of the wedge, Equation 108.

49. It would be of interest to compare the above result with the
value of $\lambda_i$ to be deduced from the $T_s$ observed in the present study.

Now, in view of the definition in Equation 101

\[
\frac{T_s}{\rho} = \frac{\lambda}{2} \int_0^1 U^2 \, d\zeta, \quad \zeta = \frac{x}{x_0}
\]

or

\[
\frac{2T_s}{\rho U_o^2} = \lambda \int_0^1 \left( \frac{U}{U_o} \right)^2 \, d\zeta = \lambda \int_0^1 \left( \frac{d\zeta}{1 - n \frac{h_s}{h_s1}} \right)^2, \quad n = \frac{h_s1}{H_o}
\]  \hspace{1cm} (116)

In the tests of the present study the average value of $n$ is 0.45 and the corresponding value of the integral in the above relation is 1.64. Then,

\[
\frac{2T_s}{\rho U_o^2} = 1.64 \lambda
\]

and as the experiments of the present study here yielded

\[
\frac{2T_s}{\rho U_o^2} = 54 \left( \frac{U_o}{H_o} \right)^{-1}
\]

the equation of the upper line in Figure 11, we have

\[
\lambda_i = 32.9 \left( \frac{H_o}{U_o} \right)^{-1}
\]  \hspace{1cm} (117)

a result not in agreement with Equation 115. At the moment we are unable to discuss the reason for this difference.
Interfacial Stress in Lock Exchange

50. In a definitive study the nature of interfacial stress in lock exchange was examined by Abraham and Eyesink (1971). The coefficient of stress \( \lambda \) may be defined as

\[
\tau_i = \frac{\lambda_i}{2} \frac{\rho U_r^2}{r}
\]  

(118)

where \( U_r \) is the relative velocity between the two layers of the exchange flow and \( \tau_i \) the interface stress. Authors use the Weisbach coefficient \( f \), \( f = 4\lambda_i \). Resorting to an energy consideration, \( 4\lambda_i \) is first evaluated utilizing the experimental data of Keulegan (1957a) and Barr (1963) on exchange flow involving laboratory flumes. To examine the effect of viscosity on the coefficient these authors performed additional experiments in two channels, one small and the other 3.45 times as great in the Delft Hydraulics Laboratory. Evaluation of the coefficient is effected using the flow equations originally formulated by Shijf and Schoenfeld (1953). Initial conditions only are considered coinciding with the densimetric Froude number \( F_r = 0.9 \), the number being defined

\[
F_r = \frac{U_r}{\sqrt{\frac{\Delta \rho}{\rho g H_0}}}
\]  

(119)

\( H_0 \) being the combined depth of the two layers. The authors present the evaluations of the coefficient as a function of Reynolds number

\[
R_e = \frac{U_2 R_2}{v}
\]  

(120)

where \( U_2 \) is the current velocity in the lower layer and \( R_2 \) is the hydraulic radius

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where \( h_s \) is the depth of the lower layer and \( B \) the width of channel.

As the depth of water \( H \) equals \( 2h_s \)

\[
h_s = \left( 2 + \frac{H_o}{B} \right) R_2
\]

For the present purpose it is desirable to express the coefficient as a function of the Reynolds number \( \frac{U_2 h_s}{\nu} \). The authors' data may be readily expressed as a function of \( \frac{U_2 h_s}{\nu} \) as well. This is done in Figure 16. Results from the authors' experiments are in agreement with

Figure 16. Relation between \( \lambda \) and \( R_e \) in exchange flow
(after Abraham and Eyesink 1971)
the evaluation from the Barr and Keulegan data. Three distinct regions are discernible. In the laminar range

\[ \lambda_i = 5.5 R_e^{-1} \quad R_e < 1000 \]  

(123)

This is also the relation derived by Ippen and Harleman (1952) for the case of an underflow, that is, flow of saline waters in an incline under a stagnant pool of fresh water. In the transition range

\[ \lambda_i = 0.055 R_e^{-1/3} \quad 1000 < R_e < 40.000 \]  

(124)

The authors point out that the coefficient is independent of Reynolds number for greater value of this number

\[ \lambda_i = 0.0017 \quad R_e < 40.000 \]  

(125)

51. Agreement with our results of the interfacial stress coefficient, as expressed in Equation 112, is lacking. Now if one would accept for a moment that the experimental procedures of our main study and the numerical analysis underlying it are admissible, then this disagreement could suggest that the breakdown of turbulence in the interfaces of an exchange flow is not similar to the breakdown in the interface of an arrested saline wedge.

**Interfacial Stress in Density Underflow**

52. A detailed study of the mechanism of interfacial flow between fresh and saline waters was given by Loftquist (1960). The case considered is the flow of saline water under a stagnant body of fresh water in a horizontal flume. The matters examined relate to the interfacial gradient and stress, entrainment, distribution of density, stress and velocity in a cross section, and relation of the length of transition
layer to the densimetric Froude number. For the present, however, only
the results of the coefficient of interfacial resistance will be con-
sidered. Loftquist defines the resistance coefficient as

\[ \lambda_m = \frac{\tau_m}{\rho u_o^2} \]  (126)

where \( \tau_m \) is in effect the stress of interface and \( u_o \) is the maximum
velocity of saline waters in a cross section. The densimetric Froude
number is defined as

\[ F_L = \frac{U^2}{g \left( \Delta \rho \right) h_r} \]  (127)

where \( h_r \) is hydraulic radius of the saline water layer

\[ h_r = \frac{bh_s}{(b + h_s)} \]  (128)

where \( h_s \) is the depth of the saline water layer, \( 2b \) the width, and
\( U \) the mean velocity in the cross section of the saline layer. In the
experiments conducted, \( h_s \) was about 18 or 19 cm and \( b \), 11.5 cm, so
that \( h_r = 0.37h_s \). The Reynolds number was defined as

\[ R_L = \frac{uh_r}{\nu_s} \]  (129)

where \( \nu_s \) is the viscosity of saline waters. The stress \( \tau_m \) is deduced
from the equation of motion of saline layer making use of observed veloc-
ity distribution and the fall of the interface. Evaluations yield for
the coefficient of interfacial resistance

\[ \lambda_m = 4.34R_L^{-1} \beta_m \]  (130)
where $\beta_m$ is a function of $R_L$ and $F_L$.

