ON THE HAAR AND WALSH SYSTEMS ON A TRIANGLE.
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ON A TRIANGLE

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ABSTRACT

In this paper we establish the Haar and Walsh systems on a triangle. These systems are complete in \( L_2(A) \). The uniform convergence of the Haar-Fourier series and the uniform convergence by group of the Walsh-Fourier series for any continuous function are proved.

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SIGNIFICANCE AND EXPLANATION

A number of papers have been concerned with developing the theories of discontinuous orthonormal systems and their applications. In particular, the Haar and Walsh systems are presently the most important examples of nonsinusoidal functions, and have proved most useful in communication.

Some authors have studied the properties of approximation from the mathematical point of view. It seems interesting and helpful for both theory and practice to investigate the Haar and Walsh functions for a multivariate setting. In fact, many signals in communications and other functions are of several variables (for instance, TV signals have two space variables and the time variable). If the domain of definition of the system is tensor product, then the existing systems are readily extended to several variables.

The problem is how to construct an orthonormal system on a triangular domain in the plane, or more generally, on a simplex in n-dimensional space. This paper defines the Haar and Walsh system on a triangle domain, proves the orthogonality and completeness in $L_2$. Also the uniform convergence for the Haar-Fourier series, uniform convergence by group for the Walsh-Fourier series are studied. All of these results can be generalized easily to n dimensions.
ON THE HAAR AND WALSH SYSTEMS ON A TRIANGLE

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1. Introduction

No doubt, it is interesting and useful to study multivariate Haar and Walsh functions either in theory or in practice. If we investigate on a domain which can be considered a Cartesian product, then the functions are readily extended to several variables from the one variable. Setting by the tensor product construct Harmuth has shown those kinds of multivariate systems in his book [5] and pointed out the applications in communication.

In this paper we attempt to focus on a triangle, or more generally on a simplex in n-dimensional space. We were unable to find any paper about it. Perhaps it puzzles some people temporarily.

The main contribution of this paper is to establish the Haar and Walsh system with two variables on a triangle. We prove their orthonormality and completeness in Hilbert space $L^2$. Moreover, the corresponding Haar-Fourier series and Walsh-Fourier series for any continuous function are uniformly convergent and convergent by group respectively.

It is easy to generalize these results to the n-dimensional simplex. For simplicity we will discuss only the two-dimensional triangle.

Now we should explain some preliminaries and notations.

The Haar functions on $[0,1]$ are defined as follows:

$$\chi_0(t) := 1 \text{ for } 0 < t < 1,$$

and

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The Walsh functions on \([0,1]\) consist of the following ones:

\[
W_0(t) := \begin{cases} 
1 & \text{for } 0 < t < 1, \\
0 & \text{elsewhere in } [0,1] 
\end{cases}
\]

\[
W_1(t) := \begin{cases} 
1 & \text{for } 0 < t < \frac{1}{2}, \\
-1 & \text{for } \frac{1}{2} < t < 1. 
\end{cases}
\]  

(1.2)

Some detailed investigation of the Haar and Walsh systems can be found in [1], [3], [5].

In order to generalize the Haar and Walsh systems to the two-dimensional case we should explain our representation in this paper. The Cartesian coordinates are not very convenient for triangular elements, and a special type of coordinate system called area coordinates should be used.

Referring to Figure 1 it is seen that the internal point \( P \) will divide the triangle \( ABC \) into three smaller triangles, and depending on the position of the point \( P \), the area of each one of the triangles \( PAB, PBC, PCA \) can vary from zero to \(|A|\), which is the area of the triangle \( ABC \). In other words, the ratios \( \frac{a}{|A|}, \frac{b}{|A|} \) and \( \frac{c}{|A|} \) will take up any value between zero and unity. Here \( a, b, c \) are the area of triangles \( PBC, PCA, PAB \) respectively.
These ratios are called area coordinates, and they are defined by

\[ l_1 = \frac{a}{|A|}, \]
\[ l_2 = \frac{b}{|A'|}, \quad l_3 = \frac{c}{|A'|}. \]

It is easy to see that

\[
\begin{pmatrix}
1 \\
x \\
y
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 1 \\
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3
\end{pmatrix}
\begin{pmatrix}
l_1 \\
l_2 \\
l_3
\end{pmatrix}.
\]

If two points \( P \) and \( Q \) are in two similar triangles respectively, and have the same area coordinates, then we denote them by \( P \sim Q \).
2. An orthonormal sequence \( X \) on a triangular domain

Suppose \( \Delta \) (or \( \Delta ABC \)) is any triangle on a plane and \( |\Delta| = 1 \) is the area of the \( \Delta ABC \). If \( D, E, F \) are midpoints of \( AB, BC, CA \) respectively, connecting \( DE, EF, FD \), we divide the \( \Delta \) into four similar small triangles \( \Delta ADF, \Delta DBE, \Delta EFC, \Delta EFD \). We call them \( \Delta_1, \Delta_2, \Delta_3, \Delta_4 \) respectively.

