AN ALTERNATIVE ESTIMATOR FOR THE MAXIMUM LIKELIHOOD ESTIMATOR $F_{\theta}$

JUN 81 F. SANEJIMA

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AN ALTERNATIVE ESTIMATOR FOR THE MAXIMUM LIKELIHOOD ESTIMATOR FOR THE TWO EXTREME RESPONSE PATTERNS

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- Latent Trait Theory

**ABSTRACT (Continue on reverse side if necessary and identify by black number)**

(Please see reverse side)
In the methods and approaches we have developed for estimating the operating characteristics of the discrete item responses, the maximum likelihood estimate of the examinee based upon the "Old Test" has an important role. When Old Test does not provide us with a sufficient amount of test information for the upper and lower part of the ability interval, however, it is likely that we obtain response patterns which are either all lowest item scores or all highest item scores, and the resultant maximum likelihood estimates are negative or positive infinity. Although such a test is undesirable for us to use as the Old Test, to a certain extent we can salvage the situation by providing some alternative estimator for these two extreme response patterns. Following RR-80-3, "Is Bayesian Estimation Proper for Estimating the Individual's Ability?", in the present paper, such an estimator is proposed and discussed.
AN ALTERNATIVE ESTIMATOR FOR THE MAXIMUM LIKELIHOOD ESTIMATOR FOR THE TWO EXTREME RESPONSE PATTERNS

ABSTRACT

In the methods and approaches we have developed for estimating the operating characteristics of the discrete item response, the maximum likelihood estimate of the examinee based upon the "Old Test" has an important role. When Old Test does not provide us with a sufficient amount of test information for the upper and lower part of the ability interval, however, it is likely that we obtain response patterns which are either all lowest item scores or all highest item scores, and the resultant maximum likelihood estimates are negative or positive infinity. Although such a test is undesirable for us to use as the Old Test, to a certain extent we can salvage the situation by providing some alternative estimator for these two extreme response patterns. Following RR-80-3, "Is Bayesian Estimation Proper for Estimating the Individual's Ability?", in the present paper, such an estimator is proposed and discussed.

The research was conducted at the principal investigator's laboratory, 409 Austin Peay Hall, Department of Psychology, University of Tennessee. Those who worked in the laboratory and helped the author in various ways for this research include Paul S. Changas, Melanie Perkins, Charles McCarter, Eva Curlee, C. I. Bonnie Chen and William J. Waldron.
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I Introduction

In a previous study (Samejima, RR-80-3), it has been observed that Bayesian estimation has a characteristic which contradicts the principle of objectivity of testing, because of its bias caused by the effect of the prior, in estimating the examinee's ability from his or her performance in the test. Thus a fairly widespread belief among psychologists that Bayesian estimation is better than the maximum likelihood estimation because of the additional information, the prior, and because of the fact that it provides us with finite estimates for all the possible response patterns, should be seriously reconsidered and dismissed. In contrast to the Bayesian estimators, the maximum likelihood estimator has the characteristic of asymptotic unbiasedness, and it has been shown (Samejima, 1977a, 1977b, 1977c, and RR-77-1) that this unbiasedness holds as a good approximation even with a relatively short test and with mediocre values of the test information function for the interval of ability, or latent trait, of our interest.

When the test information function of our test assumes reasonably high values throughout the range of ability of our interest, the probability with which an examinee obtains one of the two extreme response patterns, i.e., the set of the lowest item scores and that of the highest item scores, is negligibly small. In such a case, it is almost certain that the maximum likelihood estimator provides us with finite estimates for all the examinees and will not give us any
inconvenience in either research or practice. The fact of the matter is that we should construct and use such tests for both purposes. In practice, however, very little consideration of such nature and theoretical insight has been taken in constructing tests and the subsequent use of the tests. Many researchers and users of tests casually pick up existing tests and use them for varieties of purposes, and blame the maximum likelihood estimation for the fact that it provides us with negative and positive infinities for some examinees as their ability estimates, and turn to the Bayesian estimation simply because it does not produce infinities. Scientific examination reveals, however, that this is nothing but a disguise; the simple fact is that the test itself fails to have enough power to estimate the examinees' ability levels (cf. Samejima, RR-80-3).

We must accept the fact that every test has a finite range of ability for which it can estimate ability levels accurately enough, and avoid the pretense that it can do so outside of that range of ability by the use of such an inadequate information as a prior.

With this basic understanding in mind, a question will arise as to whether there is any way of expanding this range of ability a test has, without turning to any inappropriate information, and without sacrificing our scientific honesty and the objectivity of testing. If this is possible, then it will contribute to our research and practice, since we could use a wider range of existing tests for our purposes with our appropriate selections.
A positive answer to the above question has been given (Samejima, RR-80-3) for the situation in which the number of test items is relatively small, by proposing an alternative pair of estimates for the two extreme response patterns, which will replace negative and positive infinities resulting from the maximum likelihood estimation. The present study is a continuation of the previous study, in which the concept of these alternative estimates is expanded to cover the situation where the test has a larger number of test items.
Comparison of the New Estimate $\theta^*_V$ with Several Other Estimates

Let $\theta$ be ability, or latent trait, which assumes any real number, such that

\[ -\infty < \theta < \infty. \]

Let $g$ (=1,2,...,n) denote an item, and $x_g$ (=0,1,2,...,m) be a graded item response to item $g$. The operating characteristic, $P_{x_g}(\theta)$, of the graded item response, or item score, $x_g$ is defined as the conditional probability, given ability $\theta$, with which the examinee obtains the item score $x_g$ for item $g$. In the normal ogive model, this operating characteristic is defined by

\[ P_{x_g}(\theta) = \frac{1}{(2\pi)^{-1/2}} \int_{-\infty}^{\infty} e^{-u^2/2} \, du, \]

where $a_g$ (>0) is the item discrimination parameter and $b_x$ is the item response difficulty parameter which satisfies

\[ -\infty = b_0 < b_1 < b_2 < \ldots < b_m < b(m+1) = \infty. \]

Let $V$ denote the response pattern, or a vector of $n$ item scores such that

\[ V' = (x_1, x_2, \ldots, x_g, \ldots, x_n). \]
By the assumption of local independence (Lord and Novick, 1968), the operating characteristic of the response pattern, $P_V(0)$, or the conditional probability, given ability $\theta$, with which the examinee obtains the response pattern $V$, is the simple product of the $n$ operating characteristics of the graded item scores, such that

$$P_V(\theta) = \prod_{x \in V(x)} P_x(0).$$

The maximum likelihood estimate, $\hat{\theta}_V$, of ability $\theta$ for the examinee whose response pattern is $V$ is obtained by using this operating characteristic $P_V(\theta)$ as the likelihood function $L_V(\theta)$, or, equivalently, as the solution of $\theta$ for the equation

$$\sum_{x \in V} A_x(\theta) = 0,$$

where $A_x(\theta)$ is the basic function for the item score $x_g$, which is defined by

$$A_x(\theta) = \frac{1}{2} \log P_x(\theta).$$

The item response information function, $I_{x_g}(\theta)$, for the item score $x_g$ is obtained from the basic function, or directly from the operating characteristic, by
\[ I_x(\theta) = -\frac{\partial}{\partial \theta} \log P_x(\theta) = -\frac{\partial}{\partial \theta} \log P_x(\theta), \]

and the item information function, \( I_{\theta}(\theta) \), is defined as the conditional expectation of the response pattern information function, given \( \theta \), such that

\[ I_{\theta}(\theta) = E[I_x(\theta)|\theta] = \sum_{x=0}^{m} I_x(\theta) P_x(\theta). \]

We can write for the response pattern information function, \( I_V(\theta) \), such that

\[ I_V(\theta) = -\frac{\partial^2}{\partial \theta^2} \log P_V(\theta) = \sum_{x \in V} I_x(\theta), \]

and the test information function, \( I(\theta) \), is defined as the conditional expectation of the response pattern information function, given \( \theta \), such that

\[ I(\theta) = \sum_{V} I_V(\theta) P_V(\theta). \]

It can be shown that the test information function, which is defined by (2.11), is also the sum of the \( n \) item information functions, so that we can write

\[ I(\theta) = \sum_{i=1}^{n} I_x(\theta). \]
One of the important and useful characteristics of the maximum likelihood estimate is that, asymptotically, it distributes normally with \( \theta \) and \([I(\theta)]^{-1/2}\) as the two parameters (Samejima, 1975). It has been shown (Samejima, 1975, 1977a, 1977b, 1977c and RR-77-1) that this convergence to the normality of the conditional distribution of the maximum likelihood estimate is fairly fast, and even with a relatively small number of test items and a mediocre amount of test information this asymptotic normality can be used as a good approximation to the conditional distribution of the maximum likelihood estimate, given ability \( \theta \), when the operating characteristic of the item score \( x \) follows the normal ogive model. If the number of items is too small and so is the amount of test information, this approximation will not hold, however.

Let \( V_{\text{min}} \) and \( V_{\text{max}} \) denote the two extreme response patterns, such that

\[
\begin{align*}
V_{\text{min}}' &= (0, 0, 0, \ldots 0) \\
V_{\text{max}}' &= (m_1, m_2, m_3, \ldots, m_g, \ldots, m_n) .
\end{align*}
\]

(2.13)

In such models as the normal ogive model and the logistic model on the graded response level (Samejima, 1969, 1972), the maximum likelihood estimate for the response pattern \( V_{\text{min}} \) is negative infinity, and that for \( V_{\text{max}} \) is positive infinity. In such a situation as described in the preceding paragraph, the probability with which we obtain negative
or positive infinity as the maximum likelihood estimate is no longer negligibly small. The approximate unbiasedness of the maximum likelihood estimate, therefore, cannot be attained in such a situation. The situation will be salvaged if we define a pair of estimates, $\theta^*_{\text{V-min}}$ and $\theta^*_{\text{V-max}}$, such that

$$
\begin{align*}
\theta^*_{\text{V-min}} &= \left[ \frac{1}{2} (\theta^2 - \theta^2) - \sum_{V \not\in \text{V-min}} \hat{\delta}_V \right]_{\hat{\theta} \in \text{V-min}} \\
\theta^*_{\text{V-max}} &= \left[ \frac{1}{2} (\theta^2 - \theta^2) - \sum_{V \not\in \text{V-max}} \hat{\delta}_V \right]_{\hat{\theta} \in \text{V-max}}
\end{align*}
$$

(2.14)

where $\theta$ and $\tilde{\theta}$ are the lower and upper endpoints of an appropriately defined interval of $\theta$, and $\theta^*_c$ is a critical value of $\theta$ below which the operating characteristic $P_{\text{V-max}}(\theta)$ assumes negligibly small values and above which $P_{\text{V-min}}(\theta)$ assumes negligibly small values, and use them as the substitutes for the negative and positive infinities of the maximum likelihood estimate, respectively (cf. Samejima, RR-80-3).

Since every test has only a finite number of items, and the amount of test information is limited (Samejima, RR-79-1), any test is informative
enough in estimating the examinee's ability only when his or her ability lies within a subset of the entire range of ability. This subset may be a single, finite interval of 0, or a set of several intervals, depending upon the combination of test items and their characteristics. In many cases, however, the test information function of a test of our interest assumes high values only for a single, finite interval of ability 0, and, therefore, the subset is a finite interval, as is the case with LIS-U (Indow and Samejima, 1962, 1966), which was used in the previous study (Samejima, RR-80-3) as an example of a short test. In such a situation, the interval, (0, 0) , which was introduced in the preceding paragraph, can be considered as the subset. By virtue of the substitute estimates, \( \theta^*_{V-min} \) and \( \theta^*_{V-max} \), for the negative and positive infinities of the maximum likelihood estimate, this subset, or interval, has been enlarged, and, moreover, we can obtain an approximately unbiased estimate of ability for this range of ability. We define the estimator \( \hat{\theta}^*_V \) such that

\[
\begin{align*}
\hat{\theta}^*_V &= \theta^*_{V-min} & &\text{for } V = V-min \\
\hat{\theta}^*_V &= \theta^*_{V-max} & &\text{for } V = V-max \\
\hat{\theta}^*_V &= \hat{\theta}_V & &\text{otherwise .}
\end{align*}
\]

Hereafter, we shall call this estimator \( \hat{\theta}^*_V \) the modified maximum likelihood estimator.