For the purpose of comparing with the findings of the present study, it is desirable to express densimetric Froude number and Reynolds number in term of $h_s$ instead of $h_r$, thus

$$F_k = \frac{U^2}{g \frac{\Delta \rho}{\rho} h_s} \quad \text{and} \quad R_k = \frac{U h_s}{V} \quad (131)$$

These give

$$F_k = 0.37F_L; \quad R_k = 2.7R_L \quad (132)$$

Following the representation previously used, Equation 97, the coefficient of interfacial friction in terms of mean velocity $U$

$$\lambda_i = \frac{2T_m}{\rho U^2}$$

Comparing with Equation 126

$$\lambda_i = 2 \left( \frac{u_o}{U} \right)^2 \lambda_m$$

During velocity traverses Loftquist found $u_o^2 = 1.21U^2$ so that

$$\lambda_i = 2.42 \lambda_m$$

Introducing in Equation 130, $\lambda_m$ from the last relation and $R_L$ from Equation 132, the result is

$$\lambda_i = 28.5 R_k^{-1} \beta_m \quad (133)$$

The dependence of $\beta_m$ on Reynolds number and densimetric Froude number is given in Figure 17. Construction of the figure is original with us.
and is based on data given by Loftquist. When $F_k R_k^{1/2}$ is less than 5, $\beta_m$ equals unity and in this range the interfacial resistance coefficient is

$$\lambda_i = 28.5 R_k^{-1}$$

(134)

a result in close agreement with the findings of the present study, Equation 117. Further in the range $F R^{1/2}$ is greater than 4

$$\beta_m = 0.25 F_k R_k^{1/2}$$

which yields
\[ \lambda_i = 7.1 F_k R_k^{-1/2} \]

indicating that for larger values of densimetric Froude number the coefficient varies inversely as the one-half power of Reynolds number. The scatter of points in Figure 17 renders those results provisional. In fact, from Figure 18, where \( \beta_m \) is plotted against \( F_k \) and the scatter of points is nearly of the same extent as in Figure 17, one may also conclude that the coefficient of interfacial resistance varies directly with \( F_k \) and inversely with \( R_k \).

![Figure 18. Dependence of \( \beta_m \) on densimetric Froude number (after Loftquist 1960)](image)
PART XIII: MODIFICATION OF STRESS FORMULA DUE TO ENTRAINMENT

53. Equation 66 to evaluate the average interfacial stress \( T_s \) was based on the assumption that entrainment (that is, flow of saline waters into the freshwater current) is absent. In the presence of entrainment the coefficients \( I_n \) would undergo changes. Possibly the kinetic reaction time will register the greater change leading to the form \( I_2 + \delta I_2 \) as shown below.

54. Denoting the entrainments by \( U_m \), the vertical velocity components at the free surface and at the interface now are

\[
\omega_s = u_s \frac{dh}{dx} , \quad \omega_i = u_i \frac{dh}{dx} + U_m
\]

Then the left-hand term of Equation 21 changes to

\[
\int_{-b}^{b} \int_{h_s}^{h} \left( u \frac{\partial u}{\partial x} + \omega \frac{\partial u}{\partial z} \right) dz \ dy = \int_{-b}^{b} \int_{h_s}^{h} \frac{du}{dx}^2 \ dz \ dy + \left( \frac{2}{u_s} \frac{dh}{dx} - \frac{2}{u_i} \frac{dh}{dx} - U_m u_B \right) \]

and in the place of Equation 24 one now has

\[
\int_{-b}^{b} \int_{h_s}^{h} \left( u \frac{\partial u}{\partial x} + \omega \frac{\partial u}{\partial z} \right) dz \ dy = \frac{d}{dx} \left( \alpha U^2 A \right) - U_m u_B
\]

Integrating the right-hand member between \( x = 0 \) and \( x = x_o \) dividing by \( B \), the result is

\[
\alpha U^2 h_{w1} - \alpha U^2 H_{o} - \int_{0}^{x_o} U_m u_i \ dx
\]

(136)

where \( h_{w1} \) is the depth of fresh water at river mouth, \( h_{w1} = H_o - h_s1 \).

In view of condition of continuity
\[ U_{1} h_{w1} = U_{0} H_{o} + \int_{0}^{x} U_{m} \, dx \]

and placing \( u_{i} = mU \), \( m = 0.53 \), previously noted, and \( U_{m} = kU \)

Equation 136 reduces to

\[ \alpha U_{0}^{2} H_{o} \left( \frac{H_{o}}{h_{w1}} - 1 \right) + 2 \frac{H_{o}}{h_{w1}} \int_{0}^{x_{0}} UU_{o} \, dx - mk\int_{0}^{x_{0}} U^{2} \, dx \]

Dividing by \( U_{0}^{2} x_{o} \)

\[ \frac{H_{o}}{x_{o}} \left( \frac{H_{o}}{h_{w1}} - 1 \right) + \frac{H_{o}}{x_{o}} \left[ \frac{2}{2} \frac{x_{o}}{h_{w1}} \int_{0}^{1} \left( \frac{U}{U_{o}} \right) \, d\zeta - m \frac{x_{o}}{H_{o}} \int_{0}^{1} \left( \frac{U}{U_{o}} \right)^{2} \, d\zeta \right] \]

