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ACCELERATION WAVE PROPAGATION IN HYPERELASTIC RODS OF VARIABLE --ETC(U)
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ACCELERATION WAVE PROPAGATION IN HYPERELASTIC RODS OF VARIABLE CROSS-SECTION

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ACCELERATION WAVE PROPAGATION IN HYPERELASTIC RODS OF VARIABLE CROSS-SECTION

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ABSTRACT

It is shown that when an acceleration wave propagates in a hyperelastic rod with slowly varying cross-section, the transport equation for the wave intensity is a generalized Riccati equation. The three coefficients in the equation all depend on the material properties, but only the coefficient of the quadratic term is independent of the effect of area change. Three theorems are proved, based on the use of comparison equations, which establish that in general the acceleration wave intensity will become infinite (escape) after the wave has propagated only a finite distance along the rod. The existence of thresholds for the initial intensity are also established in certain cases, with their most notable property being that as the initial intensity decreases towards the threshold, so the distance the wave propagates to escape increases without bound.

AMS (MOS) Classifications: 73D15, 34C11

Key Words: Nonlinear Wave Propagation, Acceleration Wave, Hyperelastic Rod, Escape of Solution

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SIGNIFICANCE AND EXPLANATION

This work studies the propagation of an acceleration wave in the class of nonlinear elastic materials known as hyperelastic . These are materials in which the stress potential depends only on the displacement gradient. In particular, the effect of wave propagation along a rod of slowly varying cross-section is studied, since in many physical situations such rods are often to be found. The assumption of a slowly varying cross-section allows the problem to be approximated by a one-dimensional situation, with the area variation either modifying terms or introducing new ones.

The equation governing the acceleration wave intensity, the transport equation, is shown not to be the Bernoulli equation, which usually arises, but the generalized Riccati equation. In the case of a rod of constant cross-section this reduces to a degenerate Bernoulli equation which may be solved exactly to yield results which, in conjunction with experiment, permit the determination of the material characteristics needed if non-constant cross-section rods are to be studied.

The paper concludes with the proof of three theorems which give some insight into the behaviour of the solution of the generalized Riccati equation in terms of simple conditions placed on the coefficients and on the initial condition for the acceleration wave. It is shown that, in general, the intensity of an acceleration wave will become infinite, leading to shock wave formation, after propagating only a finite distance along the rod. An interesting new result which arises out of this analysis is the existence of a threshold for the initial wave intensity. This has the property that the closer the initial intensity is to this threshold, the further will the wave propagate before a shock wave forms, provided only that the initial intensity exceeds this positive threshold.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ACCELERATION WAVE PROPAGATION IN HYPRELASTIC RODS OF VARIABLE CROSS-SECTION

A. Jeffrey†

1. INTRODUCTION

The nonlinear elastic material studied in this paper is one in which the stress potential Σ is dependent only on the deformation, or displacement, gradient. It is thus a material which belongs to the class known as hyperelastic [1], and we shall consider a special problem concerning one-dimensional acceleration wave propagation, so that Σ will depend only on p, the displacement gradient in the direction of propagation.

Many authors have considered both static and dynamic problems for such materials, of whom we mention only Antman [2] and Antman and Jordan [3] who studied the Kirchhoff problem for nonlinearly elastic rods and qualitative properties in general, Jeffrey and Teymur [4] and Jeffrey and Suhubi [5] who considered shock wave formation and acceleration wave propagation through periodically layered media, and Antman and Liu [6] who made a detailed study of travelling waves in hyperelastic rods which were permitted various forms of deformation. In what follows we show how, when a variable area of cross-section occurs in a rod, the effect of the geometry is to produce a transport equation for the acceleration wave intensity which is more general than usual.

In Sections 2 and 3 of this paper our purpose will be to show that the transport equation for acceleration wave propagation along the axis of symmetry of a hyperelastic rod of slowly varying cross-section is a generalized Riccati equation, and to deduce the form of its coefficients. This is a generalization of the transport equation studied by Bailey and Chen

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[7,8,9] which was a variable coefficient Bernoulli equation known to apply in diverse situations.

Since the Bernoulli equation may be solved exactly, Bailey and Chen were able to make precise estimates concerning the growth and decay properties of solutions in terms of the coefficients and the initial condition. Such precise estimates are not possible in this case, so in Section 4 three representative theorems are proved concerning the unbounded nature (escape) of the acceleration wave intensity s after propagation along only a finite length X_∞ of such a rod. It is shown, for example, that some hyperelastic materials and cross-sectional area variations always lead to the escape of s, whereas in others a threshold σ_0^+ exists for s with the property that the escape can only be certain to occur if $s > \sigma_0^+$.