Table 2-1 presents the discrimination parameter, \( a_g \), and the difficulty parameter, \( b_g \), of each of the seven binary test items of LIS-U, which follows the normal ogive model on the dichotomous
TABLE 2-1

Item Discrimination Parameter, $a_g$, and Item Difficulty Parameter, $b_g$, of Each of the Seven Items of LIS-U.

<table>
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<th>Item g</th>
<th>$a_g$</th>
<th>$b_g$</th>
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<tbody>
<tr>
<td>1</td>
<td>1.031</td>
<td>-0.860</td>
</tr>
<tr>
<td>2</td>
<td>1.695</td>
<td>-0.520</td>
</tr>
<tr>
<td>3</td>
<td>1.020</td>
<td>-0.220</td>
</tr>
<tr>
<td>4</td>
<td>0.800</td>
<td>-0.030</td>
</tr>
<tr>
<td>5</td>
<td>1.111</td>
<td>0.190</td>
</tr>
<tr>
<td>6</td>
<td>1.389</td>
<td>0.470</td>
</tr>
<tr>
<td>7</td>
<td>1.370</td>
<td>0.760</td>
</tr>
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response level, and whose item characteristic function is given by (2.2).
The test information function, $I(\theta)$, and its square root, of LIS-U
are also shown as Figure 2-1, by solid and dotted lines, respectively.
Figure 2-2 presents the operating characteristics of the two extreme
response patterns, $V$-min and $V$-max, of LIS-U by solid and dotted
lines, respectively, and the position of $\theta_c$ by an arrow. This critical
value of $\theta$ is defined as the point at which the product of $P_{V\text{-min}}(\theta)$
and $P_{V\text{-max}}(\theta)$ is maximal. It turned out that $\theta_c = -0.0088$, and
$P_{V\text{-min}}(\theta_c) = 0.0027$ and $P_{V\text{-max}}(\theta_c) = 0.0031$. These values satisfy
the requirement in defining $\theta^*_v$ and $\theta^*_v$ that for $\theta < \theta_c$
$P_{V\text{-max}}(\theta)$ assumes negligibly small values and so does $P_{V\text{-min}}(\theta)$ for
$\theta > \theta_c$ (cf. Samejima, RR-80-3).

The values of $\theta^*_v$ and $\theta^*_v$ have been obtained for eleven
different intervals of $(\theta, \theta')$ in the previous study (Samejima, RR-80-3),
and the regressions of the estimate $\hat{\theta}_v$ on ability $\theta$ have also been
illustrated for these eleven cases. It has been observed that the
approximate unbiasedness of the modified maximum likelihood estimate
$\hat{\theta}_v$ holds better for smaller intervals of $(\theta, \theta')$, while the violation
of the unbiasedness becomes more conspicuous as the interval becomes
larger. Table 2-2 presents the values of $\theta^*_v$ and $\theta^*_v$ obtained
upon each of the eight smallest intervals of $(\theta, \theta')$, with the square
root of the test information function at $\theta = \tilde{\theta}$ and $\theta = \tilde{\theta}$, respectively,
together with the upper bound of the discrepancies between the regression

There is a typographical error in RR-80-3, and on page 84, line 2 from
bottom, "minimal" should be replaced by "maximal."
FIGURE 2-1

Test Information Function (Solid Line) and Its Square Root (Dotted Line) of LIS-U.
Operating Characteristics of the Two Extreme Response Patterns, \((0,0,0,0,0,0)\) (Solid Line) and \((1,1,1,1,1,1)\) (Dotted Line), of LIS-U, and the Position of the Critical Value \(\theta_c\).
TABLE 2-2

$\theta^*_{v\text{-}min}$ and $\theta^*_{v\text{-}max}$ Obtained upon Each of the Eight Smallest Intervals of $(\theta, \bar{\theta})$, the Square Root of the Test Information at Each Endpoint of Each Interval, and the Upper Bound of the Discrepancies between the Regression of $\theta^*_v$ on $\theta$ and $\theta$ Itself, for LIS-U.

<table>
<thead>
<tr>
<th>$\theta, \bar{\theta}$</th>
<th>$\sqrt{I(\theta)}$</th>
<th>$\sqrt{I(\bar{\theta})}$</th>
<th>$\theta^*_{v\text{-}min}$</th>
<th>$\theta^*_{v\text{-}max}$</th>
<th>upper bound of discrepancies</th>
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<td>$\pm1.50$</td>
<td>1.439</td>
<td>1.494</td>
<td>-1.479</td>
<td>1.522</td>
<td>0.20</td>
</tr>
<tr>
<td>$\pm1.75$</td>
<td>1.197</td>
<td>1.248</td>
<td>-1.647</td>
<td>1.656</td>
<td>0.26</td>
</tr>
<tr>
<td>$\pm2.00$</td>
<td>0.982</td>
<td>1.001</td>
<td>-1.793</td>
<td>1.776</td>
<td>0.33</td>
</tr>
<tr>
<td>$\pm2.25$</td>
<td>0.801</td>
<td>0.773</td>
<td>-1.925</td>
<td>1.892</td>
<td>0.42</td>
</tr>
<tr>
<td>$\pm2.50$</td>
<td>0.648</td>
<td>0.574</td>
<td>-2.051</td>
<td>2.008</td>
<td>0.53</td>
</tr>
<tr>
<td>$\pm3.00$</td>
<td>0.407</td>
<td>0.294</td>
<td>-2.295</td>
<td>2.241</td>
<td>0.77</td>
</tr>
<tr>
<td>$\pm3.50$</td>
<td>0.233</td>
<td>0.147</td>
<td>-2.536</td>
<td>2.480</td>
<td>1.02</td>
</tr>
<tr>
<td>$\pm4.00$</td>
<td>0.121</td>
<td>0.075</td>
<td>-2.779</td>
<td>2.723</td>
<td>1.28</td>
</tr>
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of $\hat{\theta}_V^*$ and the true value of $\theta$. We can see in this table that, even for the smallest interval, (-1.50, 1.50), the square root of the test information assumes values as low as 1.44 and 1.49, respectively, at the two endpoints of the interval, and yet the upper bound of the discrepancies is considerably low for the first, say, five intervals. This fact indicates that, in spite of the relatively low amounts of test information of LIS-U, the introduction of $\hat{\theta}_V^{\text{min}}$ and $\hat{\theta}_V^{\text{max}}$ has succeeded in providing us with an approximately unbiased estimator, i.e., the modified maximum likelihood estimator $\hat{\theta}_V^*$, which can be used for a fairly large interval of $\theta$. Since the least finite value of the maximum likelihood estimate is -1.3167 for the response pattern, $(0,0,0,1,0,0,0)$, and the greatest finite value is 1.3028 for $(1,1,1,0,1,1,1)$, the pair of values, -1.479 and 1.522, obtained for $\hat{\theta}_V^{\text{min}}$ and $\hat{\theta}_V^{\text{max}}$ upon the interval, (-1.50, 1.50), sounds reasonable enough. We could expand the interval, however, by using one of the other pairs of estimates, which are larger in absolute values than the above values of $\hat{\theta}_V^{\text{min}}$ and $\hat{\theta}_V^{\text{max}}$, and, with the trade-off of the length of interval against the upper bound of the discrepancies between the regression and the true value of $\theta$, we may conclude that we should use one of the first five intervals of $(\theta, \bar{\theta})$, the largest of which is (-2.50, 2.50).

In the present study, we choose the interval (-2.25, 2.25) for $(\theta, \bar{\theta})$, which provides us with -1.925 and 1.892 for $\hat{\theta}_V^{\text{min}}$ and $\hat{\theta}_V^{\text{max}}$, respectively. As we can see in Table 2-2, this selection assures us that the conditional expectation of our estimate, given ability
\( \theta \), does not differ from the true value of \( \theta \) by more than 0.42, at any point of the interval of \( \theta \). Figure 2-3 presents the regression of \( \theta^* \) on \( \theta \), which is given by

\[
E(\theta^*|\theta) = \sum_{V} \theta^*_V P_V(\theta),
\]

by a solid curve. In the same figure, also presented is the standard error of estimate, which is defined by

\[
\sqrt{\left( \sum_{V} (\theta^*_V - E(\theta^*_V|\theta))^2 P_V(\theta) \right)}^{1/2},
\]

and is plotted, vertically, by dots in both negative and positive directions from the regression. There is a straight line with forty-five degrees from the abscissa of the figure, which indicates the unbiasedness, and, hereafter, we shall call it the unbiasedness line. The reciprocal of the square root of the test information function, which is usually considered as the standard error in the maximum likelihood estimation, is also plotted by dotted lines, vertically, in both negative and positive directions from the unbiasedness line. It is interesting to note that, while the standard error for the maximum likelihood estimate increases as the regression diverts from the center of the interval, the counterpart for the modified maximum likelihood estimate, \( \theta^*_V \), decreases, and the two dotted curves for the latter and the regression itself converge to \( \theta^*_V^{\text{min}} \) and \( \theta^*_V^{\text{max}} \), respectively, as \( \theta \) tends to negative and positive infinites. We can also see that, for almost
Regression of the Modified Maximum Likelihood Estimate $\hat{\theta}_V$ (Solid Curve) Based upon the Interval, $-2.25 \leq \theta \leq 2.25$, on Ability $\theta$, for LIS-U. The Standard Error of Estimate is Plotted by Dots in Both Vertically Positive and Negative Directions from the Regression, As a Function of Ability $\theta$. 
the entire range of the interval, the unbiasedness line lies within
the vertical interval of the standard error for the modified maximum
likelihood estimate, \( \hat{\theta}_V \).

It has been pointed out (Samejima, RR-80-3) that, unlike the
modified maximum likelihood estimate, \( \hat{\theta}_V \), any Bayesian estimate
involves the bias caused by the prior, which contradicts the principle
of the objectivity of testing. Let \( \hat{\phi}_V \) be the Bayes modal estimate
for a specific response pattern \( V \). This estimate is defined as the
value of \( \theta \) at which the function \( B_V(\theta) \), which is given by

\[
B_V(\theta) = f(\theta) P_V(\theta),
\]

assumes the maximal value, where \( f(\theta) \) is the density function of \( \theta \),
or the prior. Figure 2-4 presents the regression of the Bayes modal
estimate \( \hat{\phi}_V \), which is obtained by replacing \( \hat{\theta}_V \) by \( \hat{\phi}_V \) in (2.16),
and the vertical interval similar to the one in Figure 2-3, with the
standard error of estimate obtained by replacing \( \hat{\theta}_V \) by \( \hat{\phi}_V \) in (2.17),
by solid and dotted lines, respectively. Comparison of this figure
with Figure 2-3 reveals that the vertical interval for the Bayes modal
estimate \( \hat{\phi}_V \) contains the unbiasedness line only for a much smaller
interval of \( \theta \), i.e., approximately \((-1.38, 1.41)\), and outside this
interval both the vertical interval and the regression converge quickly
to \( \hat{\phi}_{V-min} = -1.3617 \) and \( \hat{\phi}_{V-max} = 1.3562 \), respectively, as \( \theta \)
tends to negative and positive infinities.
Regression of the Bayes Modal Estimate $\hat{w}$ with the Prior $n(0,1)$ (Solid Curve) on Ability $\theta$, for LIS-8. The Standard Error of Estimate is Plotted by Dots in Both Vertically Positive and Sensitive Directions from the Regression, As a Function of Ability $\theta$. 
We shall observe another estimate and its regression on ability \( \theta \) and the similar vertical interval for the standard error of estimate. This is Bayes estimate, \( \hat{\theta}_V \), with the same prior, \( n(0,1) \), as we used for the Bayes modal estimate, \( \hat{\theta}_V \). This estimator is defined by

\[
(2.19) \quad \hat{\theta}_V = \int_{-\infty}^{\infty} \theta f_V(\theta) \, d\theta,
\]

where \( f_V(\theta) \) is the density function of \( \theta \) for the subgroup of examinees whose response patterns are uniformly \( V \), which is given by

\[
(2.20) \quad f_V(\theta) = f(\theta) P_V(\theta) \left[ \int_{-\infty}^{\infty} f(\theta) P_V(\theta) \, d\theta \right]^{-1}.
\]

Figure 2-5 presents, for the Bayes estimate, \( \hat{\theta}_V \), a set of functions similar to those which we have observed for both the modified maximum likelihood estimate, \( \hat{\theta}_* \), and the Bayes modal estimate, \( \hat{\theta}_V \), in Figures 2-3 and 2-4, respectively. They are the regression obtained by replacing \( \hat{\theta}_* \) by \( \hat{\theta}_V \) in (2.16) and the interval based upon the standard error of estimate obtained by the similar replacement in (2.17). We can see that this set of results is very much like the one we obtained for the Bayes modal estimate with the same prior, \( n(0,1) \), which we have observed in Figure 2-4. The interval of \( \theta \) for which the vertical interval of the standard error of estimate includes the unbiasedness line is approximately \((-1.40, 1.28)\), and outside of this small interval of \( \theta \) the three curves converge quickly to \( \hat{\theta}_{V-min} = -1.3764 \) and \( \hat{\theta}_{V-max} = 1.2695 \), respectively.
Regression of the Bayes Modal Estimate $\hat{\theta}_{IV}$ with the Prior $n(0,1)$ (Solid Curve) on Ability $\theta$, for LIS-1. The Standard Error of Estimate Is Plotted by Dots in Both Vertically Positive and Negative Directions from the Regression, As a Function of Ability $\theta$. 

