ON A DIMENSIONAL REDUCTION METHOD. SOME APPROXIMATION-THEORETIC-ETC(U)

MAY 80 M. VOGE LiUS, I BABUSKA

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SOME APPROXIMATION-THEORETIC RESULTS.

by

M. Vogelius
and
I. Babuška

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M. Vogelius

and

I. Babuška

Institute for Physical Science and Technology
University of Maryland
College Park, MD 20742

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Abstract

This paper is the second in a series of three that analyze a method of dimensional reduction. It contains some results for approximation of functions on the interval [-1,1] with elements from the nullspace of $P^N$, $N > 1$, where $P$ is a second order ordinary differential operator. A special case of this is approximation by polynomials.

The one-dimensional results are used as a tool to prove similar versions in several dimensions. These multi-dimensional results are directly related to the approximate method of dimensional reduction that was introduced in [13], and they lead to statements about the convergence properties of this approach.

The third paper, which analyzes the adaptive aspects of the method, is forthcoming.
1. Introduction

In a recent paper, [13], we introduced the concept of dimensionally reduced solutions to an elliptic boundary value problem. These are obtained by projecting (in the energy) the true solution of the boundary value problem in the n+1-dimensional domain \( \omega \times [-h,h] \) onto spaces of the form

\[
V^h_N = \{ \sum_{j=0}^{N} w_j(x) \phi_j(y/h) | w_j \text{ arbitrary} \},
\]

where \( \{\phi_j\}_{j=0}^{\infty} \) is a given set of functions on \([-1,1]\), \((x\) are coordinates on \(\omega\) and \(y\) ranges over \([-h,h]\)). For some basic ideas behind this concept, see the introduction to [13]. In that paper the focus was on the right selection of the \(\phi_j\)'s. It was shown there that for a very wide class of problems the \(\phi_j\)'s should be selected such that

\[
\text{span}\{\phi_j\}_{j=0}^{2k-1} = N(P^k),
\]

where \(P\) is a second order differential operator intrinsic to the elliptic boundary value problem.

The estimates of the error given in [13] were asymptotic in \(h \to 0\). The present paper, which was already announced there, treats convergence as \(N \to \infty\) for a fixed value of \(h\). For convenience the fixed value of \(h\) is set equal to 1.

If the bilinear form associated with the elliptic boundary value problem satisfies some kind of "inf-sup" condition, then it is well known that the rate of convergence is the same as the rate of approximation (cf.[1]).

The results proven here are hence formulated as approximation-theoretic estimates, and as such have interest regardless of the concept of dimensionally reduced solutions.
The results are all concerning approximation in the $L^2$- and $H^1$-norms, i.e. ideally suited for second order problems. This is not crucial and similar results can also be obtained, e.g., for the norms \( \left( \int_{-1}^{1} \left| \frac{du}{dy} \right|^2dy + \int_{-1}^{1} \left| A^{1/2}u \right|^2dy \right)^{1/2} \) introduced in [13]. (A here denotes a strictly positive definite (unbounded) operator in a Hilbert space $H$, and $u$ is a function with values in $H$.)

For reasons of convenience the approximation results are formulated without any boundary conditions. Various types of fixed boundary conditions can immediately be included based on the present proofs.

Estimates of the error introduced by dimensional reduction, as $N$ goes to $\infty$, do exist in the literature (cf. [5,7]). The problems considered in those two papers come from structural mechanics. The elliptic operators have constant coefficients, i.e. the $\phi_j$'s are polynomials. The results are not nearly as strong as the ones established here. In [7] the estimates are based on the degree of regularity in $C^k$-spaces; this is not very well suited to the regularity properties of solutions to elliptic boundary value problems, and therefore gives crude estimates. The estimates in [5] are based on bounding the remainder in the $N$-th order Taylor expansion. The estimates are very crude and do not give any indication of the rate of approximation.

We now give a short review of the contents of this paper. In section 2 it is shown that the set $\bigcup_{k=1}^{\infty} N(P^k)$ ($N$ denotes the nullspace) is dense in $H^1$ for any second order operator $P = b \frac{d}{dy} a \frac{d}{dy}$, where both $a$ and $b$ are bounded from above and away from 0. This is the obvious generalization of the fact that the polynomials are dense in $H^1$, and it also justifies the claim that the dimensionally reduced solutions introduced in [13] will get arbitrarily close to the true solution. In section 3 the rate of approximation using functions in $N(P^N)$, $N \geq 1$, is linked to the regularity of $u$ in spaces of the type $\mathcal{D}(P^m)$. This general result though is not always optimal, as shown, e.g., by Theorem 4.1. Section 4
is devoted to giving a necessary and sufficient condition for a certain rate of approximation by polynomials (i.e., the case where the operator $P$ is a constant-coefficient operator). In section 5 this is carried over to results in several dimensions -- directly relating to the concept of dimensional reduction. The example treated in section 6 is of the same type as the numerical examples in [13]. Finally the appendix contains the proofs of several results about the eigenvalues and eigenfunctions for two-point boundary value problems, as used in sections 2 and 3.

Note: Unless otherwise stated, all constants denoted by capital letters are generic.
2. A Density Result

Let $a$ and $b$ be two functions in $L^\infty([-1,1])$ such that $\exists$ constants $a_0, b_0$ with

$$0 < a_0 \leq a(y)$$
$$0 < b_0 \leq b(y)$$

By $P$ we denote the differential operator

$$b \frac{d}{dy} a \frac{d}{dy}.$$  

$P$ is considered as a mapping $L^2([-1,1]) \ni D(P) \to L^2([-1,1])$. $N(P^k)$ denotes the null-space of the operator $P^k$ for any integer $k \geq 1$. It is easily seen that $N(P^k) \subseteq H^1([-1,1])$. The first theorem in this section proves the density of a certain class of functions associated with the operator $P$.

**Theorem 2.1**

$\bigcup_{k=1}^\infty N(P^k)$ is dense in $H^1([-1,1])$

**Proof**

By a change of variables, $y' = \int y \frac{1}{b(s)} ds$, and multiplication by $-1$ the operator $P$ transforms into

$$-\frac{d}{dy'} a(y) \frac{d}{dy'}.$$  

We can therefore for the proof of this theorem assume that $P$ is given by

$$-\frac{d}{dy} a(y) \frac{d}{dy}$$

where $a$ satisfies: $\exists$ a constant $a_0$ with $0 < a_0 \leq a(y)$.

Define the operator $Q$ by

$$D(Q) = D(P) \cap H^1([-1,1])$$

and $Q = P$ on $D(Q)$. Let $f_0$ denote the function
\[ f_0(y) = \int_{-1}^{y} \frac{1}{a(s)} \, ds \in N(P) , \]

and define the sequence \( \{f_i\}_{i=0}^{\infty} \) by

\[ f_i = Q^{-1} f_0 \in N(P^{i+1}) . \]

\( 0 < \lambda_0 < \lambda_1 < \ldots < \lambda_m < \lambda_{m+1} \ldots \) denotes the eigenvalues of \( Q \) (repeated according to multiplicity). Let \( \{u_m\}_{m=0}^{\infty} \) be an orthonormal basis of eigenfunctions, \( u_m \) corresponding to \( \lambda_m \). \( f_0 \) can then be expanded as

\[ f_0 = \sum_{m=0}^{\infty} a_m u_m , \]

and with this notation

\[ f_i = \sum_{m=0}^{\infty} a_m \lambda^{-1}_m u_m . \]

We now proceed to prove that any eigenfunction \( u_m \) can be approximated from within \( \bigcup_{k=1}^{\infty} N(P^k) \). The proof is by induction in \( m \), and we start with \( m = 0 \).

For any \( i \geq 1 \) we have that

\[ ||u_0 - \alpha_0^{-1} \lambda_0^i f_i||_H^2 \leq \]

\[ c ||Q^{1/2} (u_0 - \alpha_0^{-1} \lambda_0^i f_i)||_L^2 \leq \]

\[ c \lambda_0 \sum_{j=1}^{\infty} \left( \frac{\alpha_j}{\alpha_0} \right)^2 \left( \frac{\lambda_0}{\lambda_j} \right)^{2i-1} \leq c \left( \frac{\lambda_0}{\alpha_0^2} \right) \left( \frac{\lambda_0}{\lambda_1} \right)^{2i-1} \sum_{j=1}^{\infty} \frac{\alpha_j^2}{\alpha_j} , \]
where we have used Lemma A.3 to guarantee that \( a_0 \neq 0 \). Since \( \sum_{j=1}^{\infty} a_j^2 < \infty \) and, by Lemma A.1, \( \lambda_0/\lambda_1 < 1 \) this shows that

\[
a_0^{-1} \lambda_0^i f_i + u_0 \quad \text{as } i \to \infty, \text{ in } H^1([-1,1]),
\]

or

\[
u_0 \in \bigcup_{k=1}^{\infty} N(p_k)
\]

( denotes the closure in \( H^1 \)).