We define the sequence \( X \) as follows:

\[
X_0(P) := \begin{cases} 1 & \text{for } P \in \Delta, \\ \chi_1(P) := \begin{cases} 1 & \text{for } P \in \Delta_1 \cup \Delta_2, \\ -1 & \text{for } P \in \Delta_3 \cup \Delta_4, \\ \end{cases} \\ \chi_2(P) := \begin{cases} \sqrt{2} & \text{for } P \in \Delta_1, \\ -\sqrt{2} & \text{for } P \in \Delta_2, \\ 0 & \text{for } P \in \Delta_3 \cup \Delta_4, \\ \end{cases} \\ \chi_3(P) := \begin{cases} \sqrt{2} & \text{for } P \in \Delta_3, \\ -\sqrt{2} & \text{for } P \in \Delta_4, \\ 0 & \text{for } P \in \Delta_1 \cup \Delta_2, \\ \end{cases} \\ \vdots \vdots \vdots \\ \chi_{n+1}(P) := \begin{cases} 2x_{n-1}^{(1)}(Q) & \text{for } P \in \Delta_{j+1}, \\ 0 & \text{for } P \in \Delta \setminus \Delta_{j+1}, \\ \end{cases}
\]

where \( Q \in \Delta, Q \neq P, j = 0, 1, 2, 3, i = 1, 2, \ldots, 3 \cdot 4^{n-2}, n = 2, 3, \ldots \).

At a point of discontinuity, let the value of these functions be the average.

Now we consider the orthogonality of the sequence \( X \). We prove the following theorem.

**Theorem.** The sequence \( X \) defined by (2.1) is orthonormal.

**Proof.** At first, it is easy to check that when \( n < 2 \) the sequence \( \{X_n^{(j)}\} \) is orthonormal. We suppose that the theorem holds for \( n < N \). For \( 2 \leq m < N + 1 \),

\[
j_1, j_2 = 0, 1, 2, 3; i_1 = 1, 2, \ldots, 3 \cdot 4^{N-1}; i_2 = 1, 2, \ldots, 3 \cdot 4^{m-2}, \text{ by (2.1) and induction}
\]

\[-4-\]
hypothesis, we get

\[ \int_{\Delta} x_{N+1} \left( p \right) x_{m} \left( p \right) dp = \delta_{j_1 j_2} \int_{\Delta} x_{N} \left( q \right) x_{m-1} \left( q \right) dp \]

\[ = \delta_{j_1 j_2} \int_{\Delta} x_{N} \left( q \right) x_{m-1} \left( q \right) dp \]

\[ = \delta_{j_1 j_2} \delta_{i_1 i_2} \delta_{N m-1} \]

It is easy to verify that

\[ \int_{\Delta} x_{N+1} \left( p \right) x_{1} \left( p \right) dp = \int_{\Delta} x_{N+1} \left( p \right) x_{0} \left( p \right) dp = 0. \]

Therefore the theorem holds for \( n = N + 1 \), and this finishes the induction.
3. Convergence properties

The triangle \( \Delta \) has been divided into four similar smaller triangles \( \Delta_i \) \((i = 1, 2, 3, 4)\). Now set
\[
\Delta_{1,i} := \Delta_i \quad (i = 1, 2, 3, 4) \, .
\]

For each \( \Delta_{1,i} \) we divide it into four similar smaller triangles in the same way as we did before. We order them as \( \Delta_{2,1}, \Delta_{2,2}, \ldots, \Delta_{2,16} \) such that
\[
\Delta_{1,i} = \Delta_{2,4i-1} \cup \Delta_{2,4i-2} \cup \Delta_{2,4i-3} \quad i = 1, 2, 3, 4 \, .
\]

