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ON OPTIMAL OPERATION OF A JOINTLY OWNED ENTERPRISE

by
DAVID GALE
and
HILTON MACHADO

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ON OPTIMAL OPERATION OF A JOINTLY OWNED ENTERPRISE†

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ABSTRACT

An enterprise is owned jointly by m agents, the i-th agent's share being $\theta_i > 0$ where $\sum_i \theta_i = 1$. The enterprise is able to produce some non-negative n-vector x of goods where x lies in some convex production set X. An *operation* consists of choosing a vector from X and distributing it among the agents. The problem is to find an operation such that the value of the i-th agent's bundle measured in a given price system is *proportional* to θ_i and such that the operation is (Pareto) optimal with respect to the agents' preferences. It is shown under standard assumptions that operations which are both optimal and proportional always exist. It is conjectured that if preferences are given by separable concave utility functions then such operations are unique. This is proved (a) when there are only two goods; (b) when X is a simplex; (c) when X represents production of a single good over n time period.

ON OPTIMAL OPERATION OF A JOINTLY OWNED ENTERPRISE

by

David Gale and Hilton Machado

1. INTRODUCTION

An enterprise such as a farm or a firm is owned jointly by m agents, the share of agent i being θ_i, where $\Sigma\, \theta_i = 1$. The enterprise is able to provide various amounts of n goods, thus, a non-negative n-vector from some *production possibility set* X (non-trivial) in R^n_+. Each agent has a preference ordering over R^n_+. The problem is then to decide which vector x in X should be produced and how this vector should be distributed among the owners of the enterprise. An obvious requirement for any such scheme is that it should be (Pareto) optimal with respect to the owners' preferences. A second requirement is that the distribution should in some way reflect the shares of the different owners. In order to formulate the latter we will assume that there is some exogenously given set of prices for the various goods. A distribution will then be called *proportional* if the values at these prices of the goods-vector distributed to each owner is proportional to his share of the enterprise. Without loss of generality we may assume that all prices are equal to one (simply define the unit of each good appropriately), so in a proportional distribution the total amount of goods received by each agent is proportional to his share. A *feasible operation* of this model consists of a choice of a vector x from X and a distribution $\chi = (x_1, \ldots, x_m)$ in R^{nm}_+ such that $\Sigma\, x_i = x$.

The purpose of this paper is to show (a) under the usual assumptions of closedness and convexity of production and preferences there always exists an optimal, proportional operation. The proof is fairly standard and is related to a much more detailed study of Balasko [2] which however does not include production. For the sake of completeness we include our own short proof. Our main concern here is with uniqueness which we conjecture to be true if preferences are given by utility functions which are increasing, strictly concave and separable. While this question is still open for general production sets we show (b) that the optimal proportional operation is unique in each of the following cases: (1) when there are only two goods, (2) when the production set X is a simplex and (3) when the model can be formulated in terms of time periods with reinvestment of collective savings and distribution of profits among the agents along the time.

We remark that this paper is a sequel to those of Gale and Sobel [3], [4] and Sobel [7] concerned with the case in which the enterprise produced only one good the amount of which was a random variable over which the owners had no control. We here eliminate the stochastic element but allow the owners to determine the output vector as well as its distribution. The interest in the present result like that of its predecessors lies in the uniqueness theorem. We have here an instance, as with the Shapley value and the Nash bargaining problem, where a natural bargaining-type problem has only one-solution satisfying certain natural requirements.

The condition of separable utility is, of course, a strong one. It is natural, however, for the special case (3) which was the original motivation for this study. Here the enterprise is to operate over n time periods and the goods can be taken to be the profits of each period. These can be controlled by the owners who must decide how much profit to distribute among themselves in each period and how much to reinvest for the next period, input and output being related via given *production functions* f_t. Under the usual assumption that each owner's utility of income is additive over time, we have an example which satisfies the separability condition (b) above.