2. Formulation of Problem

We shall consider a homogeneous hyperelastic medium [1] in the form of a semi-infinite rod of variable cross-section, which has the stress potential Σ. Furthermore, we shall assume that the cross-section is everywhere symmetrical relative to the axis of the rod and to two fixed mutually orthogonal axes that are normal to it, but that the cross-sectional area is a slowly varying function of distance measured along the rod. The material will be assumed to be incompressible.

Let us now relate the Eulerian coordinates x_k $(k = 1,2,3)$ to the Lagrangian coordinates X_k $(k = 1,2,3)$ by

$$\begin{aligned} &x = u(X,t) + X,\ x_1 = x,\ X_1 = X, \\ &x_2 = X_2, \\ &x_3 = X_3, \end{aligned} \tag{2.1}$$

where $u(X,t)$ is the displacement of a particle along the rod relative to the natural axis along which X is also measured [10]. The orientation of the X_2 and X_3 axes, which are mutually orthogonal and lie in a plane normal to the X axis, is taken so that the cross-section is symmetrically disposed relative to them.

In terms of these coordinates, the displacement gradient

$$p = \frac{\partial u}{\partial X} = u_X \ , \tag{2.2}$$

while the velocity of a particle displaced by an amount u is

$$v = \frac{\partial u}{\partial t} = u_t \ . \tag{2.3}$$

The stress-tensor is given by

$$T_{11} = T = \frac{\partial \Sigma}{\partial p} \ , \tag{2.4}$$

where because of the hyperelasticity $\Sigma = \Sigma(p)$.

Since the particle displacement u will be taken to be small, we shall regard the cross-section area variation solely as a function of X, and denote it by $S(X)$. We suppose the rod to be stressed and left in equilibrium prior to the propagation of a disturbance. Then continuity of stress normal to the (X_2,X_3)-plane, coupled with the area variation iteself, shows that the initial displacement $p_0(X)$ will be given by solving

$$\tilde{T} = \frac{S(X)}{S(0)} \left(\frac{\partial\Sigma}{\partial p}\right) , \tag{2.5}$$

where $\tilde{T} = T(0)$ is the initial constant stress at the plane boundary $X = 0$, which we take to be the end of the rod at which the cross-sectional area is $S(0)$.

The equation of motion is simply

$$\frac{\partial(TS)}{\partial X} = S\rho \frac{\partial v}{\partial t} , \tag{2.6}$$

where ρ is the constant density. Then, by virtue of equation (2.4),

$$\frac{\partial T}{\partial X} = \frac{\partial T}{\partial p}\frac{\partial p}{\partial X} = \frac{\partial^2\Sigma}{\partial p^2}\frac{\partial p}{\partial X}.$$

If we now introduce the quantity c, which has the dimensions of a velocity, through the definition

$$c = c(p) = \left(\frac{1}{\rho}\frac{\partial^2\Sigma}{\partial p^2}\right)^{1/2} , \text{ with } \frac{\partial^2\Sigma}{\partial p^2} > 0 , \tag{2.7}$$

the equation of motion (2.6) becomes

$$\frac{\partial v}{\partial t} - c^2 \frac{\partial p}{\partial X} - \frac{T}{\rho}\frac{d}{dX}(\ell n\ S) = 0 , \tag{2.8}$$

where $T = T(X)$ is the stress as a function of X, which is influenced by both the propagation of the wave and by the change of equilibrium stress caused by the variation of the cross-sectional area.

Equality of mixed derivatives then leads to the additional equation

$$\frac{\partial p}{\partial t} = \frac{\partial v}{\partial X} . \tag{2.9}$$

Thus the one-dimensional theory for the propagation of disturbances along the rod is governed by the equations (2.8) and (2.9). The last term in (2.8) represents the effect of the cross-sectional area change, and when S = const.

these equations reduce to those considered elsewhere in a somewhat different context [4,5]. This is because the one-dimensional model does not distinguish between wave propagation normal to the bounding plane of a half-space and wave propagation along a rod of constant cross-section.

3. The Transport Equation For The Acceleration Wave Amplitude

We shall suppose that a wave propagating along the the rod in the positive X direction starts from the end of the rod corresponding to $X = 0$ at time $t = 0$, and that it advances into the equilibrium state determined by (2.5). We thus assume that:

(i) p and v are continuous, with $v(X,t) = 0$ ahead of the wave,

(ii) the first and second derivatives of p and v experience at most a jump discontinuity, so that the wavefront which propagates is an acceleration wave.

