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# THE STATISTICAL THEORY OF RADIO DIRECTION FINDING

by R. L. HOLLAND

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*Technical Memorandum*

# **THE STATISTICAL THEORY OF RADIO DIRECTION FINDING**

by R. L. HOLLAND

THE JOHNS HOPKINS UNIVERSITY ■ APPLIED PHYSICS LABORATORY  
Johns Hopkins Road, Laurel, Maryland 20810  
Operating under Contract N00024-81-C-5301 with the Department of the Navy

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### ABSTRACT

The statistical theory of radio direction finding as it applies to three-dimensional geometrics is presented in this report. This theory is an extension of the two-dimensional theory presented by R. G. Stansfield in 1947. The theory has application in situations where airborne targets are maneuvering in three dimensions at high speed; and, especially when the platforms with the direction finding equipment are also aircraft.

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## SECTION I

### INTRODUCTION

It is the primary objective of this report to present the principal features and results of a statistical theory descriptive of the process of radio direction finding. The theory presented here is based upon that developed by R.G. Stansfield as documented in his paper of 1947 (Reference 1). The unique feature of the theory presented herein is its applicability to three-dimensional geometries, as opposed to the two-dimensional nature of the Stansfield theory; in this sense, then, the radio direction finding theory presented within this report is an extension of Stansfield's theory.

Interest in pursuing an extension of Stansfield's theory to three dimensions was aroused as the result of a discussion with a colleague regarding the problem of passively determining the position of a moving, radiating target. It became obvious that it would, in general, be necessary to employ several dispersed direction finding sites capable of simultaneous bearing-line measurements.

Conceptually, this circumstance leads to no difficulties until one begins to consider applying Stansfield's theory to provide analytical characterization of such a direction finding scheme. Indeed, if one contemplates the situation wherein the target is a maneuvering, high-speed aircraft, and the direction finding sites consist of several "picket" or early warning aircraft with direction finding equipment onboard, it would be very difficult if not impossible to control the picket aircraft so that they and the target aircraft stayed in the same geometric plane during the period of encounter. This would never be done operationally; but one would have to assume this geometry to apply Stansfield's theory directly to any analytical evaluation of the encounter. This is not meant to suggest that Stansfield's theory has no useful applications; it has, in fact, been the cornerstone of direction finding analysis for the last 33 years. In each instance where it has been applicable, however, the physical circumstance under investigation has allowed valid application of the two-dimensional theory (e.g., see References 2 and 3). It is still true today that the bulk of the hardware devoted exclusively to radio direction finding applications can measure bearing lines in a single plane only. The point to be made here is that there are a number of circumstances in which the ability to perform three-dimensional direction finding measurements would be extremely useful, and, to that end, a three-dimensional extension of Stansfield's two-dimensional direction finding theory is necessary in order to provide a suitable analytical tool for evaluating the effectiveness of current or prospective three-dimensional direction finding systems.

Circumspect review of this document by persons cognizant of or interested in direction finding applications and procedures is deemed essential and is heartily invited. Comments, suggestions, and criticism are encouraged not only to support the evolution of a more intelligible, useful report, but also to help in assessing the basic worth of and/or need for what is presented here. Indeed, there is some doubt as to the uniqueness of this material, i.e., even though a limited review of some current literature (e.g., see Reference 4) has failed to uncover or suggest any other three-dimensional extension of Stansfield's direction finding theory, one cannot help but feel that this should have (and certainly could have) been accomplished prior to this date.

The remainder of this memorandum will present first a brief discussion of Stansfield's two-dimensional radio direction finding theory, and then present a discussion of the three-dimensional radio direction finding theory. A precedent discussion of Stansfield's theory will obviate, at least initially, the acquisition of Stansfield's original paper by the reader and will provide a convenient point of departure from which to initiate a discussion of the three-dimensional theory. Of even more importance, however, is the sense of correspondence and/or distinction to be gained by the reader when comparing Stansfield's two-dimensional results with those of the three-dimensional theory; e.g., as a minimum requirement for credibility, one would expect the results of the three-dimensional theory to coincide with those of Stansfield's theory for planar geometries with null elevation error, and to appreciate such coincidence, the reader must be acquainted with Stansfield's theory.

In what follows, the terms "direction finding," or "direction finder" will be abbreviated as "DF," where the meaning should be clear from the context in which the abbreviation is used. Likewise "BL" and "BA" will be used to represent the words "bearing line" and "bearing angle," respectively.

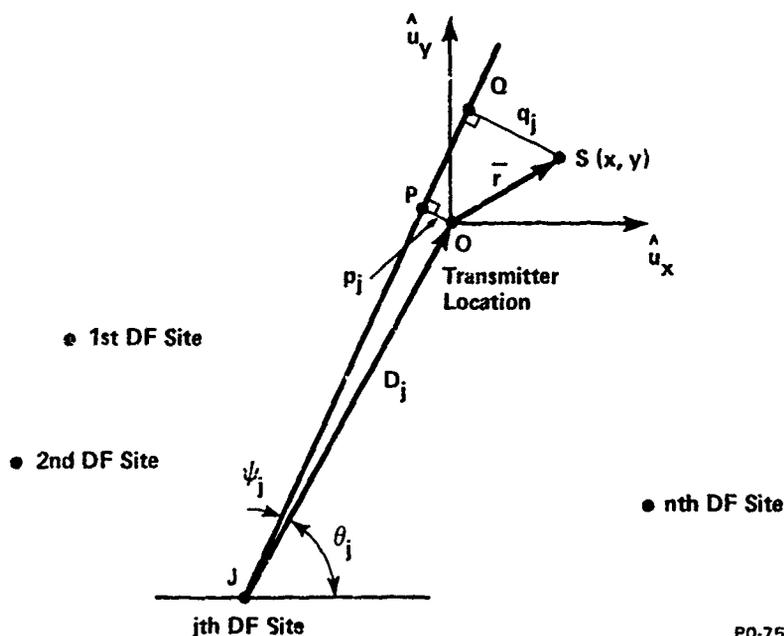
SECTION 2

STANSFIELD'S THEORY

2.1 THE TWO-DIMENSIONAL DF ENCOUNTER

Figure 2-1 illustrates the general distribution of the participants and identifies the parameters used to characterize the two-dimensional DF encounter. The target transmitter whose location is to be determined and all of the DF sites attempting to measure the angles of BLs to the transmitter are considered to be in the same geometrical plane. Points in this "encounter plane" are labeled via a fixed cartesian coordinate system with an origin  $O$  at the position of the target transmitter.

As mentioned, each DF site employs some form of directional antenna system in order to derive the angle of a BL from the DF site toward the transmitter location. The situation for the  $j$ 'th DF site is shown in detail in Figure 2-1. The BL to the actual transmitter location is the line  $\overline{JO}$  with a BA of  $\theta_j$ , and length  $D_j$ . However, due to instrumental, propagational, and operator errors, the measured BL is along the line  $\overline{JP}$  with a BA of  $\theta_j + \psi_j$ , where  $\psi_j$  is the BA error.



P0-75

Figure 2-1 Two-Dimensional Radio DF Geometry and Parameters

A fundamental assumption made by Stansfield is that the BA error  $\psi_j$  is the value of a normally distributed random variable  $\Psi_j$ , where

$$P(\psi_j \leq \Psi_j \leq \psi_j + d\psi_j) = p(\psi_j) d\psi_j = \frac{1}{\sigma_j(2\pi)^{1/2}} \exp\left\{\frac{-\psi_j^2}{2\sigma_j^2}\right\} d\psi_j \quad (1)$$

A similar assumption will be made for the azimuth and elevation BA errors when three-dimensional geometries are considered.

Rather than use  $\psi_j$  as the basic indication of the amount by which the  $j$ 'th DF site's BL measurement is in error, the length  $p_j$  of the line  $\overline{P\bar{O}}$  is used. The line  $\overline{P\bar{O}}$  is constructed from the transmitter location  $O$  perpendicular to the measured BL along  $\overline{JP}$ . The length  $p_j$  is called the bearing error. Another fundamental assumption made by Stansfield is that the standard deviation of the BA errors  $\sigma_j$  is so small that the region of uncertainty determined by the entire set of measured BLs is small in comparison with the distances  $D_j$ . In this case,  $p_j$  is related to  $\psi_j$  by the approximate relation

$$p_j = D_j \psi_j \quad (2)$$

This means that  $p_j$  is the value of a random variable  $P_j$ , where

$$\begin{aligned} P \left\{ p_j \leq P_j \leq p_j + dp_j \right\} &= P \left\{ \frac{p}{D_j} \leq \Psi_j \leq \frac{p_j}{D_j} + \frac{dp_j}{D_j} \right\} \\ &= p(p_j) dp_j = \frac{1}{\sigma_j (2\pi)^{1/2}} \exp \left[ \frac{-p_j^2}{2\sigma_j^2} \right] dp_j \end{aligned} \quad (3)$$

and where

$$\sigma_j = \sigma_j D_j \quad (4)$$

One of the  $n$  DF sites of Figure 2-1 is also a DF operations center to which all of the other DF sites send reports giving their measured BAs to the target transmitter. Having received the complete set of BA data, the personnel of the DF operations center must somehow use that data to generate an estimate of the location (the coordinates) of the transmitter. Suppose that the operations center personnel hypothesize the transmitter's location to be at the point  $S$ . Relative to the fixed reference system with origin at  $O$ , the line  $\overline{OS}$  defines a position vector  $\bar{r} = x\hat{u}_1 + y\hat{u}_2$ , so that the coordinates of the point  $S$  relative to the reference system are  $(x, y)$ . It must be emphasized here that the DF operations center personnel have no knowledge of the coordinates  $(x, y)$  of the point  $S$  relative to the reference system at  $O$ , for if they did, they could maneuver their hypothetical point  $S$  so as to diminish  $x$  and  $y$  and locate the transmitter with an arbitrarily small error. The coordinate reference system at  $O$  exists for analytical convenience and is not intended to represent any operational system of measurement.

However, when the BL for any DF sight is plotted, the DF operations center personnel can measure the "perceived error." In Figure 2-1, the measured BA of the  $j$ 'th DF site yields a BL that lies along  $\overline{JQ}$ , and the perceived error is the length of the perpendicular line  $\overline{SQ}$ , written as  $q_j$ . The perceived error  $q_j$ , in contradistinction with the actual error  $p_j$ , is the bearing error attributed to the  $j$ 'th DF site by the personnel of the DF operations center as a result of their hypothesis that the transmitter is at point  $S$ . Of course,  $q_j$  and  $p_j$  are further distinguished by the fact that  $q_j$  is a quantity known to the personnel of the DF operations center, while  $p_j$  is not known to them. Indeed, the DF operations center personnel can employ knowledge of the set of perceived errors  $\{q_k\}$ ,  $k = 1, 2, 3 \dots n$  (one for each DF site) to select a "best estimate" or "fix" for the hypothesized location  $S$  of the transmitter.

The set of perceived errors  $\{q_k\}$  accumulated at the DF operations center is considered to be a sample of  $n$  independent, normally distributed random variables  $Q_1, Q_2 \dots Q_n$ . In this case, the probability of observing any specific set of perceived errors  $\{q_k\}$  is given by:

$$P(\{q_k\}) = \frac{\left( \prod_1^n dq_k \right)}{(2\pi)^{n/2} \left( \prod_1^n \sigma_k D_k \right)} \exp \left[ -\frac{1}{2} \left( \sum_1^n \frac{q_k^2}{(\sigma_k D_k)^2} \right) \right] \quad (5)$$

The best estimate for the location of the transmitter is now taken to be the point  $S$  whose associated set of perceived errors  $\{q_k\}$  has the highest likelihood of being observed. As mentioned by Stansfield, this is an application of the statistical sampling principle known as the theorem of maximum likelihood. The expression of equation (5) will be maximized by that set of perceived errors  $\{q_k\}$  that minimizes the argument  $A$  of the exponential, where

$$A = \sum_1^n \frac{q_k^2}{(\sigma_k D_k)^2} \quad (6)$$

If we now define  $\epsilon_k = \left| \frac{q_k}{\sigma_k D_k} \right|$  (the vertical bars signify the absolute value of the ratio between them) as the relative bearing error of the  $k$ 'th DF site, the quantity  $A$  is seen to be the sum of the squares of the relative bearing errors over all the DF sites. Stansfield's maximum likelihood criterion for choosing the fix point  $S$  of the transmitter is seen to be equivalent to selecting the point  $S$  whose associated perceived errors  $\{q_k\}$  satisfy a "least squares" criterion as applied to the relative errors  $\epsilon_k$ .

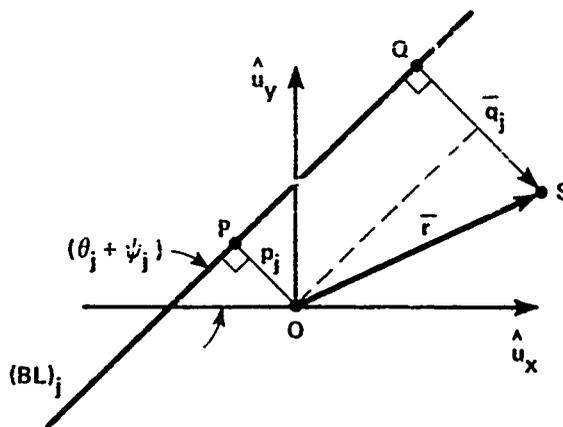
In Figure 2-2, a unit vector  $\hat{q}_j$  (horizontal bars above single letters signify a vector quantity, and a carat above a letter indicates that the quantity being described is a unit vector) is defined along the direction of the line  $QS$  and is given approximately by

$$\hat{q}_j = \sin \theta_j \hat{u}_x - \cos \theta_j \hat{u}_y \quad (7)$$

and reference to the figure yields for  $q_j$  (the length of  $\overline{QS}$ ) the expression

$$q_j = p_j + \bar{r} \cdot \hat{q}_j \quad (8)$$

The "dot" between  $\bar{r}$  and  $\hat{q}_j$  indicates a scalar product of these vectors. Employing equations (7) and (8), equation (6) can be rewritten as



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Figure 2-2 A Magnified View of the Coordinate Origin to Determine  $\bar{q}_j$

$$A = \sum_1^n \frac{(p_j + \lambda \sin \theta_j - y \cos \theta_j)^2}{(\sigma_j D_j)^2} \quad (9)$$

The sum of the squares of the relative errors is now expressed in terms of the actual bearing errors  $p_j$  and the coordinates  $(x, y)$  of the hypothesized transmitter location  $S$ . This is a critical result that allowed Stansfield to find closed form expressions for the fix coordinates  $(x, y)$  for any single attempt at locating the transmitter when the actual bearing errors are given and to describe the statistical features of the pair  $(x, y)$  when the actual bearing errors are characterized as random variables.