An interesting but as yet unsettled question would involve incorporating a stochastic element in the dynamic model of the previous paragraph. Suppose the return on investment depends on a random element as well as the amount invested. What are the appropriate investment and distribution strategies for obtaining analogues of (b) above? - the existence problem (a) is settled under fairly general hypothesis in Machado [6].

2. EXISTENCE

We assume each agent has a strictly increasing, strictly convex, closed preference ordering on R_+^n which can therefore be represented by (quasi-concave) continuous utility functions. The production set is compact and convex in R_+^n.

To prove existence of optimal proportional operations, let $X \subset R^{nm}$ be the set of all distributions $x = (x_1, \ldots, x_m)$ of the model. Choose utility functions μ_i such that $\mu_i(0) = 0$ and define $\mu : X \to R_+^m$ by $\mu(x) = (\mu_1(x_1), \ldots, \mu_m(x_m))$. If $\hat{X}$ is the set of all optimal operations, it is well known that $\hat{U} = \mu(\hat{X})$ is homeomorphic to the unit simplex Σ^{m-1} (see e.g. Arrow-Hahn [1], pp. 111-112). Further, μ restricted to $\hat{X}$ is one-to-one because of the strict convexity of preferences so μ^{-1} is a homeomorphism from $\hat{U}$ to $\hat{X}$. Define the map $E : X - \{0\} \to \Sigma^{m-1}$ by $E(x) = (e \cdot x_1, \ldots, e \cdot x_m)/(e \cdot x)$ where e is the n-vector all of whose coordinates are one and $x = \Sigma x_i$. Let $\phi = E \circ \mu^{-1}$, then ϕ is well defined because $0 \notin \hat{X}$ and continuous from $\hat{U}$ to Σ^{m-1}. Further, for any $S \subset M = \{1, \ldots, m\}$ let $\hat{U}_s = \{u \in \hat{U} ; u_i = 0 \text{ for } i \in S\}$, the image of the corresponding face of the unit simplex. By monotonicity, $\mu_i(x_i) = 0$ implies $x_i = 0$, hence $e \cdot x_i = 0$ so that ϕ maps every face of $\hat{U}$ onto the corresponding face of Σ^{m-1}. The standard homotopy result then implies that ϕ is subjective. Thus, for some optimal operation x and some positive number λ, we have $e \cdot x_i = \lambda\theta_i$ for all i, as desired.

3. THE CASE OF TWO GOODS

Assume now there are only two goods and the utility functions are differentiable (for simplicity), strictly concave, increasing and separable, i.e., $\mu_i(x_i) = \sum_{j=1}^{2} \mu_{ij}(x_{ij})$ where $x_i = (x_{i1}, x_{i2})$.

Lemma 1:

Let $x = (x^1, x^2)$ and $y = (y^1, y^2)$ be optimal productions where $x^1 > y^1$ and $x^2 < y^2$. Then for each agent i with $x_{i1} > y_{i1}$, we must have $x_{i2} \geq y_{i2}$. Symmetrically, if $x_{i2} < y_{i2}$ for some i , then $x_{i1} \leq y_{i2}$.

□ We prove the first assertion by contradiction. Assume $x_{i1} > y_{i1}$ and $x_{i2} < y_{i2}$. Define a new distribution $x' = (x_1', \ldots, x_m') = (x_1, \ldots, x_i + \delta(y-x), \ldots, x_m)$ where $0 \leq \delta \leq 1$. Since $x' = \Sigma x_i' = \Sigma x_i + \delta(y-x) = \delta x + (1-\delta)y$, we have that $x' \in X$ by convexity. Further, since $x_{i1} > 0$, we see that $x_{i1}' \geq 0$ for δ sufficiently small so that the new distribution is feasible. Now defining $\phi(\delta) = \mu_i(x_i + \delta(y-x))$, we have $\phi(\delta) \leq 0$ by optimality of $(x_1, \ldots, x_m)$. Differentiating using the chain rule gives

$$\phi'(0) = \mu_i'(x_i)(y-x) \leq 0 .$$

Symmetrically since we assumed $x_{i2} < y_{i2}$, we get

$$\mu_i'(y_i)(x-y) \leq 0 .$$

Now adding inequalities and using separability of μ_i ,

$$(x-y)(\mu_i'(x_i) - \mu_i'(y_i)) =$$

$$(x^1 - y^1)(\mu_{i1}'(x_{i1}) - \mu_{i1}'(y_{i1})) + (x^2 - y^2)(\mu_{i2}'(x_{i2}) - \mu_{i2}'(y_{i2})) \geq 0 ,$$

but notice that both of the summands above are negative, giving the desired contradiction. □