FIGURE 2-5
The similarity observed in the above two results makes us wonder if we can define an estimator which is population-free, and has an analogous meaning to the maximum likelihood estimator as Bayes estimator has to the Bayes modal estimator. Such an estimator may be a better estimator than the modified maximum likelihood estimator, \( \theta^*_v \), or may not. To find it out, we shall introduce an estimator defined by (2.19) and (2.20), where \( f(\theta) \) is a uniform density function for the interval, \((0, \theta)\). Let \( \mu^{*'}_{1V} \) denote this Bayes estimate with the uniform prior. We can write from (2.19) and (2.20) that

\[
(2.21) \quad \mu^{*'}_{1V} = \int_{0}^{\theta} \theta f_v(\theta) d\theta \{ \int_{0}^{\theta} f_v(\theta) d\theta \}^{-1}.
\]

Note that (2.21) includes only the conditional probability of response pattern \( V \), given ability \( \theta \), and, therefore, is population-free.

Figure 2-6 presents the regression of \( \mu^{*'}_{1V} \) on \( \theta \), which is obtained by replacing \( \theta^*_v \) by \( \mu^{*'}_{1V} \) in (2.16), and the vertical interval of the standard error of estimate, which is obtained by replacing \( \theta^*_v \) by \( \mu^{*'}_{1V} \) in (2.17), by solid and dotted lines, respectively. As was expected, this set of results is similar to the one obtained upon the modified maximum likelihood estimate, \( \theta^*_v \), which is shown as Figure 2-3. A close observation reveals, however, that the interval of \( \theta \) for which the vertical interval of the standard error of estimate includes the unbiasedness line is somewhat smaller, i.e., \((-1.74, 1.76)\), compared with \((-2.08, 2.06)\) for the modified maximum likelihood estimate, although
Regression of the Bayes Estimate $\hat{\theta}$ with the Uniform Prior for $-2.25 \leq \theta \leq 2.25$, on Ability $\theta$, for LIS-U. The Standard Error of Estimate is Plotted by Dots in Both Vertically Positive and Negative Directions from the Regression, as a Function of Ability $\theta$. 

FIGURE 2-6
this interval is substantially larger than the intervals found for the two Bayesian estimates. It is also noted by comparing Figures 2-3 and 2-6 that, for the entire interval of \(0, \bar{\theta}\), the regression of the modified maximum likelihood estimate, \(\hat{\theta}_V^*\), tends to be closer to the unbiasedness line than the regression of \(\hat{\mu}_{1V}^*\). The values of \(\hat{\mu}_{1V}^*\) for the two extreme response patterns, \(V\)-min and \(V\)-max, turned out to be more conservative than those of \(\hat{\theta}_{V\text{-min}}^*\) and \(\hat{\theta}_{V\text{-max}}^*\), i.e., -1.6515 vs. -1.9254 and 1.6430 vs. 1.8923, respectively.

The four sets of estimates for the total one hundred and twenty-eight response patterns of LIS-U are shown in Appendix, as Table A-1. They are also pairwisely plotted in six scatter diagrams, and presented as Figures 2-7 through 2-12. As is expected, the scatter diagram of \(\hat{\mu}_{1V}^*\) plotted against the modified maximum likelihood estimate, \(\hat{\theta}_V^*\), and that of the Bayes modal estimate, \(\hat{\theta}_V\), plotted against the Bayes estimate, \(\hat{\mu}_{1V}\), which are shown as Figures 2-7 and 2-12, respectively, are almost on the line with forty-five degrees from the abscissa, while in the other four combinations of estimates they are consistently and substantially deviated from this line. It is interesting to note that, in Figure 2-7, for all the other one hundred and twenty-six response patterns excluding \(V\)-min and \(V\)-max, the values of \(\hat{\theta}_V^*\) and \(\hat{\mu}_{1V}^*\) are very close to each other, while for these two extreme response patterns the latter are substantially smaller in absolute values than the former.

From these results, we can say that the modified maximum likelihood
FIGURE 2-7
Bayes Estimate with the Uniform Prior, $\mu_{1V}^{*}$, Plotted against the Modified Maximum Likelihood Estimate, $\theta_{V}^{*}$, for the One Hundred and Twenty-eight Possible Response Patterns of LIS-U.
FIGURE 2-8
Bayes Modal Estimate, \( \hat{\theta}_v \), with the Prior \( n(0,1) \), Plotted against the Modified Maximum Likelihood Estimate, \( \theta^*_v \), for the One Hundred and Twenty-eight Possible Response Patterns of LIS-U.
FIGURE 2-9
Bayes Estimate, $\mu_{1V}$, with the Prior $n(0,1)$, Plotted against the Modified Maximum Likelihood Estimate, $\theta_V$, for the One Hundred and Twenty-eight Possible Response Patterns of LIS-U.
**FIGURE 2-10**

Bayes Modal Estimate, $\hat{\theta}_V$, with the Prior $n(0,1)$, Plotted against the Bayes Estimate with the Uniform Prior, $\mu_{1V}$, for the One Hundred and Twenty-eight Possible Response Patterns of LIS-U.
Bayes Estimate, \( \mu_{1V}^* \), with the Prior \( \pi(0,1) \), Plotted against the Bayes Estimate with the Uniform Prior, \( \mu_{1V}^{**} \), for the One Hundred and Twenty-eight Possible Response Patterns of LIS-II.
Bayes Modal Estimate, \( \hat{\theta}_V \), with the Prior \( n(0,1) \), Plotted against the Bayes Estimate, \( \mu'_{1V} \), with the Prior \( n(0,1) \), for the One Hundred and Twenty-eight Possible Response Patterns of LIS-U.
estimator, $\hat{\theta}_V$, provides us with a better approximation to the unbiasedness and is a better estimator than $\mu_1^*$, although the latter is also population-free and is a much better estimator than the Bayesian types of estimators in satisfying the principle of objective testing.
III Sample Statistic Versions of the Alternative Estimators for the Two Extreme Response Patterns

The introduction of the two alternative estimators for \( \hat{V}_{\text{min}} \) and \( \hat{V}_{\text{max}} \) and the resultant modified maximum likelihood estimate, \( \hat{\theta}_{\text{v}} \), has enhanced the usefulness of relatively short and less informative tests, without sacrificing the objectivity of testing. When the number of items is as small as seven and all items are binary items, as is the case with LIS-U, the computation of \( \hat{\theta}_{\text{v-min}} \) and \( \hat{\theta}_{\text{v-max}} \) is relatively easy, owing to the fact that the number of all possible response patterns is as small as 128. Note, however, that the increase in the number of items, and/or in the number of item scores for each item, will soon make it practically impossible to compute these two substitute estimates, since the number of all possible response patterns will increase by gigantic steps. For example, if a test has ten binary items instead of seven, the number of all possible response patterns will be 1,024; if a test has seven three-item-score-category items, the number of all possible response patterns will be 2,187; if a test has fifteen three-item-score-category items, it will be as large as 14,348,907.

For the reason described in the preceding paragraph, it is necessary that we should invent some device in dealing with the situation in which the number of all possible response patterns is too large for us to compute \( \hat{\theta}_{\text{v-min}} \) and \( \hat{\theta}_{\text{v-max}} \) directly. By virtue of the availability of electronic computers and the Monte Carlo method, this can be done by introducing the sample statistic
versions of the two estimators.

Let $N$ be the number of examinees who were selected randomly from the uniform distribution for the interval of $\theta$, $(\frac{\theta_1}{2}, \bar{\theta})$. Let $N_L$ denote the number of examinees who belong to the above sample and whose levels of ability are lower than the critical value $\theta_c$, and $N_H$ be of that of those whose ability levels are higher than, or equal to, $\theta_c$. Thus we can write

\[(3.1) \quad N = N_L + N_H.\]

Let $N_{LV}$ and $N_{HV}$ denote the numbers of examinees who obtained the response pattern $V$, in the above two subgroups of the sample, respectively. Thus we have

\[
\begin{align*}
N_L &= \sum_V N_{LV} \\
N_H &= \sum_V N_{HV}.
\end{align*}
\]

(3.2)

It can be seen that the sample statistic corresponding to $\int_{\frac{\theta_1}{2}}^{\bar{\theta}} p_V(\theta) \, d\theta$ in the formula (2.14) is $N_L (\bar{\theta} - \theta_c) N_L^{-1}$, and also the one for $\int_{\theta_c}^{\bar{\theta}} p_V(\theta) \, d\theta$ is $N_H (\bar{\theta} - \theta_c) N_H^{-1}$. Substituting these sample statistics into (2.14) and rearranging, we obtain $\hat{\theta}_V^{\text{min}}$ and $\hat{\theta}_V^{\text{max}}$ such that

\[
\begin{align*}
\hat{\theta}_V^{\text{min}} &= \left[ \frac{1}{2} (\bar{\theta} + \theta_c) N_L - \sum_{V \neq V_{\text{min}}} \hat{\theta}_V N_{LV} \right] N_{LV}^{-1} \\
\hat{\theta}_V^{\text{max}} &= \left[ \frac{1}{2} (\bar{\theta} + \theta_c) N_H - \sum_{V \neq V_{\text{max}}} \hat{\theta}_V N_{HV} \right] N_{HV}^{-1}.
\end{align*}
\]
where \( N_{LV-min} \) and \( N_{HV-max} \) are the numbers of examinees who belong to the lower subgroup and obtained the response pattern \( V-min \), and those who belong to the upper subgroup and obtained the response pattern \( V-max \), respectively.

It can be seen that \( \hat{\theta}^{*}_{V-min} \) and \( \hat{\theta}^{*}_{V-max} \), which were defined in the preceding paragraph, are consistent, or converge in probability to \( \theta^{*}_{V-min} \) and \( \theta^{*}_{V-max} \), respectively, as the sample sizes increase. In other words, if \( N_L, N_H, N_{LV-min} \) and \( N_{HV-max} \) are large enough, the probabilities with which \( \hat{\theta}^{*}_{V-min} \) and \( \hat{\theta}^{*}_{V-max} \) assume values within the vicinities of \( \theta^{*}_{V-min} \) and \( \theta^{*}_{V-max} \), respectively, will be very high. Although the two numbers, \( N_{LV-min} \) and \( N_{HV-max} \), also depend upon the choice of the interval, \( (\hat{\theta}, \bar{\theta}) \), by virtue of the Monte Carlo method, we can control the two sample sizes, \( N_L \) and \( N_H \), as we wish.