This may be written as

\[ \alpha \frac{H_{o}}{x_{o}} I_{2} + \frac{H_{o}}{x} \delta I_{2} \]

where

\[ I_{2} = \left( \frac{H_{o}}{h_{w1}} - 1 \right) = \frac{n}{1-n} \text{ (as before)} \]

and

\[ \delta I_{2} = k \left[ \frac{2x_{o}}{h_{w1}} \int_{0}^{1} \left( \frac{U}{U_{o}} \right) \, d\zeta - m \frac{x_{o}}{H_{o}} \int_{0}^{1} \left( \frac{U}{U_{o}} \right)^{2} \, d\zeta \right] \]

66
Hence,

\[
\frac{\delta I_2}{I_2} = k \frac{x_o}{H_0} \left[ 1 - \frac{n}{1 - n} \int_0^1 \left( \frac{U}{U_0} \right) d\zeta - m \int_0^1 \left( \frac{U}{U_0} \right)^2 d\zeta \right] \frac{1 - n}{n}
\]

Assuming \( n = 0.45 \), adapting \( m = 0.53 \), assigning to \( U/U_0 \) the value

\[
\frac{U}{U_0} = 1 - n \frac{h_s}{h_{s1}}
\]

sufficiently accurate for the present purpose and carrying out the required integration using \( h_s/h_{s1} \) from Table 4, we find

\[
\frac{\delta I_2}{I_2} = 4.46k \frac{x_o}{H_0}
\]

(137)

In a previous investigation on mixing in arrested saline wedges (Keulegan 1955a), it was shown that

\[
U_m = k(U - U_c) \quad k = 2.12 \times 10^{-4}
\]

where \( U_c \) is the critical velocity for the initiation of mixing at the interface obeying the relation

\[
U_c = 7.3(vg\Delta \rho/\rho)^{1/3}
\]

In the above analysis \( U_c \) was neglected for simplicity. Using \( k \) as presently given

\[
\frac{\delta I_2}{I_2} = 9.4 \times 10^{-4} \frac{x_o}{H_0}
\]

(138)
In the tests of the present study $x_o/H_o$ is less than 200. Accordingly, the evaluation of the average interfacial stress $T_s$ on the basis of Equation 66 is nearly correct even if there be mixing at the interface.

55. On the other hand, entrainment across the interface has a much greater effect on the flow pattern in the area of the arrested saline wedge. With entrainment the assumption of similarity in the velocity profiles is no longer valid. Theoretical evaluation of $T_o$ previously made was based on the condition of similarity. The estimated values were found to be inferior to the observed values and this may in part be attributed to the fact that the effect of entrainment was not considered.
PART XIV: NEED FOR FURTHER RESEARCH

56. Expressions of the affine shape of arrested saline wedges and those of relative depth of saline water at the river mouth as established by laboratory observation appear to be equally valid for large rivers. As regards the length of a saline wedge a similar transfer to the prototype condition would not be permissible, especially when the observations are made in narrow channels. One restrictive effect in such channels is the friction of the channel walls. The summary of results from the present and previous investigations was given in Equation 5 for \( H/B = 1 \) and in Equation 6 for \( H/B = 2 \). To obtain the corresponding values for infinitely wide laboratory channels one may resort to analysis using the experimentally determined values of the interfacial and bottom stresses (Keulegan 1957b). For a small densimetric Reynolds of the order of 10 thousand

\[
\frac{L}{H_0} = 0.22 \left( \frac{V \Delta H_0}{v} \right)^{1/2} \left( \frac{2V r}{V \Delta} \right)^{-5/2} \tag{139}
\]

and for large densimetric number of order of 10 million

\[
\frac{L}{H_0} = 6 \left( \frac{V \Delta H_0}{v} \right)^{1/4} \left( \frac{2V r}{V \Delta} \right)^{-5/2} \tag{140}
\]

The latter should also apply to large rivers if the dependence of the interfacial and bottom stress on Reynolds number is the same as in laboratory small channels. This, however, is certainly open to question.

57. The coefficient of friction of the interface obtained in this study is in agreement with Loftquist data for small velocities of saline waters flowing under a pool of practically stagnant fresh water in a horizontal channel. When mixing is present, Loftquist's data indicate that the coefficient varies as \( R^{-1/2} \). This effect is absent in the data of our investigation. The Delft Laboratory tests on exchange flow of lock operation reveal that during initial motion characterized by a
constant densimetric Froude number the coefficient varies as $R^{-1/3}$ for moderately high Reynolds number and eventually at greater high Reynolds numbers above a critical value it remains constant.

58. The direct determination of the stresses, interfacial and bottom, of an arrested wedge is a difficult matter. To reduce the errors due to the side frictions it is desired that new investigation be undertaken with channels of greater width in comparison with water depths. Further, the channels should be of such depth as to allow flows of larger Reynolds number. In addition, closer attention should be paid to the matter of the velocities in the area of the wedge and in cross sections more than one, all uniformly spaced across the length of the wedge. With the larger freshwater velocities the extent of mixing should be determined and its bearing on the wedge area velocities ascertained.

59. If data are available on the saline wedges in large rivers, the question of lengths may be readily examined by assuming that the appropriate relation is

$$\frac{L}{H_o} = A \left( \frac{V H_o}{V + V} \right)^n \left( \frac{2V r}{V \Delta} \right)^{-5/2}$$

with the constants $A$ and $n$ to be determined. Taking the logarithm of the two sides of the equation, the resulting linear algebraic equation readily yields the values of the unknowns. The procedure certainly should resolve the question if the saline wedge length in rivers is independent of Reynolds number.
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Keulegan, G. H. 1955a (Jun). Interface Mixing in Arrested Saline Wedges. NBS Report 4142, Seventh Progress Report to Director, Waterways Experiment Station, Corps of Engineers, Department of Army.