Now assume it has been proven for some \( m \geq 1 \) that

\[
\{u_j\}_{j=0}^{m-1} \subseteq \bigcup_{k=1}^{\infty} N(p_k)
\]

From Lemma A.3 we know that \( a_m \neq 0 \), and hence for any \( i \geq 1 \)

\[
u_m - a_m^{-1} \lambda_m^i f_i = x_m,i - \sum_{j=m+1}^{\infty} \left( \frac{a_j}{a_m} \right) \left( \lambda_m^j / \lambda_j^i \right) u_j,
\]

where \( x_m,i = - \sum_{j=0}^{m-1} \left( \frac{a_j}{a_m} \right) \left( \lambda_m^j / \lambda_j^i \right) u_j \in \bigcup_{k=1}^{\infty} N(p_k) \)

due to the induction hypothesis. As before the \( H^1 \) - norm of the sum

\[
\sum_{j=m+1}^{\infty} \left( \frac{a_j}{a_m} \right) \left( \lambda_m^j / \lambda_j^i \right) u_j
\]

can be estimated by

\[
C \left( \lambda_m^i / a_m \right) \left( \lambda_m^i / \lambda_{m+1} \right)^{1/2} \left( \sum_{j=m+1}^{\infty} a_j^2 \right)^{1/2}
\]

Because of the facts that \( \left( \sum_{j=m+1}^{\infty} a_j^2 \right)^{1/2} < \infty \) and, by Lemma A.1, \( \lambda_m / \lambda_{m+1} < 1 \) this shows that
This finishes the induction proof, and we conclude that

\[ \{u_j\}_{j=0}^\infty \subseteq \bigcup_{k=1}^\infty N(p^k) . \]

From the definition of \( Q \) it immediately follows that

\[ \mathcal{D}(Q^{1/2}) = H^1([-1,1]) , \]

and since \( \{u_j\}_{j=0}^\infty \) is complete in \( \mathcal{D}(Q^{1/2}) \) this proves that

\[ H^1([-1,1]) \subseteq \bigcup_{k=1}^\infty N(p^k) . \]

Now if \( u \in H^1([-1,1]) \) we shall, by choosing \( c = u(-1) \) and \( d = (u(1)-u(-1))/\int_{-1}^1 a^{-1}(s) ds \), obtain that

\[ u - c - df_0 \in H^1([-1,1]) . \]

Since \( 1, f_0 \in N(P) \) we see by a combination of this and the previously proven inclusion that

\[ H^1([-1,1]) = \bigcup_{k=1}^\infty N(p^k) . \]
Based on Theorem 2.1 we can easily prove a result concerning the dimensionally reduced solutions as introduced in [13]. This result guarantees the fulfilment of the goal stating that the dimensionally reduced solutions shall be able to get arbitrarily close to the true solution.

Let \( \omega \) denote a domain in \( \mathbb{R}^n \) with a Lipschitz boundary.

**Theorem 2.2**

The set

\[
\left\{ \sum_{j=0}^{J} v_j(x) q_j(y) \mid J \in \mathbb{N}, v_j \in \mathcal{H}^1(\omega) \text{ and } q_j \in \bigcup_{k=1}^{\infty} \mathcal{N}(P^k) \text{ for } 0 \leq j \leq J \right\}
\]

is dense in \( \mathcal{H}^1(\omega \times [-1,1]) \).

**Proof**

Follows immediately from Theorem 2.1 and the fact that

\[
\left\{ \sum_{j=0}^{J} v_j(x) v_j(y) \mid J \in \mathbb{N}, v_j \in \mathcal{H}^1(\omega) \text{ and } v_j \in \mathcal{H}^1([-1,1]) \text{ for } 0 \leq j \leq J \right\}
\]

is dense in \( \mathcal{H}^1(\omega \times [-1,1]) \).
3. Estimates of the Rate of Approximation

In the previous section we proved the density of a certain class of functions associated with the operator \( P = b \frac{d}{dy} a \frac{d}{dy} \). In this section we shall prove some results concerning the rate of approximation. The first theorem is the following.

Theorem 3.1

Assume that \( a,b \in \mathcal{C}^2([-1,1]) \), and let \( m \) be an integer \( \geq 0 \). For any \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon \) such that

\[
\inf_{v \in N(P^N)} \|u - v\|_{L^2} \leq C_\varepsilon N^{-m+\varepsilon} \|u\|_{\mathcal{D}(P^m)} \quad \forall N \geq 1.
\]

Note. \( \| \cdot \|_{\mathcal{D}(P^m)} \) denotes the norm \( \|P^m(\cdot)\|_{L^2} + \|\cdot\|_{L^2} \).

One can of course combine the statement in Theorem 3.1 with interpolation by the K-method (cf. [4]). This way it follows, that if \( u \in (L^2;\mathcal{D}(P^m))_s,\infty \) for some \( 0 < s < 1 \), then for any \( \varepsilon > 0 \)

\[
\inf_{v \in N(P^N)} \|u - v\|_{L^2} \leq C_\varepsilon N^{-ms+\varepsilon} \|u\|_{(L^2;\mathcal{D}(P^m))_s,\infty}.
\]

The smoothness requirement that \( a,b \in \mathcal{C}^2([-1,1]) \) is not necessary; as it immediately will follow from this proof we only need that \( a/b \) is a \( \mathcal{C}^2 \)-function.

This last remark applies to all of the results in this section.

In order to prove Theorem 3.1 we need an auxiliary result concerning uniform approximation by polynomials. This result can be found, e.g., in chapter 6 of [6].
Lemma 3.1

Let \( \phi \) be a function in \( C^0([c,d]) \). Define \( \tilde{\phi} \) by \( \tilde{\phi}(t) = \phi\left(\frac{c-d}{2} \cos t + \frac{c+d}{2}\right) \), \( t \in [0,\pi] \).

Let \( r \) be a non-negative integer. There exists a constant \( C_r \) such that for any \( \phi \) with \( \tilde{\phi} \in C^r([0,\pi]) \) the following estimate holds

\[
\inf_{P_N} |\phi - p_N| \leq C_r (N+1)^{-r} |\tilde{\phi}|_r \quad \forall N \geq 0.
\]

The infimum here is taken over all polynomials \( P_N \) of degree \( \leq N \). \( |\cdot|_0 \) and \( |\cdot|_r \) denote the norms in \( C^0([c,d]) \) and \( C^r([0,\pi]) \) respectively.

We now continue with

Proof of Theorem 3.1

Like in the proof of Theorem 2.1 we may also here assume that \( P \) is given by \( P = - \frac{d}{dy} a(y) \frac{d}{dy} \). Let \( f_1 \geq 0 \) be defined as in that same proof. For any set of coefficients \( \{c_i\}_{i=0}^N \) we have, using Lemma A.3, that

\[
u - \sum_{i=0}^N c_if_i = \sum_{j=0}^\infty a_j (\beta_j/a_j - \sum_{i=0}^N c_i\lambda_j^{-1})u_j,
\]

where \( \sum_{j=0}^\infty \beta_j u_j \) is the expansion corresponding to \( u \). If by \( p_N \) we denote the polynomial \( p_N(x) = \sum_{i=0}^N c_i x^i \), then the above can be rewritten as

\[
u - \sum_{i=0}^N c_if_i = \sum_{j=0}^\infty a_j (\beta_j/a_j - p_N(\lambda_j^{-1}))u_j,
\]

and this leads to the equality

\[
||u - \sum_{i=0}^N c_if_i||^2_{L^2} = \sum_{j=0}^\infty a_j^2 (\beta_j/a_j - p_N(\lambda_j^{-1}))^2.
\]
As in the proof of Theorem 2.1 \( Q \) denotes the restriction of \( P \) to \( \mathcal{D}(P) \cap H^1([-1,1]) \). Let us now for a while assume that \( u \in \mathcal{D}(Q^k) \).