We continue this process. For any \( n \) we get a sequence \( \Delta_{n,1}, \Delta_{n,2}, \ldots, \Delta_{n,4^n} \) such that
\[
\Delta_{n-1,i} = \Delta_{n,4i-1} \cup \Delta_{n,4i-2} \cup \Delta_{n,4i-3} \quad i = 1, 2, 3, \ldots, 4^{n-1} \, , \quad n = 1, 2, 3, \ldots \, .
\]
\[
\Delta_{0,1} := \Delta \, .
\]

Define a function sequence \( \{f_{n,i}\} \) on the \( \Delta_i \):
\[
f_0(P) := 1 \quad \text{for} \ P \in \Delta \, ,
\]
\[
f_{1,i}(P) := \begin{cases} 
1 & \text{for} \ P \in \Delta_{1,i} \\
0 & \text{for} \ P \notin \Delta \setminus \Delta_{1,i} 
\end{cases} \quad i = 1, 2, 3, 4 \, ,
\]
\[
\vdots \quad \vdots \quad \vdots 
\]
\[
f_{n,i}(P) := \begin{cases} 
1 & \text{for} \ P \in \Delta_{n,i} \\
0 & \text{for} \ P \notin \Delta \setminus \Delta_{n,i} 
\end{cases} \quad i = 1, 2, 3, \ldots, 4^n \, , \quad n = 1, 2, 3, \ldots \, .
\]

It is obvious that the sequence \( \{f_{n,i}\} \) is orthogonal.

Let
\[
W_n := \text{span}(f_{n,1}, f_{n,2}, \ldots, f_{n,4^n}) \quad (n > 0) \, .
\]

Thus
\[
\dim W_n = 4^n \, .
\]

For convenience, sometimes we use notation
\[
X_i := x_0 \, ,
\]
\[
X_{n-1,i} := x^{(i)}_n \quad n > 1, \quad i = 1, 2, \ldots, 3 \times 4^{n-1} \, .
\]
Set
\[ H_n := \text{span}(X_1, X_2, \ldots, X_n) \] It is clear that
\[ H_{4n} = H_n, \] (3.4)
since \( H_{4n} \subseteq H_n \) and \( \dim H_{4n} = \dim H_n = 4^n \).

We define
\[ L_2(\Delta) := \{ f | \int_{\Delta} f^2 d\sigma < \infty \} \]
and
\[ L_2^2 := \int_{\Delta} f^2 d\sigma. \]

Then the Fourier series of a given function \( F \in L_2(\Delta) \) in terms of the function sequence \( \{X_n\} \) is
\[ F = \sum_{i=1}^{\infty} a_i X_i \] (3.5)
with
\[ a_i := \int_{\Delta} F(x)X_i(x) dx. \]

Let
\[ P_n F := \sum_{i=1}^{n} a_i X_i(x) \] (3.6)
be the n-th partial sum of the series (3.5).

From the orthogonality of sequence \( \{X_i\} \) we know that \( P_n F \) is the best \( L_2 \)-approximation to \( F \) from \( H_n \). Hence it is convergent to \( F \) if \( F \) is in \( L_2(\Delta) \), since \( H_n \) is dense in \( L_2(\Delta) \). Thus we get the following theorem.

Theorem 2. If \( F \in L_2(\Delta) \), then
\[ \lim_{n \to \infty} \| F - P_n F \|_2 = 0, \]
In order to study the uniform convergence we let

\[ C(\Delta) := \{ f \mid f \text{ is continuous on } \Delta \} \]

and

\[ \|f\| = \max_{P \in \Delta} |f(P)|. \]

For \( P \in C(\Delta) \) we define

\[ p_n^{j}(f) := \int_{\Delta} F_0 \, d\sigma - \int_{\Delta} F_1 \, d\sigma + \cdots + \int_{\Delta} F_n \, d\sigma + x_0^{j}. \]  

(3.7)

Set

\[ K_0(P,Q) := x_0(P)x_0(Q), \quad (\text{for } P,Q \in \Delta) \]

\[ \ldots \quad \ldots \quad \ldots \]

\[ K_n^{j}(P,Q) := x_0(P)x_0(Q) + x_1^{(1)}(P)x_1^{(1)}(Q) + \cdots + x_n^{j}(P)x_n^{j}(Q), \quad j = 1,2,\ldots,3 \cdot 4^{n-1}, \quad n = 1,2,\ldots. \]

(3.8)

Thus

\[ p_n^{j}(f) = \int_{\Delta} K_n^{j}(P,Q)f(Q) \, dQ. \]

(3.9)

Let \( A := (a_{ij}) \) \((1, j = 1,2,3,\ldots,4^n)\) be any \( 4^n \times 4^n \) \((n = 1,2,\ldots)\) matrix and \( G(P,Q) \) be any function defined on \( \Delta \times \Delta \).