Uniqueness:

Suppose that x and y are distinct optimal operations and say $x^1 > y^1$. Then $x^2 < y^2$, otherwise y would not be efficient. It follows that $x_{i1} > y_{i1}$ for some i and $x_{k2} < y_{k2}$ for some k. But then by Lemma 1 we would have $x_i \geq y_i$ and $x_k \leq y_k$. This means that $e \cdot x_i > e \cdot y_i$ and $e \cdot x_k < e \cdot y_k$ so $(e \cdot x_i)/(e \cdot x_k) > (e \cdot y_i)/(e \cdot y_k)$ and therefore one of the two operations is not proportional.

4. SIMPLICIAL PRODUCTION SET

We now prove uniqueness of optimal productional operations for simplicial sets X. We assume that $X = \left\{x \in R_+^n \; ; \; a \cdot x \leq 1\right\}$ for some positive vector $a = (a_1, \ldots, a_n)$ and that μ_i is differentiable for each i. The j-th partial differentiation will be indicated by ∂_j.

Lemma 2:

If $x = (x_1, \ldots, x_m)$ is an optimal operation, there exists a positive vector $\alpha = (\alpha_1, \ldots, \alpha_m)$ such that

(1) $\alpha_i \partial_j \mu_i(x_i) \leq a_j$ for all i and j,

(2) $\alpha_i \partial_j \mu_i(x_i) = a_j$ whenever $x_{ij} > 0$.

□ Define $\alpha_i = \min\,(a_j/\partial_j \mu_i(x_i))$. For a given i say $\alpha_i = a_1/\partial_1 \mu_i(x_i)$. Then (1) is satisfied by definition. To prove (2) suppose, say $x_{i2} > 0$. Given $\delta \geq 0$, consider the new operation $x' = (x_1', \ldots, x_m')$ where $x_j' = x_j$ for $j \neq i$ and $x_i' = (x_{i1} + \delta a_2, x_{i2} - \delta a_1, x_{i3}, \ldots, x_{in})$. For small values of δ, x' is a feasible operation. Let $\phi_i(\delta) = \mu_i'(x_i)$. By optimality of x, we have $0 \leq \phi_i'(0) = a_2 \partial_1 \mu_i(x_i) - a_1 \partial_2 \mu_i(x_i)$. It follows that $a_2/\partial_2 \mu_i(x_i) \leq \alpha_i$, hence $a_2/\partial_2 \mu_i(c_i) = \alpha_i$ as desired. □

Assuming now separability, let x and y be two optimal operations and α and β be the corresponding positive vectors given by Lemma 2. In the same spirit as for the two-good model, we will prove that x and y must coincide by direct use of the following result.

Lemma 3:

If $x_{ij} > y_{ij}$ for some agent i and some good j, then $x_{i\ell} \geq y_{i\ell}$ for all goods ℓ.

□ By Lemma 2 and strict concavity of μ_{ij}, we have that $\alpha_i \mu'_{ij}(x_{ij}) = a_j \geq \beta_i \mu'_{ij}(y_{ij}) > \beta_i \mu'_{ij}(x_{ij})$, hence $\alpha_i > \beta_i$ which means that $x_{i\ell} \geq y_{i\ell}$ for all ℓ for if $x_{i\ell} < y_{i\ell}$ for some ℓ, then the argument above would prove $\beta_i > \alpha_i$. □

Uniqueness now follows at once for we see that for each i either $x_i \geq y_i$ or $x_i \leq y_i$, but as in the previous section we cannot have $x_i \geq y_i$ $(x_i \leq y_i)$ for all i with strict inequality for some i for this would contradict efficiency, and we cannot have $x_i \gneq y_i$, $x_k \lneq y_k$ for this would contradict proportionality.