Keulegan, G. H. 1957a (Mar). An Experimental Study of the Motion of Saline Water from Locks into Fresh Water Channels. NBS Report 5168, Progress Report on Model Laws of Density Currents, Report to Director, Waterways Experiment Station, Corps of Engineers, Department of Army.

Keulegan, G. H. 1952 (Jun). Effectiveness of Salt Barriers in Rivers. NBS Report 1700, Progress Report on Model Laws of Density Currents, Report to Director, Waterways Experiment Station, Corps of Engineers, Department of Army.


### Table 1

**Arrested Saline Wedge Observed Data**

\( H_o = 45.5 \text{ cm}; \ B = 22.9 \text{ cm} \)

<table>
<thead>
<tr>
<th>No.</th>
<th>( U_o, \text{ cm/sec} )</th>
<th>( \Delta \rho, \text{ gm cm}^{-3} )</th>
<th>( h_{s1}, \text{ cm} )</th>
<th>( \theta, ^\circ\text{C} )</th>
<th>( x_o, \text{ cm} )</th>
<th>( \Delta H, \text{ cm} )</th>
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Table 2
Arrested Saline Wedge Observed Data
$H_o = 23.0$ cm; $B = 22.9$ cm

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<th>$h_s1$, cm</th>
<th>$\theta$, $^\circ C$</th>
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Table 3

Depth of Saline Water at River Mouth
(from Keulegan 1957b)

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<th>(h_{s1}/H_o)</th>
<th>(2V_r/V_\Delta)</th>
<th>(h_{s1}/H_o)</th>
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<td>1.50</td>
<td>0.232</td>
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</table>
Table 4
Affine Shape of Arrested Saline Wedges
(from Keulegan 1957b)

<table>
<thead>
<tr>
<th>ζ</th>
<th>h_s/h_s1</th>
<th>ζ</th>
<th>h_s/h_s1</th>
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<td>0.570</td>
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<td>0.20</td>
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<td>0.812</td>
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<td>0.95</td>
<td>0.885</td>
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Table 5

Affine Shape of Arrested Saline Wedges

\( H_0 = 23.0 \text{ cm}; \ B = 22.9 \text{ cm}; \ F_0 = 0.40 \)

<table>
<thead>
<tr>
<th>( \Delta \rho/\rho )</th>
<th>0.0098</th>
<th>0.0174</th>
<th>0.0396</th>
<th>0.0580</th>
<th>0.0696</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U_0 ), cm/sec</td>
<td>5.80</td>
<td>7.70</td>
<td>11.9</td>
<td>14.8</td>
<td>18.0</td>
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<tr>
<td>( \zeta )</td>
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<td></td>
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<td>0.90</td>
<td>0.95</td>
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<td></td>
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<tr>
<td>( h_s/h_{s1} )</td>
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<td>0.671</td>
<td>0.742</td>
<td>0.819</td>
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Mean
Table 6

Relative Surface Fall over Arrested Salt Wedge

\( H_0 = 23.0 \text{ cm}; B = 22.9 \text{ cm}; F_0 = 0.40 \)

<table>
<thead>
<tr>
<th>( \Delta p/p )</th>
<th>0.0098</th>
<th>0.0174</th>
<th>0.0396</th>
<th>0.0580</th>
<th>0.0696</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U_0, \text{ cm/sec} )</td>
<td>5.80</td>
<td>7.70</td>
<td>11.9</td>
<td>14.8</td>
<td>18.0</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
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<tr>
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<td>0.060</td>
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<td>0.069</td>
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<td>0.121</td>
<td>0.131</td>
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<td>0.190</td>
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<td>0.257</td>
<td>0.281</td>
<td>0.296</td>
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<td>0.336</td>
<td>0.358</td>
<td>0.396</td>
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<tr>
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<td>0.557</td>
<td>0.532</td>
<td>0.572</td>
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<td>0.770</td>
<td>0.810</td>
<td>0.870</td>
<td>0.775</td>
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<td>1.000</td>
<td>1.000</td>
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Table 7
Relative Surface Fall over Arrested Salt Wedge

\[ H_o = 45.5 \text{ cm}; \ B = 22.9 \text{ cm}; \ F_o = 0.40 \]

<table>
<thead>
<tr>
<th>( \Delta \rho/\rho )</th>
<th>0.0675</th>
<th>0.0325</th>
<th>0.0172</th>
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<tr>
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<td>22.8</td>
<td>14.5</td>
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<td>( \zeta = x/x_o )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
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<td>0.051</td>
<td>0.053</td>
<td>0.050</td>
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<td>0.107</td>
<td>0.102</td>
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<td>0.176</td>
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<td>0.621</td>
<td>0.625</td>
<td>0.592</td>
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<td>0.770</td>
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### Table 8
**Interface Velocities**

\( H_o = 23.0 \text{ cm}; B = 22.9 \text{ cm}; F_o = 0.40 \)

<table>
<thead>
<tr>
<th>( \Delta \rho/\rho )</th>
<th>0.0098</th>
<th>0.0174</th>
<th>0.0396</th>
<th>0.0580</th>
<th>0.0696</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U_o, \text{cm/sec} )</td>
<td>5.80</td>
<td>7.70</td>
<td>11.9</td>
<td>14.8</td>
<td>18.0</td>
<td></td>
</tr>
<tr>
<td>( \zeta = x/x_o )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td>0.425</td>
<td>0.472</td>
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<td>0.438</td>
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<td>0.495</td>
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<td>0.519</td>
<td>0.515</td>
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<td>0.525</td>
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<td>0.516</td>
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<td>0.522</td>
<td>0.542</td>
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<td>0.518</td>
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<td>0.511</td>
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<td>0.485</td>
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<td>0.529</td>
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### Table 9