A procedure with which we may obtain \( \hat{\theta}^{*}_{V-min} \) and \( \hat{\theta}^{*}_{V-max} \), which are defined by (3.3), can be summarized as follows.

1. Determine the interval, \( (\hat{\theta}, \bar{\theta}) \).
2. Obtain the critical value, \( \nu_c \).
3. Determine the sample size, \( N \), which makes both \( N_L \) and \( N_H \) large enough for our purpose.
4. Produce the ability levels of these \( N \) hypothetical subjects from the uniform distribution for the interval, \( (\hat{\theta}, \bar{\theta}) \). This can be done either by the Monte Carlo method, or by placing the \( N \) examinees at equally spaced points in the entire
interval, \((2, 5)\), or using one of its variations.

(5) Calibrate by the Monte Carlo method a response pattern for each of the \(N\) hypothetical examinees with respect to the \(n\) test items of our test.

(6) Find out the two frequencies, \(N_{LV}\) and \(N_{HV}\), for each response pattern \(V\).

(7) Obtain the maximum likelihood estimate \(\hat{V}\) for each response pattern whose frequencies, \(N_{LV}\) and \(N_{HV}\), are not both zero, excluding \(V\)-min and \(V\)-max.

(8) Use the above results in (3.3), and compute \(\hat{V}_{\text{min}}\) and \(\hat{V}_{\text{max}}\).

Note that the probabilities with which we obtain positive frequencies for \(N_{LV\text{-max}}\) and \(N_{HV\text{-min}}\) are both negligibly small, and this fact can be used as a checking process.

For the purpose of illustration, we shall use the five hundred hypothetical examinees, who have been used repeatedly in our previous studies of estimating the operating characteristics of the discrete item responses (Samejima, 1977c, RR-77-1, RR-78-1, RR-78-2, RR-78-3, RR-78-4, RR-78-5, RR-78-6, RR-80-2, RR-80-4), and Subtest 3, which is one of the subtests from the Old Test of thirty-five test items with three item score categories for each item. These hypothetical examinees are placed at the one hundred points of ability \(\theta\), which start from \(-2.475\) and equally spaced by the distance, \(0.05\), with
five examinees positioned together at each point. For this reason, they can be considered as a representative sample from the uniformly distributed population for the interval of \( \theta , (-2.5, 2.5) \). These fifteen test items of Subtest 3 follow the normal ogive model, whose operating characteristic is given by (2.2). The item discrimination parameter, \( a_g \), and the two item response difficulty parameters, \( b_g x \) for \( x = 1 \) and \( x = 2 \), of these items of Subtest 3 are presented in Table 3-1. The square root of the test information function of Subtest 3, which is computed through (2.12), is drawn by a solid line in Figure 3-1. We can see that the function is uni-modal, which means that Subtest 3 is more informative around the middle of the interval of \( \theta , (-2.5, 2.5) \), and less informative as \( \theta \) diverts from the middle. Figure 3-2 presents the two operating characteristics, \( P_{V-min}(\theta) \) and \( P_{V-max}(\theta) \), by solid and dotted lines, respectively, together with the critical value, \( \theta_c \), which equals -0.4146. Unlike the one we used for LIS-U in the preceding chapter, this value does not make the product of the two operating characteristics maximal, but is more deviated toward the negative side. And yet both \( P_{V-min}(\theta) \) and \( P_{V-max}(\theta) \) are practically zero at this point of \( \theta \).

Since the response pattern for each of the five hundred hypothetical examinees with respect to the Old Test was already calibrated (Samejima, 1977c), we have used the subset of this response pattern for the fifteen items of Subtest 3. The maximum likelihood estimate for each response pattern was obtained by using the basic functions, \( A_g x (\theta) \), and (2.6). It turned out that only fourteen examinees out of five hundred obtained
TABLE 3-1

Item Discrimination Parameter, $a_{g_k}$, and Item Response Difficulty Parameters, $b_{g_k}$, for $x_g = 1$ and $x_g = 2$, for Each of the Thirty-Five Test Items of the Old Test. The Fifteen Test Items Which Belong to Subtest 3 Are Marked with Crosses.

<table>
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<th>Item $g$</th>
<th>$a_{g}$</th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>Subtest 3</th>
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FIGURE 3-1

Square Root of Test Information Function (Solid Line), and Its Approximation by the Polynomial of Degree 7 (Dotted Line), of Subtest 3.
FIGURE 3-2

Operating Characteristics of the Two Extreme Response Patterns, \( P_{v\text{-min}}(\theta) \)
(Solid Line) and \( P_{v\text{-max}}(\theta) \) (Dotted Line), of Subtest 3, and the
Position of the Critical Value, \( \theta_c \).
the response pattern, $V_{\text{min}}$, and twelve obtained the response pattern, $V_{\text{max}}$. These relatively small numbers of examinees who obtained either one of the two extreme response patterns indicate that, for Subtest 3, the interval, $(-2.5, 2.5)$, is still too conservative to use as $(\tilde{\theta}, \bar{\theta})$, and it may be expanded further. The two sample sizes, $N_L$ and $N_H$, are 210 and 290, respectively. Substituting these values, together with $N_{LV_{\text{min}}} = 14$, $N_{HV_{\text{max}}} = 12$, $\theta = -2.5$, $\bar{\theta} = 2.5$, and $\theta_c = -0.4146$, into (3.3), we obtained $\hat{\theta}_{V_{\text{min}}} = -2.31453$ and $\hat{\theta}_{V_{\text{max}}} = 2.04027$.

Let $\hat{\theta}^*$ denote a new estimator, which is defined by

$$
\begin{align*}
\hat{\theta}^*_{V_{\text{min}}} &= \hat{\theta}_{V_{\text{min}}} & \text{for } V = V_{\text{min}} \\
\hat{\theta}^*_{V_{\text{max}}} &= \hat{\theta}_{V_{\text{max}}} & \text{for } V = V_{\text{max}} \\
\hat{\theta}^*_V &= \hat{\theta}_V & \text{otherwise},
\end{align*}
$$

as distinct from $\hat{\theta}_V$, which is defined by (2.15). Figure 3-3 presents the scatter diagram of the five hundred hypothetical examinees with respect to their ability levels, $\theta$, and the estimate, $\hat{\theta}_V$. We notice that, for fixed values of $\theta$, the values of $\hat{\theta}^*_V$ scatter more widely as $\theta$ departs from the middle of the interval, $(-2.5, 2.5)$, but then start having truncated distributions as $\theta$ approaches either $\tilde{\theta}$ or $\bar{\theta}$. For the purpose of comparison, Figure 3-4 presents the corresponding scatter diagram of the same five hundred hypothetical examinees, with the maximum likelihood estimate $\hat{\theta}_V$, which was based upon the original Old Test, on the ordinate. In this figure, the values of the maximum likelihood estimate for fixed ability levels scatter
FIGURE 3-3

Scatter Diagram of $\hat{\theta}$ and $\theta$ for the Five Hundred Hypothetical Examinees, Based on Subtest 3.
FIGURE 3-4
Scatter Diagram of the Maximum Likelihood Estimate $\hat{\theta}_v$ and Ability $\theta$ for the Five Hundred Hypothetical Examinees, Based Upon the Original Old Test.
approximately within the same range for the entire interval of \( \theta \), reflecting the fact that the test information function of the Old Test assumes approximately the same value throughout the interval of \( \theta \) in question.

The sample linear regression of \( \hat{\theta}_V^* \) on ability \( \theta \), or the best fitted linear function of \( \theta \) which makes the sum total of the squared discrepancies of \( \hat{\theta}_V^* \) minimal, turned out to be

\[
(3.5) \quad 0.995460 - 0.00730 .
\]

This function of \( \theta \) is shown as the straight line in Figure 3-3, which is practically indistinguishable from the unbiasedness line, or the line with forty-five degrees from the abscissa which passes the origin of the two axes, \((0,0)\). The corresponding sample linear regression for the scatter diagram, which is based upon the original Old Test and shown in Figure 3-4, proved to be \( 1.004\theta - 0.006 \) (Samejima, 1977c). We can say that these two results are practically the same.

The sample regression of \( \hat{\theta}_V^* \) on ability \( \theta \) was obtained for the one hundred ability levels, and is shown in Appendix as Figure A-1.

The mean and variance of \( \theta \) for the five hundred hypothetical examinees are 0.0000 and 2.0831, and those of \( \hat{\theta}_V^* \) turned out to be \(-0.0073\) and 2.1290, respectively. The product-moment correlation coefficient between \( \theta \) and \( \hat{\theta}_V^* \) for the five hundred hypothetical examinees is found to be 0.985.

Since we have for the uniform distribution
\[(3.6) \quad E(\theta) = \int_{\theta}^{\bar{\theta}} \theta (\bar{\theta} - \theta)^{-1} \, d\theta = (\bar{\theta} + \theta)/2\]

and

\[(3.7) \quad \text{Var.}(\theta) = \int_{\theta}^{\bar{\theta}} [\theta - E(\theta)]^2 (\bar{\theta} - \theta)^{-1} \, d\theta = (\bar{\theta} - \theta)^2/12 ,\]

in the present case of \( \theta = -2.5 \) and \( \bar{\theta} = 2.5 \), we obtain

\[(3.8) \quad E(\theta) = 0.0000\]

and

\[(3.9) \quad \text{Var.}(\theta) = 2.0833 \].

As is expected from the way we produced the ability levels of our five hundred hypothetical examinees, the above sample mean and variance are practically the same as the population mean and variance.

When an estimator, \( \lambda \), of ability \( \theta \) is conditionally unbiased, or we can write

\[(3.10) \quad E(\lambda | \theta) = \theta ,\]

the relationship such that

\[(3.11) \quad E(\lambda) = E(\theta)\]

holds in general, regardless of the distribution of \( \theta \). In such a
case, the variance of \( \lambda \) is found out to be

\[
\text{Var.}(\lambda) = \text{Var}(\theta) + E[\text{Var}(\lambda|\theta)] \geq \text{Var}(\theta)
\]

(cf. Samejima, 1977c), and the product-moment correlation coefficient between \( \theta \) and \( \lambda \) is given by

\[
\text{Corr}(\theta, \lambda) = \left[ 1 - E[\text{Var}(\lambda|\theta)] \right] \left[ \text{Var}(\lambda) \right]^{-1/2}
\]

The fact that the discrepancy of the mean of \( \hat{\theta}_V \) for our five hundred hypothetical examinees from the expectation of \( \theta \) is less than 0.001 supports this estimate for being a \( \lambda \), the unbiased estimate of \( \theta \). Since the maximum likelihood estimate, \( \hat{\theta}_V \), has a characteristic that for a fixed value of \( \theta \), it asymptotically distributes normally with \( \theta \) and \( [I(\theta)]^{-1/2} \) as the two parameters, as the amount of test information tends to infinity, and this convergence is relatively fast (Samejima, 1975, 1977a, 1977b), \( E[\text{Var}(\hat{\theta}_V|\theta)] \) can be approximated by \( E[I(\theta)^{-1}] \) for the interval, \( (\hat{\theta}, \tilde{\theta}) \). For Subtest 3, we find that

\[
E[I(\theta)^{-1}] \approx 0.0803
\]

From the above result we can see that the discrepancy of the variance of the modified maximum likelihood estimate, \( \hat{\lambda}_V \), for our sample from the population variance of \( \theta \) to be 0.0457, which is less than \( E[I(\theta)^{-1}] \) given as (3.14). If we substitute (3.14) for \( E[\text{Var}(\lambda|\theta)] \) and \( \hat{\theta}_V \) for \( \lambda \) in (3.12), we obtain for Subtest 3

\[
\hat{\theta}_V \times 2.1636.
\]
Substituting (3.14) and (3.15) into (3.13), we obtain for the product-moment correlation coefficient between $\theta$ and the maximum likelihood estimate $\hat{\theta}_V$,

$$\text{Corr}(\theta, \hat{\theta}_V) = 0.981.$$  

(3.16)

It is interesting to note that our sample variance of the modified maximum likelihood estimate, $\hat{\theta}_V$, is slightly less than the estimated population variance of the maximum likelihood estimate, $\hat{\theta}_V$, and our sample correlation coefficient between $\theta$ and $\hat{\theta}_V$ is slightly greater than the estimated population correlation coefficient between $\theta$ and the maximum likelihood estimate $\hat{\theta}_V$.

