AN INTEGRAL OF PRODUCTS OF LEGENDRE FUNCTIONS AND A CLEBSCH-GR"ETC(U)

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AN INTEGRAL OF PRODUCTS OF
LEGENDRE FUNCTIONS AND A
CLEBSCH-GORDAN SUM

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ABSTRACT

New proofs and extensions are given of a sum considered by A. M. Din
involving Clebsch-Gordan coefficients with zero magnetic quantum numbers and
of an integral involving the product of three Legendre functions, one of the
second kind.

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AN INTEGRAL OF PRODUCTS OF LEGENDRE FUNCTIONS
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Din [1] showed that

\[
S_i = \sum_{i=\min(0,\text{max}(c-b),a+b-c)}^{c+b} \frac{2i+1}{i(i+1) - a(a+1)} (\binom{c}{i+1}_0)^2 = 0
\]

(1)

when \(a, b, c\) are non-negative integers with \(a + b + c\) odd and \(|c-b| \leq a \leq c+b\). The Clebsch-Gordan coefficients with zero magnetic quantum numbers are given by

\[
\left(\binom{c}{10b}_0\right)^2 = \frac{2c+1}{2} \int_{-1}^{1} dx \frac{P_i(x)P_b(x)P_c(x)}{2}
\]

(2)

This integral was evaluated by Ferrers and others in the last century. The evaluation comes from the linearization formula

\[
P_n(x)P_m(x) = \sum_{k=0}^{\min(m,n)} \binom{m}{\frac{m-k}{2}} \binom{n}{\frac{n-k}{2}} \binom{m+n-k}{\frac{m+n-k}{2}} \binom{m+n-2k+1/2}{k} \binom{m+n-k}{\frac{m+n-k+1/2}{2}} P_{m+n-2k}(x)
\]

(3)

and the orthogonality of Legendre polynomials. See [2]. To show (1) Din reduced it to showing that

\[
I(a,b,c) := \int_{-1}^{1} dx \frac{Q_a(x)P_b(x)P_c(x)}{2} = 0
\]

(4)

when \(a, b, c > 1\) are integers, \(a + b + c\) is odd and \(|c-a| < b < c+a\).

Here \(P_1(x)\) is the Legendre polynomial and \(Q_a(x)\) is the Legendre function of the second kind on the cut \([-1, 1]\). He ended the paper by stating that I could evaluate (4) for general integers \(a, b, c\). The details follow.

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Din started with

\[ \int_{-1}^{1} \, dx \, Q_a(x) P_b(x) = \frac{1 - \cos((b-a)x)}{(b-a)(b+a+1)} \]

with a reference to [3]. A generalization of (5) is given there when \( a \) and \( b \) are complex, \( \Re a > 0, \Re b > 0 \), and the extra term which occurs vanishes when either \( a \) or \( b \) is an integer. The argument in [3] used the Legendre differential equation. Here is a second derivation of (5). Start with an expansion of Heine [4]

\[ Q_a(\cos \theta) = \frac{2 \, a!}{(\frac{3}{2})_a} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(a+\frac{3}{2})_i} \cos(a+2i+1) \theta . \]

The shifted factorial \((c)_n\) is defined by

\[ (c)_n = \Gamma(n+c)/\Gamma(c) = c(c+1)\cdots(c+n-1) \]

Since \( P_a(-x) = (-1)^a P_a(x) \) and \( Q_a(-x) = (-1)^a+1 Q_a(x) \), \( a = 0, 1, \ldots \), we may assume \( a \) and \( b \) have opposite parity, for the integral in (5) vanishes when \( a \) and \( b \) have the same parity. Then

\[ I(a,b,0) = \frac{2 \, a!}{(\frac{3}{2})_a} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(a+\frac{3}{2})_i} \int_0^\pi \cos(a+2i+1) \theta \sin \theta \, P_b(\cos \theta) \]

\[ = \frac{2 \, a!}{(\frac{3}{2})_a} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!(a+\frac{3}{2})_i} \frac{1}{(1/2)^{(i+1)(i+a+1)}} \]

by a special case of an integral of Gegenbauer which is equivalent to [5]

\[ C_n^\mu(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(\mu)_{n-k}(\mu-\lambda)_{k}(n-2k+\lambda)}{(\lambda+1)_{n-k} \lambda^k} C_n^{\lambda-k}(x) \]

where \( C_n^{\lambda}(x) \) is the ultraspherical polynomial.