5. THE MULTIPERIOD PRODUCTION MODEL

Here a single good is produced in each of T consecutive periods and agent i owns θ_i shares of the enterprise, θ_i a (positive) fraction of the initial input $x_o = 1$. The production output or profit associated with input x_{t-1} at the beginning of period t is $y_t = f_t(x_{t-1})$. The *production functions* f_t have positive non-increasing derivatives and $f_t(0) = 0$.

The output of period t is split into individual consumption c_{it} by the different agents i and savings x_t, input for the next period. Now $\mu_{it}(c)$ measures the utility to agent i of consumption or income c in period t and it is assumed that μ_{it} has positive strictly decreasing derivative on R^+. We are also given a price sequence $p = (p_1, \ldots, p_T) > 0$, where p_t stands for the unit price of the commodity in period t, but appropriate scaling of production and utility functions allows for the usual simplification, $p_t = 1$ for all t, which is assumed here.

A *feasible operation* is now a matrix $s = (c_1, \ldots, c_m, x)$ where $c_i = (c_{i1}, \ldots, c_{iT})$ and $x = (x_1, \ldots, x_T)$ satisfy the following conditions

(1) $c_{it} \geq 0$, $x_t \geq 0$ for all i, t, $x_T = 0$

(2) $\sum_i c_{it} + x_t = f_t(x_{t-1})$ for all $t \geq 1$.

For short, we will write $s = (c,x)$ where c is the $m \times T$ consumption matrix (c_{it}) and x is the savings-vector; we occasionally refer simply to c rather than to s.

As before, agent i judges a given operation or scheme s on the basis of two quantities, $U_i(s) = \sum_t \mu_{it}(c_{it})$ and $V_i(s) = e \cdot c_i$ and the problem is to find in the set S of all feasible operations one which is Pareto optimal and proportional, i.e., the m-vector $U(s)$ is a maximal element in the partial ordering of R_+^m and $V(s)$ is a positive multiple of the share-vector θ .

While here we use a different notation, better adapted to the model, it is easy to verify that this is a particular case of the general problem described in the Introduction. In particular, optimality of a scheme $\bar{s}$ is equivalent to the following property

(3) for some $\alpha \in \Sigma^{m-1}$, $\bar{s}$ maximizes the function $\alpha \cdot U(s)$ over the set S .

Furthermore, the existence of optimal proportional schemes follows from our result in Section 2 by taking for production set $X = \{ec \; ; \; (c,x) \in S\}$.

To prove uniqueness we first establish some necessary conditions for optimality. Roughly they say that an agent may decide to sacrifice part of his present income for the sake of some agent's present, past or future consumption but that no increase of the critical value $\alpha \cdot U(s)$ will result from this.

<u>Lemma 4</u>:

Let (c,x) be an optimal operation associated with the vector α in Σ^{m-1} . If for some agent i and period t , $c_{it} > 0$, then the following "backward inequality"

$$(4) \quad \alpha_i \mu'_{it}(c_{it}) \prod_{v=s+1}^{t} f'_v(x_{v-1}) \geq \alpha_j \mu'_{js}(c_{js})$$

holds for every agent j and every period $s \leq t$ (we convention the value one for "empty" products). On the other hand, if $c_{js} > 0$, j and s as before, then we have the "forward inequality"

$$(5)\quad \alpha_j \mu'_{js}(c_{js}) \geq \alpha_i \mu'_{it}(c_{it}) \prod_{v=s+1}^{t} f'_v(x_{v-1}) .$$

□ We prove (4) when $s < t$, the other cases are simpler. Consider the operation (c',x') obtained from (c,x) as follows

$$(6)\quad \begin{array}{lll} c'_{js} = c_{js} + \Delta_s & & (\Delta_s > 0) \\ x'_s = x_s - \Delta_s & & \\ x'_{s+1} = x_{s+1} - \Delta_{s+1} & \text{where} & \Delta_{s+1} = f_{s+1}(x_s) - f_{s+1}(x'_s) \\ \cdots & & \cdots \\ x'_{t-1} = x_{t-1} - \Delta_{t-1} & & \Delta_{t-1} = f_{t-1}(x_{t-2}) - f_{t-1}(x'_{t-2}) \\ c'_{it} = c_{it} - \Delta_t & & \Delta_t = f_t(x_{t-1}) - f_t(x'_{t-1}) \end{array}$$

(all other entries as before).