## 2.2 THE TWO-DIMENSIONAL DF ANALYTICAL RESULTS

If one uses equation (9) to evaluate  $\left(\frac{\partial A}{\partial x}\right)$  and  $\left(\frac{\partial A}{\partial y}\right)$ , then the equations  $\left(\frac{\partial A}{\partial x}\right) = 0$  and  $\left(\frac{\partial A}{\partial y}\right) = 0$  constitute two equations in the two unknowns  $x$  and  $y$  which can be solved for  $x$  and  $y$ . Solving these equations one finds

$$x = \frac{1}{(\lambda\mu - \nu^2)} \left[ \sum \frac{p_j}{(\sigma_j D_j)^2} (\nu \cos \theta_j - \mu \sin \theta_j) \right] \quad (10)$$

$$y = \frac{1}{(\lambda\mu - \nu^2)} \left[ \sum \frac{p_j}{(\sigma_j D_j)^2} (\lambda \cos \theta_j - \nu \sin \theta_j) \right] \quad (11)$$

where

$$\lambda = \sum \frac{\sin^2 \theta_j}{(\sigma_j D_j)^2} \quad (12)$$

$$\mu = \sum \frac{\cos^2 \theta_j}{(\sigma_j D_j)^2} \quad (13)$$

$$\nu = \sum \frac{\sin \theta_j \cos \theta_j}{(\sigma_j D_j)^2} \quad (14)$$

Although not represented explicitly, the summations are over the index  $j$  for  $j = 1, 2 \dots n$ ; i.e., the summations contain one term associated with each DF site

Equations (10) and (11) are Stansfield's equations for the fix coordinates that would be determined by the personnel of the DF operations center for any single location attempt; wherein the actual bearing errors are the set  $\{p_j\}$ ,  $j = 1, 2 \dots n$ . Other than the set of actual bearing errors, the values for the fix coordinates depend upon the geometrical distribution of the DF sites relative to the actual transmitter location (represented by the parameter sets  $\{\theta_j\}$  and  $\{D_j\}$ ) and the instrumental, propagational, and operator error characteristics (represented by the parameter set  $\{\sigma_j\}$ ).

Notice that if we do not wish to or cannot describe the distribution of the bearing errors as normally distributed about the actual target position, we can use a more general form of equation (6) given by

$$A = \sum_k \frac{q_k^2}{\langle Q_k^2 \rangle} \quad (15)$$

where  $\langle Q_k^2 \rangle$  is the mean squared value of the random variable  $Q_k$  which characterizes the bearing error

of the  $k$ 'th DF site. If  $Q_k$  is other than normally distributed about the actual target position, then, in general,  $\langle Q_k^2 \rangle$  will not be equal to  $\langle \sigma_k D_k \rangle^2$ . In this case, equations (10) through (14) are still useful when  $(\sigma_k D_k)^2$  is replaced by  $\langle Q_k^2 \rangle$ .

If one rewrites equations (10) and (11) as

$$x = \frac{1}{(\lambda\mu - \nu^2)} \left[ \nu \left\langle \sum \frac{p_j \cos \theta_j}{(\sigma_j D_j)^2} \right\rangle - \mu \left\langle \sum \frac{p_j \sin \theta_j}{(\sigma_j D_j)^2} \right\rangle \right] \quad (16)$$

$$y = \frac{1}{(\lambda\mu - \nu^2)} \left[ \lambda \left\langle \sum \frac{p_j \cos \theta_j}{(\sigma_j D_j)^2} \right\rangle - \nu \left\langle \sum \frac{p_j \sin \theta_j}{(\sigma_j D_j)^2} \right\rangle \right] \quad (17)$$

one sees that these equations take the form

$$x = \frac{\nu}{(\lambda\mu - \nu^2)} u - \frac{\mu}{(\lambda\mu - \nu^2)} t \quad (18)$$

$$y = \frac{\lambda}{(\lambda\mu - \nu^2)} u - \frac{\nu}{(\lambda\mu - \nu^2)} t \quad (19)$$

where

$$u = \sum \frac{p_j \cos \theta_j}{(\sigma_j D_j)^2} \quad (20)$$

$$t = \sum \frac{p_j \sin \theta_j}{(\sigma_j D_j)^2} \quad (21)$$

Since  $p_j$  is the value of a random variable  $P_j$ , it is clear that  $u$  and  $t$  are the values of random variables we will designate as  $Y$  and  $\Theta$ , respectively, and thus  $x$  and  $y$  are the values of random values to be designated as  $X$  and  $Y$ , respectively. Equations (18) and (19) mean that our random variables satisfy similar equations; i.e.,

$$X = \frac{\nu}{(\lambda\mu - \nu^2)} Y - \frac{\mu}{(\lambda\mu - \nu^2)} \Theta \quad (22)$$

$$Y = \frac{\lambda}{(\lambda\mu - \nu^2)} Y - \frac{\nu}{(\lambda\mu - \nu^2)} \Theta \quad (23)$$

Recall that, consistent with Stansfield's assumptions, the random variables  $\{ P_j \}$  are all distributed in a similar fashion with zero means. Thus, the central limit theorem can be invoked to contend that the random variables  $Y$  and  $\Theta$  are normally distributed with zero means (Reference 5). This being the case, we also know that  $X$  and  $Y$  are normally distributed (reference 6).

If  $Y$  and  $\Theta$  are normally distributed, then their joint density will be given, in general, by (References 5, 6, and 7)

$$p(u,t) = \frac{\exp\left[\frac{-1}{2(1-R^2)}\left(\frac{u^2}{\sigma_u^2} - \frac{2Rut}{\sigma_u\sigma_t} + \frac{t^2}{\sigma_t^2}\right)\right]}{2\pi\sigma_u\sigma_t(1-R^2)^{1/2}} \quad (24)$$

where  $\sigma_u$  and  $\sigma_t$  are the standard deviations of  $Y$  and  $\Theta$ , respectively, and  $R$  is the correlation coefficient of  $Y$  and  $\Theta$ . Evaluation of  $\sigma_u$ ,  $\sigma_t$ , and  $R$  give

$$\sigma_u = \mu^{1/2} \quad (25)$$

$$\sigma_t = \lambda^{1/2} \quad (26)$$

$$R = \frac{\nu}{(\mu\lambda)^{1/2}} \quad (27)$$

If equations (18) and (19) are inverted, they yield

$$u = \mu y - \nu x \quad (28)$$

$$t = \nu y - \lambda x \quad (29)$$

and substitution of equations (25) through (29) into (24) gives

$$p'(x,y) = \frac{\exp\left[-\frac{1}{2}(\lambda x^2 - 2\nu xy + \mu y^2)\right]}{2\pi(\lambda\mu - \nu^2)^{1/2}} \quad (30)$$

When changing coordinates from the  $(u,t)$  space to the  $(x,y)$  space, we have in general that

$$p(x,y) = p'(x,y)J(u,t/x,y) \quad (31)$$

where  $J(u,t/x,y)$  is the Jacobian determinant for the transformation. In this case,

$$J(u,t/x,y) = \lambda\mu - \nu^2 \quad (32)$$

so with (31) and (32), equation (30) becomes

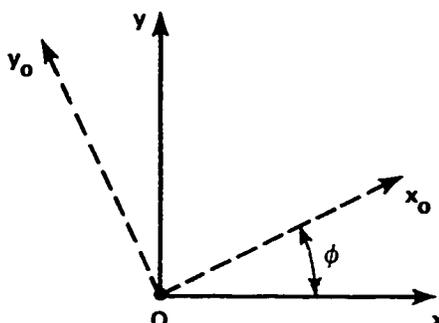
$$p(x,y) = \frac{(\lambda\mu - \nu^2)^{1/2}}{2\pi} \exp\left[-\frac{1}{2}(\lambda x^2 - 2\nu xy + \mu y^2)\right] \quad (33)$$

Following equations (10) and (11), equation (33) constitutes the second significant analytical result of Stansfield's two-dimensional radio DF theory. Equation (33) is the joint probability density of the random variables  $X$  and  $Y$ , the coordinates of the DF position fix for the target transmitter. When multiplied by the differential area of a neighborhood about the point  $(x,y)$ , it determines the likelihood that the DF fix resulting from any given attempt to locate the target transmitter will lie within that differential neighborhood of the point  $(x,y)$ . The quadratic nature of the argument of the exponential and the fact that  $\lambda$  and  $\mu$  are greater than or equal to zero indicates that the contours of constant likelihood are ellipses in the encounter plane centered on the actual target position (the point  $O$  in Figures 2-1 and 2-2). It is, of course, possible to specify a system of coordinates rotated about  $O$  by the angle  $\phi$  relative to the system  $x,y$  in terms of which these elliptical contours can be expressed as a simple sum of squares of the coordinates  $(x_0, y_0)$ . If in Figure 2-3 we suppose that the locus of points which

constitute a given elliptical contour of constant likelihood can be expressed in canonical form as

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = k^2 \quad (34)$$

where  $k^2$  is a positive constant, then use of the coordinate transformation between the  $x_0y_0$  and  $xy$  coordinate systems with equation (34) yields an expression for the locus as expressed in the  $xy$  coordinate system



PO-75

Figure 2-3 The Coordinate System  $x_0y_0$  Rotated About O Relative to the System  $xy$

$$\left( \frac{\cos^2 \phi}{a^2} + \frac{\sin^2 \phi}{b^2} \right) x^2 + 2 \sin \phi \cos \phi \left( \frac{1}{a^2} - \frac{1}{b^2} \right) xy + \left( \frac{\sin^2 \phi}{a^2} + \frac{\cos^2 \phi}{b^2} \right) y^2 = c^2 \quad (35)$$

Comparison with the argument of the exponential in equation (33) yields

$$\lambda = \frac{\cos^2 \phi}{a^2} + \frac{\sin^2 \phi}{b^2} \quad (36)$$

$$\nu = -\sin \phi \cos \phi \left( \frac{1}{a^2} - \frac{1}{b^2} \right) \quad (37)$$

$$\mu = \frac{\sin^2 \phi}{a^2} + \frac{\cos^2 \phi}{b^2} \quad (38)$$

From equations (36) through (38) we find

$$-2\nu = \sin 2\phi \left( \frac{1}{a^2} - \frac{1}{b^2} \right) \quad (39)$$

$$\lambda - \mu = \cos 2\phi \left( \frac{1}{a^2} - \frac{1}{b^2} \right) \quad (40)$$

thus

$$\tan 2\phi = \frac{-2\nu}{\lambda - \mu} \quad (41)$$

This is another of Stansfield's results and allows one to determine the angle  $\phi$   $\left( -\frac{\pi}{4} \leq \phi \leq \frac{\pi}{4} \right)$  which specifies the orientation of the  $x_0, y_0$  axes relative to the  $x, y$  axes.

Use of equations (39) and (40) also gives the result

$$\left( \frac{1}{a^2} - \frac{1}{b^2} \right) = \pm \left[ (\lambda - \mu)^2 + 4\nu^2 \right]^{1/2} \quad (42)$$

and since

$$\lambda + \mu = \frac{1}{a^2} + \frac{1}{b^2} \quad (43)$$

then from equations (42) and (43) we have

$$\frac{2}{a^2} = \lambda + \mu \pm \left[ (\lambda - \mu)^2 + 4\nu^2 \right]^{1/2} \quad (44)$$

$$\frac{2}{b^2} = \lambda + \mu \pm \left[ (\lambda - \mu)^2 + 4\nu^2 \right]^{1/2} \quad (45)$$

where the upper signs apply for  $\left( \frac{1}{a^2} - \frac{1}{b^2} \right) > 0$ , and the lower signs apply for  $\left( \frac{1}{a^2} - \frac{1}{b^2} \right) < 0$ . The sign option arises here because the form (35) results whether the semimajor axes of the ellipse lies along the  $x_0$  axes or the  $y_0$  axes.

Using equations (36) through (38) once again, we find

$$(\lambda\mu - \nu^2)^{1/2} = \frac{1}{ab} \quad (46)$$

so equation (33) can be rewritten for the  $x_0, y_0$  coordinate system as

$$p(x_0, y_0) = \frac{1}{2\pi ab} \exp \left[ -\frac{1}{2} \left( \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} \right) \right] \quad (47)$$

This joint probability density function for the random variables  $X_0$  and  $Y_0$ , when multiplied by a differential area, determines the likelihood that the fix coordinates resulting from any given attempt to locate the target transmitter will fall within a differential neighborhood of the point  $(x_0, y_0)$ .

Notice that using equations (39) and (43) one can derive the ancillary results,

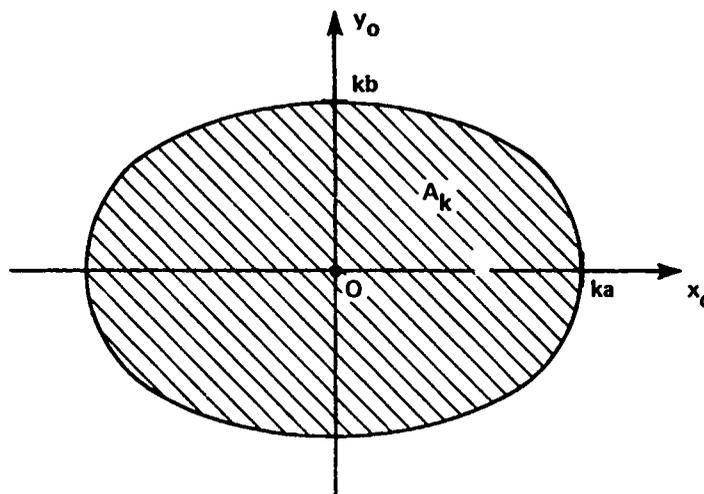
$$\frac{1}{a^2} = \frac{(\lambda + \mu)}{2} - \frac{\nu}{\sin 2\phi} \quad (48)$$

$$\frac{1}{b^2} = \frac{(\lambda + \mu)}{2} + \frac{\nu}{\sin 2\phi} \quad (49)$$

These provide an additional and, perhaps, a more convenient way to calculate  $\frac{1}{a^2}$  and  $\frac{1}{b^2}$  as compared with equations (44) and (45). However, equations (48) and (49) are mentioned here primarily as replacements for Stansfield's equations (15) and (16) which are in error [direct addition of Stansfield's equations (15) and (16) yields the result  $\frac{1}{a^2} + \frac{1}{b^2} = 2(\lambda + \mu)$ , which is contrary to equation (43)].