**Surface Velocities**

\( H_o = 23.0 \text{ cm}; \ B = 22.9 \text{ cm}; \ F_o = 0.40 \)

<table>
<thead>
<tr>
<th>( \Delta p/p )</th>
<th>0.0098</th>
<th>0.0174</th>
<th>0.0396</th>
<th>0.0580</th>
<th>0.0696</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U_o, \text{ cm/sec} )</td>
<td>5.80</td>
<td>7.70</td>
<td>22.9</td>
<td>14.8</td>
<td>18.0</td>
</tr>
<tr>
<td>( \xi = x/x_o )</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>0.2</td>
<td>0.3</td>
<td>0.4</td>
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<td>0.7</td>
<td>0.8</td>
<td>0.9</td>
</tr>
<tr>
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<td>1.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
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<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
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<td>1.080</td>
<td>1.090</td>
<td>1.100</td>
<td>1.095</td>
<td>1.100</td>
<td>1.095</td>
<td>1.100</td>
<td>1.095</td>
<td>1.100</td>
</tr>
</tbody>
</table>
Table 10
Velocity Distribution in Salt Wedge
\( H_o = 23.0 \text{ cm}; B = 22.9 \text{ cm}; F_o = 0.40 \)

<table>
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<th>Series</th>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta p/\rho )</td>
<td>0.0098</td>
<td>0.0174</td>
<td>0.0396</td>
<td>0.0580</td>
<td>0.0696</td>
<td></td>
</tr>
<tr>
<td>( h_w, \text{ cm} )</td>
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<td>14.9</td>
<td>15.6</td>
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<td>17.4</td>
<td>15.8</td>
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<td>7.1</td>
<td>6.7</td>
<td>6.2</td>
<td>7.0</td>
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<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>( u/U )</td>
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<td></td>
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<td></td>
<td></td>
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<td>0.527</td>
<td>0.530</td>
</tr>
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<td>0.383</td>
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<td>0.245</td>
<td>0.253</td>
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<tr>
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<td>0.235</td>
<td>0.212</td>
<td>0.184</td>
<td>0.199</td>
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<td>-0.20</td>
<td>0.152</td>
<td>0.129</td>
<td>0.195</td>
<td>0.171</td>
<td>0.140</td>
<td>0.157</td>
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<td>0.070</td>
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<td>-0.156</td>
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<td>-0.152</td>
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<td>0.000</td>
<td>0.000</td>
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</tr>
</tbody>
</table>

* \( z' \) is measured from interface upward.
### Table 11

**Velocity Distribution**

**in Fresh Water over the Salt Wedge**

\[ H_o = 23.0 \text{ cm}; \ B = 22.9 \text{ cm}; \ F_o = 0.40 \]

<table>
<thead>
<tr>
<th>Series</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta p/\rho )</td>
<td>0.0098</td>
<td>0.0174</td>
<td>0.0396</td>
<td>0.0580</td>
<td>0.0696</td>
<td></td>
</tr>
<tr>
<td>( h_w, \text{ cm} )</td>
<td>15.1</td>
<td>14.9</td>
<td>15.6</td>
<td>16.4</td>
<td>17.4</td>
<td>15.8</td>
</tr>
<tr>
<td>( h_s )</td>
<td>7.4</td>
<td>7.7</td>
<td>7.1</td>
<td>6.7</td>
<td>6.2</td>
<td>7.0</td>
</tr>
</tbody>
</table>

| \( z'/h^* \) | | | | | | | \( u/U \) |
|-----|-----|-----|-----|-----|-----|-----|
| 0.00 | 0.540 | 0.536 | 0.567 | 0.480 | 0.527 | 0.530 |
| 0.04 | 0.648 | 0.666 | 0.665 | 0.580 | 0.646 | 0.641 |
| 0.08 | 0.736 | 0.746 | 0.735 | 0.651 | 0.708 | 0.715 |
| 0.12 | 0.801 | 0.792 | 0.785 | 0.706 | 0.755 | 0.768 |
| 0.16 | 0.846 | 0.827 | 0.822 | 0.746 | 0.789 | 0.806 |
| 0.20 | 0.880 | 0.857 | 0.852 | 0.780 | 0.830 | 0.840 |
| 0.24 | 0.908 | 0.883 | 0.876 | 0.810 | 0.852 | 0.866 |
| 0.32 | 0.950 | 0.921 | 0.915 | 0.855 | 0.886 | 0.905 |
| 0.40 | 0.982 | 0.952 | 0.944 | 0.889 | 0.922 | 0.938 |
| 0.56 | 1.023 | 0.997 | 0.987 | 0.956 | 0.966 | 0.986 |
| 0.72 | 1.045 | 1.026 | 1.015 | 1.018 | 0.994 | 1.020 |
| 1.04 | 1.062 | 1.064 | 1.052 | 1.056 | 1.036 | 1.054 |
| 1.20 | 1.060 | 1.070 | 1.064 | 1.064 | 1.050 | 1.062 |
| 1.36 | 1.052 | 1.072 | 1.071 | 1.074 | 1.062 | 1.066 |
| 1.52 | 1.040 | 1.077 | 1.070 | 1.080 | 1.074 | 1.068 |
| 1.68 | 1.038 | 1.058 | 1.064 | 1.080 | 1.078 | 1.060 |
| 1.84 | 1.026 | 1.046 | 1.053 | 1.078 | 1.078 | 1.056 |
| 2.00 | 1.011 | 1.038 | 1.038 | 1.072 | 1.070 | 1.047 |
| 2.32 | 0.991 | 1.038 | 1.021 | 1.072 | 1.063 | 1.037 |

* \( z' \) is measured from interface upward.*
Table 12
Evaluation of Total Interfacial and Bottom Frictional Stress Forces