The error score, $e_s$, for each individual examinee $s$ is defined by

$$e_s = [\hat{\theta}_s^V - \theta_s][I(\theta_s)]^{-1/2},$$  

(3.17)

where $\theta_s$ is the ability level of the examinee $s$, and $V_s$ indicates the response pattern obtained by the examinee $s$. Note in this definition of the error score the discrepancy of the estimated ability from the true ability is divided by the reciprocal of the square root of the amount of test information. Thus, if $\hat{\theta}_V$ distributes, approximately, normally with the true ability $\theta$ and the reciprocal of the square root of the test information function as the two parameters, then the error score $e_s$ will conditionally distribute, approximately, normally with zero and unity as its two parameters, regardless of the ability level $\theta_s$. 
Figure 3-5 presents the cumulative frequency function of the error score \( e_s \) for the five hundred hypothetical examinees, which was obtained upon Subtest 3, together with the distribution function of the standard normal distribution. We can see in this figure that this cumulative frequency function is fairly close to the standard normal distribution function, the result which supports, though weakly, the approximate normality for the conditional distribution of \( e_s \), given \( \theta_s \). The corresponding cumulative frequency function of \( e_s \), which was obtained upon the original Old Test, is presented in Figure 3-6, together with the standard normal distribution function. Comparison of these two results reveals that the two cumulative frequency functions are very similar to each other, regardless of the fact that the former error score is defined for the modified maximum likelihood estimate \( \hat{\theta}_V \), which includes twenty-six substitute estimates for negative and positive infinities, and the latter is with respect to the maximum likelihood estimate \( \hat{\theta}_V \) itself, and that the square root of the test information function of Subtest 3 is unimodal while that of the original Old Test is constant (±4.65). The mean of the error score for Subtest 3 turned out to be -0.025 and the one for the original Old Test is -0.027, both of which are close to zero. The standard deviation, or the estimated second parameter, proved to be 0.972 for Subtest 3, and 0.995 for the original Old Test, in comparison with unity for the standard normal distribution. The variance of the error score is 0.944 for Subtest 3, and the one for the original Old Test is 0.990. It is interesting to note that
FIGURE 3-5

Cumulative Frequency Function of the Error Score $e_5$, Which Is Obtained upon Subtest 3, for the Five Hundred Hypothetical Examinees, Together with the Standard Normal Distribution Function.
Cumulative Frequency Function of the Error Score $e_s$, Which Is Obtained upon the Original Old Test, for the Five Hundred Hypothetical Examinees, Together with the Standard Normal Distribution Function.
the error score for Subtest 3 has a less dispersion than the one for the original Old Test. This fact is inconsistent with the finding in the preceding chapter about the reduction of the standard error of estimation for $\frac{0^4}{V}$ compared with the reciprocal of the square root of the test information function of LIS-U.

The sample linear regression of the error score $e_s$ for Subtest 3 on ability $\theta$ turned out to be $0.00342\theta + 0.00009$, which is practically indistinguishable from the abscissa. This is even stronger support for the independence of the error score and ability $\theta$ than the result for the original Old Test, whose sample linear regression turned out to be $0.045796 + 0.00124$.

The frequency distribution of the five hundred error scores was obtained using the category width of 0.2, for both Subtest 3 and the original Old Test. Figures 3-7 and 3-8 present these two results in the form of histogram, together with the standard normal density function. The chi-square test for the goodness of fit was made for each histogram against the standard normal density function, by combining all the categories below $e = -2.8$ and those above $e = 2.8$ into single categories, respectively. It turned out that $\chi^2_0 = 35.2252$ with 29 degrees of freedom, which provides us with $.20 < p < .30$, for Subtest 3, and $\chi^2_0 = 34.2248$ with 29 degrees of freedom, which also gives us $.20 < p < .30$, for the original Old Test, respectively. In this process of the chi-square test, there are fourteen categories for which the theoretical frequencies are less than ten. If we combine them appropriately
FIGURE 3-7

Frequency Distribution of the Error Score $e_s$, Which Is Based upon Subtest 3, for the Five Hundred Hypothetical Examinees, Compared with the Standard Normal Density Function.
FIGURE 3-8

Frequency Distribution of the Error Score $e_s$, Which Is Based upon the Original Old Test, for the Five Hundred Hypothetical Examinees, Compared with the Standard Normal Density Function.
so that all the frequencies should become greater than, or equal to, ten, then the fits will be even better.

In the preceding chapter, we introduced and defined by (2.21) another population-free estimator, $\mu_{1V}^*$. We notice that, although it is practically impossible to compute the values of this estimate when the number of possible response patterns is too large, as is the case with Subtest 3, we can compute these values for the two extreme response patterns, $V_{\text{min}}$ and $V_{\text{max}}$, and used them as substitutes for negative and positive infinities of the maximum likelihood estimate.

The rationale behind these two estimates, $\mu_{1V-\text{min}}^*$ and $\mu_{V-\text{max}}^*$, may be given as follows. Let $\theta_{\text{V}}^{**}$ be an unknown estimator, which makes the integral of the conditional expectation of the squared error of estimation, given $\theta$, for the interval, $(\bar{\theta}, \bar{\theta})$, minimum.

We define $Q$ such that

$$Q = \int_{\bar{\theta}}^{\bar{\theta}} E[(\theta_{\text{V}}^{**} - \theta)^2 | \theta] \ d\theta$$

$$= \frac{1}{V} \int_{0}^{V} (\theta_{\text{V}}^{**} - \theta)^2 p_{\text{V}}(\theta) \ d\theta .$$

Differentiating (3.18) with respect to $\theta_{\text{V}}^{**}$ and setting the result equal to zero, we obtain

$$\theta_{\text{V}}^{**} = \int_{0}^{V} p_{\text{V}}(\theta) \ d\theta \left[ \int_{0}^{V} p_{\text{V}}(\theta) \ d\theta \right]^{-1}$$

$$= \mu_{1V}^* .$$
Thus we have found that the estimator $\hat{\mu}_{IV}$ is the one which makes the integral of the conditional expectation of the squared error of estimation, given $\hat{\sigma}$, for the interval, $(\hat{\sigma}, \bar{\sigma})$, minimum. Note that (3.19) includes no other response patterns, and is given as a function of $\hat{\sigma}$ and the interval, $(\hat{\sigma}, \bar{\sigma})$ only. This implies that $\hat{\mu}_{IV}$ can be used as the estimate for a specific response pattern when those for any other response patterns are already given. For example, if we define $\hat{\sigma}_{\chi}^{***}$ in such a way that

$$
\hat{\sigma}_{\chi}^{***} = \begin{cases} 
\hat{\sigma}_{\chi}^{***} & \text{for } \hat{\sigma} = \hat{\sigma}_{\chi} - \min \\
\hat{\sigma}_{\chi}^{***} & \text{for } \hat{\sigma} = \hat{\sigma}_{\chi} - \max \\
\hat{\sigma} & \text{otherwise }
\end{cases}
$$

(3.20)

and search for $\hat{\sigma}_{\chi}^{***}$ and $\hat{\sigma}_{\chi}^{***}$ following the above principle, then we will obtain

$$
\hat{\sigma}_{\chi}^{***} = \int_{\hat{\sigma}}^{\hat{\sigma}_{\chi}} \hat{P}_{\chi}^{\min}(\hat{\sigma}) \, d\hat{\sigma} \left[ \int_{\hat{\sigma}}^{\hat{\sigma}_{\chi}} \hat{P}_{\chi}^{\min}(\hat{\sigma}) \, d\hat{\sigma} \right]^{-1} 
$$

(3.21)

and

$$
\hat{\sigma}_{\chi}^{***} = \int_{\hat{\sigma}}^{\hat{\sigma}_{\chi}} \hat{P}_{\chi}^{\max}(\hat{\sigma}) \, d\hat{\sigma} \left[ \int_{\hat{\sigma}}^{\hat{\sigma}_{\chi}} \hat{P}_{\chi}^{\max}(\hat{\sigma}) \, d\hat{\sigma} \right]^{-1} 
$$

(3.22)

This is exactly the case with the present situation, which provides us with the justification for using $\hat{\mu}_{IV}^{\chi_{\min}}$ and $\hat{\mu}_{IV}^{\chi_{\max}}$ for the two extreme response patterns, $\chi_{\min}$ and $\chi_{\max}$, which substitute
for the negative and positive infinities, respectively, of the maximum likelihood estimate. An advantage of these two estimates, \( \hat{\mu}_{1V-min} \) and \( \hat{\mu}_{1V-max} \), over \( \hat{\theta}_{V-min} \) and \( \hat{\theta}_{V-max} \), is that they are theoretical values, and obtainable without depending upon the Monte Carlo method. Note, however, that no consideration for the conditional unbiasedness of the estimator is given in adopting \( \hat{\mu}_{1V-min} \) and \( \hat{\mu}_{1V-max} \). We computed these values for Subtest 3 using the interval of \( \theta \), \((-2.5, 2.5)\), and they turned out to be \(-2.2684\) and \(2.2884\), respectively. Comparison of these values with \( \hat{\theta}_{V-min} \) \((-2.3145\)) and \( \hat{\theta}_{V-max} \) \((= 2.0403)\) reveals that they are not too far from each other.

It should be noted that the above values of \( \hat{\theta}_{V-min} \) and \( \hat{\theta}_{V-max} \) are not the least and greatest values of \( \hat{\theta}_{V} \). In fact, among the four hundred and seventy-four finite values of the maximum likelihood estimate, we find such values as \(-2.4698\) (8), \(-2.3887\) (5), \(-2.3846\) (2) and \(-2.3585\) (6), which are less than \(-2.3145\), and \(2.4651\) (7), \(2.3526\) (12), \(2.3454\) (7), \(2.3359\) (2), \(2.2762\) (3), \(2.0885\) (5) and \(2.0789\) (3), which are greater than \(2.0403\), with the integer attached to each value in parenthesis indicating the corresponding frequency. Considering that fact that \( \hat{\theta}_{V-min} \) and \( \hat{\theta}_{V-max} \) substitute for negative and positive infinities of the maximum likelihood estimate, it may be hard to accept these values. This fact indicates that the effectiveness of Subtest 3 as an instrument for measuring ability \( \theta \) extends itself for a greater interval of \( \theta \) than \((-2.5, 2.5)\). From the definition of the
substitute estimates, $\hat{\theta}_{V-min}^*$ and $\hat{\theta}_{V-max}^*$, and also from the findings in the preceding chapter with respect to $\hat{\theta}_{V-min}^*$ and $\hat{\theta}_{V-max}^*$, we can expect to be able to find an optimal interval of $\theta$ for which the modified maximum likelihood estimate, $\hat{\theta}_V^*$, is, approximately, conditionally unbiased, with $\hat{\theta}_{V-min}^*$ and $\hat{\theta}_{V-max}^*$ being the least and the greatest values of the estimate $\hat{\theta}_V^*$. 
IV Modified Maximum Likelihood Estimate for the Transformed Latent Trait

We have seen in the preceding chapter that, for Subtest 3, the two alternative estimates, \( \hat{\theta}_V^{\text{min}} \) and \( \hat{\theta}_V^{\text{max}} \), for the two extreme response patterns may be too small in absolute values, if we take the interval of \( \theta, (-2.5, 2.5) \), for \((\hat{\theta}, \bar{\theta})\), to accept as the substitutes for the negative and positive infinities of the maximum likelihood estimate. The logical step we should take next will, therefore, be to search for an optimal interval for \((\hat{\theta}, \bar{\theta})\) for this purpose. This can be done by expanding the interval as far as possible in both negative and positive directions, with the restriction that the resultant \( \hat{\theta}_V^{\text{min}} \) and \( \hat{\theta}_V^{\text{max}} \) provide us with an, approximately, conditionally unbiased estimator \( \hat{\theta}_V \) for that interval of \( \theta \).