Consider now Figure 2-4 and the problem of determining the likelihood of the event  $\{x_0', y_0'\} \in A_k$  that the coordinates  $\{x_0', y_0'\}$  of a DF fix will fall within the shaded region  $A_k$ . The locus of points defining the boundary of this elliptical region satisfy the equation

$$\frac{x_0'^2}{a^2} + \frac{y_0'^2}{b^2} = k^2 \quad (50)$$



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Figure 2-4 A Contour of Equal Likelihood Bounding the Region  $A_k$

A family of concentric ellipses about the point  $O$  to which the ellipse of equation (50) belongs is defined by

$$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = K^2, \quad 0 < K < \infty \quad (51)$$

The probability that the fix point  $(x_0', y_0')$  will lie between the elliptical contours defined by  $K$  and  $K + dK$  is, by using equation (47),

$$\int_{K, K+dK} p(x_0, y_0) dx_0 dy_0 = K \left[ \exp \left[ -\frac{K^2}{2} \right] \right] dK \quad (52)$$

Thus, the desired probability can be evaluated by

$$P(x_0', y_0' \in A_k) = \int_0^k K \left[ \exp \left[ -\frac{K^2}{2} \right] \right] dK \quad (53)$$

so

$$P(x_0', y_0' \in A_k) = 1 - \left[ \exp \left[ -\frac{k^2}{2} \right] \right] \quad (54)$$

Since the coordinates of the locus of points defining the boundary of the region  $A_k$  satisfy equation (50), equation (54) can be rewritten as

$$P(x_0', y_0' \in A_k) = 1 - \left[ \exp \left[ -\frac{1}{2} \left( \frac{x_0'^2}{a^2} + \frac{y_0'^2}{b^2} \right) \right] \right] \quad (55)$$

Subtracting 1 from both sides and taking the natural logarithm,  $(\ln)$  of both sides gives another of Stansfield's results

$$\frac{x_0'^2}{a^2} + \frac{y_0'^2}{b^2} = -2 \ln \left[ 1 - P(x_0', y_0' \in A_k) \right] \quad (56)$$

Equation (56) expresses a convenient relationship between the locus of points bounding an elliptical region  $A_k$  and the probability that the coordinates of a given DF fix will fall inside  $A_k$ . It is often desirable to specify a required value of  $P(x_0', y_0' \in A_k)$  and to use (56) to determine the resultant dimensions of the region  $A_k$ . Indeed, the lengths of the semimajor and semiminor axes (or vice versa if  $b > a$ ) are

$$|x_{0m}| = a \left[ \ln \left( \frac{1}{1 - P(x_0', y_0' \in A_k)} \right) \right]^{1/2} \quad (57)$$

$$|y_{0m}| = b \left[ \ln \left( \frac{1}{1 - P(x_0', y_0' \in A_k)} \right) \right]^{1/2} \quad (58)$$

and the area  $\Omega_{2k}$  of the region  $A_k$  is

$$\Omega_{2k} = \pi ab \ln \left( \frac{1}{1 - P(x_0, y_0 \in A_k)} \right)^2 \quad (59)$$

The final consideration we shall give to Stansfield's two-dimensional, radio DF analysis is directed toward reproduction of Stansfield's expression for the root-mean-square error  $\rho_2$  to be expected in the DF position fixes in which the statistics are defined by equation (33) and, equivalently, (47). Given a position fix specified by the coordinates  $(x_0, y_0)$ , the absolute positioning error is defined as

$$e_2 = (x_0^2 + y_0^2)^{1/2} \quad (60)$$

The mean square error is then

$$\rho_2^2 = \left( \frac{1}{2\pi ab} \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_0^2 + y_0^2) \left\{ \exp \left[ -\frac{1}{2} \left( \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} \right) \right] \right\} dx dy \quad (61)$$

Evaluation of the integral by expanding the integrand gives

$$\rho_2^2 = \frac{8(a^2 + b^2)}{\pi} \left( \int_0^{\infty} e^{-u^2} du \right) \left( \int_0^{\infty} u^2 e^{-u^2} du \right) \quad (62)$$

Thus:

$$\rho_2 = (a^2 + b^2)^{1/2} \quad (63)$$

Use of equations (43) and (46) with equation (63) gives

$$\rho_2 = \left( \frac{\lambda + \mu}{\lambda\mu - \nu^2} \right)^{1/2} \quad (64)$$

## SECTION 3

### THE THREE-DIMENSIONAL EXTENSION OF STANSFIELD'S THEORY

#### 3.1 THE THREE-DIMENSIONAL DF ENCOUNTER

##### 3.1.1 INITIAL CONCEPTS

Figure 3-1 illustrates the general distribution of the participants and identifies most of the parameters used to characterize the three-dimensional DF encounter. It should be understood that, in general, the DF sites may be placed anywhere in the three-dimensional space surrounding the actual target position at  $O$ .

As in the two-dimensional case, it is the objective of the network of DF sites to make BL measurements and process these measurements in an operations center to determine a best estimate of the location of the target transmitter. However, contrary to the approach employed in the two-dimensional case, most of the fundamental quantities of interest will be represented as vectors.

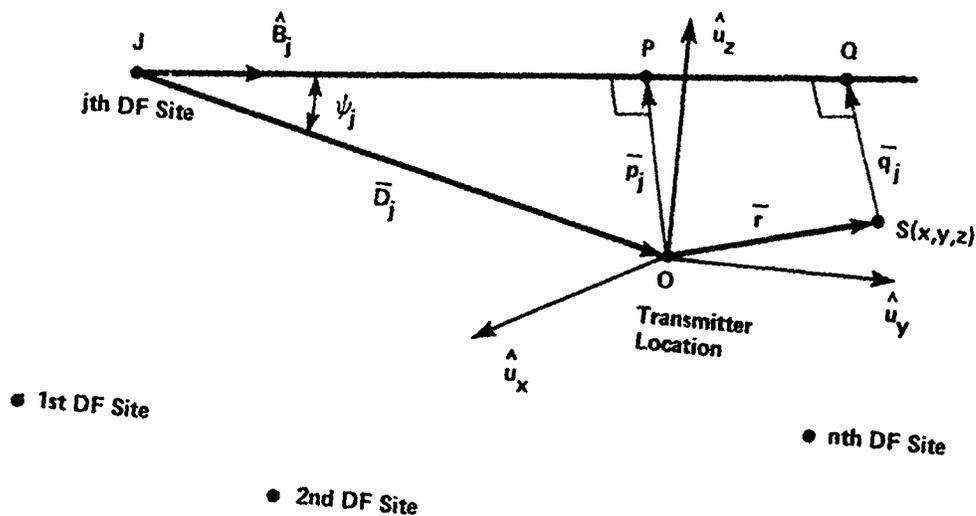
This could have been done in the two-dimensional case, but it was desirable to follow Stansfield's procedure using scalar quantities only in order to facilitate comparison of Section 2 with Stansfield's original paper. In addition to considering the three-dimensional theory, this section will present the most important of Stansfield's results as special cases of the three-dimensional theory applied to a planar space. Where appropriate, these will be presented in vector notation.

The points of the "encounter volume" throughout which the DF sites are distributed are labeled via a fixed, right-handed Cartesian coordinate system whose origin is coincident with the actual position of the target transmitter. It is assumed that each DF site employs some form of directional antenna system from which a unit "bearing vector" ( $\hat{B}$ ) along a line toward the target can be determined. The situation for the  $j$ 'th DF site is shown in detail in Figure 3-1. The BL to the actual transmitter location is the line  $\overline{JO}$  which is represented by the vector  $\overline{D}_j$ . The length of  $\overline{JO}$  is the magnitude of  $\overline{D}_j$ , and is represented by the symbol  $D_j$ . A similar convention will apply for all vector quantities: e.g.,  $p_j$  is the magnitude of the vector  $\overline{p}_j$ , thus  $\overline{p}_j = p_j \hat{p}_j$ .

As in the two-dimensional DF encounter, one of the DF sites in the three-dimensional DF encounter is considered to be an operations center at which the BL data from each DF site are gathered and processed in order to determine the best estimate for the location of the target transmitter. If the point  $S$  with coordinates  $(x, y, z)$  in Figure 3-1 is hypothetically offered as the location of the target transmitter by the personnel of the DF operations center, then the perceived error associated with the target transmitter bearing measurement  $\hat{B}_j$  from the  $j$ 'th DF site is the vector  $\overline{q}_j$ . The actual bearing error in the  $j$ 'th DF site's measurement, which is unknown to the personnel of the DF operations center, is the vector  $\overline{p}_j$ . The vector  $\overline{r}$  is a position vector describing the point  $S$  relative to the fixed coordinate system with its origin at  $O$ , i.e., the components of  $\overline{r}$  are  $(x, y, z)$ . Since it is not necessarily clear from inspection of Figure 3-1, it will be emphasized here that the vectors  $\hat{B}_j$ ,  $\overline{D}_j$ ,  $\overline{p}_j$ ,  $\overline{q}_j$ , and  $\overline{r}$  need not, in general, be coplanar.

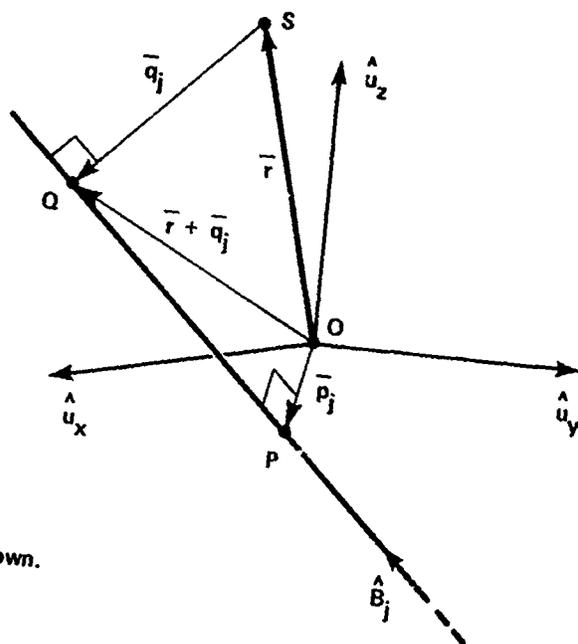
##### 3.1.2 THE CONSTRUCTION OF $\overline{p}_j$ AND $\overline{q}_j$

We embark here upon a digression required in order to lend credence to concepts employed later in the fundamental statistical characterization of the three-dimensional DF encounter. As indicated in Figures 3-1 and 3-2, we have thus far assumed the lines  $\overline{OP}$  and  $\overline{SQ}$  to be constructed so that they are perpendicular to the line along  $\overline{JQ}$ , i.e., the vectors  $\overline{p}_j$  and  $\overline{q}_j$  are orthogonal to the vector  $\hat{B}_j$ . This approach is consistent with the construction of analogous quantities in Stansfield's two-dimensional DF theory and is sensible when attempting a logical extension of that theory to three dimensions. However, the criterion of constructing  $\overline{p}_j$  and  $\overline{q}_j$  orthogonal to the vector  $\hat{B}_j$  is otherwise arbitrary. Indeed, it will soon be convenient to think of  $\overline{p}_j$  as being orthogonal to  $\overline{D}_j$ , and  $\overline{q}_j$  as being orthogonal to  $\overline{D}_j + \overline{r}$  as illustrated in Figure 3-3. It is the objective of this subsection to demonstrate that the assumption that the linear dimensions of the volume of uncertainty associated with the target transmitter's location are small when compared with the distances  $D_j$  (as per Stansfield's assumption in two-dimensions) leads to the same essential analytical conclusions whether we assume  $\overline{p}_j$  and  $\overline{q}_j$  orthogonal to  $\hat{B}_j$ , or that  $\overline{p}_j$  is orthogonal to  $\overline{D}_j$ , and  $\overline{q}_j$  is orthogonal to  $\overline{D}_j + \overline{r}$ .



PO-75

Figure 3-1 Three-Dimensional Radio DF Geometry and Parameters



Note:  
 The vector  $\bar{D}_j$  is  
 intentionally not shown.