The above sum can be written as a generalized hypergeometric series and then summed by a formula of Dougall [6]. A more general sum of Dougall will be stated below. A routine reduction shows that (5) holds when \( a, b = 0, 1, \ldots \), with the integral equal to zero when \( a = b \).
To compute the evaluation of (4) use the Ferrers-Adams linearization formula (3) and (5) to obtain

$$I(a,b,c) = \sum_{k=0}^{\min(b,c)} \frac{(b-k)(c-k)I_1(b-c-k+1/2)}{b-k}(b+c-k+1/2)$$

$$\frac{[1-\cos(b+c-2k-a)\pi]}{(b+c-a-2k)(b+c+a+1-2k)}$$

$$\frac{[1-\cos(b+c-a)\pi]}{(b+c-a)(b+c+a+1)bc I}(b+c)$$

$$\gamma F_6\left\{b+c-1, b/2, c/2, (b+c)/2, (b+c-a)/2, (b+c-a-2k)/2 \right\}$$

Dougall's sum of the very well poised 2-balanced $\gamma F_6$ [7],

$$\gamma F_6\left\{a, 1+a/2, b, c, d, e, -n \right\}$$

$$\gamma F_6\left\{a/2, 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a+n \right\}$$

$$\gamma F_6\left\{(1+a)\frac{(1+a-b-c)(1+a-b-d)(1+a-c-d)}{(1+a-b)(1+a-c)(1+a-d)(1+a-c-d)} \right\}$$

when $1+2a = b+c+d+e-n$, can be used and the result is

$$\int_{-1}^{1} dx \frac{Q_a(x)P_b(x)P_c(x)}{1-1}$$

$$\frac{[1-\cos(b+c-a)\pi](b+c+a)/a}{((b-c-a-1)/2)}$$

when $0 \leq b \leq c$, $a+b+c$ odd, and zero when $a+b+c$ is even. Since this integral vanishes when $b+c+a$ is even, we may write $a = b+c+1+2k$. The integral is then
\[ \int_{-1}^{1} dx \; Q_{b+c+1+2k}(x) \; P_{b}(x)P_{c}(x) \]

\[ = -\frac{\Gamma(k+b+c+\frac{3}{2})\Gamma(k+b+1)^{2}\Gamma(k+c+1)^{2}\Gamma(k+\frac{1}{2})}{2\Gamma(k+b+c+2)^{2}\Gamma(k+b+\frac{3}{2})\Gamma(k+c+\frac{3}{2})\Gamma(k+1)} \]  

(10)

This integral vanishes when \( k = -1, -2, \ldots, -\min(b,c) \) as was shown by Din.

Since (5) holds when \( a \) is not an integer, and the rest of the above argument only used the integrality of \( b \) and \( c \), formula (8) continues to hold when \( \text{Re} \; a > 0 \). In this case it is better to write it as

\[ \int_{-1}^{1} dx \; Q_{a}(x)P_{b}(x)P_{c}(x) = \frac{[1-\cos(b+c-a)] \cdot \Gamma\left(\frac{c-b-a}{2}\right)\Gamma\left(\frac{b-c-a}{2}\right)}{(b+c-a)(b+c+a+1)\Gamma\left(\frac{c-b-a+1}{2}\right)\Gamma\left(\frac{b-c-a+1}{2}\right)} \]

(11)

\[ \frac{\Gamma\left(\frac{b+c-a+1}{2}\right)\Gamma\left(\frac{b-c-a+1}{2}\right)}{\Gamma\left(\frac{b+c-a}{2}\right)\Gamma\left(\frac{b-c-a}{2}\right)} , \; \text{Re} \; a > 0, \; b,c = 0,1,\ldots, \]

with an appropriate limit taken when one of the gamma functions has a pole.

The sum in (1) can be evaluated in exactly the same way, only the details are easier. One only needs to use (2) to replace the Clebsch-Gordan coefficients by a known integral, rewrite the series as a generalized hypergeometric series and use Dougall's sum (8). Fortunately Din was unaware of Dougall's sum, for the integral in (11) seems to be a fundamental result, and it does not seem to have been evaluated before. I was surprised by this, since Hobson [8] wrote that F. E. Neumann had evaluated this integral. However it is not given in the book of Neumann that Hobson mentions nor in the other book of Neumann that I have looked at.

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REFERENCES


5. ibid. (4.10.27)


7. ibid, 4.3(5).


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