Let us now generalize an argument used by Gurtin [11] in connection with water waves, and show that it may also be used in this case, though we shall see that the transport equation which results differs significantly.

In what follows we denote by a superscript minus sign the value of a quantity immediately behind the advancing wavefront, that is to say in the disturbed region, and by a superscipt plus sign the corresponding value of the wavefront in the equilibrium state.

We then conclude from (i) above that

$$p^- = p^+ \quad \text{and} \quad v^- = v^+ = 0 \ , \tag{3.1}$$

while from (2.8) it may be seen that the time-independent solution p^+ must satisfy

$$c_0^2 \frac{\partial p^+}{\partial X} + \frac{T_0}{\rho} \frac{d}{dX} (\ell n \ S) = 0 \ , \tag{3.2}$$

where the suffix zero signifies the form of the function on the wavefront.

Differentiating v^- parallel to and immediately behind the wavefront we obtain

$$\frac{\partial v^-}{\partial X} + \frac{dt}{dX} \frac{\partial v^-}{\partial t} = 0 \ , \tag{3.3}$$

to which result may be supplemented the form taken by (2.9) immediately behind the wavefront

$$\frac{\partial p^-}{\partial t} = \frac{\partial v^-}{\partial X} . \tag{3.4}$$

Next, differentiating $p^- = p^+$ in similiar fashion, and noting that p^+ is a function only of X, we find

$$\frac{\partial p^-}{\partial X} + \frac{dt}{dX}\frac{\partial p^-}{\partial t} = \frac{\partial p^+}{\partial X} , \tag{3.5}$$

which by virtue of (3.2) becomes

$$\frac{\partial p^-}{\partial X} + \frac{dt}{dX}\frac{\partial p^-}{\partial t} + \frac{T_0}{\rho c_0^2}\frac{d}{dX}(\ln S) = 0 . \tag{3.6}$$

This must also be supplemented by the form taken by (2.8) immediately behind the wavefront

$$\frac{\partial v^-}{\partial t} - c_0^2\frac{\partial p^-}{\partial X} - \frac{T_0}{\rho}\frac{d}{dX}(\ln S) = 0. \tag{3.7}$$

We are now in a position to interpret dX/dt, the speed of wavefront propagation, in terms of the material constants. Combining equations (3.3), (3.4), (3.6) and (3.7) and solving for dX/dt we find that in fact

$$\left(\frac{dX}{dt}\right)^2 = c_0^2 . \tag{3.8}$$

Thus the quantity c introduced in (2.7) is merely the propagation speed of the disturbance (acceleration wave) characterized by (i) and (ii) above. We notice that $T_0 = T_0(X)$ and $c_0 = c_0(X)$, since the initial displacement given by solving (2.5) is $p_0 = p_0(X)$.

Hereafter we confine attention to the situation immediately behind the wavefront. Differentiating equations (2.8) and (2.9) partially with respect to t and X, respectively, and eliminating $\partial^2 p^-/\partial t\partial X$ gives

$$\frac{\partial^2 v^-}{\partial t^2} - c_0^2\frac{\partial^2 v^-}{\partial X^2} - 2c_0\left(\frac{\partial c}{\partial p^-}\right)\left(\frac{\partial p^-}{\partial t}\right)\left(\frac{\partial p^-}{\partial X}\right)$$

$$- \left(\frac{\partial T}{\partial p^-}\right)\left(\frac{\partial p^-}{\partial t}\right)\frac{1}{\rho}\frac{\partial}{\partial X}(\ln S) = 0 . \tag{3.9}$$

Now it follows from (3.8) that on the wavefront $\partial/\partial t \equiv c_0\, d/dX$, which leads at once to the identity

$$\frac{\partial^2 v^-}{\partial t^2} \equiv c_0 \frac{d(v_t^-)}{dX} . \tag{3.10}$$

It is also true that on the wavefront

$$\frac{\partial^2 v^-}{\partial X^2} \equiv \frac{\partial(v_X^-)}{dX} , \tag{3.11}$$

so combining (3.10) and (3.11) brings us to the final identity

$$\frac{\partial^2 v^-}{\partial t^2} - c_0^2 \frac{\partial^2 v^-}{\partial X^2} \equiv c_0 \frac{d(v_t^-)}{dX} - c_0^2 \frac{d(v_X^-)}{dX} . \tag{3.12}$$