\( H_o = 23.0 \text{ cm}; \ B = 22.9 \text{ cm}; \ v = 0.00919 \)

<table>
<thead>
<tr>
<th>No.</th>
<th>( \Delta p/\rho )</th>
<th>( 2\nu_0/\Delta )</th>
<th>( h_s/H_o )</th>
<th>( U_{H_o}/v )</th>
<th>( 2T_s/\rho U_o^2 )</th>
<th>( 2T_o/\rho U_o^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0730</td>
<td>1.001</td>
<td>0.370</td>
<td>49.2 \times 10^3</td>
<td>10 \times 10^{-4}</td>
<td>33 \times 10^{-4}</td>
</tr>
<tr>
<td>2</td>
<td>0.0745</td>
<td>0.891</td>
<td>0.405</td>
<td>45.2</td>
<td>7</td>
<td>31</td>
</tr>
<tr>
<td>3</td>
<td>0.0730</td>
<td>0.698</td>
<td>0.480</td>
<td>35.4</td>
<td>16</td>
<td>22</td>
</tr>
<tr>
<td>4</td>
<td>0.0720</td>
<td>0.588</td>
<td>0.523</td>
<td>16.8</td>
<td>20</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>0.0386</td>
<td>0.972</td>
<td>0.380</td>
<td>35.9 \times 10^3</td>
<td>19 \times 10^{-4}</td>
<td>30 \times 10^{-4}</td>
</tr>
<tr>
<td>6</td>
<td>0.0368</td>
<td>0.806</td>
<td>0.425</td>
<td>28.8</td>
<td>43</td>
<td>32</td>
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<tr>
<td>7</td>
<td>0.0382</td>
<td>0.682</td>
<td>0.485</td>
<td>25.1</td>
<td>56</td>
<td>13</td>
</tr>
<tr>
<td>8</td>
<td>0.0357</td>
<td>0.710</td>
<td>0.475</td>
<td>25.1</td>
<td>44</td>
<td>11</td>
</tr>
<tr>
<td>9</td>
<td>0.0362</td>
<td>0.601</td>
<td>0.530</td>
<td>21.6</td>
<td>27</td>
<td>27</td>
</tr>
<tr>
<td>10</td>
<td>0.0355</td>
<td>0.542</td>
<td>0.540</td>
<td>18.5</td>
<td>48</td>
<td>15</td>
</tr>
<tr>
<td>11</td>
<td>0.0184</td>
<td>1.015</td>
<td>0.372</td>
<td>25.8 \times 10^3</td>
<td>20 \times 10^{-4}</td>
<td>43 \times 10^{-4}</td>
</tr>
<tr>
<td>12</td>
<td>0.0181</td>
<td>0.773</td>
<td>0.450</td>
<td>21.8</td>
<td>12</td>
<td>68</td>
</tr>
<tr>
<td>13</td>
<td>0.0182</td>
<td>0.756</td>
<td>0.455</td>
<td>18.0</td>
<td>48</td>
<td>13</td>
</tr>
<tr>
<td>14</td>
<td>0.0172</td>
<td>0.616</td>
<td>0.510</td>
<td>15.4</td>
<td>27</td>
<td>52</td>
</tr>
<tr>
<td>15</td>
<td>0.0172</td>
<td>0.523</td>
<td>0.545</td>
<td>12.9</td>
<td>130</td>
<td>26</td>
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<tr>
<td>16</td>
<td>0.0174</td>
<td>0.479</td>
<td>0.570</td>
<td>11.9</td>
<td>38</td>
<td>22</td>
</tr>
<tr>
<td>17</td>
<td>0.0092</td>
<td>1.001</td>
<td>0.370</td>
<td>18.1 \times 10^3</td>
<td>56 \times 10^{-4}</td>
<td>50 \times 15^{-4}</td>
</tr>
<tr>
<td>18</td>
<td>0.0097</td>
<td>0.803</td>
<td>0.435</td>
<td>14.7</td>
<td>56</td>
<td>41</td>
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<tr>
<td>19</td>
<td>0.0089</td>
<td>0.616</td>
<td>0.510</td>
<td>10.7</td>
<td>56</td>
<td>49</td>
</tr>
<tr>
<td>20</td>
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<td>0.532</td>
<td>0.510</td>
<td>9.3</td>
<td>80</td>
<td>23</td>
</tr>
<tr>
<td>21</td>
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<td>0.472</td>
<td>0.570</td>
<td>8.3</td>
<td>32</td>
<td>24</td>
</tr>
</tbody>
</table>
Table 13
Evaluation of Total Interfacial and Bottom Frictional Stress Forces

\[ H_0 = 45.5 \text{ cm}; B = 22.9 \text{ cm}; \nu = 0.00950 \]