Choose \( \Lambda \) so that \( \{ \lambda_j^{-1} \}_{j=0}^\infty \subseteq [0,\Lambda] \). Define a sequence of functions \( \phi_M \in C^\infty([0,\Lambda]) \), \( 1 \leq M \), with the following properties

\[
\phi_M(\lambda_j^{-1}) = \frac{\beta_j}{a_j} \quad 0 \leq j \leq M-1
\]

\[
\phi_M(x) = 0 \quad \text{on } [0,\lambda_M^{-1}]
\]

Let \( \phi \) denote the mapping

\[
\phi(t) = \frac{A}{2} (1-\cos t): [0,\pi] \to [0,\Lambda],
\]

it then follows from Lemma A.1 and Lemma A.2 that

\[
|\phi^{-1}(\lambda_j^{-1}) - \phi^{-1}(\lambda_{j-1}^{-1})| \geq C/j^2 \quad \text{for any } j \geq 1.
\]

This estimate tells us that it is possible to construct the \( \phi_M \)'s such that \( \forall M, j \geq 1 \)

\[
|\phi_M(\phi(t))| \leq C_r \sup_{0 \leq j \leq M-1} |(j+1)^2 r \beta_j/a_j|.
\]

Now since \( u \in \mathcal{D}(Q^k) \) we know that \( |\beta_j| \leq C_k (j+1)^{-2k} ||Q^k u||_{L^2} \), and combining this with Lemma A.4 we get

\[
|\beta_j/a_j| \leq C_k (j+1)^{-2k+1} ||Q^k u||_{L^2}.
\]
i.e., we have for any \( r \geq k \) and \( M \geq 1 \)

\[
| \phi_M(\phi(t)) |_r \leq C_r M^{2(r-k)+1} ||Q^k u||_{L^2} .
\]

Because of Lemma 3.1 we can now, for any \( r \geq k , \ M \geq 1 \), find a polynomial
\( p_N \) of degree \( \leq N \) such that

\[
| \phi_M - p_N |_0 \leq C_r (N+1)^{-r} M^{2(r-k)+1} ||Q^k u||_{L^2} .
\]

We now go back to estimate the right hand side of (1)

\[
\sum_{j=0}^{M} \alpha_j^2 (\beta_j / \alpha_j - p_N (\lambda_j^{-1}))^2 \leq \sum_{j=0}^{M-1} \alpha_j^2 ((\phi_M - p_N (\lambda_j^{-1}))^2 + 2 \sum_{j=M}^{\infty} \beta_j^2 + 2 \sum_{j=M}^{\infty} \alpha_j^2 ((\phi_M - p_N (\lambda_j^{-1}))^2
\]

The first and the third sum can be estimated by

\[
C_r (N+1)^{-2r} M^{4(r-k)+2} \left( \sum_{j=0}^{\infty} \alpha_j^2 \right) ||Q^k u||_{L^2}^2 \leq C_r (N+1)^{-2r} M^{4(r-k)+2} ||Q^k u||_{L^2}^2
\]

the second by

\[
C_k \sum_{j=M}^{\infty} (j+1)^{-4k} ||Q^k u||_{L^2}^2 \leq C_k N^{-4k+1} ||Q^k u||_{L^2}^2
\]
In summary we have therefore proven

\[ ||u - \sum_{i=0}^{N} c_i f_i||_{L^2}^2 \leq C_k((N+1)^{-2r}M^4(r-k)+2 + M^{-4k+1})||Q^k u||_{L^2}^2 \]

for any \( r \geq k \), \( N \geq 0 \) and \( M \geq 1 \). Taking \( M = [\sqrt{N}] + 1 \) ([·] denotes the integer part) this estimate gives

\[ ||u - \sum_{i=0}^{N} c_i f_i||_{L^2}^2 \leq C_k(N+1)^{-2k+1}||Q^k u||_{L^2}^2, \tag{2} \]

all provided that \( u \in D(Q^k) \). Let \( \Pi_N \) denote the \( L^2 \)-projection onto linear combinations of the functions \( f_0, \ldots, f_N \).

(2) expresses that

\[ ||u - \Pi_N u||_{L^2} \leq C_k(N+1)^{-k+1/2}||Q^k u||_{L^2}, \]

at the same time it is clear that

\[ ||u - \Pi_N u||_{L^2} \leq ||u||_{L^2}. \]

Applying interpolation by the K-method we get for any \( 0 \leq m \leq k \)

\[ ||u - \Pi_N u||_{L^2} \leq C_k(N+1)^{-(k-1/2)m/k}||Q^m u||_{L^2}. \]

Now let \( m \) be fixed and \( k \to \infty \), from the previous inequality we then get

\[ \forall \epsilon > 0 \exists C_\epsilon \text{ such that} \]
\begin{equation}
|\|u - \Pi_N u\|_2 \leq C_\varepsilon (N+1)^{-m+\varepsilon} |\|Q^m u\|_2, \end{equation}

provided \( u \in \mathcal{D}(Q^m) \). If we only know that \( u \in \mathcal{D}(P^m) \), then choose \( \{g_j\}_{j=1}^m \subseteq N(P^m) \) such that

\[ p^{j-1} g_j = p^{j-1} u \text{ for } y = \pm 1 \]
\[ p^i g_j = 0 \text{ for } y = \pm 1 \]

and any \( i \neq j-1 \), (this is obviously possible). This way

\[ u - \sum_{j=1}^m g_j \in \mathcal{D}(Q^m) \text{ and} \]

\begin{equation}
\sum_{j=1}^m g_j \in N(P^m). \end{equation}

From (3) and (4) it now follows that

\[ |\|u - \Pi_N(u - \sum_{j=1}^m g_j) - \sum_{j=1}^m g_j\|_2 \leq C_\varepsilon (N+1)^{-m+\varepsilon}. \]

\[ \cdot |\|Q^m(u - \sum_{j=1}^m g_j)\|_2 = C_\varepsilon (N+1)^{-m+\varepsilon} |\|P^m u\|_2. \]

Since the image under \( \Pi_N \) is contained in \( N(P^{N+1}) \), this estimate yields the desired result for \( N \geq m \). There are only a finite number of \( N \)'s \( < m \), and hence the result can be obtained for all \( N \) by possibly increasing \( C_\varepsilon \).
Based on Theorem 3.1 we can prove the following result concerning approximation in the $H^1$-norm.

**Theorem 3.2**

Assume that $a, b \in C^2([-1, 1])$, and let $m$ be an integer $\geq 1$. For any $\varepsilon > 0$ there exists a constant $C_\varepsilon$ such that

$$\inf_{v \in \mathcal{N}(P^N)} \|u-v\|_{H^1} \leq C_\varepsilon N^{-m+1+\varepsilon} \|u\|_{D(P^m)} \quad \forall N \geq 1$$

**Proof**

From Theorem 3.1 it follows that there exist $\tilde{v}_N \in \mathcal{N}(P^N)$ such that

$$\|P u - \tilde{v}_N\|_{L^2} \leq C N^{-m+1+\varepsilon} \|u\|_{D(P^m)} .$$

Now choose $v_N \in \mathcal{N}(P^{N+1})$ with $P v_N = \tilde{v}_N$, and such that $v_N = u$ for $y = \pm 1$.

It then follows that

$$\|u - v_N\|_{H^1} \leq C \|P u - v_N\|_{L^2} = C \|P u - \tilde{v}_N\|_{L^2} \leq C N^{-m+1+\varepsilon} \|u\|_{D(P^m)} .$$

We can also easily prove a result relating to the dimensionally reduced solutions as introduced in [13]. Let $\omega \subseteq \mathbb{R}^n$ be a domain with a Lipschitz boundary.

$x = (x_1, \ldots, x_n)$ denotes coordinates in $\omega$ and $y$ ranges over $[-1, 1]$. $\mathcal{P}$ denotes $b(y) \frac{\partial}{\partial y} a(y) \frac{\partial}{\partial y}$ considered as an operator $L^2(\omega \times [-1, 1]) \rightarrow \mathcal{D}(\mathcal{P}) \rightarrow L^2(\omega \times [-1, 1])$. 

Theorem 3.3

Assume that \( a, b \in C^2([-1,1]) \) and let \( m \) be an integer \( \geq 1 \). Let \( u \) be an element of \( L^2(\omega \times [-1,1]) \) with \( \frac{\partial}{\partial x_1} u, \ldots, \frac{\partial}{\partial x_n} u \), \( u \in D(\pi^m) \). Then for any \( \epsilon > 0 \) there exist \( C_\epsilon \) (independent of \( u \)) such that

\[
\inf_{v \in V_N} ||u - v||_{L^2(\omega \times [-1,1])} \leq C_\epsilon (N+1)^{-m+1+\epsilon} \left( \sum_{i=1}^{n} \left| \frac{\partial}{\partial x_1} u \right|_{D(\pi^m)} + ||u||_{D(\pi^m)} \right)
\]

Here \( V_N \) denotes the set \( \{ \sum_{j=0}^{N} w_j(x) \phi_j(y) | w_j \in H^1(\omega) \}, \) where \( \{ \phi_j \}_{j=0}^{2k-1} \) is a basis for \( N(\pi^k) \subseteq H^1([-1,1]) \).

