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MULTIPLE COMPARISONS FOR ORTHOGONAL CONTRASTS: EXAMPLES AND TAB--ETC(U)
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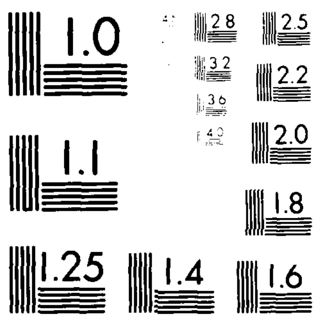

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MULTIPLE COMPARISONS FOR ORTHOGONAL CONTRASTS:

EXAMPLES AND TABLES

by

Robert E. Bechhofer
and
Charles W. Dunnett*

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and

*Natural Sciences and Engineering Research Council of Canada
at McMaster University

*Department of Clinical Epidemiology and Biostatistics, and Department of Mathematical Sciences, McMaster University.

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Abstract

In many experimental situations the pertinent inferences are made on the basis of orthogonal contrasts among the treatment means (as in 2^n factorial experiments). In this setting a particularly useful form of inference is one involving multiple comparisons. The present paper describes situations in which such inferences are meaningful, gives examples of their use, and provides an extensive set of tables of constants needed to implement such multiple comparison procedures. The procedures can also be used for statistically legitimate "data snooping" (in the sense of Scheffé (1959), p. 80) to help decide which contrasts within a specified set warrant further study.

KEY WORDS: Multiple comparisons, orthogonal contrasts, joint confidence intervals, experimentwise error rates, Studentized maximum modulus, simultaneous inference.

1. INTRODUCTION

The results of many planned experiments can be analyzed in terms of meaningful orthogonal contrasts among treatment means. This is the situation, for example, in 2^n factorial experiments, and in experiments with quantitative factors for which orthogonal polynomials (Fisher and Yates (1938-1963); Davies (1978), Appendix 8C) are employed to fit a regression curve or surface; Cochran and Cox (1957), Sections 3.4-3.5, discuss orthogonal contrasts in some detail. Ackermann (1979) proved a general formula to calculate the values of orthogonal polynomials for the case of nonequidistant levels and unequal numbers of observations.

For hypothesis testing the orthogonality of the treatment contrasts makes it possible to partition the sum of squares for treatments into a set of one-degree-of-freedom sums of squares which add up to the sum of squares for treatments: under normality each individual sum of squares associated with one of the treatment contrasts is distributed as χ^2 with one degree of freedom, independently of all of the other individual sums of squares. The orthogonality of the treatment contrasts also guarantees that their usual best linear unbiased estimators are normally and independently distributed.

If only <u>one</u> contrast is of interest, then Student's t can be used for hypothesis testing or for interval estimation. However, in most experiments there is more than one contrast which is of interest. The problem then becomes somewhat more complicated for <u>joint</u> inferences are now involved, and the experimenter may desire to control the <u>experimentwise</u> error rate. Thus, e.g., when two or more hypotheses are each tested separately at level of significance α using (say) Student's t, the experimentwise error rate (which controls the probability of at least one false positive) is greater than α. To help compensate for this effect some experimenters use, for each test, a common value of α which is smaller than the one that would typically be employed if only one hypothesis were being tested; others use critical values based on Bonferoni inequalities. (The problem is further complicated by the fact that the individual tests are not independent since each usually employs the same residual mean square as the estimator of the underlying variance.) Analogous problems arise when two or more orthogonal contrasts are to be estimated jointly using one-sided or two-sided confidence intervals.

In the present paper we shall mainly consider joint two-sided confidence interval estimation of orthogonal contrasts; the joint confidence interval approach is usually more relevant than joint hypothesis testing. In any case, joint interval estimation can easily be re-formulated as joint hypothesis testing. We shall describe situations in which the joint interval estimation approach would appear to be appropriate, and show how to make exact confidence statements concerning the joint interval estimates of the orthogonal contrasts which are of interest. In Section 2 we provide an extensive set of tables of constants which are needed to implement the procedure. The constants are based on a special case of the multivariate Student's t of Dunnett and Sobel (1954) and Cornish (1954). The theory underlying these tables is given in Section 2 along with a description of the method underlying their construction, and comments on the accuracy of the entries therein. Section 3 describes some examples of the application of the tables.

2. DISTRIBUTION THEORY AND TABLES

2.1 Distribution theory

We assume that the random variables Y_{ij} $(1 \leq i \leq k;\ j = 1,2,\ldots,N)$ are independent and normally distributed with $E\{Y_{ij}\} = \mu_i$ and $\mathrm{Var}\{Y_{ij}\} = \sigma^2$, the $\{\mu_i\}$ and σ^2 being unknown. Define $\theta_m = \sum_{i=1}^{k} c_{mi}\mu_i$ $(1 \leq m \leq p)$ where the c_{mi} $(1 \leq i \leq k,\ 1 \leq m \leq p)$ are specified constants such that $\sum_{i=1}^{k} c_{mi} = 0$ $(1 \leq m \leq p)$. The θ_m represent a family of p contrasts among the μ_i; we suppose that the experimenter is interested in obtaining two-sided interval estimates of the θ_m with specified joint confidence coefficient $1-\alpha$. Such interval estimators with the required joint confidence coefficient are given by

$$B_m = \left\{\theta_m :\ \theta_m \in \hat{\theta}_m \pm h\sqrt{\sum_{i=1}^{k} c_{mi}^2 s^2/N}\right\} \quad (1 \leq m \leq p)$$