The notation \( G(P,Q) \leftrightarrow A \) means that the value of \( G(P,Q) \) is \( a_{ij} \) when \( P \in \Delta, j \)

\( Q \in \Delta \). It leads to the following relationship:

\[ x_0(P)x_0(Q) \leftrightarrow a_0 := \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}, \]

\[ x_1^{(1)}(P)x_1^{(1)}(Q) \leftrightarrow a_1 := \begin{pmatrix}
1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 \\
-1 & -1 & 1 & 1
\end{pmatrix}. \]

(3.10)
In order to write those more shortly, we use the notation

\[
\text{diag block}(A_1, A_2, \ldots, A_m) := \begin{bmatrix}
A_1 & 0 \\
0 & A_2 \\
& & \ddots \\
0 & 0 & & A_m
\end{bmatrix},
\]

where \( A_1 \) is square submatrix.

Using (3.11) we get

\[
X_1^{(1)}(P, Q) + \sigma_0 + \sigma_1 = \text{diag block} \begin{bmatrix}
2 & 2 \\
2 & 2
\end{bmatrix},
\]

\[
X_1^{(2)}(P, Q) + \sigma_0 + \sigma_1 + \sigma_2 = \text{diag block} \begin{bmatrix}
2 & 2 \\
2 & 2
\end{bmatrix},
\]

\[
X_1^{(3)}(P, Q) + \sigma_0 + \sigma_1 + \sigma_2 + \sigma_3 = \text{diag block}(4, 4, 4, 4),
\]

where \( \text{diag block}(4, 4, 4, 4) = 4I_4 \otimes I_n \) is \( n \times n \) identity matrix. We denote the \( m \times m \) zero-element matrix by \( O_m \) below.

Since

\[
X_2^{(1)}(P) X_2^{(1)}(Q) + \text{diag}(4\sigma_1, O_4, O_4, O_4), \quad (i = 1, 2, 3)
\]

we get
\[ K^{(1)}_2(P, Q) \leftrightarrow \text{diag block}(4 \sum_{j=0}^{1} \sigma_j, 4I_4, 4I_4, 4I_4), \]
\[ K^{(3+1)}_2(P, Q) \leftrightarrow \text{diag block}(4^2I_4, 4 \sum_{j=0}^{1} \sigma_j, 4I_4), \]
\[ K^{(6+1)}_2(P, Q) \leftrightarrow \text{diag block}(4^2I_4, 4^2I_4, 4 \sum_{j=0}^{1} \sigma_j, 4I_4), \]

and
\[ K^{(9+1)}_2(P, Q) \leftrightarrow \text{diag block}(4^3I_4, 4^2I_4, 4^2I_4, 4 \sum_{j=0}^{1} \sigma_j), \quad (i = 1, 2, 3) \]

especially
\[ K^{(12)}_2(P, Q) \leftrightarrow 4^2I_{16} = \text{diag block}(4^2I_4, 4^2I_4, 4^2I_4, 4^2I_4). \]

Suppose in the general case that
\[ K^{(3+4n-1)}_n(P, Q) \leftrightarrow 4^nI_{4^n}. \quad (3.12) \]

By definition of (2.1) and (3.8),
\[ K^{(t+1)}_{t+1}(P, Q) \leftrightarrow \text{diag block}(K_1, K_2, \ldots, K_{4^n}, 4^n) \]

where each \( K_i \) is a \( 4 \times 4 \) matrix. More precisely
\[ K^{(1)}_{n+1}(P, Q) \leftrightarrow \text{diag block}(4^n \sum_{t=0}^{1} \sigma_t, 4^nI_4, \ldots, 4^nI_4), \quad (i = 1, 2, 3), \]
\[ \ldots \ldots \ldots \]
\[ K^{(3+1)}_{n+1}(P, Q) \leftrightarrow \text{diag block}(4^{n+1}I_4, \ldots, 4^n \sum_{t=0}^{1} \sigma_t, \ldots, 4^nI_4) \]

where the term \( 4^n \sum_{t=0}^{1} \sigma_t \) is the \((j+1)\)-th block, \((j = 1, 2, \ldots, 4^n - 1)\) in particular
\[ K_{n+1}^{(3^n-1)}(P,Q) \rightarrow 4^{n+1} \sum_{n+1}^{n} \]

therefore for any \( n = 1,2, \ldots \) (3.12) holds.