By definition, $s' = (c',x')$ satisfies *the feasibility condition (2)*. As $c_{it} > 0$ we have $f_t(x_{t-1}) > 0$, so $x_{t-1} > 0$ and $f_{t-1}(x_{t-2}) > 0$. Recursively, $x_v > 0$ for all v in the interval $s \leq v \leq t-1$. Continuity and monotonicity of the production functions now guarantee that, for small values of Δ_s, we have $x'_v \geq 0$ and $c'_{it} \geq 0$ so that condition (1) also holds and $s' \in \mathcal{S}$. By optimality of s, we have

$$\begin{aligned}(7)\quad 0 \geq \alpha\cdot(U(s') - U(s)) &= \alpha_j(\mu_{js}(c'_{js}) - \mu_{js}(c_{js})) + \\ &\alpha_i(\mu_{it}(c'_{it}) - \mu_{it}(c_{it})) = \\ &\alpha_j \delta\mu_{js}(c_{js}\,;\,\Delta_s)\Delta_s - \alpha_i \delta\mu_{it}(c_{it}\,;\,-\Delta_t)\Delta_t\end{aligned}$$

where we use the standard notation

$$\delta F(x;z) = (F(x+z) - F(x))/z .$$

On the other hand, with the same δ notation,

$$(8)\quad \Delta_t = \delta f_t(x_{t-1}\ ;\ -\Delta_{t-1})\Delta_{t-1}$$

$$\cdots$$

$$\Delta_{s+1} = \delta f_{s+1}(x_s\ ;\ -\Delta_s)\Delta_s$$

so that, by multiplication,

$$(9)\quad \Delta_t = \left(\prod_{s+1}^{t} \delta f_v(x_{v-1}\ ;\ -\Delta_{t-1})\right)\Delta_s\ .$$

Substituting this in (7), dividing by $\Delta_s > 0$ and passing to the limit as $\Delta_s \to 0$ (all $\Delta_v \to 0$, necessarily) we get (4), as wanted. □

Lemma 5:

Let (c,x) and (d,y) be optimal operations, $\alpha > 0$ and $\beta > 0$ the associated vectors in Σ^{m-1}, $1 \leq i$, $j \leq m$ and $1 \leq s \leq t \leq T$. If we have

$$(10)\quad (c_{it} - d_{it})(c_{js} - d_{js}) < 0 \quad \text{and}$$

$$(11)\quad (c_{it} - d_{it})(x_v - y_v) \geq 0 \quad \text{for all} \quad s \leq v \leq t-1\ ;$$

then

$$(12)\quad (c_{it} - d_{it})(\alpha_i/\beta_i - \alpha_j/\beta_j) > 0\ .$$

□ Because of the symmetry we may as well assume that $c_{it} > d_{it}$, $c_{js} < d_{js}$, $x_v \geq y_v$ for all the v's and prove that $\alpha_i/\beta_i > \alpha_j/\beta_j$. As $c_{it} > 0$ and $d_{js} > 0$, Lemma 4 implies

$$(13)\quad \alpha_i\mu'_{it}(c_{it}) \prod_{s+1}^{t} f'_v(x_{v-1}) \geq \alpha_j\mu'_{js}(c_{js})$$

$$(14) \quad \beta_j \mu'_{js}(d_{js}) \geq \beta_i \mu'_{it}(d_{it}) \prod_{s+1}^{t} f'_v(y_{v-1}) .$$