Proof

Let \( v_N \) denote the orthogonal projection of \( \hat{u} \) onto \( \{ \sum_{j=0}^{N} w_j(x) \phi_j(y) | w_j \in L^2(\omega) \} \) in the \( L^2(\omega \times [-1,1]) \) inner product. Then it is clear that \( \frac{\partial}{\partial x_1} v_N \) is the \( L^2 \) projection of \( \frac{\partial}{\partial x_1} u \) onto the same subspace. From Theorem 3.1 we immediately get

\[
\sum_{i=1}^{n} \left| \frac{\partial}{\partial x_1} (u - v_N^*) \right|_{L^2(\omega \times [-1,1])} + \left| \frac{\partial}{\partial x_1} (u - v_N^*) \right|_{L^2(\omega \times [-1,1])} \leq \left( \sum_{i=1}^{n} \left| \frac{\partial}{\partial x_1} u \right|_{D(\pi^m)} + ||u||_{D(\pi^m)} \right)
\]

for any function \( v_N^* \in V_{N+3} \) with \( \hat{v}_N^* = v_N \) (if \( N \) is odd such a \( v_N^* \) will be contained in \( V_{N+2} \), but this is not necessarily so for \( N \) even). Now choosing \( v_N^* \) so that also \( v_N^* = u \) for \( y = \pm 1 \) (this is obviously possible), it follows that
\[
\sum_{i=1}^{n} \left| \frac{\partial}{\partial x_i} (u - v_N^\#) \right|_{L^2(\omega \times [-1,1])} + \left| \frac{\partial}{\partial y} (u - v_N^\#) \right|_{L^2(\omega \times [-1,1])} + \left| u - v_N^\# \right|_{L^2(\omega \times [-1,1])} \leq \\
C \left( \sum_{i=1}^{n} \left| \frac{\partial}{\partial x_i} (u - v_N^\#) \right|_{L^2(\omega \times [-1,1])} + \left| \frac{\partial}{\partial y} (u - v_N^\#) \right|_{L^2(\omega \times [-1,1])} \right) \leq \\
C \varepsilon (N+1)^{-m+1+\varepsilon} \left( \sum_{i=1}^{n} \left| \frac{\partial}{\partial x_i} u \right|_{D(\mathbb{R}^m)} + ||u||_{D(\mathbb{R}^m)} \right) .
\]
4. The Constant Coefficient Case - 1 Dimensional Results

The following two sections are devoted to the case where the operator \( P \) has constant coefficients. In the previous section we estimated the rate of approximation for general \( P \)'s, but the estimates established there do not have exact inverse counterparts nor are they always optimal. As will be shown in this section and the next, the question of approximation rate can be much further clarified when \( P \) is a constant coefficient operator. We start with an analysis of the 1 dimensional problem. The space \( N(P^k) \), \( k \geq 1 \), consists simply of all polynomials of degree \( \leq 2k-1 \). Theorem 3.1 combined with interpolation says that if \( u \in H^t([-1,1]) \), then there exist polynomials \( p_N \), of degree \( N \), such that \[ \left\| u - p_N \right\|_{L^2} \leq C_t(N+1)^{-t/2+\epsilon} \]. Under the present simplified circumstances we can prove a better result. In the formulation of this result we use the Besov spaces \( B^t_{2,\infty} \), \( t > 0 \), (cf. [4]), instead of the ordinary Sobolev spaces \( H^t \). For an interpretation in terms of the spaces \( H^t \) use the inclusions \( H^t \subseteq B^t_{2,\infty} \subseteq H^{t-\epsilon} \) valid for any \( t > 0 \), \( \epsilon > 0 \).

Theorem 4.1

Let \( t \) be a given positive number. There exists a constant \( C_t \), such that for any \( u \in B^t_{2,\infty}([-1,1]) \) one can find a sequence of polynomials \( \{ p_N \}_{N=0}^\infty \), the degree of \( p_N \leq N \), with

\[ \left\| u - p_N \right\|_{L^2} \leq C_t(N+1)^{-t} \left\| u \right\|_{B^t_{2,\infty}} \]

Note. A similar theorem is also valid for the \( H^1 \)-norm. The estimate here becomes (for \( t > 1 \))

\[ \left\| u - p_N \right\|_{H^1} \leq C_t(N+1)^{-t+1} \left\| u \right\|_{B^t_{2,\infty}} \]
The rate of approximation established in Theorem 4.1 is optimal in the following sense.

**Theorem 4.2**

If \( u \in L^2([-1,1]) \) and there exist a constant \( C \) and a sequence of polynomials \( \{p_N\}_{N=0}^\infty \), the degree of \( p_N \leq N \), such that

\[
\|u - p_N\|_2 \leq C(N+1)^{-t} \quad \text{for some } t > 0,
\]

then \( u \in (B^{t/2}_{2,\infty})_{\text{loc}} \cap B^{t/2}_{2,\infty} \).

**Note.** Theorem 4.2 is not an exact inverse of Theorem 4.1 since it only guarantees that \( u \in B^{t/2}_{2,\infty}([-1,1]) \). But based on Theorem 4.2 we conclude that for a general type function in \( B^t_{2,\infty}([-1,1]) \) we cannot expect more than an approximation rate of \((N+1)^{-t}\).

Theorem 4.2 is optimal in that one can find \( u \) such that \( \|u - p_N\| \leq C(N+1)^{-t+\varepsilon} \) and \( u \notin B^{t/2+\varepsilon}_{2,\infty} \), \( u \notin (B^{t+\varepsilon}_{2,\infty})_{\text{loc}} \) for any \( \varepsilon > 0 \) (cf. [12]).

The proof of Theorem 4.1 is very simple, based on transforming \( u \) into a periodic function and estimating the remainder of the \( k \)'th order Fourier expansion. Details can be found, e.g., in [2].

The proof of Theorem 4.2 is not quite as simple. The cornerstone is the so-called Bernstein's inequality

\[
\| (\frac{d}{dy}) p_N \|_2 \leq C N^2 \|p_N\|_2,
\]

valid for any polynomial of degree \( \leq N \). For more details see [2] or [10].

As already noted Theorem 4.2, although optimal, is not an exact inverse of Theorem 4.1. This can be taken as evidence that the standard Sobolev or Besov spaces are not very good for expressing the kind of regularity needed for a
certain rate of approximation by polynomials. They do not take into account the well known fact, already noted by Timan (cf. [9]), that the polynomials have a certain ability to absorb singularities at the end points of an interval.

Let $L$ denote the operator $-\frac{d}{dy} (1-y^2) \frac{d}{dy}$ with domain of definition

$$D(L) = \{ u \in L^2([-1,1]) | Lu \in L^2([-1,1]) \}.$$ 

Now introduce the Besov spaces $H^t$, $t > 0$, by

$$H^t = (D(L^p); D(L^q))_{s, \infty}$$

where $p, q$ are two integers with $0 < p < t < q$ and $0 < s < 1$ is selected so that $p(1-s) + qs = t$. Because of Theorem 14.1 in [8], which says that $(D(L^b); D(L^a))_{s, 2} = D(L^{b(1-s)+a})$, and the reiteration theorem on p.50 of [4] it follows that modulo equivalent norms $H^t$ is independent of the choice of $p$ and $q$.

We are now in a position to characterize completely the regularity needed for a certain order of approximation by polynomials.

**Theorem 4.3**

Let $t$ be a positive number. For any $u \in H^t$ we can find a sequence of polynomials $\{ p_N \}_{N=0}^\infty$, the degree of $p_N \leq N$, such that

$$||u-p_N||_{L^2} \leq (N+1)^{-2t} ||u||_{H^t}.$$ 

On the other hand if $u \in L^2([-1,1])$ and there exists a constant $C_u$ and a sequence of polynomials $\{ p_N \}_{N=0}^\infty$, the degree of $p_N \leq N$, such that
then \( u \in H^t \) and

\[
\|u\|_t \leq C\left(\frac{\|u\|_2}{\|u\|_2}\right)
\]

for some constant \( C \) independent of \( u \).