Setting $\partial p^-/\partial X = s$, the acceleration wave intensity, and combining (3.9) and (3.12), brings us to the result

$$\frac{ds}{dX} = -\mu s + \beta s^2 + \gamma , \tag{3.13}$$

where

$$\mu(X) = \frac{1}{2\rho c_0^3}\left[\frac{3c_0}{\rho}\left(\frac{\partial^2\Sigma}{\partial p^2}\right)_0 + \frac{T_0}{2\rho c_0}\left(\frac{\partial^3\Sigma}{\partial p^3}\right)_0 \frac{d}{dX}\right](\ell n\ S(X)), \tag{3.14}$$

$$\beta(X) = \frac{-5}{4\rho c_0^2}\left(\frac{\partial^3\Sigma}{\partial p^3}\right)_0 , \tag{3.15}$$

$$\gamma(X) = \frac{T_0}{2\rho c_0^3}\left[2c_0 \frac{d^2}{dX^2}(\ell n\ S(X)) + \frac{1}{\rho c_0}\left(\frac{\partial^2\Sigma}{\partial p^2}\right)_0 \left\{\frac{d}{dX}(\ell n\ S(X))\right\}^2\right]. \tag{3.16}$$

Equation (3.13) is the transport equation for the acceleration wave intensity, and it has been expressed in this form as this is an obvious generalization of the canonical form introduced by Bailey and Chen [7,8,9]. However, on account of the geometrical effect introduced by the change of cross-sectional area, (3.13) is now a generalized Riccati equation [12], rather than the Bernoulli equation found by Bailey and Chen.

We conclude this section with some general remarks about the coefficients μ, β, and γ. As observed by Bailey and Chen, the coefficient β is soley determined by the material behaviour, but μ and the new coefficient γ involve both the material behaviour and the geometry. Now $T_0 > 0$, and it has already been observed in (2.7) that for c to be real it is necessary to have

$\partial^2\Sigma/\partial p^2 > 0$. However the *sign* of $\partial^3\Sigma/\partial p^3$ will be determined by the particular material that is involved.

Some special cases of area variation arise which are worthy of note. If S is such that $\frac{d^2}{dX^2}(\ell n\ S) > 0$ the graph of $\ell n\ S$ is concave-up, and it will be increasing if $\frac{d}{dX}(\ell n\ S) > 0$ and decreasing if $\frac{d}{dX}(\ell n\ S) < 0$. Conversely, if S is such that $\frac{d^2}{dX^2}(\ell n\ S) < 0$ the graph of $\ell n\ S$ is concave-down, and it will be increasing if $\frac{d}{dX}(\ell n\ S) > 0$ and decreasing if $\frac{d}{dX}(\ell n\ S) < 0$.

When $S \equiv$ const. both μ and γ vanish and $\beta = \beta_0 =$ const. Equation (3.13) may then be integrated to give

$$s(X) = \frac{s(0)}{1-\beta_0 s(0)X}\ , \tag{3.17}$$

where $s(0)$ is the initial *acceleration* wave intensity when $X = 0$. In the event that $\beta_0 s(0) > 0$ the acceleration wave intensity will become infinite after it has propagated a finite distance X_∞, which we shall call the escape distance. In this case this is given by [see 4]

$$X_\infty = 1/\beta_0 s(0)\ . \tag{3.18}$$

When $\beta_0 s(0) < 0$ the acceleration wave intensity will simply decay to zero as the wave propagates. The position $X = X_\infty$ may be interpreted as the point at which a shock first forms, but we shall not pursue this aspect further and refer instead to Chen [9].

The transport equation (3.13) can only be used to study general wave propagation if $\partial^2\Sigma/\partial p^2$ and $\partial^3\Sigma/\partial p^3$ are known, and the simple case of a constant cross-section rod of *fixed length* L may be used to provide this information. If the sound speed $c(p)$ in such a rod is measured for different displacement gradients p, then $\partial^2\Sigma/\partial p^2$ follows by use of (2.7). Similiarly, if for such a rod of length L the initial intensity $s(0)$ and transmitted intensity $s(L)$ are measured for different p, then $\partial^3\Sigma/\partial p^3$

follows by solving (3.17) for β_0 to get

$$\beta_0 = \frac{s(L) - s(0)}{Ls(0)}, \tag{3.19}$$

and then using (3.15). This pre-supposes, of course, that the length L is sufficiently short to avoid escape of the intensity s before the end of the rod is reached.