Since utilities were assumed to be strictly concave, (13) gives

$$(15) \quad \alpha_i \mu'_{it}(d_{it}) \prod_{s+1}^{t} f'_v(y_{v-1}) > \alpha_j \mu'_{js}(d_{js}) .$$

Multiplying now (14) and (15) together, we get $\alpha_i/\beta_i > \alpha_j/\beta_j$ as wanted. □

The proof of uniqueness relies essentially on the impossibility of certain specific patterns in the order-relationship of two schemes, an idea introduced in [7]. From now on we assume that (c,x) and (d,y) are two given optimal proportional schemes with, say $V(c) = \lambda\theta$, $V(d) = \eta\theta$ where $\lambda \geq \eta > 0$, and α , β (necessarily positive) are the corresponding vectors in Σ^{m-1} . Furthermore, we assume that the agents are so arranged to make the quotient α_i/β_i an increasing function of i . One immediate consequence of this set-up is that, if $c_{it} > d_{it}$ and $c_{js} < d_{js}$ for $i \leq j$ and some pair t , s then *none* of the following may occur

(16) $t \leq s$ and $x_v \leq y_v$ for $t \leq v \leq s-1$

(17) $t \geq s$ and $x_v \geq y_v$ for $s \leq v \leq t-1$.

(The proof is a straightforward application of Lemma 5.) Figure 1 indicates sketchly the five impossibility situations. It is worth noticing the central role played by Lemma 4 whose "give-away" technique goes back to [3]. Case (c) is explicit and basic in [7].

$$\begin{array}{l} c\,,d \ \text{ relation} \\ \\ \\ x\,,y \ \text{ relation} \end{array} \begin{pmatrix} > & . & . & < \\ & & & \\ & & & \\ \leq & \leq & \leq & . \end{pmatrix} \begin{pmatrix} > & . & . & . \\ & & & \\ . & . & . & < \\ \leq & \leq & \leq & . \end{pmatrix}$$

(a) (b)

$$\begin{pmatrix} > \\ < \\ . \end{pmatrix} \qquad \begin{pmatrix} < & . & . & > \\ & & & \\ & & & \\ \geq & \geq & \geq & . \end{pmatrix} \begin{pmatrix} . & . & . & > \\ < & . & . & . \\ \geq & \geq & \geq & . \end{pmatrix}$$

(c) (d) (e)

FIGURE 1

<u>Uniqueness</u>:

□ Consider the smallest i such that $c_{it} > d_{it}$ for some t (if none exists, then $c = d$ as remarked previously). Observe first that $c_{js} \leq d_{js}$ for all $j \leq i$ and s so that $\lambda\theta_j \leq \eta\theta_j$, hence by the hypothesis on λ and η, we have equality and $c_{js} = d_{js}$ for all such pairs.

Assume first that $f_t(x_{t-1}) \leq f_t(y_{t-1})$. The impossibility case (c) described above applies to guarantee that $c_{jt} \geq d_{jt}$ for all j, hence $\sum_j c_{jt} > \sum_j d_{jt}$ and $x_t < y_t$. Cases (a) and (b) now apply to force $c_{jt+1} \geq d_{jt+1}$ for all $j \geq i$, hence for all j and $\sum_j c_{jt+1} \geq \sum_j d_{jt+1}$. As $x_t < y_t$ we have $f_{t+1}(x_t) < f_{t+1}(y_t)$ so that, as before, $x_{t+1} < y_{t+1}$. Recursion leads now to the conclusion that in the last period T, $c_{iT} \geq d_{iT}$ for all i and $f_T(x_{T-1}) < f_T(y_{T-1})$, a contradiction since there are no savings in period T $(x_T = y_T = 0)$.

If we assume instead that $f_t(x_{t-1}) > f_t(y_{t-1})$, we can use impossibility cases (d) and (c) in a backward step-by-step procedure similar to the one above to conclude that, in period 1, $c_{i1} \geq d_{i1}$ for all agents i and $x_1 > y_1$, a contradiction since the initial output is fixed $(f_1(1))$. □

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