**Note.** \( C_u \) here is not generic, it is the same constant in the two inequalities.

**Proof**

We start by proving the direct part. It is well known that the eigenfunctions of \( L \) are the Legendre polynomials \( \{\ell_k\}_{k=0}^\infty \). Also

\[
L(\ell_k) = k(k+1)\ell_k.
\]

Let now \( u \) be an element of \( D(L^p) \) and let \( \sum \alpha_m \ell_m \) be the Legendre series for \( u \). Since \( u \in D(L^p) \) we know that \( \sum \alpha_m 4^p < \|u\|_2^2 \). Define \( p_N = \sum_{m=0}^N \alpha_m \ell_m \), then

\[
\|u - p_N\|_2^2 = \sum_{m=N+1}^\infty \alpha_m^2 \leq (N+1)^{-4p} \sum_{m=N+1}^\infty \alpha_m 4^p \leq (N+1)^{-4p} \|u\|_2^2 \quad D(l^p),
\]

i.e.,

\[
\|u - p_N\|_2 \leq (N+1)^{-2p} \|u\|_2 \quad D(l^p).
\]

Interpolation applied to this gives the desired result.

We now turn to the inverse. Assume that there exists polynomials \( p_N \) of degree \( \leq N \) such that
\[ \| u - p_N \|_2 \leq C_u (N+1)^{-2t} \]

Define \( r_0 = p_1, \ r_m = p_{2^m} - p_{2^{m-1}} \quad m \geq 1 \).

Then

\[ \| r_0 \|_2 \leq C_u + \| u \|_2 \quad \text{and} \]
\[ \| r_m \|_2 \leq \| u - p_{2^m} \|_2 + \| u - p_{2^{m-1}} \|_2 \leq C_t \cdot C_u \cdot 2^{-2tm} \quad m \geq 1 \]  

Since \( \| L p_n \|_2 \leq C n^2 \| p_n \|_2 \) for any polynomial \( p_n \) of degree \( \leq n \), it follows from above that for any non-negative integer \( q \)

\[ \| L^q r_0 \|_2 \leq C_q (C_u + \| u \|_2) \quad \text{and} \]
\[ \| L^q r_m \|_2 \leq C_q \cdot C_t \cdot C_u \cdot 2^{(q-t)m} \quad m \geq 1 \]  

Now define \( v_k = \sum_{m=0}^{k} r_m \). We then get

\[ \| v_k \|_D(L^q) \leq \sum_{m=0}^{k} (\| L^q r_m \|_2 + \| r_m \|_2) \]
\[ \leq C_{q,t} (C_u + \| u \|_2 + C_u \sum_{m=1}^{k} 2^{2(q-t)m}) \]
\[ \leq C_{q,t} \cdot 2^{2(q-t)k} (C_u + \| u \|_2) \]

provided \( q > t \). At the same time
\[ ||u-v_k||_{L^2} = ||u-p||_{L^2} = C \cdot 2^{-2tk} \]

By defining \( s_k = 2^{-2kq} \) we therefore have

\[
-\frac{t}{q} \left( ||u-v_k||_{L^2} + s_k ||v_k||_{\mathcal{D}(L^q)} \right) \leq C_{q,t} \left( C_u + ||u||_{L^2} \right),
\]

and since \( s_k \to 0 \) for \( k \to \infty \) this proves that

\[ u \in \mathcal{L}^2; \mathcal{D}(L^q) \]

\[ \frac{t}{q} \to \infty \]

with

\[ \frac{||u||}{H^t} \leq C_{q,t} \left( C_u + ||u||_{L^2} \right) \]

This theorem also allows a version formulated by using the spaces \( \mathcal{D}(L^t) \) instead of the corresponding Besov spaces. It is derived from the inclusions

\[ \mathcal{D}(L^t) \subseteq H^t \subseteq \mathcal{D}(L^{t-\varepsilon}) \]

valid for any \( t > 0, \varepsilon > 0 \).

A theorem similar to Theorem 4.3 but concerning approximation in the \( H^1 \) norm can be derived immediately based on Theorem 4.3.

For practical purposes, in determining the rate of approximation, the following characterization of \( \mathcal{D}(L^q) \) (cf. [3]) will often be convenient:

\[ \mathcal{D}(L^q) = \{ u \in L^2([-1,1]) | u \in H^q([-1,1]) , (1-y^2)^q u \in H^2([-1,1]) \} \]
Let us now give one simple example that shows how a result similar to Theorem 4.3 can be established also in a case with non-constant coefficient.

**Example 4.1**

Let \( P \) denote the operator \( a^{-1} \frac{d}{dy} a \frac{d}{dy} \), where the function \( a \) is given by

\[
a(y) = \begin{cases} 
a_+ & \text{for } y > 0 \\
_+ & \text{for } y < 0
\end{cases}
\]

with \( a_+ \) and \( a_- \) being two positive constants. (This is the operator arising in the numerical examples of [13].)

It is not difficult to see that the following set of functions is a basis for \( N(P^N) \):

\[
\phi_0 = 1, \quad \phi_1 = \frac{1}{a(y)} y
\]

\[
\phi_{2k} = \int_{-1}^{y} \ell_{2k-1}(t) dt, \quad \phi_{2k+1} = \frac{1}{a(y)} \int_{-1}^{y} \ell_{2k}(t) dt \quad 1 \leq k \leq N-1.
\]

Here \( \ell_k \) denotes the Legendre polynomial of order \( k \). Performing the Gram-Schmidt orthogonalization on the set \( \phi_0, \phi_1, \ldots, \phi_{2N-2}, \phi_{2N-1} \) (in that sequence), in the inner-product \( u, v \rangle = \int_{-1}^{1} u(y)v(y) a(y) dy \), we end up with a new set of functions \( \psi_0, \psi_1, \ldots, \psi_{2N-2}, \psi_{2N-1} \). \( \psi_k \) is a piecewise polynomial of degree \( k \). Let \( L_a \) denote the operator \( -a^{-1}(y) \frac{d}{dy} a(y)(1-y^2) \frac{d}{dy} \), it is then clear that

\[
L_a \psi_{2k} = \sum_{j=0}^{k} \lambda_j, k \psi_{2j} \quad 0 \leq k \leq N-1,
\]
\[
L_a \psi_{2k-1} = \sum_{j=1}^{k} \lambda_{j,k} \psi_{2j-1} \quad 1 \leq k \leq N
\]

Now we have because of the orthogonality of the \( \psi_k \)'s and the fact that \( L_a \) is self-adjoint in \( L^2([-1,1], a(y)dy) \)

\[
\langle L_a \psi_k, \psi_j \rangle_a = \langle \psi_k, L_a \psi_j \rangle_a = 0 \quad \text{for} \quad j < k
\]
i.e.,

\[
L_a \psi_k = \lambda_k \psi_k \quad \text{for any} \quad k
\]

It immediately follows that \( \lambda_k = k(k+1) \).

It also follows, since \( \{\psi_k\}_{k=0}^\infty \) is dense in \( L^2([-1,1]) \), that \( \{\lambda_k, \psi_k\}_{k=0}^\infty \) is a complete set of eigenvalues and eigenfunctions for \( L_a \).

As in the proof of Theorem 4.3 we now get that

\[
\inf_{v \in N(P)} ||u-v||_2 \leq C N^{-2t}
\]

for any \( 0 \leq p < t < q \), and \( 0 < s < 1 \) chosen such that \( t = p(1-s) + qs \).

In summary, we have found a singular operator \( L_a \) that characterizes the rate of approximation with functions in \( N(P^N) \) the same way that the Legendre operator does with polynomials.
5. The Constant Coefficient Case - Dimensions Higher Than 1

In this section we prove a result relating to the dimensionally reduced solutions introduced in [13]. We give a characterization of the regularity needed for a certain rate of approximation. The approximating functions are of the form \( \sum_{j=0}^{N} w_j(x)p_j(y) \) where \( w_j \in H^1(\omega) \) and \( p_j \) is a polynomial of degree \( j \), i.e., the operator \( P \) has constant coefficients. \( \omega \) as before denotes a domain in \( \mathbb{R}^n \) with a Lipschitz boundary and \( y \) ranges over \([-1,1]\).

In the proof of the main result in this section the following lemma will be very useful.

**Lemma 5.1**

Let \( H_1 \subseteq H_0 \) be two Banach spaces with norms \( \| \cdot \|_1 \) and \( \| \cdot \|_0 \) respectively.

Let \( \{V_N\}_{N=0}^{\infty} \) be an increasing sequence of subspaces of \( H_1 \) and let \( \theta \) be a positive number. We assume that the following implication holds

\[
\inf_{q \in V_N} \| u - q \|_0 \leq C_u (N+1)^{-\theta} , \quad \forall N \geq 0
\]

\[
u \in H_1 \quad \text{and} \quad \| u \|_1 \leq C(C_u + \| u \|_0)
\]

for some \( C \) independent of \( u \). (\( C_u \) here is not generic, it is the same constant in the two inequalities.)