4. Escape of Solutions to the Transport Equation

In this final section we offer some results concerning the conditions necessary for the escape of the solution of the transport equation (3.13). These are obtained by making simple assumptions about the coefficients α, β and γ of equation (3.13), and then using elementary comparison equation methods. No systematic study of the qualitative properties of the transport equation has been attempted, since our objective is merely to establish that, in general, an acceleration wave in a hyperelastic material will have a finite escape distance. However, in addition, we also establish the existence of thresholds $\sigma_0^+ > 0$ and $\sigma_0^- < 0$. These have the property that in certain circumstances escape only occurs at an infinite distance when the initial acceleration wave intensity $s(0)$ is such that $s(0) \downarrow \sigma_0^+$, while when $s(0) \uparrow \sigma_0^-$ the solution s is monotonic increasing with $\lim_{X \to \infty} s(X) > \sigma_0^-$.

We now prove three theorems, the first two being related, and begin by supposing $\gamma(X) > 0$ for $X \geq 0$, for then

$$-\mu s + \beta s^2 + \gamma > -\mu s + \beta s^2 .$$

Now consider the equation

$$\frac{ds}{dX} = -\mu s + \beta s^2 + \gamma , \tag{4.1}$$

and the comparison equation

$$\frac{d\sigma}{dX} = -\mu\sigma + \beta\sigma^2 , \tag{4.2}$$

with $s(0) = \sigma(0) > 0$, where we now assume $\mu(x) < 0$, $\beta(X) > 0$ for $X \geq 0$. Then

$$\frac{d}{dX}(s - \sigma)\Big|_{X=0} = \gamma(0) > 0 ,$$

and it follows directly that

$$\frac{d}{dX}(s - \sigma) > 0 \quad \text{for} \quad X \geq 0 , \tag{4.3}$$

unless the derivative becomes unbounded for some finite escape distance X_∞^+.

This shows that $s > \sigma$ as long as the derivative in (4.3) is defined. If σ has a finite escape distance $\tilde{X}_\infty^+$, then so also has s, say X_∞^+, where $0 < X_\infty^+ < \tilde{X}_\infty^+$ showing that $\tilde{X}_\infty^+$ provides an upper bound for X_∞^+. To find $\tilde{X}_\infty^+$ we may use the results of Bailey and Chen [7, Theorem 4.3(i) or 9, Theorem 3.2.4(i)] in connection with the comparison equation (4.2) with $\sigma(0) = s(0) > 0$.

The result asserts that if $s(0) > \alpha$, where

$$\alpha = 1/\{\int_0^\infty \beta(X)\, \exp(-\int_0^X \mu(\tau)d\tau)dX\}\ , \tag{4.4}$$

then there exists a finite escape distance $\tilde{X}_0^+ > 0$ such that

$$\int_0^{\tilde{X}_\infty^+} \beta(X)\, \exp(-\int_0^X \mu(\tau)d\tau)dX = 1/s(0). \tag{4.5}$$

We conclude that, when a finite $\tilde{X}_\infty^+$ exists, there is some X_∞^+ with $0 < X_\infty^+ < \tilde{X}_\infty^+$ such that

$$\lim_{X\uparrow X_\infty^+} s(X) = +\infty\ .$$

The second part of the theorem by Bailey and Chen cannot be used to establish the decay of $s(X)$ to zero as $X\uparrow+\infty$ when $0 < s(0) < \alpha$. This is because we are working with the inequality $s > \sigma$ and the decay of σ does not imply the decay of s.

Let us now consider equation (4.1) and the comparison equation (4.2) when $\mu(X) < 0$, $\beta(X) < 0$, $\gamma(X) < 0$ and $s(0) = \sigma(0) < 0$. Then the same form of reasoning as before again establishes that escape is possible for σ at some escape distance $\tilde{X}_\infty^- > 0$, determined by the results of Bailey and Chen when a threshold condition on $s(0)$ is satisfied, and that this time

$$\lim_{X\uparrow \tilde{X}_\infty^-} \sigma(X) = -\infty\ .$$

Since the stated conditions ensure that $s < \sigma$, we conclude that when σ has an escape distance $\tilde{X}_\infty^-$, then so has s, say X_∞^-, where $0 < X_\infty^- < \tilde{X}_\infty^-$, and that

$$\lim_{X \uparrow \tilde{X}_\infty^-} s(X) = -\infty$$

For the same reason as before, no conclusion may be reached from the second part of the theorem by Bailey and Chen concerning the decay of s to zero when the threshold condition on $s(0)$ is not satisfied and $-\alpha < s(0) < 0$.

We have proved the following result.