PO-75

Figure 3-2 A Magnified View of the Coordinate Origin to Determine  $\bar{q}_j$   
 When  $\bar{q}_j$  and  $\bar{p}_j$  are Orthogonal to  $\hat{\beta}_j$



Now, let us attempt the same thing except let us begin by considering Figure 3-3 and the case where  $\bar{p}_i$  is orthogonal to  $\bar{D}_i$  and  $\bar{q}_i$  is orthogonal to  $\bar{D}_i + \bar{r}$ . From the figure, we see that

$$(\bar{r} + \bar{q}_i) - \bar{p}_i = \alpha \hat{B}_i \quad (69)$$

Taking the scalar product with  $\hat{B}_i$  on both sides gives

$$\alpha = [(\bar{r} + \bar{q}_i) - \bar{p}_i] \cdot \hat{B}_i \quad (70)$$

so equation (69) can be written in the symmetrical form

$$\bar{q}_i - (\bar{q}_i \cdot \hat{B}_i) \hat{B}_i = (\bar{p}_i - \bar{r}) - [(\bar{p}_i - \bar{r}) \cdot \hat{B}_i] \hat{B}_i \quad (71)$$

Equation (71) has a solution of the form

$$\bar{q}_i = \bar{p}_i - \bar{r} + k_i \hat{B}_i \quad (72)$$

where  $k_i$  is an arbitrary constant. However, we also have the condition that  $\bar{q}_i$  is orthogonal to  $\bar{D}_i + \bar{r}$  expressed as

$$(\bar{D}_i + \bar{r}) \cdot \bar{q}_i = 0 \quad (73)$$

Using equation (72) in equation (73) leads to an expression for  $k_i$  given by

$$k_i = \frac{\bar{r} \cdot (\bar{D}_i + \bar{r} - \bar{p}_i)}{(\bar{D}_i + \bar{r}) \cdot \hat{B}_i} \quad (74)$$

Thus, the exact solution for  $\bar{q}_i$  is given by

$$\bar{q}_i = \bar{p}_i - \bar{r} + \left[ \frac{\bar{r} \cdot (\bar{D}_i + \bar{r} - \bar{p}_i)}{(\bar{D}_i + \bar{r}) \cdot \hat{B}_i} \right] \hat{B}_i \quad (75)$$

Comparison of equation (75) with equation (67) gives the impression that the two are distinct which is the case in general. However, we may rewrite equation (74) as

$$k_i = \frac{\bar{r} \cdot \hat{D}_i}{\left[ \hat{D}_i + \left[ \frac{r}{D_i} \right] \hat{r} \right] \cdot \hat{B}_i} \left\{ 1 + \left[ \frac{r}{D_i} \right] \left[ \frac{\hat{r} \cdot \left[ \frac{\hat{r} - \bar{p}_i}{r} \right]}{\hat{r} \cdot \hat{D}_i} \right] \right\} \quad (76)$$

so that when  $\left[ \frac{r}{D_i} \ll 1 \right]$  and  $(\hat{D}_i \approx \hat{B}_i)$ , we see that

$$k_i \approx \bar{r} \cdot \hat{D}_i \quad (77)$$

so that equation (75) becomes

$$\hat{q}_i \approx \bar{p}_i - \bar{r} + \left[ \bar{r} \cdot \hat{D}_i \right] \hat{D}_i \quad (78)$$

which is identical to equation (67).

What we have demonstrated here is that when the linear dimensions of the volume of uncertainty of the transmitter position fix are small relative to the distances  $D_j$ , construction of the vectors  $\bar{p}_i$  and  $\bar{q}_i$  orthogonal to the vectors  $\hat{D}_i$  and  $\hat{D}_i + \bar{r}$ , respectively, yields the same approximate analytical relationship between  $\bar{q}_i$  and  $\bar{p}_i$  as when  $\bar{p}_i$  and  $\bar{q}_i$  are constructed orthogonal to  $\hat{B}_i$ .

Equation (68) plays the same important role in the three-dimensional DF theory as that played by equation (8) in Stansfield's two-dimensional theory. Although equation (78) is written as a function of the vectors  $\bar{p}_i$ ,  $\bar{r}$ , and  $\hat{D}_i$ , it can also be expressed in terms of  $\hat{B}_i$ ,  $\hat{D}_i$ , and  $\bar{r}$ , since  $\bar{p}_i$  is completely specified if  $\hat{B}_i$  and  $\hat{D}_i$  are known. Indeed, when  $\bar{p}_i$  is constructed orthogonal to  $\hat{B}_i$ , we have the relations

$$\hat{p}_i \cdot \hat{B}_i = 0 \quad (79)$$

$$(\hat{B}_i \times \hat{p}_i) \cdot \hat{D}_i = 0 \quad (80)$$

$$(\hat{B}_i \times \hat{p}_i) \cdot (\hat{D}_i \times \hat{B}_i) = \left| \left[ (\hat{D}_i \times \hat{B}_i) \cdot (\hat{D}_i \times \hat{B}_i) \right]^{1/2} \right| = H_i \quad (81)$$

which when solved yield

$$p_{i\alpha} = \frac{-(1 - B_{i\alpha}^2) D_{i\alpha} + (B_{i\alpha} B_{i\alpha}) D_{i\alpha} + (B_{i\alpha} B_{i\alpha}) D_{i\alpha}}{H_i} \quad (82)$$

$$p_{i\beta} = \frac{(B_{i\beta} B_{i\alpha}) D_{i\alpha} - (1 - B_{i\beta}^2) D_{i\beta} + (B_{i\beta} B_{i\alpha}) D_{i\alpha}}{H_i} \quad (83)$$

$$p_{i\gamma} = \frac{(B_{i\gamma} B_{i\alpha}) D_{i\alpha} + (B_{i\gamma} B_{i\alpha}) D_{i\beta} - (1 - B_{i\gamma}^2) D_{i\gamma}}{H_i} \quad (84)$$

where

$$H_i = \left| \left[ (B_{i\alpha} D_{i\alpha} - B_{i\alpha} D_{i\alpha})^2 + (B_{i\beta} D_{i\alpha} - B_{i\alpha} D_{i\beta})^2 + (B_{i\gamma} D_{i\alpha} - B_{i\alpha} D_{i\gamma})^2 \right]^{1/2} \right| \quad (85)$$

and  $(p_{i\alpha}, p_{i\beta}, p_{i\gamma})$  are the components of  $\hat{p}_i$  while  $(B_{i\alpha}, B_{i\beta}, B_{i\gamma})$  and  $(D_{i\alpha}, D_{i\beta}, D_{i\gamma})$  are the components of  $\hat{B}_i$  and  $\hat{D}_i$ , respectively. As before, the vertical bars around terms in equations (81) and (85) indicate that the absolute value of each term is to be used. Thus  $\hat{p}_i$  is known when  $\hat{B}_i$  and  $\hat{D}_i$  are known and, since the magnitude of  $\bar{p}_i$  is given by

$$p_i \approx D_i H_i \quad (86)$$

then the components of  $\bar{p}_i = p_i \hat{p}_i$  are

$$(\bar{p}_j)_x = -(1 - B_{jx}^2) D_{jx} D_j + (B_{jx} B_{jy}) D_{jx} D_j + (B_{jx} B_{jz}) D_{jz} D_j \quad (87)$$

$$(\bar{p}_j)_y = (B_{jx} B_{jy}) D_{jx} D_j - (1 - B_{jy}^2) D_{jy} D_j + (B_{jy} B_{jz}) D_{jz} D_j \quad (88)$$

$$(\bar{p}_j)_z = (B_{jx} B_{jz}) D_{jx} D_j + (B_{jz} B_{jy}) D_{jy} D_j - (1 - B_{jz}^2) D_{jz} D_j \quad (89)$$

Now, when  $\hat{p}_j$  is constructed orthogonal to  $\hat{D}_j$ , we have the relations

$$\hat{p}_j \cdot \hat{D}_j = 0 \quad (90)$$

$$(\hat{p}_j \times \hat{D}_j) \cdot \hat{B}_j = 0 \quad (91)$$

$$(\hat{D}_j \times \hat{p}_j) \cdot (\hat{D}_j \times \hat{B}_j) = \left| \left[ (\hat{D}_j \times \hat{B}_j) \cdot (\hat{D}_j \times \hat{B}_j) \right] \right| = H_j \quad (92)$$

which when solved yield the components of  $\hat{p}_j$  as

$$p_{jx} = \frac{(1 - D_{jx}^2) B_{jx} - (D_{jx} D_{jy}) B_{jy} - (D_{jx} D_{jz}) B_{jz}}{H_j} \quad (93)$$

$$p_{jy} = \frac{-(D_{jx} D_{jy}) B_{jx} + (1 - D_{jy}^2) B_{jy} - (D_{jy} D_{jz}) B_{jz}}{H_j} \quad (94)$$

$$p_{jz} = \frac{-(D_{jx} D_{jz}) B_{jx} - (D_{jz} D_{jy}) B_{jy} + (1 - D_{jz}^2) B_{jz}}{H_j} \quad (95)$$

where, as before,  $H_j$  is given by equation (85). The components of  $\bar{p}_j$  are therefore given by [since equation (86) is still applicable]

$$(\bar{p}_j)_x = (1 - D_{jx}^2) B_{jx} D_j - (D_{jx} D_{jy}) B_{jy} D_j - (D_{jx} D_{jz}) B_{jz} D_j \quad (96)$$

$$(\bar{p}_j)_y = -(D_{jx} D_{jy}) B_{jx} D_j + (1 - D_{jy}^2) B_{jy} D_j - (D_{jy} D_{jz}) B_{jz} D_j \quad (97)$$

$$(\bar{p}_j)_z = -(D_{jx} D_{jz}) B_{jx} D_j - (D_{jz} D_{jy}) B_{jy} D_j + (1 - D_{jz}^2) B_{jz} D_j \quad (98)$$

### 3.1.3 THE FUNDAMENTAL STATISTICAL CHARACTERIZATION OF THE THREE-DIMENSIONAL DF THEORY

In Stansfield's two-dimensional DF theory, the actual bearing errors  $p_i$  were assumed to be the values of random variables  $P_i$  that were normally distributed with zero mean. In the three-dimensional case, our bearing errors are represented by the vectors  $\bar{p}_i$  which vary in some stochastic fashion from one DF location attempt to another. In contrast to the two-dimensional theory wherein only azimuthal bearing errors were considered, we must consider both azimuthal and elevation bearing error components in the three-dimensional theory. In order to facilitate the characterization of this slightly more complicated circumstance, we imagine the construction of a right-handed triad of orthogonal unit vectors at the point  $O$  for each DF site. For each DF site, these unit vectors are labeled  $\hat{\alpha}$ ,  $\hat{\beta}$ , and  $\hat{\gamma}$ ; and the specific triad for the  $j$ 'th DF site is thus composed of the vectors  $\hat{\alpha}_j$ ,  $\hat{\beta}_j$ ,  $\hat{\gamma}_j$ . These vectors define a coordinate system  $(\alpha, \beta, \gamma)_j$  in terms of which the actual bearing error  $\bar{p}_i$  will be resolved for analytic convenience.

Figure 3-4 is representation of the  $(\alpha, \beta, \gamma)$  coordinate axes associated with the  $j$ 'th DF site. The vector  $\hat{\alpha}$ , is collinear with the vector  $\hat{D}_j$  (or  $\bar{D}_j$ ) but opposite in direction. The unit vectors  $\hat{\beta}$ , and  $\hat{\gamma}$ , are constructed so they lie in a plane perpendicular to the vector  $\hat{\alpha}$ , and such that  $\hat{\beta}$ , lies in the  $x$ - $y$  plane while the projection of  $\hat{\gamma}$ , onto  $\hat{u}_z$  is always greater than or equal to zero. Mathematically, these conditions may be expressed by the equations

$$\hat{\alpha}_j \cdot \hat{\gamma}_j = 0 \quad (99)$$

$$(\hat{\alpha}_j \times \hat{\gamma}_j) \cdot \hat{u}_z = 0 \quad (100)$$

$$(\hat{\alpha}_j \times \hat{\gamma}_j) \cdot (\hat{\alpha}_j \times \hat{u}_z) = [\alpha_j^2 + \alpha_n^2] \quad (101)$$

Solving for  $\hat{\gamma}_j$  from these in terms of  $\hat{\alpha}_j$ , and using  $\hat{\beta}_j = (\hat{\gamma}_j \times \hat{\alpha}_j)$  and  $\hat{\alpha}_j = -\hat{D}_j$ , we have

$$\hat{\alpha}_j = -(D_{jx})\hat{u}_x - (D_{jy})\hat{u}_y - (D_{jz})\hat{u}_z \quad (102)$$

$$\hat{\beta}_j = \frac{-(D_{jy})\hat{u}_x + (D_{jx})\hat{u}_y}{(D_{jx}^2 + D_{jy}^2)^{1/2}} \quad (103)$$

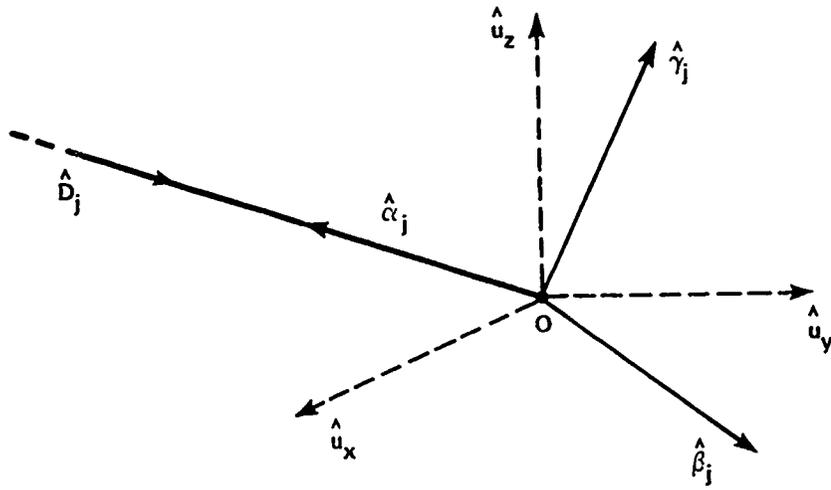
$$\hat{\gamma}_j = \frac{-(D_{jx}D_{jz})\hat{u}_x - (D_{jy}D_{jz})\hat{u}_y + (D_{jx}^2 + D_{jy}^2)\hat{u}_z}{(D_{jx}^2 + D_{jy}^2)^{1/2}} \quad (104)$$

These expressions will be of value in later discussions.