Suppose
\[ a \in \Delta_{n, \Delta(4i-1)+1} \cap \Delta_{n-1, \Delta^i} \quad i = 1,2,3,4 \]

By (3.9) we know
\[ P_n(\Delta, F) = \int_{\Delta} P(a, \Delta) F(a) \, da \]

By (3.13), (3.14) we obtain
\[ P_n(\Delta, F) = 4^{n-1} \int_{\Delta_{n-1, \Delta^i}} P(Q) \, dQ = \frac{1}{|\Delta|} \int_{\Delta_{n-1, \Delta^i}} P(Q) \, dQ \]

for \( j \leq 3(\Delta - 1) \) and
\[ P_n(\Delta, F) = 4^n \int_{\Delta_{n-1, \Delta^i}} P(Q) \, dQ = \frac{1}{|\Delta|} \int_{\Delta_{n-1, \Delta^i}} P(Q) \, dQ \]

for \( j > 3\Delta \).

For \( j = 3\Delta - 2, 3\Delta - 1 \) we have
\[ P_n(\Delta, F) = \frac{2}{|\Delta|} \left( \int_{\Delta_{n, 4(i-1)+1}} P(Q) \, dQ + \int_{\Delta_{n, 4(i-1)+1}} P(Q) \, dQ \right) \]

(\( i = 1 \) or 3) and

\[ P_n(\Delta, F) = \left\{ \begin{array}{ll}
\frac{1}{|\Delta|} \int_{\Delta_{n, 4(i-1)+1}} P(Q) \, dQ & a \in \Delta_{n, 4(i-1)+1}, \\
\frac{2}{|\Delta|} \left( \int_{\Delta_{n, 4(i-1)+3}} P(Q) \, dQ + \int_{\Delta_{n, 4(i-1)+4}} P(Q) \, dQ \right) & i = 3,4 \end{array} \right. \]

In any case, from (3.15) to (3.18) we conclude
\[ \lim_{n \to \infty} P_n(\Delta, F) = \frac{1}{|\Delta|} \int_{\Delta} P(Q) \, dQ = P(a), \]

where \( \Delta \in \Delta_{n, i} \) and \( a \in \Delta_n, |\Delta| \to 0 \) when \( n \to \infty \).
It is easy to check that

$$\int_{\Delta} \left| K_n^{(j)} (p,q) \right| dq = 1. \quad (3.20)$$

Now (3.19), (3.20) imply the following theorem.

**Theorem 3.** For $F \in C(\Delta)$ \( \lim_{n \to \infty} \| P_n^{(j)} F - P_n^{(j)} \|_n = 0 \) \( (j = 1, 2, \ldots, 3 \cdot 4^{n-1}) \).
4. On the Walsh system

Naturally there exist some different forms of definition which are equivalent. We use
area coordinates to define the two-variable Walsh function. Some notations follow the case
of the Haar functions.

\[ W_0(P) = 1 \quad \text{for } P \in A; \]
\[ W_n^{(i)}(P) = W_n^{(i)}(Q) \quad \text{for } P \in A, \quad i = 1,2,\ldots,4^n, \]
\[ W_{n+1}^{(4^n+1)}(P) = \begin{cases} \lambda & \text{for } P \in \delta_1 \cup \delta_3, \\ -\lambda & \text{for } P \in \delta_2 \cup \delta_4, \end{cases} \]
\[ W_{n+1}^{(2\cdot4^n+1)}(P) = \begin{cases} \lambda & \text{for } P \in \delta_1 \cup \delta_2, \\ -\lambda & \text{for } P \in \delta_3 \cup \delta_4, \end{cases} \]
\[ W_{n+1}^{(3\cdot4^n+1)}(P) = \begin{cases} \lambda & \text{for } P \in \delta_1 \cup \delta_4, \\ -\lambda & \text{for } P \in \delta_2 \cup \delta_3, \end{cases} \]

(4.1)

where

\[ \lambda := w_n^{(i)}(Q), \quad Q \in \delta, \quad Q \sim P, \quad i = 1,2,3,\ldots,4^n, \quad n = 0,1,2,\ldots. \]
\[ W_0^{(1)}(P) := W_0(P) = 1 \quad \text{for } P \in A. \]

This finishes the definition of the sequence \( W \).