<u>Theorem 1</u>

Consider the differential equation

$$\frac{ds}{dX} = -\mu s + \beta s^2 + \gamma ,$$

where μ, β and γ are integrable functions of X on every finite subinterval of $[0,\infty)$; and let

$$\alpha = 1/ \{ \int_0^\infty |\beta(X)| \exp(-\int_0^X \mu(\tau)d\tau)dX\} .$$

Then, if $\left\{\begin{matrix}\mu(X) < 0, \ \beta(X) > 0, \ \gamma(X) > 0 \\ \mu(X) < 0, \ \beta(X) < 0, \ \gamma(X) < 0\end{matrix}\right\}$ and $\left\{\begin{matrix} s(0) > 0 \\ s(0) < 0\end{matrix}\right\}$ with $|s(0)| > \alpha$,

there exists a unique finite escape distance $\left\{\begin{matrix} X_\infty^+ > 0 \\ X_\infty^- > 0\end{matrix}\right\}$ for $s(X)$, with upper bounds $\left\{\begin{matrix} \tilde{X}_\infty^+ \\ \tilde{X}_\infty^-\end{matrix}\right\}$, such that

$$\int_0^{\tilde{X}_\infty^\pm} \beta(X) \exp(-\int_0^X \mu(\tau)d\tau)dX = 1/s(0) ,$$

and $\left\{\begin{matrix} \lim_{X \uparrow X_\infty^+} s(X) = +\infty \\ \lim_{X \uparrow X_\infty^-} s(X) = -\infty \end{matrix}\right\}$.

Now suppose constants μ_0, β_0 exist such that $\mu(X) < \mu_0 < 0$, $\beta(X) > \beta_0 > 0$, and suppose also that $\gamma(X) > 0$ and $s(0) > 0$. Then

$$- \mu s + \beta s^2 + \gamma > - \mu_0 s + \beta_0 s^2 ,$$

so that considering equation (4.1) and the constant coefficient comparison equation

$$\frac{d\sigma}{dX} = - \mu_0 \sigma + \beta_0 \sigma^2 \tag{4.6}$$

with $\sigma(0) = s(0) > 0$ we see, as before, that $s > \sigma$. Applying Theorem 1 to (4.6) we find $\alpha = 0$, so that in this case an escape distance $\tilde{X}_\infty^+$ always exists for σ, and hence for s. A trivial integration of (4.6), or of the condition determining $\tilde{X}_\infty^+$ in Theorem 1, shows

$$\tilde{X}_\infty^+ = \frac{1}{\mu_0} \ell n\left(\frac{s(0)}{s(0)-\lambda_0}\right) > 0 , \tag{4.7}$$

where $\lambda_0 = \mu_0/\beta_0 > 0$. Hence, if X_∞^+ is the escape distance for s, we have $0 < X_\infty^+ < \tilde{X}_\infty^+$, where

$$\lim_{X \uparrow X_\infty^+} s(X) = +\infty .$$

A similar form of argument applies when $\mu(X) \leq \mu_0 < 0$, $\beta(X) \leq \beta_0 < 0$, $\gamma(X) < 0$ and $s(0) < 0$, only now the solution s escapes negatively at some escape distance X_∞^- where $0 < X_\infty^- < \tilde{X}_\infty^-$ and $\tilde{X}_\infty^-$ is still given by equation (4.7). We have thus arrived at our next result.

<u>Theorem 2</u>

Let s satisfy the differential equation

$$\frac{ds}{dX} = - \mu s + \beta s^2 + \gamma .$$

Then if constants μ_0, β_0 exist such that

$$\left\{\begin{matrix} \mu(X) < \mu_0 < 0,\ \beta(X) > \beta_0 > 0 \\ \mu(X) < \mu_0 < 0,\ \beta(X) < \beta_0 < 0 \end{matrix}\right\} \text{ and } \left\{\begin{matrix} \gamma(X) > 0,\ s(0) > 0 \\ \gamma(X) < 0,\ s(0) < 0 \end{matrix}\right\} \text{ for } X \geq 0, \text{ it}$$

follows that there exists a finite escape distance $\left\{\begin{matrix} X_\infty^+ > 0 \\ X_\infty^- > 0 \end{matrix}\right\}$ for s such that

$$\left\{\begin{matrix} \lim_{X \uparrow X_\infty^+} s(X) = +\infty \\ \lim_{X \uparrow X_\infty^-} s(X) = -\infty \end{matrix}\right\}, \text{ where an upper bound } \left\{\begin{matrix} \tilde{X}_\infty^+ \\ \tilde{X}_\infty^- \end{matrix}\right\} \text{ on } \left\{\begin{matrix} X_\infty^+ \\ X_\infty^- \end{matrix}\right\} \text{ is provided by}$$

$$\tilde{X}_\infty^{\pm} = \frac{1}{\mu_0} \ell n \left(\frac{s(0)}{s(0)-\lambda_0}\right) > 0 ,$$

with

$$\lambda_0 = \begin{Bmatrix} \mu_0/\beta_0 < 0 \\ \mu_0/\beta_0 > 0 \end{Bmatrix} .$$