Figure 3-5 shows how the actual bearing error vector  $\hat{p}_i$  can be resolved in the  $(\alpha, \beta, \gamma)_j$  coordinate system. Since we shall choose to construct  $\hat{p}_i$  orthogonal to the vector  $\hat{D}_j$ ,  $\hat{p}_i$  lies in the plane of  $\hat{\gamma}_j$  and  $\hat{\beta}_j$ . A fundamental assumption of the three-dimensional analysis pursued here is that the elevation and azimuth components of  $(\hat{p}_i)_\alpha$ ,  $(\hat{p}_i)_\beta$ , and  $(\hat{p}_i)_\gamma$ , respectively, are the values of normally distributed random variables with zero means and equal variance; i.e.,  $(P_i)_\gamma$  has the probability density

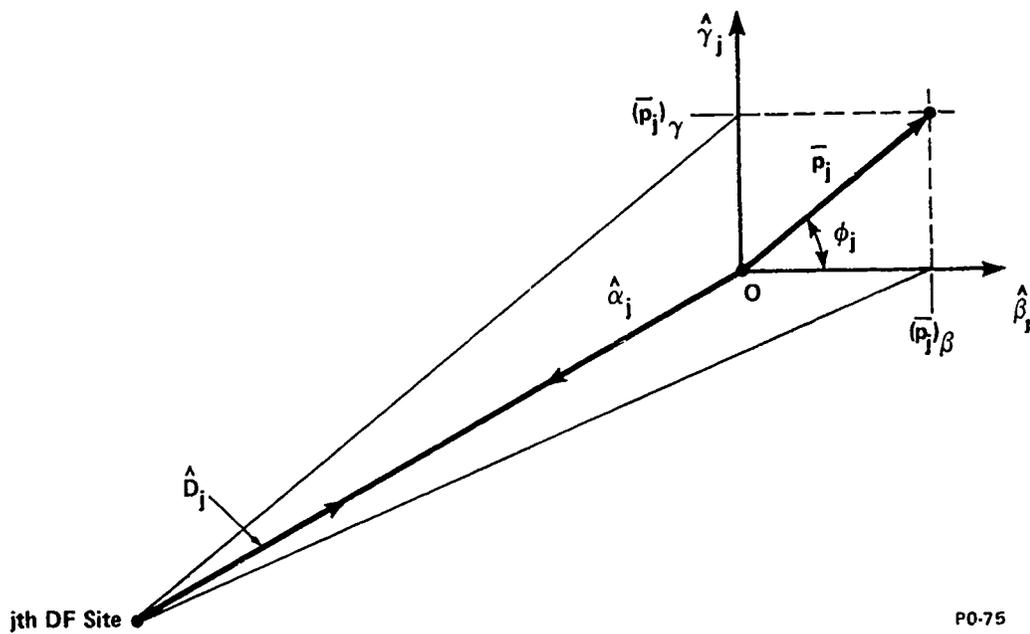
$$p \left[ (\hat{p}_i)_\gamma \right] = \frac{\exp \left[ \frac{-(\hat{p}_i)_\gamma^2}{2\sigma_j^2} \right]}{\sigma_j(2\pi)^{1/2}} \quad (105)$$

and  $(P_i)_\beta$  has the probability density



P0-75

Figure 3-4 The  $(\alpha, \beta, \gamma)$ , Coordinate Axes Relative to the  $(x, y, z)$  Coordinate Axes



P0-75

Figure 3-5 The Resolution  $\bar{p}_j$  in the  $(\beta - \gamma)$ , Plane of the  $(\alpha, \beta, \gamma)$ , Coordinate System

$$p \left[ (\hat{p}_i)_\beta \right] = \frac{\exp \left[ \frac{-(p_i)^2}{2\sigma_J^2} \right]}{\sigma_J (2\pi)^{1/2}} \quad (106)$$

where

$$\sigma_J = (\sigma_i)_\gamma D_i = (\sigma_i)_\beta D_i = \sigma_i D_i \quad (107)$$

Thus, the quantity  $p_i$ , which is the magnitude of the vector  $\bar{p}_i$ , and is given by the expression

$$p_i = \left[ (\hat{p}_i)_\gamma^2 + (\hat{p}_i)_\beta^2 \right]^{1/2} \quad (108)$$

is the value of a random variable  $P_i$  that is Rayleigh distributed (References 7 and 8); i.e.,  $P_i$  has the probability density

$$p(p_i) = \frac{2(p_i) \exp \left[ \frac{-(p_i)^2}{2\sigma_J^2} \right]}{2\sigma_J^2} \quad (109)$$

Also, the angle  $\phi_i$ , is the value of a random variable  $\Phi_i$ , that is uniformly distributed; i.e.,  $\Phi_i$ , has the probability density

$$p(\phi_i) = \left( \frac{1}{2\pi} \right) \quad (110)$$

These results characterize the actual bearing error vector  $\bar{p}_i = p_i \hat{p}_i$  as being composed of two distinct stochastic factors, a magnitude  $p_i$  that is Rayleigh distributed, and a unit vector  $\hat{p}_i$  which assumes random orientations about the line along  $\hat{D}_i$  with uniform likelihood. We now have a statistical representation of our actual bearing error  $\bar{p}_i$ , analogous to the statistical characterization given to Stansfield's actual bearing error by equation (3).

At this point it is possible to characterize the set of perceived bearing errors  $\{\bar{q}_i\}$  resulting from a hypothesized target transmitter location  $S$  as a statistical sample whose likelihood of being witnessed is maximum for the best estimate of the target transmitter's location.

This is a procedure analogous to Stansfield's approach for finding the best estimate of the target transmitter's location. However, we will circumvent this lengthier alternative and, instead, invoke the least squares criterion directly. Thus, we shall stipulate that the personnel of the DF operations center select as their DF fix the point which minimizes the sum of the squares of the relative errors, where the relative error for the  $j$ 'th DF site is given by

$$\epsilon_j = \left| \frac{q_j}{2\sigma_j D_j} \right| \quad (111)$$

analogous to the definition of relative error given for the two-dimensional case. The 2 in the denominator appears because  $Q_j$  will be Rayleigh distributed in the three-dimensional case as opposed to normally distributed in the two-dimensional case. The sum of the squares of the relative errors now has the form

$$A = \sum_j \frac{(\bar{q}_j \cdot \bar{q}_j)}{2(\sigma_j^2 D_j^2)} \quad (112)$$

analogous to that of equation (6).

Now, with the aid equation (68), equation (112) can be written as

$$A = \sum_j \frac{[(\bar{r} - \bar{p}_j) \cdot (\bar{r} - \bar{p}_j) - (\bar{r} \cdot \hat{D}_j)^2]}{2(\sigma_j D_j)^2} \quad (113)$$

or

$$A = \sum_j \frac{[p_j^2 + (x^2 + y^2 + z^2) - 2(xp_{jx} + yp_{jy} + zp_{jz}) - (xD_{jx} + yD_{jy} + zD_{jz})^2]}{2(\sigma_j D_j)^2} \quad (114)$$

Equation (114) is analogous to the result obtained by Stansfield as expressed in equation (9). Just as equation (9) was crucial to Stansfield's further development of the two-dimensional DF theory, so is equation (114) crucial as a prerequisite to the development of the substantial analytical results of the three-dimensional DF theory. Beginning with equation (114), we will derive closed-form expressions for the DF fix coordinates  $(x,y,z)$  for any single attempt to locate the target transmitter, once the set of actual bearing errors is known  $\bar{p}_j$ , and we will describe the statistical features of the triple  $(x,y,z)$  when the actual bearing errors are characterized as random variables.

### 3.2 THE THREE-DIMENSIONAL DF ANALYTICAL RESULTS

#### 3.2.1 EXPRESSION FOR THE DF FIX COORDINATES

We will choose as the best estimate of the target transmitter's location that point  $S$  with coordinates  $(x,y,z)$  that minimizes  $A$  as given in equation (114). In order to find closed form expressions for the coordinates of  $S$ , we evaluate  $\frac{\partial A}{\partial x}$ ,  $\frac{\partial A}{\partial y}$ , and  $\frac{\partial A}{\partial z}$  and set each to zero in order to produce three equations in three unknowns  $x, y$ , and  $z$ . The three resultant equations are

$$\sum_j \left[ \frac{(p_j p_{jx})}{(\sigma_j D_j)^2} \right] = \left\{ x \sum_j \left[ \frac{(1 - D_{jx}^2)}{(\sigma_j D_j)^2} \right] + y \sum_j \left[ \frac{-(D_{jx} D_{jy})}{(\sigma_j D_j)^2} \right] + z \sum_j \left[ \frac{-(D_{jx} D_{jz})}{(\sigma_j D_j)^2} \right] \right\} \quad (115)$$

$$\sum_j \left[ \frac{(p_j p_{jy})}{(\sigma_j D_j)^2} \right] = \left\{ x \sum_j \left[ \frac{-(D_{jx} D_{jy})}{(\sigma_j D_j)^2} \right] + y \sum_j \left[ \frac{(1 - D_{jy}^2)}{(\sigma_j D_j)^2} \right] + z \sum_j \left[ \frac{-(D_{jy} D_{jz})}{(\sigma_j D_j)^2} \right] \right\} \quad (116)$$

$$\sum_j \left[ \frac{(p_j p_{jz})}{(\sigma_j D_j)^2} \right] = \left\{ x \sum_j \left[ \frac{-(D_{jx} D_{jz})}{(\sigma_j D_j)^2} \right] + y \sum_j \left[ \frac{-(D_{jy} D_{jz})}{(\sigma_j D_j)^2} \right] + z \sum_j \left[ \frac{(1 - D_{jz}^2)}{(\sigma_j D_j)^2} \right] \right\} \quad (117)$$

Next we define a set of fundamental parameters as follows:

$$\lambda = \sum_i \left[ \frac{(1 - D_{jx})^2}{(\sigma, D_i)^2} \right] \quad (118)$$

$$\mu = \sum_i \left[ \frac{(1 - D_{jy})^2}{(\sigma, D_i)^2} \right] \quad (119)$$

$$\xi = \sum_i \left[ \frac{(1 - D_{jz})^2}{(\sigma, D_i)^2} \right] \quad (120)$$

$$\nu = \sum_i \left[ \frac{-(D_{jx} D_{jy})}{(\sigma, D_i)^2} \right] \quad (121)$$

$$\eta = \sum_i \left[ \frac{-(D_{jx} D_{jz})}{(\sigma, D_i)^2} \right] \quad (122)$$

$$\zeta = \sum_i \left[ \frac{-(D_{jy} D_{jz})}{(\sigma, D_i)^2} \right] \quad (123)$$

Now, solving equations (115) through (117) for  $x$ ,  $y$  and  $z$  gives

$$x = \frac{(\mu\xi - \zeta^2) \sum_i \left[ \frac{p_j p_{jx}}{(\sigma, D_i)^2} \right] + (\eta\xi - \nu\xi) \sum_i \left[ \frac{p_j p_{jy}}{(\sigma, D_i)^2} \right] + (\nu\xi - \mu\eta) \sum_i \left[ \frac{p_j p_{jz}}{(\sigma, D_i)^2} \right]}{\lambda(\mu\xi - \zeta^2) + \nu(\zeta\eta - \nu\xi) + \eta(\nu\xi - \eta\mu)} \quad (124)$$

$$y = \frac{(\zeta\eta - \nu\xi) \sum_i \left[ \frac{p_j p_{jx}}{(\sigma, D_i)^2} \right] + (\lambda\xi - \eta^2) \sum_i \left[ \frac{p_j p_{jy}}{(\sigma, D_i)^2} \right] + (\eta\nu - \lambda\xi) \sum_i \left[ \frac{p_j p_{jz}}{(\sigma, D_i)^2} \right]}{\lambda(\mu\xi - \zeta^2) + \nu(\zeta\eta - \nu\xi) + \eta(\nu\xi - \eta\mu)} \quad (125)$$

$$z = \frac{(\nu\xi - \eta\mu) \sum_i \left[ \frac{p_j p_{jx}}{(\sigma, D_i)^2} \right] + (\eta\nu - \lambda\xi) \sum_i \left[ \frac{p_j p_{jy}}{(\sigma, D_i)^2} \right] + (\lambda\mu - \nu^2) \sum_i \left[ \frac{p_j p_{jz}}{(\sigma, D_i)^2} \right]}{\lambda(\mu\xi - \zeta^2) + \nu(\zeta\eta - \nu\xi) + \eta(\nu\xi - \eta\mu)} \quad (126)$$

Equations (124) through (126) are the coordinates of the best estimate of the location of the target transmitter as determined by the personnel of the DF operations center for a given set of actual bearing errors  $\{\bar{p}_j = p_j p_{jx} \hat{u}_x + p_j p_{jy} \hat{u}_y + p_j p_{jz} \hat{u}_z\}$ . These are analogous to Stansfield's equations for the fix coordinates given by equations (10) and (11). Indeed, if we apply these equations to the case where all of the DF encounter participants lie in the  $x$ - $y$  plane and we set the elevation error to zero, then  $\eta = \zeta = p_{jz} = 0$ , and equations (124) through (126) become

$$x = \frac{1}{[\lambda\mu - \nu^2]} \sum_j \left[ \frac{p_j}{(\sigma, D_i)^2} \right] \{\mu p_{jx} - \nu p_{jy}\} \quad (127)$$

$$y = \frac{1}{[\lambda\mu - \nu^2]} \sum_j \left[ \frac{p_j}{(\sigma_j D_j)^2} \right] \{\lambda p_{j\lambda} - \nu p_{j\nu}\} \quad (128)$$

$$z = 0$$

Now, as per Stansfield's theory, the vector  $\bar{p}_j$  is given by  $\bar{p}_j = p_j \hat{p}_j$ , where  $p_j$  is the value of a normally distributed random variable  $P_j$  of zero mean (thus it can assume positive and negative values with equal likelihood) and  $\hat{p}_j$  is a fixed unit vector very nearly orthogonal to  $\hat{D}_j$ ; i.e., (see, for instance, Figure 2-1)

$$\hat{p}_j = -\sin \theta_j \hat{u}_\lambda + \cos \theta_j \hat{u}_\nu \quad (129)$$

Equations (127) and (128) can now be written as

$$x = \frac{1}{(\lambda\mu - \nu^2)} \left[ \nu_s \left[ \sum_j \frac{p_j \cos \theta_j}{(\sigma_j D_j)^2} \right] - \mu \left[ \sum_j \frac{p_j \sin \theta_j}{(\sigma_j D_j)^2} \right] \right] \quad (130)$$

$$y = \frac{1}{(\lambda\mu - \nu^2)} \left[ \lambda \left[ \sum_j \frac{p_j \cos \theta_j}{(\sigma_j D_j)^2} \right] - \nu_s \left[ \sum_j \frac{p_j \sin \theta_j}{(\sigma_j D_j)^2} \right] \right] \quad (131)$$

where  $\nu_s = -\nu$ .