We notice, however, that we need a transformation of \( \theta \) to \( \tau \) in order to use a test like Subtest 3, which does not have a constant test information function, as the Old Test in our methods of estimating the operating characteristics of discrete item responses (Samejima, RR-80-2, RR-80-4). The search for the alternative estimates for the two extreme response patterns, \( V^{\text{min}} \) and \( V^{\text{max}} \), will, therefore, become more meaningful if we do it with respect to \( \tau \), which provides us with a constant test information function, \( I^*(\tau) \), for Subtest 3.

Let \( P^*(\tau) \) be the operating characteristic of the response pattern \( V \), which is defined as a function of the transformed
latent trait \( \tau \). This conditional probability, given ability \( \theta \), stays the same as the original operating characteristic, \( P_V(\cdot) \), as far as \( \tau \) is a strictly increasing, or decreasing, function of \( \theta \). Thus we can write

\[
P_v^*(\tau) = P_v(\tau)
\]

Let \( T_{V-min}^* \) and \( T_{V-max}^* \) denote the estimates of \( \tau \) which are analogous to \( \theta_{V-min}^* \) and \( \theta_{V-max}^* \) defined by (2.14). We can write

\[
\tau_{V-min}^* = \left[ \frac{1}{2}(\tau_c^2 - \tau_i^2) - \Sigma \int_{\tau_i}^{T_c} P_v^*(\tau) \, d\tau \right] \quad \text{and} \quad \tau_{V-max}^* = \left[ \frac{1}{2}(\tau_i^2 - \tau_c^2) - \Sigma \int_{T_i}^{\tau_c} P_v^*(\tau) \, d\tau \right]
\]

where \( \tau_i \) and \( \tau_c \) indicate the lower and upper endpoints of an appropriately selected interval of \( \tau \), \( \tau_c \) is a critical value of \( \tau \) below which \( P_{V-max}^*(\tau) \) assumes negligibly small values, and above which so does \( P_{V-min}^*(\tau) \), and \( \tau_v \) is the maximum likelihood estimate of \( \tau \) which is assigned to the response pattern \( V \). Let us assume that the first three values, \( \tau_i \), \( \tau_c \) and \( \tau_v \), are
directly transformed from \( \theta \), \( \tilde{\theta} \) and \( \theta_c \), respectively, through the strictly increasing transformation

\[
(4.3) \quad \tau = \tau(\theta) = \sum_{k=0}^{8} \alpha_k \theta^k \quad \text{for} \quad -4.0 \leq \theta \leq 4.0,
\]

whose nine coefficients are given in Table 4-2. The critical value, \( \tau_c \), which was transformed from \( \theta_c = -0.4146 \) through (4.3), turned out to be -0.5455. Again, this value of \( \tau \) does not make the product of the two operating characteristics, \( P^*_{V\text{-min}}(\tau) \) and \( P^*_{V\text{-max}}(\tau) \), maximal, but is deviated toward the negative side. It should be noted that the maximum likelihood estimate, \( \hat{\theta}_V \), of the transformed ability \( \tau \) is also given as the direct transformation of \( \hat{\theta}_V \) (Samejima, 1969) through (4.3), for every response pattern \( V \).

Thus we can rewrite (4.2) in the form,

\[
(4.4) \left\{ \begin{array}{l}
\tau^*_{V\text{-min}} = \frac{1}{2}(\tau(\theta)^2 - \tau(\hat{\theta}_V)^2) - \sum_{V \neq V\text{-min}} \sum_{V \neq V\text{-max}} \tau(\hat{\theta}_V) \int_{\theta_c}^{\theta_v} P_V(\hat{\theta}_V) \frac{d\tau}{d\hat{\theta}_V} d\hat{\theta}_V \\
\tau^*_{V\text{-max}} = \frac{1}{2}(\tau(\theta)^2 - \tau(\hat{\theta}_V)^2) - \sum_{V \neq V\text{-min}} \sum_{V \neq V\text{-max}} \tau(\hat{\theta}_V) \int_{\theta_c}^{\theta_v} P_V(\hat{\theta}_V) \frac{d\tau}{d\hat{\theta}_V} d\hat{\theta}_V
\end{array} \right.
\]

It is obvious from (4.4) that, in general, we have...
The sample statistic versions of $\hat{\tau}_{V\text{-}\min}$ and $\hat{\tau}_{V\text{-}\max}$, which will be denoted by $\hat{\tau}_{V\text{-}\min}^*$ and $\hat{\tau}_{V\text{-}\max}^*$, respectively, are defined by

$$
\begin{align*}
\hat{\tau}_{V\text{-}\min}^* &= \left[ \frac{1}{2}(\hat{\tau}_c + \hat{\tau}) \right] N_L - \sum_{V\#V\text{-}\min} \hat{\tau}_V N_{LV\text{-}\min}^{-1} \\
\hat{\tau}_{V\text{-}\max}^* &= \left[ \frac{1}{2}(\hat{\tau}_c + \hat{\tau}) \right] N_H - \sum_{V\#V\text{-}\max} \hat{\tau}_V N_{HV\text{-}\max}^{-1} 
\end{align*}
$$

(4.6)

where $N_L$, $N_H$, $N_{LV}$, $N_{HV}$, $N_{LV\text{-}\min}$ and $N_{HV\text{-}\max}$ are as defined in the preceding chapter. From (4.6) and (3.3), it is obvious that, again, in general, we have

$$
\begin{align*}
\hat{\tau}_{V\text{-}\min}^* &\neq \tau(\hat{\omega}_{V\text{-}\min}) \\
\hat{\tau}_{V\text{-}\max}^* &\neq \tau(\hat{\omega}_{V\text{-}\max}) .
\end{align*}
$$

(4.7)

We must make our choice, therefore, as to which of the two sets of the alternative estimates should be taken. In this chapter, our choice is to take $\hat{\tau}_{V\text{-}\min}^*$ and $\hat{\tau}_{V\text{-}\max}^*$.

The transformation of $\theta$ to $\tau$ for Subtest 3 starts from the approximation of the square root of the test information function by a polynomial, using the method of moments (Elderton and Johnson, 1969, Johnson and Kotz, 1970), as was done for some other subtests of the original Old Test in a previous study (Samejima, RR-80-4).

Note that such a polynomial is the best fitted polynomial of a given
degree in the least squares principle (Samejima and Livingston, RR-79-2).

The degree of the polynomial selected here is seven, and the interval of \( \theta \) for which the method of moments was applied is \((-4.0, 4.0)\). The eight coefficients of the resultant polynomial, \( \sum_{k=0}^{7} a_k \theta^k \), are presented in Table 4-1 in the ascending order of \( k \), and the polynomial is shown by a dotted line in Figure 3-1 of the preceding chapter, together with the original square root of the test information function of Subtest 3. We can see in this figure that the polynomial thus obtained provides us with an extremely good approximation to the square root of the test information function of Subtest 3. Using this approximated polynomial, the transformation of ability \( \theta \) to \( \tau \) is also given in the form of another polynomial (Samejima, RR-80-2), such that

\[
\tau(\theta) = \sum_{k=0}^{8} a_k^* \theta^k,
\]

where

\[
a_k^* \begin{cases} 
  = d & k = 0 \\
  = (Ck)^{-1} a_{k-1} & k = 1, 2, \ldots, 8
\end{cases}
\]

with \( d \) indicating an arbitrarily set constant, and \( C \) being the constant which the square root of the test information function, \([I^*(\tau)]^{1/2}\), assumes for the interval of \( \tau \) of our interest. In the present study, we use \( d = 0 \) and \( C = 3.5 \). The coefficients, \( a_k^* \), of the resultant polynomial, which transforms \( \theta \) to \( \tau \), are
TABLE 4-1

Coefficients of the Polynomial of Degree 7 Obtained by the Method of Moments Using the Interval of \( 0, (-4.0, 4.0) \) to Approximate the Square Root of the Test Information Function of Subtest 3.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( a_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.46408884D+01</td>
</tr>
<tr>
<td>1</td>
<td>0.60789659D-01</td>
</tr>
<tr>
<td>2</td>
<td>-0.41482735D+00</td>
</tr>
<tr>
<td>3</td>
<td>0.14684659D-01</td>
</tr>
<tr>
<td>4</td>
<td>0.51686862D-02</td>
</tr>
<tr>
<td>5</td>
<td>-0.36903316D-02</td>
</tr>
<tr>
<td>6</td>
<td>0.21313602D-03</td>
</tr>
<tr>
<td>7</td>
<td>0.15726020D-03</td>
</tr>
</tbody>
</table>
shown in the ascending order of $k$ in Table 4-2, and the functional relationship between $\tau$ and $t$ is observed in Figure 4-1. The two operating characteristics, $P_\text{V-min}(\tau)$ and $P_\text{V-max}(\tau)$, are shown by solid and dotted lines, respectively, together with the position of the critical value, $\tau_c (= -0.5455)$, in Figure 4-2.

As for the interval, $(\bar{r}, \bar{r})$, seven different cases were chosen more or less arbitrarily, and are shown as Cases 1 through 7 in Table 4-3. The intervals were selected in such a way that we set the values of $\min \{\sqrt{I(r)}\}$, the lower bound of the square root of the test information function, and then corresponding intervals, $(\bar{r}, \bar{r})$, were determined, and, finally, the pairs of values, $\bar{\tau}$ and $\bar{\tau}$, were obtained through (4.8). In addition to these seven cases, another interval of $\tau$, $(-3.0, 3.0)$, was added as Case 8.

The number of hypothetical examinees for each of the eight cases was determined in the following way. It was intended that these numbers should be substantially larger than five hundred, which was used in the preceding chapter, in order to decrease the error caused by the Monte Carlo method. For Case 8, we use five thousand hypothetical examinees, or $N = 5,000$. They were placed at the one thousand positions of $\bar{\tau}$, which start from $-2.997$ and end with $2.997$, with equal steps of $0.006$, with five hypothetical examinees sharing each position. For Case 7, out of these five thousand hypothetical examinees, those who were located outside of the interval, $(-2.8267, 2.8095)$, were excluded. Thus the total number of the hypothetical examinees is 4,695 in Case 7.
TABLE 4-2

Coefficients of the Polynomial of Degree 8 to Transform \( \theta \) to \( \tau \) for Subtest 3.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( a_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00000000D 00</td>
</tr>
<tr>
<td>1</td>
<td>0.13259652D 01</td>
</tr>
<tr>
<td>2</td>
<td>0.86842420D-02</td>
</tr>
<tr>
<td>3</td>
<td>-0.39506409D-01</td>
</tr>
<tr>
<td>4</td>
<td>0.10489276D-02</td>
</tr>
<tr>
<td>5</td>
<td>0.29536370D-03</td>
</tr>
<tr>
<td>6</td>
<td>-0.17572918D-03</td>
</tr>
<tr>
<td>7</td>
<td>0.86989735D-05</td>
</tr>
<tr>
<td>8</td>
<td>0.56164139D-05</td>
</tr>
</tbody>
</table>
FIGURE 4-1

Functional Relationship Between $\theta$ and $\tau$ Based upon Subtest 3. The Positions of $\theta_c$ and $\tau_c$ Are Also Indicated.
FIGURE 4-2

Operating Characteristics of $V_{\text{min}}$ (Solid Line) and $V_{\text{max}}$ (Dotted Line) Given As Functions of the Transformed Latent Trait $\tau$, Together With the Position of the Critical Value, $\tau_c$. 

$\tau_c = -0.5455$
with the exclusion of the first 145 examinees and the last 160 examinees from the original five thousand. In each of the remaining six cases, the number of hypothetical examinees was determined in the similar manner, and it is presented in the last column of Table 4-3. From this total number of examinees, \( N \), the two sample sizes, \( N_L \) and \( N_H \), were determined in each case, depending upon how many examinees were positioned below and above the critical value, \( t_c (=-0.5455) \). These numbers are also presented in Table 4-3.

Thus, in each case, these hypothetical examinees can be considered as a sample from the uniform distribution for the interval, \( (\bar{t}, \bar{t}) \).