As a result of this it follows that for any \( 0 < \theta < 1 \)

\[
u \in H_0 \quad \text{and} \quad \inf_{q \in V_N} \| u - q \|_0 \leq C_u (N+1)^{-\theta_0} , \quad \forall N \geq 0
\]

\[
u \in (H_0, H_1)_{\theta, \infty} \quad \text{and} \quad \| u \|_{\theta, \infty} \leq C(C_u + \| u \|_0)
\]
for some $C$ independent of $u$. (As before $C_u$ is not generic.)

**Proof**

Let $0 < \theta < 1$ and assume that there exists a sequence of elements $q_N \in \mathcal{V}_N$, $N \geq 0$, such that

$$||u-q_N||_0 \leq C_u(N+1)^{-\theta}$$. 

Define

$$r_0 = q_0$$

$$r_m = q_{2m} - q_{2m-1} \quad m \geq 1$$

then

$$||u - \sum_{m=0}^{k} r_m||_0 = ||u-q_k||_0 \leq C_u 2^{-k\theta}$$.

At the same time

$$||r_0||_0 \leq C_u + ||u||_0$$ and

$$||r_m||_0 \leq ||u-q_{2m}||_0 + ||u-q_{2m-1}|| \leq C_{\theta,\beta} C_u 2^{-m\theta} \quad m \geq 1$$.

That is, $r_m \in \mathcal{V}_{2^m} \quad m \geq 1$, and

$$||2^{m\theta}(\theta-1)r_m||_0 \leq C_{\theta,\beta}(C_u + ||u||_0)2^{-m\theta}$$.

From the first implication in the statement of this theorem it follows that

$$||2^{m\theta}(\theta-1)r_m||_1 \leq C_{\theta,\beta}(C_u + ||u||_0 + ||2^{m\theta}(\theta-1)r_m||_0)$$.
or

\[ \| \kappa_m \|_1 \leq C_{\theta, \beta}(C_u + \| u \|_0) \cdot 2^{mB(1-\theta)}. \]

We therefore get

\[ \| \sum_{m=0}^{k} \kappa_m \|_1 \leq \sum_{m=0}^{k} \| \kappa_m \|_1 \leq C_{\theta, \beta}(C_u + \| u \|_0) \cdot 2^{kB(1-\theta)}. \]

If we define \( s_k = 2^{-kB} \) the following inequality now holds

\[ s_\theta^k (\| u - \sum_{m=0}^{k} \kappa_m \|_0 + s_k \sum_{m=0}^{k} \| \kappa_m \|_1) \leq C_{\theta, \beta}(C_u + \| u \|_0). \]

Since \( s_k \to 0 \) for \( k \to \infty \) this proves that

\[ u \in (H_0; H^1)_{\theta, \infty} \text{ with } \| u \|_{\theta, \infty} \leq C_{\theta, \beta}(C_u + \| u \|_0). \]

\[ \square \]

Let us introduce the spaces

\[ k^R = \{ u \in D(\mathcal{L}^R) \mid \frac{\partial}{\partial x_i} u \in D(\mathcal{L}^R) \quad i = 1, \ldots, n \}, \]

\( \mathcal{L} \) here denotes \(-\frac{\partial}{\partial y} (1-y^2) \frac{\partial}{\partial y} \) considered as an operator \( L^2(\omega \times [-1,1]) \supset \mathcal{D}(\mathcal{L}) \supset L^2(\omega \times [-1,1]), \) and \( R \) is a non-negative integer.

\( V_N \) denotes the space \( \{ \sum_{j=0}^{N} w_j(x)p_j(y) \mid w_j \in H^1(\omega) \} \) where \( p_j \) is a polynomial of degree \( j \), \( j \geq 0 \). We are then able to give the following characterization of approximation by the spaces \( V_N \) in the \( H^1 \)-norm.
Theorem 5.1

Let $\alpha$ be a given positive number. If $u \in (H^1(\omega \times [-1,1]); K^R_{\alpha/R,\infty})$ for some integer $R > \alpha$ and $R > 2\alpha/\varepsilon$, where $\varepsilon$ is a positive number, then there exists a constant $C$ such that

$$\inf_{q \in V_N} |u-q|_{H^1(\omega \times [-1,1])} < C(N+1)^{-2\alpha + \varepsilon}, \quad \forall N > 0.$$ 

On the other hand if for some $\varepsilon > 0$ there exist a constant $C$ such that

$$\inf_{q \in V_N} |u-q|_{H^1(\omega \times [-1,1])} < C(N+1)^{-2\alpha - \varepsilon}, \quad \forall N > 0,$$

then

$$u \in \bigcap_{R \in \mathbb{N}, R > \alpha} (H^1(\omega \times [-1,1]); K^R_{\alpha/R,\infty}).$$

Before we proceed with the proof of Theorem 5.1, let us state a corollary that immediately follows from this theorem.

Modulo $\varepsilon$ this is the equivalent of Theorem 4.3 in more than one dimension.

Corollary 5.1

Let $\alpha$ be a given positive integer. If $u \in \bigcap_{R \in \mathbb{N}, R > \alpha} (H^1(\omega \times [-1,1]); K^R_{\alpha/R,\infty})$, then for any $\varepsilon > 0$ there exist a constant $C_\varepsilon$ such that

$$\inf_{q \in V_N} |u-q|_{H^1(\omega \times [-1,1])} < C_\varepsilon(N+1)^{-2\alpha + \varepsilon}, \quad N > 0.$$ 

On the other hand if for some $\varepsilon > 0$ there exists a constant $C$ such that
\[ \inf_{q \in V_N} ||u-q||_{H^1(\omega \times [-1,1])} \leq C(N+1)^{-2\alpha-\varepsilon}, \quad \forall N \geq 0, \]

then

\[ u \in \bigcap_{R \in \mathbb{N}, R > \alpha} (H^1(\omega \times [-1,1]); K^R_{\alpha/R}) . \]

**Proof of Theorem 5.1**

Assume that \( u \in \mathcal{K}^R \) and \( R \) is an integer \( > 1 \). Let \( v_N \) denote the orthogonal projection of \( Lu \) onto \( \{ \sum_{j=0}^{N} w_j(x)p_j(y) | w_j \in L^2(\omega) \} \) in the \( L^2(\omega \times [-1,1]) \) inner product. Then it is clear that \( \frac{\partial}{\partial x_1} v_N \) is the \( L^2 \) projection of \( L \frac{\partial}{\partial x_1} u \) onto the same subspace. From Theorem 4.3 we immediately get

\[ \sum_{i=1}^{n} ||L \frac{\partial}{\partial x_1} (u-v_N^*)||_{L^2(\omega \times [-1,1])} + ||L(u-v_N^*)||_{L^2(\omega \times [-1,1])} \leq C R^{-2} \left( \sum_{i=1}^{n} ||\frac{\partial}{\partial x_1} u||_{D(L^R)} + ||u||_{D(L^R)} \right) \]

for any function \( v_N^* \in V_N \) with \( L v_N^* = v_N \). Now choosing \( v_N^* \) so that also \( \int_{-1}^{1} v_N^*(x,y)dy = \int_{-1}^{1} u(x,y)dy \) for any \( x \in \omega \) (this is obviously possible) it follows that

\[ \sum_{i=1}^{n} ||L \frac{\partial}{\partial x_1} (u-v_N^*)||_{L^2(\omega \times [-1,1])} + ||u-v_N^*||_{L^2(\omega \times [-1,1])} + ||\frac{\partial}{\partial y} (u-v_N^*)||_{L^2(\omega \times [-1,1])} \leq C \left( \sum_{i=1}^{n} ||\frac{\partial}{\partial x_1} u||_{D(L^R)} + ||u||_{D(L^R)} + ||\frac{\partial}{\partial y} u||_{D(L^R)} \right) \]

\[ \leq C R^{-2} \left( \sum_{i=1}^{n} ||\frac{\partial}{\partial x_1} u||_{D(L^R)} + ||u||_{D(L^R)} \right) = C R^{-2} ||u||_{\mathcal{K}^R} \]
Using interpolation on this result we get that

$$\inf_{q \in V_N} \|u-q\|_{H^1(\omega \times [-1,1])} \leq C R N^{-2\alpha+\frac{2\alpha}{\alpha}} \|u\|_{\alpha/R,\infty}$$

where $\|u\|_{\alpha/R,\infty}$ denotes the norm on $(H^1(\omega \times [-1,1]; K^R)_{\alpha/R,\infty}$.

Since $2\alpha/\epsilon < R$, i.e. $2\alpha/R < \epsilon$, the direct part of this theorem immediately follows.