Equations (130) and (131) are Stansfield's equations (16) and (17) determined directly from the three-dimensional fix equations (124) through (126). Using equations (93) through (95) for the case when  $B_{j\lambda} = D_{j\lambda} = 0$ , we can express Stansfield's equations entirely in terms of the set of  $D_j$  and the components of  $\hat{B}_j$  and  $\hat{D}_j$ ,

$$x = \frac{1}{(\lambda\mu - \nu^2)} \sum_j \left[ \frac{1}{\sigma_j^2 D_j} \right] \frac{(B_{j\lambda} D_{j\nu} - B_{j\nu} D_{j\lambda})(\mu D_{j\nu} + \nu D_{j\lambda})}{(D_{j\lambda}^2 + D_{j\nu}^2)} \quad (132)$$

$$y = \frac{1}{(\lambda\mu - \nu^2)} \sum_j \left[ \frac{1}{\sigma_j^2 D_j} \right] \frac{(B_{j\nu} D_{j\lambda} - B_{j\lambda} D_{j\nu})(\nu D_{j\nu} + \lambda D_{j\lambda})}{(D_{j\lambda}^2 + D_{j\nu}^2)} \quad (133)$$

which can be expressed in vector notation as

$$x = \frac{1}{(\lambda\mu - \nu^2)} \sum_j \left[ \frac{1}{\sigma_j} \right]^2 \left[ \frac{(\hat{B}_j \times \bar{D}_j) \cdot \hat{u}_z}{\bar{D}_j \cdot \bar{D}_j} \right] \left[ \mu(\hat{u}_\nu \cdot \hat{D}_j) + \nu(\hat{u}_\lambda \cdot \hat{D}_j) \right] \quad (134)$$

$$y = \frac{1}{(\lambda\mu - \nu^2)} \sum_j \left[ \frac{1}{\sigma_j} \right]^2 \left[ \frac{(\bar{D}_j \times \hat{B}_j) \cdot \hat{u}_z}{\bar{D}_j \cdot \bar{D}_j} \right] \left[ \nu(\hat{u}_\nu \cdot \hat{D}_j) + \lambda(\hat{u}_\lambda \cdot \hat{D}_j) \right] \quad (135)$$

### 3.2.2 DF FIX COORDINATE STATISTICS

Let us define the variables  $r_1$ ,  $r_2$ , and  $r_3$  by the equations

$$r_1 = \sum_j \left[ \frac{p_j p_{j1}}{(\sigma_j D_j)^2} \right] \quad (136)$$

$$r_2 = \sum_j \left[ \frac{p_j p_{j2}}{(\sigma_j D_j)^2} \right] \quad (137)$$

$$r_3 = \sum_j \left[ \frac{p_j p_{j3}}{(\sigma_j D_j)^2} \right] \quad (138)$$

We have changed the subscripts  $x$  to 1,  $y$  to 2, and  $z$  to 3 for computational and notational convenience; i.e.,  $p_{jx} = p_{j1}$ ,  $p_{jy} = p_{j2}$ ,  $p_{jz} = p_{j3}$ .

Since  $p_j$ ,  $p_{j1}$ ,  $p_{j2}$ , and  $p_{j3}$  are all the values of random variables, where  $P_j$  is Rayleigh distributed as per equation (109) and  $p_{j1}$ ,  $p_{j2}$ , and  $p_{j3}$  are all distributed alike (we will see exactly how they are distributed later), then the quantities  $r_1$ ,  $r_2$ , and  $r_3$  are the values of random variables we will denote as  $R_1$ ,  $R_2$ , and  $R_3$ . Note that the random variables  $V_1 = P_j p_{j1}$ ,  $V_2 = P_j p_{j2}$ , and  $V_3 = P_j p_{j3}$  are all similarly distributed. Inspection of equations (124) through (126) reveals that the fix coordinates  $x$ ,  $y$ , and  $z$ , are linear functions of  $r_1$ ,  $r_2$ , and  $r_3$  and thus the values of random variables we will denote as  $X$ ,  $Y$ , and  $Z$ . It is our objective in this section to determine the joint probability density for the random variables  $X$ ,  $Y$ , and  $Z$  whose values are the fix coordinates for any given attempt to locate the target transmitter.

We will begin by assuming that the number of DF sites  $n$  is large enough that, by virtue of the central limit theorem (Reference 5), the random variables  $R_1$ ,  $R_2$ , and  $R_3$  are normally distributed. The joint probability density for  $R_1$ ,  $R_2$ , and  $R_3$  is then given by (References 6 and 9)

$$\rho(r_1, r_2, r_3) = \frac{\exp \left[ -\frac{1}{2} \{ [R]^T [C^{-1}] [R] \} \right]}{(2\pi)^{3/2} (\text{Det} [C])^{1/2}} \quad (139)$$

where  $[C]$  is the covariance matrix,  $[C^{-1}]$  the inverse of the covariance matrix,  $[R]$  is the deviation matrix, and  $[R]^T$  is the transpose of the matrix  $[R]$ . The determinant of the matrix  $[C]$  is written as  $\text{Det} [C]$ .

The elements of  $[C]$  are given by

$$C_{mn} = \text{Cov} (R_n, R_m) \quad (140)$$

where  $n = 1, 2, \text{ or } 3$  and  $m = 1, 2, \text{ or } 3$ ; i.e.,  $C_{mn}$  is in the  $m$ th row and  $n$ th column of the 3-by-3 matrix  $[C]$ . The matrix  $[R]$  is a column vector given by

$$[R] = \begin{bmatrix} r_1 - \langle R_1 \rangle \\ r_2 - \langle R_2 \rangle \\ r_3 - \langle R_3 \rangle \end{bmatrix} \quad (141)$$

where  $\langle R_1 \rangle$ ,  $\langle R_2 \rangle$ , and  $\langle R_3 \rangle$  are the mean values of the random variables  $R_1$ ,  $R_2$ , and  $R_3$ , respectively. To determine the explicit form of the joint density of equation (139), one must find  $\langle R_1 \rangle$ ,  $\langle R_2 \rangle$ , and  $\langle R_3 \rangle$  and the covariance elements  $C_{mn}$ ,  $n = 1, 2, 3$  and  $m = 1, 2, 3$ .

From the definitions of the mean and covariance

$$\langle R_n \rangle = \sum_i \left[ \frac{\langle P_i P_m \rangle}{(\sigma_i D_i)^2} \right], \quad n=1,2,3 \quad (142)$$

and

$$C_{mm} = \sum_k \sum_i \frac{[\langle P_i P_k P_m P_{kn} \rangle - \langle P_i P_m \rangle \langle P_k P_{kn} \rangle]}{[\sigma_i \sigma_k D_i D_k]^2}, \quad n \text{ and } m=1,2,3 \quad (143)$$

It now becomes apparent that we must know the probability density for each of the random variables in equations (142) and (143). We already know that  $P_i$  is Rayleigh distributed as per equation (109), and we will now determine the probability density for each random variable  $P_m$ ,  $n = 1, 2, 3$ , whose values are the components of the unit vector  $\hat{p}_i$ . Before proceeding, however, we shall impose the assumption that  $P_i$  and  $P_m$ ,  $n = 1, 2, 3$ , are statistically independent quantities; i.e., we suppose that the magnitude of  $\hat{p}_i$  and its direction are independent. We further suppose that the measurements at one DF site are independent of those at another DF site. Equations (142) and (143) can now be written as

$$\langle R_n \rangle = \sum_i \left[ \frac{\langle P_i \rangle \langle P_m \rangle}{(\sigma_i D_i)^2} \right], \quad n \text{ and } m=1,2,3 \quad (144)$$

$$C_{mm} = \sum_i \frac{[\langle P_i^2 \rangle \langle P_i \rangle \langle P_m \rangle - \langle P_i \rangle^2 \langle P_m \rangle \langle P_m \rangle]}{(\sigma_i D_i)^4}, \quad n \text{ and } m=1,2,3 \quad (145)$$

Now, notice from Figure 3-5 that  $\hat{p}_i$  can be written as

$$\hat{p}_i = \cos \phi_i \hat{\beta}_i + \sin \phi_i \hat{\gamma}_i \quad (146)$$

and, since  $\hat{\beta}_i$  and  $\hat{\gamma}_i$  are given by equations (103) and (104) respectively, equation (146) can be written as

$$\hat{p}_i = \frac{1}{(D_{i1}^2 + D_{i2}^2)^{1/2}} \left[ (D_{i2} \cos \phi_i - D_{i1} D_{i3} \sin \phi_i) \hat{u}_1 \right. \\ \left. - (D_{i1} \cos \phi_i + D_{i2} D_{i3} \sin \phi_i) \hat{u}_2 + (D_{i1}^2 + D_{i2}^2) \sin \phi_i \hat{u}_3 \right] \quad (147)$$

from which we see

$$p_{i1} = \frac{(D_{i2} \cos \phi_i - D_{i1} D_{i3} \sin \phi_i)}{(D_{i1}^2 + D_{i2}^2)^{1/2}} \quad (148)$$

$$p_{i2} = \frac{-(D_{i1} \cos \phi_i + D_{i2} D_{i3} \sin \phi_i)}{(D_{i1}^2 + D_{i2}^2)^{1/2}} \quad (149)$$

$$p_{i3} = (D_{i1}^2 + D_{i2}^2)^{1/2} \sin \phi_i \quad (150)$$

Each of the components of  $\hat{p}_i$  varies in a random manner because the angle  $\phi_i$  is the value of a

uniformly distributed random variable  $\Phi$ , [see equation (110)]. The statistical nature of the random variables  $P_{j1}$ ,  $P_{j2}$ , and  $P_{j3}$  is more readily recognized if equations (148) and (149) are written as

$$p_{j1} = \left\{ \frac{D_{j2}^2 + D_{j1}^2 D_{j3}^2}{D_{j1}^2 + D_{j2}^2} \right\}^{1/2} \sin [\phi_j + \delta_1] \quad (151)$$

$$p_{j2} = \left\{ \frac{D_{j1}^2 + D_{j2}^2 D_{j3}^2}{D_{j1}^2 + D_{j2}^2} \right\}^{1/2} \sin [\phi_j + \delta_2] \quad (152)$$

where

$$\delta_1 = \tan^{-1} \left[ \frac{(-D_{j1} D_{j3})}{D_{j2}} \right] \quad (153)$$

$$\delta_2 = \tan^{-1} \left[ \frac{D_{j2} D_{j3}}{D_{j1}} \right] \quad (154)$$

Thus, the random variables  $P_{jn}$ ,  $n = 1, 2, 3$ , have probability densities of the form (References 7 and 8)

$$p(p_{jn}) = \frac{1}{\pi (F_{jn}^2 - p_{jn}^2)^{1/2}} \quad (155)$$

where  $F_{jn}$  is the coefficient of the sine factor of  $p_{jn}$  in the forms of equations (150) through (152); e.g., for  $p_{j1}$ ,  $F_{j1} = \left\{ \frac{D_{j2}^2 + D_{j1}^2 D_{j3}^2}{D_{j1}^2 + D_{j2}^2} \right\}^{1/2}$ . Thus,  $|p_{jn}| \leq F_{jn}$ . With equation (155) we are able to evaluate  $\langle P_{jn} \rangle$ ; i.e.,

$$\langle P_{jn} \rangle = \int_{-F_{jn}}^{F_{jn}} p_{jn} p(p_{jn}) dp_{jn} = \frac{1}{\pi} \int_{-F_{jn}}^{F_{jn}} \frac{p_{jn} dp_{jn}}{(F_{jn}^2 - p_{jn}^2)^{1/2}} = 0 \quad (156)$$

a result that follows from the fact that the integrand in the second integral is an odd function of  $p_{jn}$ . Equations (144) and (145) now reduce to

$$\langle R_n \rangle = 0 \quad (157)$$

$$C_{mn} = \sum_i \frac{\langle P_i^2 \rangle \langle P_m P_{jm} \rangle}{(\sigma_i D_i)^4} \quad (158)$$

From equation (109) it follows that

$$\langle P_i^2 \rangle = 2(\sigma_i D_i)^2 \quad (159)$$

thus we are left with evaluating  $\langle P_m P_{jm} \rangle$ . In order to do this, we first express  $p_{jn}$  as

$$p_{jn} = a_{jn} \cos \phi_j + b_{jn} \sin \phi_j \quad (160)$$

a simplified representation of equations (148) through (150), where

$$a_{j1} = \frac{D_{j2}}{(D_{j1}^2 + D_{j2}^2)^{1/2}} \quad (161)$$

$$a_{j2} = \frac{-D_{j1}}{(D_{j1}^2 + D_{j2}^2)^{1/2}} \quad (162)$$

$$a_{j3} = 0 \quad (163)$$

$$b_{j1} = \frac{-D_{j1}D_{j3}}{(D_{j1}^2 + D_{j2}^2)^{1/2}} \quad (164)$$

$$b_{j2} = \frac{-D_{j2}D_{j3}}{(D_{j1}^2 + D_{j2}^2)^{1/2}} \quad (165)$$

$$b_{j3} = (D_{j1}^2 + D_{j2}^2)^{1/2} \quad (166)$$

In general, then,

$$p_{jn}p_{jm} = a_{jn}a_{jm} \cos^2 \phi_j + (a_{jn}b_{jm} + a_{jm}b_{jn}) \cos \phi_j \sin \phi_j + b_{jn}b_{jm} \sin^2 \phi_j \quad (167)$$

thus use of equation (110) gives

$$\langle P_{jn}P_{jm} \rangle = \int_0^{2\pi} p_{jn}(\phi_j) p_{jm}(\phi_j) p[\phi_j] d\phi_j = \frac{1}{2\pi} \int_0^{2\pi} p_{jn}(\phi_j) p_{jm}(\phi_j) d\phi_j \quad (168)$$

Therefore, use of equation (167) in equation (168) yields

$$\langle P_{jn}P_{jm} \rangle = \frac{1}{2\pi} \left\{ a_{jn}a_{jm} \int_0^{2\pi} \cos^2 \phi_j d\phi_j + (a_{jn}b_{jm} + a_{jm}b_{jn}) \int_0^{2\pi} \cos \phi_j \sin \phi_j d\phi_j + b_{jn}b_{jm} \int_0^{2\pi} \sin^2 \phi_j d\phi_j \right\} \quad (169)$$

or

$$\langle P_{jn}P_{jm} \rangle = \frac{a_{jn}a_{jm} + b_{jn}b_{jm}}{2} \quad (170)$$

Now, substitution of equations (159) and (170) into equation (158) gives

$$C_{nm} = \sum_i \frac{(a_{in}a_{im} + b_{in}b_{im})}{(\sigma_i D_i)^2} \quad (171)$$