For our final theorem we begin by supposing constants α_0, β_0, γ_0 exist such that $\mu(X) < \mu_0 < 0$, $\beta(X) < \beta_0 < 0$ and $\gamma_0 < \gamma(X) < 0$ for $X > 0$, and that s satisfies equation (4.1) with an appropriate initial condition $s(0)$ which ensures

$$-\mu s + \beta s^2 + \gamma > -\mu_0 s + \beta_0 s^2 + \gamma_0 > 0 . \tag{4.8}$$

Clearly condition (4.8) will be true at $X = 0$ if $|s(0)|$ is sufficiently large, thereby ensuring that when comparing equation (4.1) with the comparison equation

$$\frac{d\sigma}{dX} = -\mu_0 \sigma + \beta_0 \sigma^2 + \gamma_0 , \tag{4.9}$$

the inequality will remain true for $X > 0$ so that $s > \sigma$. Hence the escape of σ will ensure the escape of s.

Now the right hand side of (4.9) will be positive provided σ does not lie between its two zeros

$$\sigma_0^{\pm} = [\mu_0 \pm (\mu_0^2 - 4\beta_0\gamma_0)^{1/2}]/2\beta_0 .$$

Consequently, inequality (4.8) will be true for $X > 0$, so that $s > \sigma$, provided either (i) $s(0) = \sigma(0) > \sigma_0^+ > 0$, or (ii) $s(0) = \sigma(0) < \sigma_0^- < 0$.

Suppose (i) is true and write (4.9) in the form

$$\frac{1}{\beta_0} \frac{dY}{dX} = Y^2 - a^2 , \tag{4.10}$$

where $Y = \sigma - (\mu_0/2\beta_0)$ and $a^2 = (\mu_0/2\beta_0)^2 - (\gamma_0/\beta_0)$ or, equivalently,

$a = \sigma_0^+ - (\mu_0/2\beta_0)$ with $a > 0$. Then (4.10) may be integrated to give

$$X = \frac{1}{2a\beta_0}\left\{\ln\left(\frac{Y-a}{Y+a}\right) - \ln\left(\frac{Y_0-a}{Y_0+a}\right)\right\}, \tag{4.11}$$

where $Y_0 = \sigma(0) - (\mu_0/2\beta_0)$.

This may be solved for σ when we find

$$\sigma = \frac{\mu_0}{2\beta_0} + a\left(\frac{1 + A\exp(2\beta_0 aX)}{1 - A\exp(2\beta_0 aX)}\right), \tag{4.12}$$

where

$$A = \frac{Y_0-a}{Y_0+a} = \frac{\sigma(0) - \sigma_0^+}{\sigma(0) - \sigma_0^-} \quad \text{and} \quad 0 < A < 1.$$

Inspection of (4.12) shows that escape of σ occurs at $\tilde{X}_\infty^+$, when $1 = A\exp(2\beta_0 a\,\tilde{X}_\infty^+)$. This leads to the required upper bound $\tilde{X}_\infty^+$ for the escape distance for s

$$\tilde{X}_\infty^+ = \frac{1}{2\beta_0 a}\ln\left(\frac{\sigma(0) - \sigma_0^-}{\sigma(0) - \sigma_0^+}\right) > 0\,. \tag{4.13}$$

The significance of the threshold σ_0^+ is seen by examining result (4.13). The closer the initial value $s(0) = \sigma(0)$ is to σ_0^+, the further the comparison solution propagates before the escape of σ occurs.

Repetition of the same form of argument for case (ii) leads to the results

$$\sigma = \frac{\mu_0}{2\beta_0} + b\left(\frac{1 + B\exp(2\beta_0 bX)}{1 - B\exp(2\beta_0 bX)}\right), \tag{4.14}$$

where $b = (\mu_0/2\beta_0) - \sigma_0^-$ with $b > 0$, and

$$B = \frac{Y_0 - b}{Y_0 + b} = \frac{\sigma(0) - \sigma_0^+}{\sigma(0) - \sigma_0^-} \quad \text{with} \quad B > 1\,. \tag{4.15}$$

Inspection of (4.14) shows that since $B > 1$ and $\beta_0 b > 0$, there can be no escape of σ as $X \to +\infty$. In fact, as $X \to +\infty$ so $\sigma \to (\mu_0/2\beta_0) - b = \sigma_0^-$, showing that as $X \to +\infty$ s must exceed the value σ_0^-.