<table>
<thead>
<tr>
<th>No.</th>
<th>( \Delta p/\rho )</th>
<th>( 2V/V )</th>
<th>( h_{SI} / H_o )</th>
<th>( U H_o / \nu )</th>
<th>( 2T_s / \rho U_o^2 )</th>
<th>( 2T_o / \rho U_o^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.066</td>
<td>1.08</td>
<td>0.382</td>
<td>( 14.3 \times 10^4 )</td>
<td>( 4 \times 10^{-4} )</td>
<td>( 20 \times 10^{-4} )</td>
</tr>
<tr>
<td>2</td>
<td>0.063</td>
<td>0.89</td>
<td>0.425</td>
<td>13.7</td>
<td>6</td>
<td>40</td>
</tr>
<tr>
<td>3</td>
<td>0.068</td>
<td>0.86</td>
<td>0.472</td>
<td>11.4</td>
<td>6</td>
<td>52</td>
</tr>
<tr>
<td>4</td>
<td>0.066</td>
<td>0.85</td>
<td>0.459</td>
<td>11.0</td>
<td>4</td>
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<tr>
<td>5</td>
<td>0.058</td>
<td>0.84</td>
<td>0.483</td>
<td>10.2</td>
<td>2</td>
<td>18</td>
</tr>
<tr>
<td>6</td>
<td>0.067</td>
<td>0.76</td>
<td>0.485</td>
<td>9.8</td>
<td>6</td>
<td>24</td>
</tr>
<tr>
<td>8</td>
<td>0.032</td>
<td>0.95</td>
<td>0.427</td>
<td>( 8.6 \times 10^4 )</td>
<td>( 4 \times 10^{-4} )</td>
<td>( 42 \times 10^{-4} )</td>
</tr>
<tr>
<td>9</td>
<td>0.031</td>
<td>0.91</td>
<td>0.450</td>
<td>8.2</td>
<td>4</td>
<td>36</td>
</tr>
<tr>
<td>10</td>
<td>0.031</td>
<td>0.89</td>
<td>0.461</td>
<td>7.8</td>
<td>2</td>
<td>28</td>
</tr>
<tr>
<td>11</td>
<td>0.031</td>
<td>0.87</td>
<td>0.493</td>
<td>7.4</td>
<td>8</td>
<td>32</td>
</tr>
<tr>
<td>12</td>
<td>0.033</td>
<td>0.80</td>
<td>0.493</td>
<td>6.9</td>
<td>10</td>
<td>32</td>
</tr>
<tr>
<td>13</td>
<td>0.030</td>
<td>0.75</td>
<td>0.502</td>
<td>6.7</td>
<td>16</td>
<td>18</td>
</tr>
<tr>
<td>14</td>
<td>0.030</td>
<td>0.76</td>
<td>0.521</td>
<td>6.3</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>15</td>
<td>0.016</td>
<td>1.03</td>
<td>0.395</td>
<td>( 6.7 \times 10^4 )</td>
<td>( 4 \times 10^{-4} )</td>
<td>( 30 \times 10^{-4} )</td>
</tr>
<tr>
<td>17</td>
<td>0.015</td>
<td>0.92</td>
<td>0.450</td>
<td>5.8</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>18</td>
<td>0.018</td>
<td>0.80</td>
<td>0.461</td>
<td>5.5</td>
<td>20</td>
<td>28</td>
</tr>
<tr>
<td>19</td>
<td>0.016</td>
<td>0.80</td>
<td>0.477</td>
<td>5.3</td>
<td>8</td>
<td>16</td>
</tr>
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<td>0.498</td>
<td>4.9</td>
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<td>28</td>
</tr>
<tr>
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<td>0.65</td>
<td>0.526</td>
<td>4.4</td>
<td>12</td>
<td>60</td>
</tr>
</tbody>
</table>
### Table 14
Values of the Definite Integral $F_1$

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>0.05</th>
<th>0.10</th>
<th>0.15</th>
<th>0.20</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>4.74</td>
<td>4.26</td>
<td>3.91</td>
<td>3.60</td>
</tr>
<tr>
<td>0.7</td>
<td>4.94</td>
<td>4.19</td>
<td>3.80</td>
<td>3.46</td>
</tr>
<tr>
<td>0.6</td>
<td>5.33</td>
<td>4.70</td>
<td>4.25</td>
<td>3.87</td>
</tr>
<tr>
<td>0.5</td>
<td>5.99</td>
<td>5.25</td>
<td>4.73</td>
<td>4.28</td>
</tr>
<tr>
<td>0.4</td>
<td>7.07</td>
<td>6.15</td>
<td>5.51</td>
<td>4.97</td>
</tr>
<tr>
<td>0.3</td>
<td>8.88</td>
<td>7.71</td>
<td>6.88</td>
<td>6.19</td>
</tr>
</tbody>
</table>

### Table 15
Values of the Definite Integral $F_2$

<table>
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<tr>
<th>$\varepsilon$</th>
<th>0.05</th>
<th>0.10</th>
<th>0.15</th>
<th>0.20</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.8</td>
<td>3.95</td>
<td>3.85</td>
<td>3.74</td>
<td>3.63</td>
</tr>
<tr>
<td>0.7</td>
<td>3.17</td>
<td>3.02</td>
<td>2.91</td>
<td>2.80</td>
</tr>
<tr>
<td>0.6</td>
<td>2.67</td>
<td>2.57</td>
<td>2.46</td>
<td>2.30</td>
</tr>
<tr>
<td>0.5</td>
<td>2.39</td>
<td>2.28</td>
<td>2.17</td>
<td>2.06</td>
</tr>
<tr>
<td>0.4</td>
<td>2.19</td>
<td>2.08</td>
<td>1.97</td>
<td>1.87</td>
</tr>
<tr>
<td>0.3</td>
<td>2.10</td>
<td>1.98</td>
<td>1.87</td>
<td>1.76</td>
</tr>
</tbody>
</table>
APPENDIX A: NOTATION

b  Half width of channel
B  Width of channel

\( F_k \)  Densimetric Froude number \( F_k = \frac{U_k^2}{\rho g h} \) for saline underflow

\( F_L \)  Densimetric Froude number \( F_L = \frac{U_L^2}{\rho g h} \) for saline underflow below stagnant fresh water

\( F_o \)  Densimetric Froude number \( F_o = \frac{U_o^2}{\rho g h} \) arrested saline wedge. \( F_o = \frac{U_o^2}{\rho g h} \) for lock exchange flow

h  Total depth of water in the area of the arrested wedge

\( h_r \)  Hydraulic radius of saline waters of a saline underflow, \( h_r = \frac{bh_s}{b + h_s} \)

\( h_s \)  Depth of saline water in an arrested saline wedge; depth of saline water of an underflow or the depth of saline water in a lock exchange flow