Let us now give a proof of the second part of the theorem. If

$$\inf_{q \in V_N} \|u-q\|_{H^1(\omega \times [-1,1])} \leq C N^{-2R} \frac{R\epsilon}{\alpha}$$

for some $R > \alpha$, then as in the proof of Theorem 4.3 it easily follows that

$$u \in D(L^R) \quad \text{and} \quad \frac{\partial}{\partial x_i} u \in D(L^R),$$

i.e. $u \in K^R$. If we apply Lemma 5.1 with $H_1 = K^R$, $H_0 = H^1(\omega \times [-1,1])$ and $\theta = \alpha/R$, we then get that

$$\inf_{q \in V_N} \|u-q\|_{H^1(\omega \times [-1,1])} \leq C(N+1)^{-2\alpha-\epsilon}$$

implies

$$u \in (H^1(\omega \times [-1,1]); K^R)_{\alpha/R,\infty},$$

for any integer $R > \alpha$, i.e.

$$u \in \bigcap_{R \in \mathbb{N}, R > \alpha} (H^1(\omega \times [-1,1]); K^R)_{\alpha/R,\infty}.$$
For the conclusion of this section let us give a simple example that shows the practical usefulness of Theorem 5.1 (or corollary 5.1).

Example 5.1

Let \( \omega \) be the interval \([0,1]\). Let \( \gamma \) be a positive number and let \((\kappa, \theta)\) denote polar coordinates around the point \((1,1)\). We then consider functions of the type \( u = \kappa \gamma \phi(\theta) \), where \( \phi \) is an element of \( C^\infty([0,\pi/2]) \).

It is not difficult to prove that

\[
\inf_{q \in V_N} \|u-q\|_{H^1(\omega \times [-1,1])} \leq C \epsilon (N+1)^{-2\gamma + \epsilon}, \quad \forall N \geq 0
\]

for any \( \epsilon > 0 \), and at the same time that for a general choice of \( \phi \) (or for any \( \phi \neq 0 \) in the case \( \gamma \notin \mathbb{N} \)) there exist no \( \epsilon > 0 \) and \( C \epsilon \) such that

\[
\inf_{q \in V_N} \|u-q\|_{H^1(\omega \times [-1,1])} \leq C \epsilon (N+1)^{-2\gamma - \epsilon}, \quad \forall N \geq 0
\]

A function of the type \( \kappa \gamma \phi(\theta) \) is a typical example of a corner-singularity as arising from the solution of an elliptic boundary value problem.

Theorem 5.1 (or corollary 5.1) is thus well suited to predict the optimal order of convergence (modulo \( \epsilon \)) that one can in general expect by dimensional reduction of elliptic boundary value problems.
A result like this could not have been obtained by using the a-priori knowledge of the regularity of solutions to elliptic boundary value problems in terms of standard Sobolev spaces.
6. A Simple Example of Dimensional Reduction

Let us consider the boundary value problem

\[
\begin{align*}
\Delta u &= 0 \quad \text{in } [0,1] \times [-1,1]\n\quad u &= 0 \quad \text{for } x = 0, 1
\frac{\partial u}{\partial n} &= g(x) \quad \text{for } y = \pm 1
\end{align*}
\]

(\(n\) is the outward normal).

From [13] it follows that the optimal choice of basis functions for dimensional reduction in this case is the polynomials. \(V_N\) denotes the set

\[
\{ \sum_{j=0}^{N} w_j(x) p_j(y) | w_j \in H^{1}([0,1]) \}, \text{ where } p_j \text{ is a polynomial of degree } j.
\]

Let \(u_N\) denote the projection of \(u\) onto \(V_N\) in the inner product

\[
B(\phi, \psi) = \int_{0}^{1} \int_{-1}^{1} \left( \frac{\partial \phi}{\partial y} \frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial x} \frac{\partial \psi}{\partial x} \right) dydx.
\]

It is clear that

\[
\inf_{q \in V_N} ||u - q||^2_{H^1([0,1] \times [-1,1])} \leq C B(u - u_N, u - u_N) \leq \inf_{q \in V_N} ||u - q||^2_{H^1([0,1] \times [-1,1])},
\]

and hence that the energy error \(B(u - u_N, u - u_N)\) is asymptotically in \(N\) equivalent to the square of the distance (in \(H^1\)) from \(u\) to \(V_N\).

If \(g\) has the Fourier series

\[
g(x) = \sum_{k=1}^{\infty} g_k \sin k\pi x,
\]
then it immediately follows that $u$ is given by

$$u(x,y) = \sum_{k=1}^{\infty} \frac{\cosh(k\pi y)}{\sinh(k\pi)} \frac{g_k}{k^{\alpha}} \sin k\pi x.$$ 

In terms of regularity of $u$ it is not difficult to prove that this formula leads to the following three results:

1) $\forall \alpha \geq 0$: $\sum_{k=1}^{\infty} g_k^2 k^{2\alpha} < \infty \iff u \in H^{3/2+\alpha}([0,1] \times [-1,1])$

2) $\forall \alpha > 0$: $\sum_{k=1}^{\infty} g_k^2 k^{2\alpha} < \infty \quad \Rightarrow \quad \begin{cases} u \in \bigcap_{R \in \mathbb{N}, R > \theta} (H^1([0,1] \times [-1,1]); K^R)_{R,\infty} \\
\text{with } \theta = \alpha + 1/2 \end{cases}$

3) $\forall \alpha > 0, \varepsilon > 0$: $u \in \bigcap_{R \in \mathbb{N}, R > \theta} (H^1([0,1] \times [-1,1]); K^R)_{R,\infty} \quad \Rightarrow \quad \sum_{k=1}^{\infty} g_k^2 k^{2\alpha} < \infty \quad \text{with } \theta = \alpha + 1/2 + \varepsilon.$

We consider two different choices for $g$

$$g(x) = \pi/4$$
$$g(x) = x(x-1)$$

For the first choice of $g$ it follows that

$$\sum_{k=1}^{\infty} g_k^2 k^{2\theta} < \infty \quad \text{for any } \theta < 1/2,$$

$$\sum_{k=1}^{\infty} g_k^2 k = \infty.$$
similarly for the second choice

\[ \sum_{k=1}^{\infty} g_k^2 k^{2\theta} < \infty \text{ for any } \theta < 5/2, \text{ and} \]

\[ \sum_{k=1}^{\infty} g_k^5 k^5 = \infty. \]

Corollary 5.1 together with the regularity results ii) and iii) now ensure that

In the case \( g(x) = \pi/4 \)

\( B(u-u_N, u-u_N) \) will converge to zero faster than \( N^{-4+\epsilon}, \forall \epsilon > 0 \), but
on the other hand slower than \( N^{-4-\epsilon}, \forall \epsilon > 0 \).

In the case \( g(x) = x(x-1) \)

\( B(u-u_N, u-u_N) \) will converge to zero faster than \( N^{-12+\epsilon}, \forall \epsilon > 0 \), but
on the other hand slower than \( N^{-12-\epsilon}, \forall \epsilon > 0 \).

Figures 1 and 2 show the actual computed values of \( B(u-u_N, u-u_N) \) as a function of \( N \) in the two different cases. Note that the asymptotic rate of convergence is obtained already for a fairly small number of polynomials.

For details concerning the computation of the \( u_N \)'s see [13].

Instead of using Corollary 5.1 to obtain information about the rate of convergence we could have used the regularity result i) and a 2-dimensional version of Theorem 4.1. This way we could at most have predicted convergence of the order of \( N^{-2+\epsilon} \) and \( N^{-6+\epsilon} \) respectively, i.e. only half the actual convergence rate.

In [13] we considered the same boundary value problem as here, only it was on the domain \([0,1]x[-h,h]\) for some \( h > 0 \), and not on \([0,1]x[-1,1]\). From
Fig. 1  Energy error $\times 10^6$ with $g(x) = \pi/4$, $h = 1$. 
Fig. 2 Energy error $x \times 10^{14}$ with $g(x) = x(x-1)$, $h = 1$. 
the computational results there it follows that for a fixed $N \geq 2$, $B(u-u_N^h, u-u_N^h)$ behaves like $h^2, h + 0$, in the case where $g(x) = \pi/4$.

($u_N^h$ is the projection of $u$ onto $\{ \sum_{j=0}^{N} \psi_j(x) p_j(y/h) \mid \psi_j \in H^1([0,1]) \}$).

Comparing this to the result obtained here for $g(x) = \pi/4$ it is seen that using $N$ polynomials, $y \in [-1,1]$, is in some sense equivalent to having a domain of thickness $1/N^2$. A similar feature has been noticed by comparison of the standard $h$-version of the F.E.M. with the so-called $p$-version (cf. [2]).