We have thus proved our final result.

Theorem 3

Let s satisfy the differential equation

$$\frac{ds}{dX} = -\mu s + \beta s^2 + \gamma ,$$

subject to the initial condition $s = \sigma(0)$, where $\mu(X) < \mu_0 < 0$, $\beta(X) > \beta_0 > 0$, $\gamma_0 < \gamma(X) < 0$ for $X \geq 0$, and set

$$\sigma_0^{\pm} = [\mu_0 \pm (\mu_0^2 - 4\beta_0\gamma_0)^{1/2}]/2\beta_0 .$$

Then it follows that:

(i) if $\sigma(0) > \sigma_0^+$ the solution s will escape at a finite escape distance $X_\infty^+ > 0$, where an upper bound $\tilde{X}_\infty^+$ on X_∞^+ is provided by

$$\tilde{X}_\infty^+ = \frac{1}{\left(2\beta_0\sigma_0^+ - \mu_0\right)} \ell n\left(\frac{\sigma(0) - \sigma_0^-}{\sigma(0) - \sigma_0^+}\right) ,$$

and

$$\lim_{X \uparrow X_\infty^+} s(X) = +\infty ;$$

(ii) if $\sigma(0) < \sigma_0^-$ the solution $s(X)$ is monotonic increasing and

$$\lim_{X \to \infty} s(X) > \sigma_0^- .$$

In conclusion, we remark that other similar theorems may be formulated, but these three will suffice to indicate the behaviour of the solution to the transport equation (3.13) in some typical circumstances. Here again we will not pursue further the matter of shock wave formation associated with the escape of the acceleration wave intensity s.

REFERENCES

[1] Eringen, A. C., and Suhubi, E. S., Elastodynamics 1. Finite Motions, Academic Press, New York, 1974.

[2] Antman, S. S., Kirchhoff's problem for nonlinearly elastic rods. Quart. Appl. Math. 32(1974), 221-240.

[3] Antman, S. S., and Jordan, K. B., Qualitative aspects of the spatial deformation of non-linearly elastic rods. Proc. Roy. Soc. Edinburgh 73A(1975), 85-105.

[4] Jeffrey, A., and Teymur, M., Formation of shock waves in hyperelastic solids, Acta Mechanica 20(1974), 133-149.

[5] Jeffrey, A., and Suhubi, E. S., Propagation of weak discontinuities in a layered hyperelastic half-space. Proc. Roy. Soc. Edinburgh 75A(1975/76), 209-221.

[6] Antman, S. S. and Liu, T. P., Travelling waves in hyperelastic rods. Quart. Appl. Math. 37(1979), 377-399.

[7] Bailey, P. B., and Chen, P. J., On the local and global behaviour of acceleration waves. Arch Rat. Mech. Anal. 41(1971), 121-131.

[8] Bailey, P. B., and Chen, P. J., On the local and global behaviour of acceleration waves: Addendum, Asymptotic behaviour. Arch. Rat. Mech. Anal. 44(1971), 212-216.

[9] Chen, P. J. Selected Topics in Wave Propagation, Noordhoff International Publishing, Leyden, The Netherlands, 1976.

[10] Collins, W. D., One-dimensional nonlinear wave propagation in incompressible elastic materials. Quart. J. Mech. and Appl. Math. 19(1966), 259-328.

[11] Gurtin, M., On the breaking of water waves on a sloping beach of arbitrary shape. Quart. Appl. Math. 33(1975), 187-189.

[12] Kamke, E., Differentialgleichungen Lösungsmethoden und Lösungen. Vol. 1. Chelsea Publishing Co., New York, 1959 (Section 4.9).

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It is shown that when an acceleration wave propagates in a hyperelastic rod with slowly varying cross-section, the transport equation for the wave intensity is a generalized Riccati equation. The three coefficients in the equation all depend on the material properties, but only the coefficient of the quadratic term is independent of the effect of area change. Three theorems are proved, based on the use of comparison equations, which establish that in general the acceleration wave intensity will become infinite (escape) after the wave has

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propagated only a finite distance along the rod. The existence of thresholds for the initial intensity are also established in certain cases, with their most notable property being that as the initial intensity decreases towards the threshold, so the distance the wave propagates to escape increases without bound.

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