Explicit substitution of the  $a$ 's and  $b$ 's from equations (161) through (166) gives for the covariance elements (upon changing back to the subscripts  $x, y, z$  for 1, 2, 3, respectively)

$$C_{xx} = \sum_i \left[ \frac{(1 - D_{ix}^2)}{(\sigma_i D_i)^2} \right] \quad (172)$$

$$C_{yy} = \sum_i \left[ \frac{(1 - D_{iy}^2)}{(\sigma_i D_i)^2} \right] \quad (173)$$

$$C_{xx} = \sum_i \left[ \frac{(1 - D_i^2)}{(\sigma_i D_i)^2} \right] \quad (174)$$

$$C_{xy} = C_{yx} = \sum_i \left[ \frac{-(D_{ix} D_{iy})}{(\sigma_i D_i)^2} \right] \quad (175)$$

$$C_{xz} = C_{zx} = \sum_i \left[ \frac{-(D_{ix} D_{iz})}{(\sigma_i D_i)^2} \right] \quad (176)$$

$$C_{yz} = C_{zy} = \sum_i \left[ \frac{-(D_{iy} D_{iz})}{(\sigma_i D_i)^2} \right] \quad (177)$$

Comparison of the covariance matrix elements with equations (118) through (123) shows that the coefficients of the fix coordinates  $(x, y, z)$  in equations (115) through (117) are, in fact, these same covariance elements. Also, equation (157) means that the deviation matrix of equation (151) reduces to

$$[R] = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} \quad (178)$$

Thus, if we define the column vector  $[X]$  as

$$[X] = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (179)$$

then equations (115) through (117) can be expressed by the matrix equation

$$[R] = [C][X] \quad (180)$$

This important result provides us with the capability to use equation (139) for the joint probability density of the random variables  $R_x$ ,  $R_y$ , and  $R_z$  in order to derive the joint probability density for the DF fix random variables  $X$ ,  $Y$ ,  $Z$ . Indeed, since  $r_x$ ,  $r_y$ , and  $r_z$  are each functions of the fix coordinates  $(x, y, z)$  as expressed by (180), then the joint probability density for the fix coordinates is given by

$$\rho(x, y, z) = \{\rho(r_x[x, y, z], r_y[x, y, z], r_z[x, y, z])\} \{J(r_x, r_y, r_z; x, y, z)\} \quad (181)$$

where  $J(r_x, r_y, r_z; x, y, z)$  is the Jacobian determinant for the transformation from the coordinates  $(r_x, r_y, r_z)$  to the coordinates  $(x, y, z)$ . Use of equations (115) through (117) yields

$$J(r_x, r_y, r_z; x, y, z) = \text{Det}[C] \quad (182)$$

From equations (139) and (180) we have

$$\rho(r_x[x, y, z], r_y[x, y, z], r_z[x, y, z]) = \frac{\exp\left\{-\frac{1}{2}[X^T][C^T][C^{-1}][C][X]\right\}}{(2\pi)^{3/2}(\text{Det}[C])^{1/2}} \quad (183)$$

or, since  $[C^{-1}][C][X] = [X]$  and the symmetry of  $[C]$  gives  $[C^T] = [C]$ , then

$$\rho(r_1[x,y,z], r_2[x,y,z], r_3[x,y,z]) = \frac{\exp\left\{-\frac{1}{2}[X^T][C][X]\right\}}{(2\pi)^{3/2}(\text{Det}[C])^{1/2}} \quad (184)$$

Use of the parameter definitions in equations (118) through (123) and substitution of equations (182) and (184) into equation (181) gives the three-dimensional joint probability density for the DF fix coordinates as

$$\rho(x,y,z) = \left\{ \frac{\left[ \lambda(\mu\xi - \zeta^2) + \nu(\zeta\eta - \nu\xi) + \eta(\nu\zeta - \eta\mu) \right]^{1/2}}{(2\pi)^{3/2}} \right\} \left\{ \exp\left[ -\frac{1}{2}(\lambda x^2 + \mu y^2 + \xi z^2 + 2\nu xy + 2\eta xz + 2\zeta yz) \right] \right\} \quad (185)$$

After the expressions given for the fix coordinates in equations (124) through (126), equation (185) is the second significant analytical result of the three-dimensional DF theory. When multiplied by the differential volume of the neighborhood about a point  $(x,y,z)$ , the joint density of equation (185) gives the probability that the DF fix resulting from any single attempt to locate the target transmitter will fall within that neighborhood of the point  $(x,y,z)$ . The quadratic nature of the argument of the exponential and the fact that  $\lambda$ ,  $\mu$ , and  $\xi$  are all greater than or equal to zero indicates that the surfaces of constant likelihood are ellipsoids centered on the actual target position (the point  $O$  in Figure 3-1).

Equation (185) is, of course, analogous to that of Stansfield's two-dimensional theory as given by equation (33). Indeed, if one applies the statistical analysis procedure employed to generate equation (185) to the case where all the participants of the DF encounter lie in the  $x$ - $y$  plane and the elevation error  $p_z = 0$  for each DF site, then the matrix  $[R]$  becomes reduced to

$$[R] = \begin{bmatrix} r_x \\ r_y \end{bmatrix} \quad (186)$$

where  $r_x$  and  $r_y$  are given by equations (136) and (137). One also considers that the entire random variation of the bearing errors is embodied in normally varying random magnitudes  $P_j$ , the unit vector components  $p_{jm}$  being deterministically related to the vectors  $D_j$ . The random variables  $R_x$  and  $R_y$  are still normally varying, so the joint probability density for the random variables  $R_x$  and  $R_y$  is given by a modification of equation (139) for the two-dimensional case; i.e.,

$$\rho(r_x, r_y) = \frac{\exp\left\{-\frac{1}{2}([R^T][C^{-1}][R])\right\}}{(2\pi)^{1/2}(\text{Det}[C])^{1/2}} \quad (187)$$

The covariance matrix is now a 2-by-2 matrix whose elements are given by equations (172), (173), and (175), where it must be understood that the vector  $\hat{D}_j$  has only the components  $D_{jx}$  and  $D_{jy}$  in this two-dimensional case. We see then that if we now define

$$[X] = \begin{bmatrix} x \\ y \end{bmatrix} \quad (188)$$

then we have an analogous result in two-dimensions to that given by equation (180) for three-dimensional; i.e.,

$$[R] = [C][X] \quad (189)$$

Substitution of equation (189) into equation (187), use of the parameter definitions from equations (118), (119), and (121), and multiplication of the result by the Jacobian determinant for the variable change from the  $(r_x, r_y)$  space to the  $(x, y)$  space gives

$$\rho(x, y) = \left\{ \frac{(\lambda\mu - \nu^2)^{1/2}}{2\pi} \right\} \left\{ \exp \left[ -\frac{1}{2}(\lambda x^2 + 2\nu xy + \mu y^2) \right] \right\} \quad (190)$$

This is, of course, Stansfield's result and is identical to equation (33) [recall that  $\nu$  as defined by equation (121) is opposite in sign to Stansfield's  $\nu$  as defined by equation (14)].

### 3.2.3 FURTHER CONSIDERATIONS OF THE JOINT DENSITY OF THE DF FIX COORDINATES

Now that it has been shown that the surfaces of constant likelihood for the DF fix of a three-dimensional DF encounter are ellipsoids centered on the actual target transmitter location, it is useful to derive ancillary results which allow us to describe the dimensions and orientation of these ellipsoids relative to our reference coordinate system as well as to compute the probability that a DF fix will fall within the region bounded by these surfaces.

From equations (181), (182), and (184) we have

$$\rho(x, y, z) = \frac{\{\text{Det}[C]\}^{1/2} \exp\{-\frac{1}{2}[X']\{C\}[X]\}}{(2\pi)^{3/2}} \quad (191)$$

which is an alternate form of equation (185) for the joint probability density of the DF fix coordinates. The scalar quantity

$$Q = [X']\{C\}[X] \quad (192)$$

which apart from a factor of  $(-1/2)$  is the argument of the exponential in equation (191), is a quadratic form because of the fact that  $\{C\}$  is a symmetric matrix. Since  $\{C\}$  is also real, it can be diagonalized (Reference [10]) by an appropriate transformation. In particular, there is an orthogonal transformation matrix  $[A]$  which transforms the coordinates  $(x, y, z)$  of a point  $P$  in the encounter space relative to our reference coordinate system to the coordinates  $(x_0, y_0, z_0)$  of the same point  $P$  relative to a coordinate frame whose origin is still at the actual target position but whose axes are, in general, rotated relative to the axes of the reference system. Thus, there is a matrix  $[A]$  such that

$$\{X_0\} = [A]\{X\} \quad (193)$$

where

$$\{X_0\} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \quad (194)$$

Since the transpose of an orthogonal transformation matrix  $[A]$  is also its inverse, then equation (193) implies the results

$$[X'] = \{X_0\}'[A], [X] = [A']\{X_0\} \quad (195)$$

so equation (192) becomes

$$Q = \{X_0\}'[A]\{C\}[A']\{X_0\} \quad (196)$$

where we require

$$[A][C][A^T] = [D] \quad (197)$$

where  $[D]$  is a diagonal matrix. If we denote the matrix element in the  $j$ 'th row and  $k$ 'th column as  $[D]_{jk}$ , then equation (191) becomes

$$\rho(x_0, y_0, z_0) = \frac{\{\text{Det}[C]\}^{1/2} J(x, y, z; x_0, y_0, z_0)}{(2\pi)^{3/2}} \exp \left\{ -\frac{1}{2} \left( x_0^2 [D]_{11} + y_0^2 [D]_{22} + z_0^2 [D]_{33} \right) \right\} \quad (198)$$

since

$$J(x, y, z; x_0, y_0, z_0) = 1 \quad (199)$$

for orthogonal transformations to rotated coordinate systems (with no inversion), then we may write equation (198) as

$$\rho(x_0, y_0, z_0) = \frac{\{\text{Det}[C]\}^{1/2}}{(2\pi)^{3/2}} \exp \left\{ -\frac{1}{2} \left( \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} \right) \right\} \quad (200)$$

where to allow comparison with Stansfield's results we have defined

$$\frac{1}{a^2} = [D]_{11} \quad (201)$$

$$\frac{1}{b^2} = [D]_{22} \quad (202)$$

$$\frac{1}{c^2} = [D]_{33} \quad (203)$$

Using the definitions in equations (201) through (203) and taking the determinant of both sides of equation (197) gives

$$\text{Det}[C] = \left( \frac{1}{a^2 b^2 c^2} \right) \quad (204)$$

so equation (200) becomes

$$\rho(x_0, y_0, z_0) = \frac{\exp \left\{ -\frac{1}{2} \left( \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} \right) \right\}}{(2\pi)^{3/2} (abc)} \quad (205)$$

The result of equation (205) is analogous to that of equation (47) from Stansfield's two-dimensional theory. The joint probability density  $\rho(x_0, y_0, z_0)$  for the random variables  $X_0, Y_0, Z_0$  determines, when multiplied by a differential volume, the likelihood that the DF fix coordinates resulting from any single attempt to locate the target transmitter will fall within a differential neighborhood of the point  $(x_0, y_0, z_0)$ . Thus, it is possible to find a coordinate system with the actual target location at its origin and relative to which the surfaces of constant likelihood are ellipsoids expressed in canonical form; i.e., each of the locus of points for which the argument of the exponential in equation (205) is a constant value  $k^2$  satisfies the equation

$$\frac{\lambda_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = k^2 \quad (206)$$

Equation (206) is analogous to equation (34) discussed in association with Stansfield's elliptic contours of equal likelihood. It was subsequently shown in equations (44) and (45) that the quantities  $\frac{1}{a^2}$  and  $\frac{1}{b^2}$  could be defined in terms of the fundamental parameters  $\lambda$ ,  $\mu$ , and  $\nu$ . In order to accomplish a similar sort of result for the three-dimensional theory, the following identity was considered:

$$[A][C][A^T] - [A]r[1][A^T] = [D] - r[1] \quad (207)$$

where [1] is the identity matrix. The validity of equation (207) follows from equation (197). Using the distributive property of matrix multiplication over addition, equation (207) can be written as

$$[A]([C] - r[1])[A^T] = [D] - r[1] \quad (208)$$

and taking determinants on both sides gives

$$\text{Det}([C] - r[1]) = \prod_{j=1}^3 ([D]_{jj} - r) \quad (209)$$

Equation (209) is the equation for the three-dimensional theory that is analogous to equations (44) and (45) from the two-dimensional theory. Equation (209) shows that the diagonal elements of [D] are the roots of the third-degree polynomial in  $r$  on the left-hand side of the equation. Once the elements  $[D]_{jj}$ ,  $j = 1, 2, 3$ , are found, then equations (201) through (203) yield  $\frac{1}{a^2}$ ,  $\frac{1}{b^2}$ , and  $\frac{1}{c^2}$ . Indeed, when set equal to zero, the left-hand side of equation (209) is the characteristic polynomial of the problem and is expressed in terms of the fundamental parameters of equations (118) through (123) as

$$r^3 + (\mu + \lambda + \xi)r^2 - (\nu^2 + \eta^2 + \zeta^2 - \mu\xi - \mu\lambda - \lambda\xi)r - \lambda(\mu\xi - \zeta^2) + \nu(\zeta\eta - \nu\xi) + \eta(\nu\zeta - \eta\mu) = 0 \quad (210)$$

As mentioned, each of the matrix elements  $[D]_{jj}$  is set equal to one of the roots of the characteristic polynomial in equation (210), one root is associated with only one matrix element at a time. However, when there are distinct roots, there are six unique ways in which the matrix elements  $[D]_{jj}$  can be assigned values. This circumstance merely reflects the fact that there are six unique ways in which the axes of the reference coordinate system can be rotated and aligned with the principal axes of an equal likelihood ellipsoid so as to yield a reference system relative to which the equation of the locus of the ellipsoid in canonical form. For most applications, the ambiguity in the assignment of values to the elements  $[D]_{jj}$  will be of no consequence, although it is important for the analyst to understand how his assignment of these values affects the characterization of his particular problem.