\( h_{sl} \)  Depth of saline water of an arrested wedge at the river mouth

\( h_w \)  Depth of fresh water in the layer over the arrested wedge

\( h_{wl} \)  Depth of fresh water at the river mouth

\( H_o \)  Depth of fresh water at the tip of an arrested saline wedge; total depth of the fresh and the saline water layers of a lock exchange flow

L  Total length of arrested saline wedge. Same as \( x_o \)

M  Momentum of body liquid of arrested saline wedge, and of body of fresh water resting on the entire length of arrested saline wedge

\( M_1 + M_o \)  Value of \( M \) at time \( t \)

\( M_2 + M_o \)  Value of \( M \) at time \( t + \Delta t \)

n  Ratio \( h_s / h_{sl} \)

\( P_i \)  Pressure total force acting on the entire interface

\( P_1, P_2 \)  Pressure forces on the upstream and the downstream faces in freshwater layer

Q  Discharge of river waters

R  Reynolds number \( \frac{U_o H_o}{v} \) for arrested wedge
Reynolds number $Re = \frac{U_2 R_2}{v}$ for exchange flow

Reynolds number $R_{kL} = \frac{U_h}{v}$ for saline underflow

Reynolds number $R_{L} = \frac{U_h}{v}$ for saline underflow

Hydraulic radius of lock exchange flow, $R_2 = \frac{h_B}{2(h_s + B)}$

Mean bottom stress averaged along the entire length of the arrested saline wedge $T_o$

Mean interfacial stress averaged along the entire length of the arrested saline wedge $T_s$

Retarding force from the sidewalls in the entire freshwater layer $T_w^1$

Retarding force from the sidewalls in the entire saltwater layer $T_w^2$

Particle velocity; root mean square of velocities in strip plane $u$

Maximum velocity in the saline layer of an underflow $u_o$

Mean velocity of fresh water in the layer over an arrested saline wedge; mean velocity of saline water in the layer under the stagnant water of an underflow $U$

Velocity of points at the interface $U_1$

Velocity of transport across the interface due to mixing $U_m$

Mean velocity of fresh water at the tip of an arrested saline wedge. Same as $U_r$

Relative velocity of layers in a lock exchange flow $U_r$

Velocity of surface waters in an arrested saline wedge $U_s$

Mean velocity of fresh water at the river mouth $U_1$

Mean velocity of saline layer in a lock exchange flow $U_2$

River velocity ($V_r = U_o$) $V_r$

Densimetric velocity, $V_\Delta = \sqrt{\frac{\rho}{\rho_0}} g h_o$ $V_\Delta$

Coordinate in the direction of freshwater flow, measured from the tip of an arrested saline wedge $x$

Length of an arrested saline wedge. Same as $L$ $x_o$

Coordinate in the channel transverse direction, measured from channel midvertical plane $y$

Coordinates in the vertical; $z$ measured from the bottom of horizontal channel; $z'$ from the interface $z, z'$

Boussinesq coefficient of velocity distribution in relation to freshwater flow over arrested saline wedge $\alpha$
\beta \quad \text{Boussinesq coefficient in relation to saline waters of arrested wedge}

\beta_m \quad \text{Numerical factor in Loftquist formula for } \lambda_w

\Delta_h \quad \text{Fall of free surface in the area of an arrested saline wedge}

\Delta H \quad \text{Fall of free surface at the river mouth, } H_o = \Delta H + h_{wl} + h_s

\eta \quad \text{Relative depth of salt wedge, } h_s / H_o

\eta_L \quad \text{Relative depth of salt wedge at river mouth, same as } \eta = h_{sl} / H_o

\theta \quad \text{Temperature of water expressed in centigrade}

\phi \quad \text{A function of } \eta \text{ defined by Equation 107}

\lambda \quad \text{Coefficient of resistance of river channel}

\lambda_i \quad \text{Interfacial coefficient of resistance. } \lambda_i = 2 \tau_i / \rho U^2 \text{ for arrested saline wedges and for saline underflow. } \lambda_i = 2 \tau_i / \rho U_r^2 \text{ for lock exchange flow}

\lambda_m \quad \text{Interfacial coefficient of resistance } \lambda_m = \tau_m / \rho U_o^2 \text{ for saline underflow}

\lambda_o \quad \text{Coefficient of resistance defined by Equation 65}

\mu \quad \text{Viscosity of water}

\nu \quad \text{Kinematic viscosity of water}

\nu_s \quad \text{Kinematic viscosity of saline water}

\tau_i \quad \text{Interfacial stress for arrested saline wedge or for lock exchange flow or for underflow}

\tau_m \quad \text{Maximum value of stress in underflow, practically same as } \tau_i

\tau_o \quad \text{Bottom stress of an arrested saline wedge}

\tau_s \quad \text{Interfacial stress}

\bar{\tau}_s \quad \text{Average value of interfacial stress along the channel width}

\tau_w \quad \text{Stress of vertical wall}

\bar{\tau}_w \quad \text{Average value of wall stress across span } h - h_s
In accordance with letter from DAEN-RDC, DAEN-ASI dated 22 July 1977, Subject: Facsimile Catalog Cards for Laboratory Technical Publications, a facsimile catalog card in Library of Congress MARC format is reproduced below.

Keulegan, Garbis H.

Significant stresses of arrested saline wedges / by Garbis H. Keulegan (Hydraulics Laboratory, U.S. Army Engineer Waterways Experiment Station). -- Vicksburg, Miss. : The Station ; Springfield, Va. : available from NTIS, 1981.

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Final report.
Bibliography: p. 71.

1. Fresh water. 2. Saltwater encroachment. I. United States. Army. Corps of Engineers. Office of the Chief of Engineers. II. Title III. Series: Miscellaneous paper (U.S. Army Engineer Waterways Experiment Station) ; HL-81-5.

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