Sometimes we prefer \( W^i \) (\( i = 1,2,3,\ldots,4^{n+1} \)) to \( W_{n+1}^{(j\cdot4^n+i)} \) with \( i = j\cdot4^n + i \)
\( (i = 1,2,\ldots,4^n, \quad j = 0,1,2,3) \).

At a point of discontinuity, the values of these functions are taken as the average.

Figures 2 and 3 show the Walsh sequence when \( n = 0,1 \).

Before the discussion of the orthogonality of the Walsh system we introduce the
Hadamard matrix ([2],[4], p. 207).

The Hadamard matrix is a square array whose elements consist only of +1 and -1 and
whose rows (and columns) are orthogonal to one another. Obviously the lowest order
nontrivial Hadamard matrix is of the order two, viz.
Higher order matrices whose orders are powers of two can be obtained from the recurrent relationship

\[ H_n = H_{n/2} \otimes H_2 \]  

(4.3)

where \( \otimes \) denotes the direct or Kronecker product and \( n \) is a power of two. The direct product means replacing each element in the matrix by the \( H_2 \). With the help of the Hadamard matrix the one-dimensional Walsh function can be defined \([1]\). In the two-dimensional case we should use the \( 4 \times 4 \) matrix

\[ H_4 = H_2 \otimes H_2 \]

and get the recurrent relationship

\[ H_n = H_{n/4} \otimes H_4 \]  

(4.4)

The Hadamard matrix (4.4) corresponds to the Walsh sequence \( \{W_i\} \) \((i = 1, 2, \ldots, 4^n)\) for a given \( N \).

Figures 2 and 3 show the Walsh sequence associated with the Hadamard matrix.

In Figures 2 and 3 black areas represent \(+1\), white areas \(-1\). The following triangle shows a certain order.
2. (1) (3) (4)

-1 + + -

\[ W_{12} + W_{14} + \text{I} \]

\[ \text{where we omit I in these elements of the Hadamard matrix} \]

**Figure 2**

\[
\begin{align*}
  w_1^{(1)} & \leftrightarrow \begin{pmatrix} + & + & + \\ + & + & - \\ + & - & - \\ + & - & + \\ \end{pmatrix} = H_4 \\
  w_1^{(2)} & \leftrightarrow \begin{pmatrix} + & + & + \\ + & + & - \\ + & - & - \\ + & - & + \\ \end{pmatrix} \\
  w_1^{(3)} & \leftrightarrow \begin{pmatrix} + & + & + \\ + & + & - \\ + & - & - \\ + & - & + \\ \end{pmatrix} \\
  w_1^{(4)} & \leftrightarrow \begin{pmatrix} + & + & + \\ + & + & - \\ + & - & - \\ + & - & + \\ \end{pmatrix}
\end{align*}
\]

(\text{where we omit I in these elements of the Hadamard matrix})
Figure 3
Now we notice that

$$P_n F := \sum_{i=1}^{n} a_i W_i$$

is the best $L_2$-approximation to a given function $F$ from

$$M_n := \text{span}(W_i)_{i=1}^{n}$$

where

$$a_i := \int_{\Delta} F(p) W_i(p) d\alpha.$$ 

Hence it is convergent to $F$ if $F$ is in $L_2$, since $M_n$ is dense in $L_2$. I.e.

**Theorem 4.** If $F \in L_2(\Delta)$, then $\lim_{n \to \infty} \| F - P_n F \|_2 = 0$.

Since $M_4^n = M_n$ and from Theorem 3 we get the following theorem.

**Theorem 5.** Let $F \in C(\Delta)$, $P_n$ be $L_2$-projector onto $M_4^n$ on $\Delta$, then

$$\lim_{n \to \infty} \| F - P_n F \|_2 = 0.$$ 

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In this paper we establish the Haar and Walsh systems on a triangle. These systems are complete in $L^2(\Delta)$. The uniform convergence of the Haar-Fourier series and the uniform convergence by group of the Walsh-Fourier series for any continuous function are proved.