Note, however, that, because of the way the examinees were selected, the values \( \bar{t} \) and \( \bar{t} \) were slightly shifted for Cases 1 through 7. The new endpoints of the interval of \( \tau \) are presented in Table 4-5 for the four cases, Cases 4 through 7.

It turned out that for the first three cases, Cases 1 through 3, the two frequencies, \( N_{LV-min} \) and \( N_{HV-max} \), are so small, i.e., 1 and 3 for Case 1, 1 and 10 for Case 2, and 8 and 19 for Case 3, respectively. This is due to the fact that these three intervals of \( \tau \) are relatively small, and the probability with which the examinee, whose ability level is within each interval, obtains either \( V-min \) or \( V-max \) is low. With these small frequencies substituted in (4.6), we obtained such absurd results for \( \hat{t}_{V-min} \) and \( \hat{t}_{V-max} \) as 7.7998 and -2.2507 for Case 1, 11.3745 and 0.1132 for Case 2, and -0.8183 and 1.4841 for Case 3, respectively. It is obvious that we should not take these results
### Table 4-3

Lower Bound of the Square Root of the Test Information Function, \( \text{Min} (\sqrt{I(\theta)}) \), of Subtest 3, the Two Endpoints of the Interval, \( \theta \) and \( \bar{\theta} \), the Transformed Endpoints, \( \bar{t} \) and \( \bar{\bar{t}} \), the Numbers of Hypothetical Subjects, \( N_L \) and \( N_H \), for the Two Intervals, \( (t_c, \tau_c) \) and \( (\tau_c, \bar{\tau}) \), and the Total Number of Subjects, \( N \), in Each of the Eight Cases.

<table>
<thead>
<tr>
<th>Case</th>
<th>( \text{Min} \sqrt{I(\theta)} )</th>
<th>( \theta )</th>
<th>( \bar{\theta} )</th>
<th>( \bar{t} )</th>
<th>( \bar{\bar{t}} )</th>
<th>( N_L )</th>
<th>( N_H )</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.50</td>
<td>-1.5</td>
<td>1.7</td>
<td>-1.8456</td>
<td>2.0771</td>
<td>1,085</td>
<td>2,185</td>
<td>3,270</td>
</tr>
<tr>
<td>2</td>
<td>3.25</td>
<td>-1.7</td>
<td>1.9</td>
<td>-2.0521</td>
<td>2.2668</td>
<td>1,255</td>
<td>2,345</td>
<td>3,600</td>
</tr>
<tr>
<td>3</td>
<td>3.00</td>
<td>-1.9</td>
<td>2.1</td>
<td>-2.2461</td>
<td>2.4373</td>
<td>1,415</td>
<td>2,485</td>
<td>3,900</td>
</tr>
<tr>
<td>4</td>
<td>2.75</td>
<td>-2.1</td>
<td>2.3</td>
<td>-2.4273</td>
<td>2.5860</td>
<td>1,570</td>
<td>2,610</td>
<td>4,180</td>
</tr>
<tr>
<td>5</td>
<td>2.50</td>
<td>-2.2</td>
<td>2.4</td>
<td>-2.5131</td>
<td>2.6516</td>
<td>1,640</td>
<td>2,665</td>
<td>4,305</td>
</tr>
<tr>
<td>6</td>
<td>2.25</td>
<td>-2.4</td>
<td>2.6</td>
<td>-2.6757</td>
<td>2.7636</td>
<td>1,775</td>
<td>2,760</td>
<td>4,535</td>
</tr>
<tr>
<td>7</td>
<td>2.00</td>
<td>-2.6</td>
<td>2.7</td>
<td>-2.8267</td>
<td>2.8095</td>
<td>1,900</td>
<td>2,795</td>
<td>4,695</td>
</tr>
<tr>
<td>8</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>-3.0000</td>
<td>3.0000</td>
<td>2.045</td>
<td>2,955</td>
<td>5,000</td>
</tr>
</tbody>
</table>
seriously, and we must conclude that these three intervals of \( \tau \) are too small for our purpose of obtaining the two estimates, \( \hat{\tau}_{V\text{-min}}^* \) and \( \hat{\tau}_{V\text{-max}}^* \). In all eight cases, both \( N_{LV\text{-max}} \) and \( N_{HV\text{-max}} \) turned out to be zero, the fact that indicated the success in selecting the critical value, \( \tau_c \).

Table 4-4 presents the resultant values of \( \hat{\tau}_{V\text{-min}}^* \) and \( \hat{\tau}_{V\text{-max}}^* \), together with the two frequencies, \( N_{V\text{-min}} \) and \( N_{V\text{-max}} \), for each of Cases 4 through 8. These two estimates increase in absolute values as the interval becomes larger, as is expected from their definitions.

The sample regressions of \( \hat{\tau}_V^* \) on \( \tau \) for Cases 4 through 8 are presented in Figures 4-3 through 4-7, respectively. In each graph, the mean of five \( \hat{\tau}_V^* \)'s corresponding to a fixed value of \( \tau \) is plotted as one point, to make the total number of points 836 for Case 4, 861 for Case 5, 907 for Case 6, 939 for Case 7, and 1,000 for Case 8, respectively. We can see that, in each case, these points of sample regression cluster around the unbiasedness line, or the straight line with forty-five degrees from the abscissa passing the origin, \((0,0)\), which is shown in each of the five figures.

Table 4-5 presents the sample mean and variance of \( \tau \), those of \( \hat{\tau}_V^* \), and the sample product-moment correlation coefficient of \( \tau \) and \( \hat{\tau}_V^* \), together with the two endpoints of the interval, \((\bar{\tau}, \bar{\tau})\), for each of Cases 4 through 8. In the same table, also presented in parentheses are the corresponding expectations, population variances and population correlation coefficients. These values
TABLE 4-4

Two Estimates, $\tilde{\theta}_{V_{\text{min}}}$ and $\tilde{\theta}_{V_{\text{max}}}$, and the Numbers of Hypothetical Subjects, $N_{V_{\text{min}}}$ and $N_{V_{\text{max}}}$, Who Obtained Either of the Two Extreme Response Patterns, $V_{\text{min}}$ and $V_{\text{max}}$, Respectively, in Each of the Five Cases, Cases 4 through 8.

<table>
<thead>
<tr>
<th>Case</th>
<th>$\tilde{\theta}<em>{V</em>{\text{min}}}$</th>
<th>$N_{V_{\text{min}}}$</th>
<th>$\tilde{\theta}<em>{V</em>{\text{max}}}$</th>
<th>$N_{V_{\text{max}}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>-1.6061</td>
<td>23</td>
<td>2.0856</td>
<td>32</td>
</tr>
<tr>
<td>5</td>
<td>-2.0651</td>
<td>39</td>
<td>2.2750</td>
<td>42</td>
</tr>
<tr>
<td>6</td>
<td>-2.4788</td>
<td>81</td>
<td>2.5455</td>
<td>74</td>
</tr>
<tr>
<td>7</td>
<td>-2.6867</td>
<td>145</td>
<td>2.6865</td>
<td>93</td>
</tr>
<tr>
<td>8</td>
<td>-2.8214</td>
<td>258</td>
<td>2.8596</td>
<td>196</td>
</tr>
</tbody>
</table>
FIGURE 4-3

Sample Regression of $\hat{Y}_v$ on $\tau$, for 836 Fixed Values of $\tau$.

Case 4
Sample Regression of $\hat{\tau}_V$ on $\tau$, for 861 Fixed Values of $\tau$.
Case 5
FIGURE 4-5

Sample Regression of $\hat{t}_v$ on $\tau$, for 907 Fixed Values of $\tau$,
Case 6
Sample Regression of $i^* \hat{v}$ on $t$, for 939 Fixed Values of $t$

Case 7
FIGURE 4-7
Sample Regression of $\hat{i}_v$ on $\tau$, for 1,000 Fixed Values of $\tau$.
Case 8
TABLE 4-5

Mean and Variance of $\tau$, Those of $\hat{\tau}_V$, the Product-Moment Correlation Coefficient of $\tau$ and $\hat{\tau}_V$, and the Two Endpoints of the Interval, $(\bar{\tau}, \overline{\tau})$, in Each of the Five Cases.

<table>
<thead>
<tr>
<th>Case</th>
<th>mean $\tau$</th>
<th>variance</th>
<th>mean $\hat{\tau}_V$</th>
<th>variance</th>
<th>Corr.$(\tau, \hat{\tau}_V)$</th>
<th>$\overline{\tau}$</th>
<th>$\overline{\tau}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.07800</td>
<td>2.09669</td>
<td>0.07879</td>
<td>2.16160</td>
<td>0.98081</td>
<td>-2.430</td>
<td>2.586</td>
</tr>
<tr>
<td></td>
<td>(0.075)</td>
<td>(2.09669)</td>
<td>(0.078)</td>
<td>(2.17832)</td>
<td>(0.98108)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.06900</td>
<td>2.22396</td>
<td>0.06929</td>
<td>2.28554</td>
<td>0.98254</td>
<td>-2.514</td>
<td>2.652</td>
</tr>
<tr>
<td></td>
<td>(0.069)</td>
<td>(2.22396)</td>
<td>(0.069)</td>
<td>(2.30560)</td>
<td>(0.98214)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.04500</td>
<td>2.46794</td>
<td>0.04458</td>
<td>2.52176</td>
<td>0.98475</td>
<td>-2.676</td>
<td>2.766</td>
</tr>
<tr>
<td></td>
<td>(0.045)</td>
<td>(2.46795)</td>
<td>(0.045)</td>
<td>(2.54958)</td>
<td>(0.98386)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>-0.00900</td>
<td>2.64516</td>
<td>-0.00844</td>
<td>2.70365</td>
<td>0.98589</td>
<td>-2.826</td>
<td>2.808</td>
</tr>
<tr>
<td></td>
<td>(-0.009)</td>
<td>(2.64516)</td>
<td>(-0.009)</td>
<td>(2.72680)</td>
<td>(0.98492)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.00000</td>
<td>3.00000</td>
<td>0.00027</td>
<td>3.05110</td>
<td>0.98762</td>
<td>-3.000</td>
<td>3.000</td>
</tr>
<tr>
<td></td>
<td>(0.000)</td>
<td>(3.00000)</td>
<td>(0.000)</td>
<td>(3.08163)</td>
<td>(0.98667)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
were obtained by replacing \( \tau \) for \( \nu \) in (3.6), (3.7), (3.10), (3.11), (3.12) and (3.13), and replacing \( \hat{i}_V^* \) for \( \lambda \) in the last four of these six formulas, and by using \( C^{-2} (\doteq 0.081633) \) for E\{Var, \( (\hat{i}_V^* | \tau) \) \}. We can see that, in each case, these sample means, variances and correlation coefficients are very close to the corresponding population parameters. It is interesting to note, however, that there is a mild tendency that the sample variance of \( \hat{i}_V^* \) is less than the population variance, and the sample correlation coefficient between \( \tau \) and \( \hat{i}_V^* \) is greater than the population correlation coefficient.

Table 4-6 presents the two coefficients of the linear regression, \( a + b \), of \( \hat{i}_V^* \) on \( \tau \), or the best fitted line in the least squares principle, for each of Cases 4 through 8. We can see that the first coefficient, \( a \), is very close to unity, and the second coefficient, \( b \), is very close to zero, in each of the five cases, and the linear regression is practically the same as the unbiasedness line, or the line with forty-five degrees from the abscissa passing the origin, (0,0). Evidently, the two alternative estimates, \( \hat{i}_{V-\text{min}}^* \) and \( \hat{i}_{V-\text{max}}^* \), turned out to be suitable substitutes for the negative and positive infinities of the maximum likelihood estimate so that the resultant \( \hat{i}_V^* \) be, approximately, conditionally unbiased for the interval, \( (\tau, \bar{\tau}) \), in each of the five cases. If we extend the interval beyond \( (\tau, \bar{\tau}) \), however, the approximate unbiasedness of \( i_V^* \) does not necessarily hold. The expansion of the interval to \( (-1.0, 1.0) \) for Case 7, for example, makes the linear regression
TABLE 4-6

Two Coefficients of the Sample Linear Regression of $\frac{\tau^*}{\tau}$ on $\tau$, in Each of the Five Cases.