In this example we used slight variations of the approximation results proved in sections 4 and 5, namely with fixed boundary conditions $\equiv 0$ at $x = 0,1$. The proofs of these results follow immediately from the proofs of the similar results with no boundary conditions.
Appendix

In sections 2 and 3, we used some results concerning the eigenvalues and eigenfunctions of the boundary value problem

\[- \frac{d^2}{dy^2} a \frac{du}{dy} = \lambda u\]

\[u(-1) = u(1) = 0\]

\[a\] here is a function in \(L^\infty([-1,1])\) such that \(\exists\) a constant \(a_0\) with \(0 < a_0 < a(y)\). From the theory of Sturm-Liouville systems it immediately follows that the eigenvalues (repeated according to multiplicity) form a sequence:

\[0 < \lambda_0 < \lambda_1 < \cdots < \lambda_m < \lambda_{m+1} < \cdots\]

with \(+\infty\) as the only limit point.

Lemma A.1

With notation as above

\[\lambda_m \neq \lambda_{m'}, \text{ for } m \neq m',\]

i.e. the eigenvalues are all simple.

Proof

Assume that for some \(m \neq m'\) \(\lambda_m = \lambda_{m'}\). This means that the eigenvalue \(\lambda = \lambda_m(= \lambda_{m'})\) has multiplicity \(> 1\). Let \(u\) and \(\tilde{u}\) be two linearly independent eigenvectors corresponding to \(\lambda\), and let \(v = cu + d\tilde{u}\) be a nontrivial linear combination with the property that \(a \frac{dv}{dy} = 0\) for \(y = -1\) (such one obviously
exists). The function \( v \) is then a solution to the initial-value problem

\[
- \frac{d}{dy} a \frac{d}{dy} v = \lambda v \quad \text{in } [-1,1]
\]

\[
v = a \frac{d}{dy} v = 0 \quad \text{for } y = -1,
\]

and because of the uniqueness of solutions to this problem, it follows that \( v = 0 \). Since \( v \) is a nontrivial linear combination of \( u \) and \( \tilde{u} \) this shows that \( u \) and \( \tilde{u} \) are linearly dependent. We therefore have arrived at a contradiction, i.e., \( \lambda_m \neq \lambda_{m'} \) for \( m \neq m' \).

It is well known that there exist constants \( 0 < C_1 \) and \( 0 < C_2 \) such that

\[
C_1 (m+1)^2 \leq \lambda_m \leq C_2 (m+1)^2.
\]

By imposing an extra smoothness requirement on \( a \) we can obtain a much more detailed statement.

**Lemma A.2**

If \( a \in C^2([-1,1]) \), then

\[
\lambda_m = (\pi/k)^2 \cdot (m+1)^2 + O(1),
\]

where \( \lambda = \int_{-1}^{1} (a(y))^{-1/2} dy \).

A proof of Lemma A.2 is found in chapter 4 of [113], and shall not be repeated here.

Let \( \{u_m\}_{m=0}^{\infty} \) denote a sequence of normalized eigenfunctions, \( u_m \) corresponding to \( \lambda_m \). Let \( f_0 \) be given as in section 2, namely

\[
f_0(y) = \int_{-1}^{y} \frac{1}{a(s)} ds.
\]
Lemma A.3

The function $f_0$ has the expansion

$$f_0 = \sum_{m=0}^{\infty} a_m u_m,$$

where $a_m \neq 0$ for every $m$.

Proof

That $f_0$ has a unique expansion is well-known. The coefficient $a_m$ is given by

$$a_m = \int_{-1}^{1} \int_{-1}^{1} \frac{1}{a(s)} ds \ u_m(y) \ dy.$$

Now assume that for some value of $m = m_0$, $a_{m_0} = 0$, i.e.

$$\int_{-1}^{1} \int_{-1}^{1} \frac{1}{a(s)} ds \ u_{m_0}(y) dy = 0.$$

Since $u_{m_0}(y) = -\frac{1}{\lambda_{m_0}} \frac{d}{dy} a \frac{d}{dy} u_{m_0}$ we get that

$$\int_{-1}^{1} \int_{-1}^{1} \frac{1}{a(s)} ds \left[ \frac{d}{dy} a \frac{d}{dy} u_{m_0} \right](y) \ dy = 0.$$

Performing an integration by parts this yields

$$\int_{-1}^{1} \frac{1}{a(s)} ds \left[ a \frac{d}{dy} u_{m_0} \right](y) - \int_{-1}^{1} \frac{d}{dy} u_{m_0}(y) \ dy = 0.$$
and the last integral here vanishes due to the fact that \( u_m(1) = u_m(-1) = 0 \).

We therefore conclude that

\[
a \frac{d}{dy} u_m = u_m = 0 \quad \text{for } y = 1.
\]

On the other hand \( u_m \) satisfies the differential equation

\[
\frac{d}{dy} a \frac{d}{dy} u_m + \lambda u_m = 0 \quad \text{in } [-1,1].
\]

Because of uniqueness of solutions to the initial-value problem, this implies that \( u_m = 0 \). We have thus arrived at a contradiction, meaning that \( \alpha_m \not= 0 \) for every \( m \).

\[\square\]

Again, by imposing an extra smoothness requirement on \( a \), we obtain a very detailed result concerning the decay-properties of the \( \alpha_m \)'s.

**Lemma A.4**

If \( a \in C^2([-1,1]) \), then \( \exists \) constants \( 0 < C_1 \) and \( 0 < C_2 \) such that

\[
\frac{C_1}{m+1} \leq |\alpha_m| \leq \frac{C_2}{m+1} \quad \text{for all } m.
\]

**Proof**

From [11] p. 176 we get the following asymptotic formula for \( u_m(y) \)

\[
u_m(y) = D_m(a(y))^{-1/4} \left[ \sin \left( \frac{(m+1)\pi}{\ell} \xi \right) - \right.
\]

\[
- \frac{1}{m+1} T(\xi) \cos \left( \frac{(m+1)\pi}{\ell} \xi \right) \left] + O((m+1)^{-2}) \right.
\]
where $\xi(y) = \int_{-1}^{y} (a(s))^{-1/2} \, ds$ and $\xi = \xi(1) = \int_{-1}^{1} (a(s))^{-1/2} \, ds$. The function $T$ is in $C^1$ and the constants $D_m$ satisfy

$$\exists \, D \text{ (independent of } m \text{)} \text{ such that }$$

$$\frac{1}{D} \leq |D_m| \leq D \quad \text{for all } m .$$

Also, $O(\cdot)$ here means uniformly in $y$. Let us now calculate $\alpha_m$:

$$\alpha_m = \int_{-1}^{1} f_0(y) u_m(y) \, dy = \int_{-1}^{1} \frac{1}{a(s)} ds \int_{-1}^{1} u_m(y) \, dy =$$

$$I_1 + I_2 + O((m+l)^{-2}) .$$

$I_2$ denotes the integral

$$- \frac{1}{m+1} D_m \int_{-1}^{1} \left( \int_{-1}^{1} \frac{1}{a(s)} ds \right) (a(y))^{-1/4} T(\xi) \cos \left( \frac{(m+l)\pi}{\xi} \xi \right) \, dy .$$

By a change of variables from $y$ to $\xi$ and an integration by parts, it immediately follows that $I_2$ is $O((m+l)^{-2})$.

$I_1$ is given as

$$D_m \int_{-1}^{1} \left( \int_{-1}^{1} \frac{1}{a(s)} ds \right) (a(y))^{-1/4} \sin \left( \frac{(m+l)\pi}{\xi} \xi \right) \, dy .$$

By a change of variables from $y$ to $\xi$ and an integration by parts we get that
\[ I_1 = D_m \left( \int_{-1}^{1} \frac{1}{a(s)} ds \right) & (a(1))^{1/4} \cdot \frac{1}{(m+1)^{m+1}} \cdot (-1)^{m+1} + O((m+1)^{-2}) \].

This immediately implies the existence of two constants \( 0 < C_1 \) and \( 0 < C_2 \) such that

\[
\frac{C_1}{m+1} \leq |a_m| \leq \frac{C_2}{m+1}
\]

for \( m \) sufficiently large. Now combining with Lemma A.3 and possibly changing the constants \( C_1 \) and \( C_2 \) we get the desired result.
References


The Laboratory for Numerical Analysis is an integral part of the Institute for Physical Science and Technology of the University of Maryland, under the general administration of the Director, Institute for Physical Science and Technology. It has the following goals:

- To conduct research in the mathematical theory and computational implementation of numerical analysis and related topics, with emphasis on the numerical treatment of linear and nonlinear differential equations and problems in linear and nonlinear algebra.

- To help bridge gaps between computational directions in engineering, physics, etc. and those in the mathematical community.

- To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.

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Further information may be obtained from Professor I. Babuška, Chairman, Laboratory for Numerical Analysis, Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742.