Once the matrix [D] is known, one can also specify the unit vectors that define the axes of the  $(x_0, y_0, z_0)$  coordinate system and thereby determine the orientation of a given family of ellipsoids of constant likelihood. Starting with equation (197), we find

$$[C][A^T] = [A^T][D] \quad (211)$$

If, for a given square matrix [M], we let  $[M]_j$  denote a column vector formed from the  $j$ 'th column of [M], then from equation (211) we have

$$[C][A^T]_{,,} = [C][A^T]_j, = [A^T]_{,j}[D]_{,,} \quad (212)$$

Now, since

$$[A^T]_{,,} = [1][A^T]_j, \quad (213)$$

then equation (213) becomes

$$[C][A^T]_j, = [D]_{,,}[1][A^T]_j, \quad (214)$$

or

$$[C][A^T]_j, = [D]_{,,}[1][A^T]_j, \quad (215)$$

and thus

$$[C] - [D]_{,,}[1] [A^T]_j, = 0 \quad (216)$$

Equation (216) is a matrix representation of three equations for the three unknown  $[A^T]_{1,j}$ ,  $[A^T]_{2,j}$ , and  $[A^T]_{3,j}$ . The equation will have a nontrivial solution only if

$$\text{Det} [C] - [D]_{,,}[1] = 0 \quad (217)$$

Starting again with (197) one can show that

$$[A] [C] - [D]_{,,}[1] [A^T] = [D] - [D]_{,,}[1] \quad (218)$$

where  $[D] - [D]_{,,}[1]$  is a diagonal matrix for which

$$[D] - [D]_{,,}[1]_{,,} = 0 \quad (219)$$

so that taking determinants on both sides of equation (218) yields (217). Thus, non-trivial solutions of equation (216) are assured.

We now have a way of finding each of the columns of the matrix  $[A^T]$ , and thus we have  $[A^T]$  and the orthogonal transformation matrix  $[A]$  (which is the transpose of  $[A^T]$ ). Notice from equation (195) that

$$[X] = [A^T][X_0] \quad (220)$$

A unit vector along the  $x_0$  axis can be written in the  $(x_0, y_0, z_0)$  system as

$$\hat{u}_{x_0} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (221)$$

so the coordinates of this vector in the  $(x, y, z)$  system are given by substitution of equation (221) into equation (220). A similar procedure is used to express  $\hat{u}_{y_0}$  and  $\hat{u}_{z_0}$ , yielding

$$\hat{u}_{x_0} = [A^T]_1^j \quad (222)$$

$$\hat{u}_{y_0} = [A^T]_2^j \quad (223)$$

$$\hat{u}_{z_0} = [A']_j \quad (224)$$

Thus, the axes of the  $(x_0, y_0, z_0)$  coordinate system are known, and, because these are along the principal axes of the ellipsoids of constant likelihood, we know the orientation of these principal axes relative to the axes of the  $(x, y, z)$  coordinate system. Indeed, to specify the orientation of any given ellipsoid of constant likelihood we now need only to determine the points at which the ellipsoid intersects each of the principal axes, and this is easily done with the use of equation (206) since each of the matrix elements  $[D]_{jj}$  is associated with one column vector  $[A']_j$ ; and is related to the coefficients of equation (206) via equations (201) through (203).

We are now in a position to consider the problem of determining the probability of the event  $(x_0', y_0', z_0') \in V_k$  that the coordinates of a DF fix  $(x_0', y_0', z_0')$  will fall within the region  $V_k$  enclosed by the ellipsoid of constant likelihood whose locus is given by

$$\frac{x_0'^2}{a^2} + \frac{y_0'^2}{b^2} + \frac{z_0'^2}{c^2} = k^2 \quad (225)$$

A family of ellipsoids concentric about the point  $O$  of the actual target position is given by the equation

$$\frac{x_0'^2}{a^2} + \frac{y_0'^2}{b^2} + \frac{z_0'^2}{c^2} = K^2, \quad 0 < K < \infty \quad (226)$$

The ellipsoid of equation (225) is one member of this family. The probability that the fix point  $(x_0', y_0', z_0')$  lies between the ellipsoidal surfaces defined by,  $K$  and  $K + dK$  is, using equation (205),

$$\int_{K, K+dK} \rho(x_0, y_0, z_0) dx_0 dy_0 dz_0 = \left(\frac{2}{\pi}\right)^{3/2} K^2 \exp\left[-\frac{K^2}{2}\right] dK \quad (227)$$

Thus, the desired probability can be evaluated by

$$P(x_0', y_0', z_0' \in V_k) = \frac{4}{(\pi)^{3/2}} \int_0^{k/(2)^{1/2}} u^2 \exp[-u^2] du \quad (228)$$

or

$$P(x_0', y_0', z_0' \in V_k) = \operatorname{erf}\left[\frac{k}{(2)^{1/2}}\right] - \left(\frac{2}{\pi}\right)^{1/2} k \exp\left[-\frac{k^2}{2}\right] \quad (229)$$

Equation (229) is analogous to equation (54), derived for the two-dimensional DF encounter. We cannot solve equation (229) explicitly for  $k$  in terms of  $P(x_0', y_0', z_0' \in V_k)$  as was possible for equation (54) in two dimensions. However, we can still derive results in the three-dimensional case that are equivalent to equations (57) and (58) which give the principal axes intercepts in the two-dimensional case. Indeed, if one plots the function  $P(x_0', y_0', z_0' \in V_k)$  as given by equation (229) on "probability paper" (see Figure 3-6), then one can graphically invert the equation to solve for  $k$  given a desired value of  $P(x_0', y_0', z_0' \in V_k)$ . Thus, if one wishes to determine the principal axis intercepts of an ellipsoid within which the DF fix point has a probability  $\gamma$  of falling, one enters the axis of  $P(x_0', y_0', z_0' \in V_k)$  at  $\gamma$

and traces back to find the associated value of  $k, k_\gamma$ . This procedure is illustrated in Figure 3-6. The principal axes intercepts of the ellipsoid within which the DF fix has probability  $\gamma$  of falling are then

$$x_{0m} = \pm k_\gamma a \tag{230}$$

$$y_{0m} = \pm k_\gamma b \tag{231}$$

$$z_{0m} = \pm k_\gamma c \tag{232}$$

$$P(x'_0, y'_0, z'_0 \in V_k) = \text{erf} \left[ \frac{k}{(2)^{1/2}} \right] - \left( \frac{2}{\pi} \right)^{1/2} k \exp \left[ -\frac{k^2}{2} \right].$$

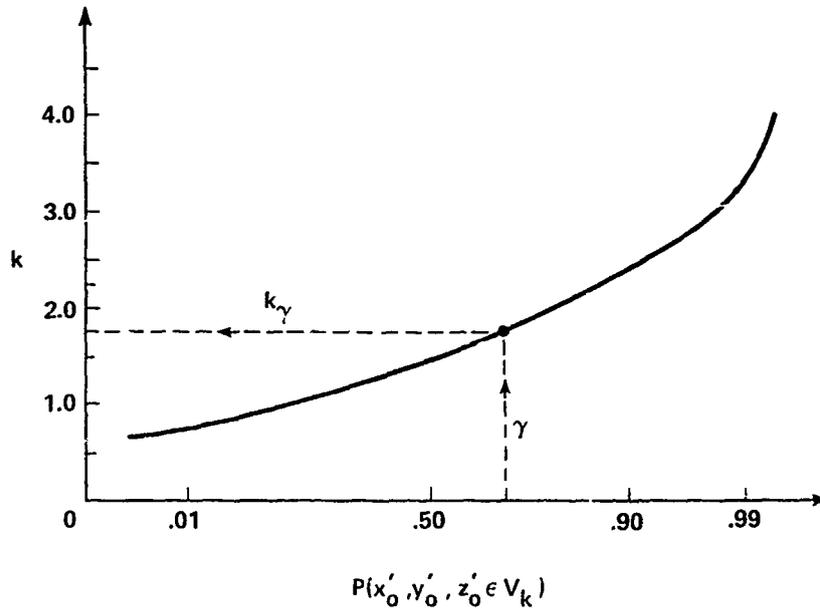


Figure 3-6 Graphical Inversion of  $P(x'_0, y'_0, z'_0 \in V_k)$

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One can, in fact, reduce computation time for two-dimensional DF problems by graphical inversion of equation (54) instead of using equations (57) and (58) directly.

Of course, in light of equations (230) through (232), the volume  $\Omega_{3_k}$  of the region  $V_k$  contained within the ellipsoid is

$$\Omega_{3_k} = \frac{4\pi}{3} abck_3^3 \quad (233)$$

If, when given a DF fix specified by the coordinates  $(x_0, y_0, z_0)$ , we define the absolute positioning error as

$$e_3 = (x_0^2 + y_0^2 + z_0^2), \quad (234)$$

then the mean square error is given by

$$\rho_3^2 = \frac{1}{(2\pi)^{3/2} abc} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x_0^2 + y_0^2 + z_0^2) \exp\left\{-\frac{1}{2}\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2}\right)\right\} dx_0 dy_0 dz_0 \quad (235)$$

Evaluation of this integral by expanding the integrand gives

$$\begin{aligned} \rho_3^2 = & \frac{\delta}{(2\pi)^{3/2} abc} \left\{ \left[ 2bc \int_0^{\infty} \exp(-u^2) du \right]^2 \left[ 2^{3/2} a^3 \int_0^{\infty} u^2 \exp(-u^2) du \right] \right. \\ & + \left[ 2ac \int_0^{\infty} \exp(-u^2) du \right]^2 \left[ 2^{3/2} b^3 \int_0^{\infty} u^2 \exp(-u^2) du \right] \\ & \left. + \left[ 2ab \int_0^{\infty} \exp(-u^2) du \right]^2 \left[ 2^{3/2} c^3 \int_0^{\infty} u^2 \exp(-u^2) du \right] \right\} \quad (236) \end{aligned}$$

and thus

$$\rho_3 = (a^2 + b^2 + c^2)^{1/2} \quad (237)$$

This is, of course, the three-dimensional analog of equation (6.3). Using equations (201) through (203) we see that equation (237) can be written as

$$\rho_3 = \left\{ \frac{1}{[D]_{11}^2} + \frac{1}{[D]_{22}^2} + \frac{1}{[D]_{33}^2} \right\} \quad (238)$$

The methods of this section can be applied to the two-dimensional DF encounter as well. In the two-dimensional case the matrix  $[C]$  is a 2-by-2 matrix whose elements are given by equations (172), (173), and (175), where the vector  $\hat{D}_i$  has only the components  $D_{1i}$  and  $D_{2i}$ . The characteristic equation of equation (209) then becomes a second-degree polynomial given by

$$r^2 - (\lambda + \mu)r + (\lambda\mu - \nu^2) = 0 \quad (239)$$

from which we derive

$$\{D\}_{11} = \frac{1}{a^2} = \frac{1}{2} \left\{ (\lambda + \mu) \pm \left[ (\lambda - \mu)^2 + 4\nu^2 \right]^{1/2} \right\} \quad (240)$$

$$\{D\}_{22} = \frac{1}{b^2} = \frac{1}{2} \left\{ (\lambda + \mu) \pm \left[ (\lambda - \mu)^2 + 4\nu^2 \right]^{1/2} \right\} \quad (241)$$

Equations (240) and (241) reproduce the results of equations (44) and (45) as derived by Stansfield. The upper signs in equations (240) and (241) set the unit vector  $\hat{u}_{x_0}$  along the semiminor axes and the unit vector  $\hat{u}_{y_0}$  along the semimajor axes of the given ellipse of constant likelihood. The lower signs place  $\hat{u}_{x_0}$  along the semimajor axes and  $\hat{u}_{y_0}$  along the semiminor axes of the ellipse of constant likelihood. Indeed, using equations (240) and (241) to solve equation (216), and then using equations (222) through (224) gives the unit vectors of the axes of the rotated reference system (relative to which the equations for the ellipses of constant likelihood are in canonical form) as

$$\hat{u}_{x_0} = \frac{-\nu \hat{u}_x + \left[ \frac{1}{2} \left[ (\lambda - \mu) \pm \left[ (\lambda - \mu)^2 + 4\nu^2 \right]^{1/2} \right] \right] \hat{u}_y}{\left\{ \left[ \frac{1}{2} \left[ (\lambda - \mu) \pm \left[ (\lambda - \mu)^2 + 4\nu^2 \right]^{1/2} \right] \right]^2 + \nu^2 \right\}^{1/2}} \quad (242)$$

$$\hat{u}_{y_0} = \frac{-\nu \hat{u}_y + \left[ \frac{1}{2} \left[ (\lambda - \mu) \pm \left[ (\lambda - \mu)^2 + 4\nu^2 \right]^{1/2} \right] \right] \hat{u}_x}{\left\{ \left[ \frac{1}{2} \left[ (\lambda - \mu) \pm \left[ (\lambda - \mu)^2 + 4\nu^2 \right]^{1/2} \right] \right]^2 + \nu^2 \right\}^{1/2}} \quad (243)$$

where the choice of signs has the same effect described for equations (240) and (241).

## REFERENCES

1. R. G. Stansfield, "The Statistical Theory of DF Fixing," *Journal of the IEE*, Volume 94, part IIIa, pp. 762-770, 1947.
2. R. L. Holland, "Persep C<sup>3</sup> Test Objectives Met," JHU/APL FS-78-015 (Preliminary), January 1978.
3. S. E. Brown, "Computer Program for Evaluating the Performance of Airborne DF Systems Against a Stationary Emitter," JHU/APL Memorandum F4B-1-79-007, March 19, 1979.
4. P. J. D. Gething, "Radio Direction Finding," London, Peter Peregrinus Ltd., 1978; ISBN 0-901223-71-9.
5. M. D. Springer, "The Algebra of Random Variables," New York, John Wiley and Sons, 1979; ISBN 0-471-C1406-0.
6. William Feller, "An Introduction to Probability Theory and Its Applications," New York, John Wiley and Sons, 1966; ISBN 0-471-257095.
7. A. Papoulis, "Probability Random Variables, and Stochastic Processes," New York, McGraw-Hill Book Company, 1965; Library of Congress Card Number 64-22956.
8. Peter Beckmann, "Probability in Communications Engineering," New York, Harcourt, Brace and World, Inc., 1967; Library of Congress Card Number 67-12495.
9. A. A. Sveschnikov, "Problems in Probability Theory, Mathematical Statistics, and Theory of Random Variables," New York, Dover Publications, 1968; ISBN 0-486-63717-4, Library of Congress Card Number 78-57171.
10. J. W. Detman, "Mathematical Methods in Physics and Engineering," New York, McGraw-Hill Book Company, 1969 (Second Edition); Library of Congress Card Number 68-284112.

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