<table>
<thead>
<tr>
<th>Case</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.99588</td>
<td>0.00111</td>
</tr>
<tr>
<td>5</td>
<td>0.99605</td>
<td>0.00057</td>
</tr>
<tr>
<td>6</td>
<td>0.99542</td>
<td>-0.00022</td>
</tr>
<tr>
<td>7</td>
<td>0.99673</td>
<td>0.00053</td>
</tr>
<tr>
<td>8</td>
<td>0.99599</td>
<td>0.00027</td>
</tr>
</tbody>
</table>
0.98339T + 0.00043, that for Case 6 makes it 0.96849T + 0.00563, and that for Case 5 makes it 0.93922T + 0.01638, all of which are flatter than the original linear regressions.

It is observed that, in all five cases, the least value of the finite maximum likelihood estimates which our hypothetical examinees obtained is -2.6518, and the greatest value 2.7683. These two values are larger in absolute values than $\hat{T}_{V-\text{min}}$ and $\hat{T}_{V-\text{max}}$, respectively, for the four cases, Cases 4 through 7, and only Case 8 provides us with $\hat{T}_{V-\text{min}}$ and $\hat{T}_{V-\text{max}}$ which are larger in absolute values than these two finite maximum likelihood estimates. This fact implies that, out of the five sets of intervals of $T$ and the corresponding pairs of alternative estimates, those of Case 8 may be the most suitable ones for Subtest 3. This set of alternative estimates also gives us an approximate conditional unbiasedness of $\hat{T}_V$ for truncated intervals. Figure 4-8 presents the sample regression of $\hat{T}_V$ on $T$ in Case 8, for the truncated interval of $T$, (-2.430, 2.586), which is the same as the interval, $(\tau, \bar{\tau})$, in Case 4. We can see in this figure that the sample regression still clusters around the unbiasedness line for this truncated interval. In contrast to this, Figure 4-9 presents the sample regression in Case 4 for the extended interval of $T$, (-3.0, 3.0). The awkward shapes of clusters around the two endpoints of the extended interval of $\tau$ indicates that the two alternative estimates in Case 4 fail to provide us with an approximate conditional unbiasedness of $\hat{T}_V$ for this extended interval of $\tau$. 
FIGURE 4-8
Sample Regression of $\hat{\tau}$ on $\tau$: Case 8, Using the Interval $(-2.430, 2.586)$, instead of $(-3.000, 3.000)$.

FIGURE 4-9
Sample Regression of $\hat{\tau}$ on $\tau$: Case 4, Using the Interval $(-1.000, 3.000)$, instead of $(-2.430, 2.586)$. 
The error score for each individual examinee $s$ is defined as was done in the preceding chapter, by replacing $\theta_s$ by $\tau_s$, $\hat{\theta}_s^*$ by $\tau_s^*$, and $I(0)$ by $I^*(\tau)$ in (3.17). For convenience, the same symbol, $e_s$, will be used for the error score defined for $\tau_s^*$. Note that, in the present situation, the test information function, $I^*(\tau)$, is constant ($= 3.5^2$) for the interval of $\tau$ of our interest, instead of being a unimodal function. The error score $e_s$ was computed for each of the 4,180 examinees of Case 4, and each of the 5,000 examinees of Case 8. The frequency distributions of these error scores are presented as histograms in Figures 4-10 and 4-11, respectively, with the category width of 0.2, together with the standard normal density function. We can see that these two histograms are much closer to the standard normal density function, in comparison with those obtained in the preceding chapter upon the five hundred hypothetical examinees. It is also noted that these two resultant histograms are substantially different from each other, in spite of the fact that 4,125 error scores are common for these two histograms. The chi-square test for the goodness of fit of these two frequency distributions against the standard normal density function gives $\chi^2_0 = 23.3491$ with 29 degrees of freedom ($0.70 < p < 0.80$) for Case 4, and $\chi^2_0 = 55.6856$ with the same number of degrees of freedom ($0.001 < p < 0.01$) for Case 8. The mean, variance and standard deviation of the error score, $e_s$, are 0.0028, 1.0070 and 1.0035 for Case 4, and 0.0009, 0.9206 and 0.9595 for Case 8, respectively. It is interesting to note that
FIGURE 4-10

Frequency Distribution of the Error Score, $e_s$, Which Is Based upon Subtest 3, for the 4,180 Hypothetical Examinees of Case 4, Compared with the Standard Normal Density Function.
Figure 4-11

Frequency Distribution of the Error Score, $e_s$, Which Is Based upon Subtest 3, for the 5,000 Hypothetical Examinees of Case 8, Compared with the Standard Normal Density Function.
the dispersion of the error score in Case 8 is substantially less than unity. The correlation coefficient between $\tau$ and $\hat{\tau}_v$ is -0.021 for Case 4, and -0.025 for Case 8. The sample linear regression of the error score $e_s$ on $\tau$ is $-0.01444\tau + 0.00387$ in Case 4, and $-0.01403\tau + 0.00093$ in Case 8, both of which are very close to zero.

The pair of estimates, $\mu^*_{IV-min}$ and $\mu^*_{IV-max}$, which were introduced in the preceding chapter, were also obtained with respect to $\tau$, for each of Cases 1 through 8. These results are presented in Table 4-7. We can see that for larger intervals of $\tau$, like those in Cases 6 through 8, the resultant estimates are similar to the corresponding values of $\hat{\tau}^*_{V-min}$ and $\hat{\tau}^*_{V-max}$, respectively.
Table 4-7

Bayes Estimates with the Uniform Prior for the Two Extreme Response Patterns, $\mu^{t}_{IV-min}$ and $\mu^{t}_{IV-max}$, Obtained with Respect to $\tau$, for Each of the Eight Cases.

<table>
<thead>
<tr>
<th>Case</th>
<th>$\mu^{t}_{IV-min}$</th>
<th>$\mu^{t}_{IV-max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1.7434</td>
<td>1.9980</td>
</tr>
<tr>
<td>2</td>
<td>-1.9286</td>
<td>2.1810</td>
</tr>
<tr>
<td>3</td>
<td>-2.0965</td>
<td>2.3457</td>
</tr>
<tr>
<td>4</td>
<td>-2.2464</td>
<td>2.4905</td>
</tr>
<tr>
<td>5</td>
<td>-2.3143</td>
<td>2.5551</td>
</tr>
<tr>
<td>6</td>
<td>-2.4364</td>
<td>2.6684</td>
</tr>
<tr>
<td>7</td>
<td>-2.5402</td>
<td>2.7171</td>
</tr>
<tr>
<td>8</td>
<td>-2.7527</td>
<td>2.7805</td>
</tr>
</tbody>
</table>
Discussion and Conclusions

The modified maximum likelihood estimate, \( \hat{\theta}_V \), and its variation, \( \hat{\theta}_V^* \), have been introduced and investigated, in comparison with Bayesian estimates and another population-free estimate, \( \mu^* \). The former of these two newly proposed estimates is effective when a given test is short, like LIS-U, and the latter is useful when it is longer and more informative, like Subtest 3.

The basic idea behind this research is to admit that each test has a certain limited range of ability for which it is effective in estimating the examinee's ability. Although this is a self-evident fact, for some reason, the idea has not fully been accepted by many researchers, and people tend to use tests for overly wide ranges of ability, and turn to inappropriate methods like Bayesian estimation, in order to make the result plausible. This is evidently a false solution, using the pretense that the test has measured something, while it has failed in so doing, and it is mainly an arbitrarily set prior which has given the examinee his ability score. The greatest fault of the Bayesian estimation may be that it is against the principle of objective testing, since it contaminates the resulting ability estimate by something other than the examinee's performance in the test.

The conditional unbiasedness of the ability estimate, given ability, is by far the most important in order to sustain the principle of the objectivity of testing. Taking this fact in mind, we can still try to enhance the usefulness of a given test, by expanding the range
of ability for which the test is effective. One way of doing this is to provide a suitable estimator.

The maximum likelihood estimator has a useful characteristic of asymptotic conditional unbiasedness. For less informative tests, however, the conditional probability with which the examinee obtains one of the two extreme response patterns, V-min and V-max, given ability, is substantially high, and this asymptotic characteristic cannot be used as an approximation. Thus, in such a case, we will see extreme values like negative and positive infinities among the maximum likelihood estimates of our examinees: the fact that restricts the effectiveness of the test.

The two modified maximum likelihood estimates, $\hat{\theta}_V^*$ and $\hat{\theta}_V$, which were proposed and discussed in the present paper, were conceived with the following considerations in mind.

(1) We follow the principle of the objectivity of testing, and, in estimating his ability, we use nothing but the examinee's performance on the test.

(2) The resultant estimate provides us with an approximate conditional unbiasedness of estimation.

(3) The range of ability for which the test is effective is enhanced.

This has been done by replacing the maximum likelihood estimates for the two extreme response patterns, V-min and V-max, by $\hat{\theta}_{\text{V-min}}^*$ and $\hat{\theta}_{\text{V-max}}^*$, or by $\hat{\theta}_{\text{V-min}}$ and $\hat{\theta}_{\text{V-max}}$, respectively. The results
proved to be promising.

One distinct feature of the present study may be the use of the Monte Carlo method in obtaining \( \hat{\theta}_{V_{-}\text{min}} \) and \( \hat{\theta}_{V_{-}\text{max}} \). One may argue that, because of this, we cannot avoid the sampling fluctuations of \( \hat{\theta}_{V_{-}\text{min}} \) and \( \hat{\theta}_{V_{-}\text{max}} \). While it is true, this can be minimized by using a large enough sample size. In the present study, we used as large a sample size as 5,000, and this can be made even larger, if we wish. In any case, even if we use, say, ten thousand hypothetical examinees, it is still a reduction, considering that even a relatively short test like Subtest 3 contains as many as 14,348,907 different response patterns.

An alternative way may be the use of the estimates, \( \mu_{V_{-}\text{min}}^{*} \) and \( \mu_{V_{-}\text{max}}^{*} \). Since these estimates do not depend upon the Monte Carlo method, or any samples, they do not have the problem of sampling fluctuations. The approximate conditional unbiasedness may not be reached just as well, however, if we use \( \mu_{V_{-}\text{min}}^{*} \) and \( \mu_{V_{-}\text{max}}^{*} \) instead of \( \hat{\theta}_{V_{-}\text{min}}^{*} \) and \( \hat{\theta}_{V_{-}\text{max}}^{*} \).
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(14) Samejima, F. Estimation of the operating characteristics when the test information of the old test is not constant I: Rationale. Office of Naval Research, RR-80-2, 1980.


(16) Samejima, F. Estimation of the operating characteristics when the test information of the old test is not constant II: Simple sum procedure of the conditional P.D.F. approach/normal approach method using three subtests of the old test. Office of Naval Research, RR-80-4, 1980.
### Table A-1

Comparison of Two Population-Free Estimates, $\bar{\theta}_{IV}$ and $\bar{\mu}_{IV}'$, and Two Bayesian Estimates, $\bar{\mu}_{IV}$ and $\bar{\delta}_{IV}$, for Each of the One Hundred and Twenty-Eight Response Patterns of LIS-U.

<table>
<thead>
<tr>
<th>ID</th>
<th>$\bar{\theta}_{IV}$</th>
<th>$\bar{\mu}_{IV}'$</th>
<th>$\bar{\mu}_{IV}$</th>
<th>$\bar{\delta}_{IV}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1.2564</td>
<td>-1.4527</td>
<td>-1.3764</td>
<td>-1.6617</td>
</tr>
<tr>
<td>2</td>
<td>-1.7262</td>
<td>-1.2535</td>
<td>-3.0209</td>
<td>-1.7267</td>
</tr>
<tr>
<td>3</td>
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FIGURE A-1
Sample Regression of the Modified Maximum Likelihood Estimate, $\hat{\theta}_v^*$, for the One Hundred Ability Levels:
Each Point is the Mean of Five Values of $\hat{\theta}_v^*$.
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