where $\hat{\theta}_m = \sum_{i=1}^{k} c_{mi}\hat{\mu}_i$; $\hat{\mu}_i = \sum_{j=1}^{N} Y_{ij}/N$, s^2 is the usual unbiased estimate of σ^2 based on ν d.f. ($\nu S^2/\sigma^2$ is distributed as χ^2_ν, independently of $\hat{\underset{\sim}{\theta}} = (\hat{\theta}_1,\ldots,\hat{\theta}_p)$), and h is a constant chosen to satisfy

$$P\{\bigcap_{m=1}^{p} B_m\} = 1-\alpha.$$

In order to determine the value of h to be used, we note that $\hat{\underset{\sim}{\theta}}$ has a p-variate normal distribution $N(\hat{\underset{\sim}{\theta}}|\underset{\sim}{\theta},\underset{\sim}{\Sigma})$ where the elements of $\underset{\sim}{\Sigma}$ are given by

$$\sigma_{m_1 m_2} = \left(\sum_{i=1}^{k} c_{m_1 i} c_{m_2 i}\right)\sigma^2/N \quad (1 \leq m_1, m_2 \leq p).$$

Then the distribution of $\underset{\sim}{T} = (T_1,\ldots,T_p)$ where

$$T_j = (\hat{\theta}_j - \theta_j) / \sqrt{\sum_{i=1}^{p} c_{mi}^2 S^2 / N}$$

is a central p-variate Student t-distribution with ν d.f. Its joint density function (see Dunnett and Sobel (1954) or Cornish (1954)) is given by

$$f_\nu(t_1,\ldots,t_p;\underset{\sim}{R}) = \frac{|A|^{1/2}\Gamma[(\nu+p)/2]}{(\nu\pi)^{p/2}\Gamma(\nu/2)}\left[1+\frac{1}{\nu}\sum_{j_1=1}^{p}\sum_{j_2=1}^{p} a_{j_1 j_2} t_{j_1} t_{j_2}\right]^{-\frac{(\nu+p)}{2}} \qquad (2.1)$$

where the $\{a_{j_1 j_2}\}$ are the elements of $\underset{\sim}{A} = \underset{\sim}{R}^{-1}$. Let $|\underset{\sim}{T}| = (|T_1|,\ldots,|T_p|)$ where $\underset{\sim}{T}$ has the joint density function (2.1), and let

$$\begin{aligned} G_\nu(h;p,\underset{\sim}{R}) &= P\{|T_m| \leq h \quad (1 \leq m \leq p)\} \\ &= \int_{-h}^{h}\cdots\int_{-h}^{h} f_\nu(t_1,\ldots,t_p;\underset{\sim}{R})dt_1\ldots dt_p \qquad (2.2) \\ &= \int_0^\infty \left[\underset{\substack{-hs<x_m<hs \\ m=1,\ldots,p}}{\int\cdots\int} \phi(x_1,\ldots,x_p;\underset{\sim}{R})dx_1\ldots dx_p\right] g_\nu(s)ds \end{aligned}$$

where $\phi(x_1,\ldots,x_p;\underset{\sim}{R})$ is the standard p-variate normal probability density function (pdf) with correlation matrix $\underset{\sim}{R} = \{\rho_{m_1 m_2}\}$, and $g_\nu(s)$ is the pdf of S. If $\rho_{m_1 m_2} = \rho$ $(m_1 \neq m_2,\ 1 \leq m_1, m_2 \leq p)$ then the multiple integral within the square brackets of (2.2) can be expressed as an iterated integral (thus simplifying the problem of evaluating it numerically), and (2.2) reduces to

$$H_\nu(h;p,\rho) = \int_0^\infty\left[\int_{-\infty}^{\infty}\left\{\Phi\left(\frac{hs+\rho^{1/2}y}{(1-\rho)^{1/2}}\right) - \Phi\left(\frac{-hs+\rho^{1/2}y}{(1-\rho)^{1/2}}\right)\right\}^p \times d\Phi(y)dy\right] g_\nu(s)ds \qquad (2.3)$$

where $\Phi(\cdot)$ is the standard normal cdf.

The constants $h = h_\nu(p,\rho,\alpha)$ satisfying $H_\nu(h;p,\rho) = 1-\alpha$ have been tabulated to two decimal places by Dunnett (1964) for $\rho = 0.5$; $p = 2(1)12,15,20$; $\alpha = 0.05,0.01$; $\nu = 5(1)20,24,30,40,60,120,\infty$. Hahn and Hendrickson (1971) tabulated h to three decimal places for $\rho = 0.0,0.2,0.4,0.5$; $p = 1(1)6(2)12,15,20$; $\alpha = 0.10,0.05,0.01$; $\nu = 3(1)12,15(5)30,40,60$. Other earlier tabulations are cited in Hahn and Hendrickson. Krishnaiah and Armitage (1970) tabulated h^2 to two decimal places for $\rho = 0.1(0.1)0.9$; $p = 1(1)10$; $\alpha = 0.05,0.01$; $\nu = 5(1)35$.

For the special case of orthogonal contrasts with which we are concerned here, we have $\sum_{i=1}^{k} c_{m_1 i} c_{m_2 i} = 0$ $(m_1 \neq m_2,\ 1 \leqq m_1, m_2 \leqq p)$; hence $\rho = 0$ and (2.3) becomes

$$H_\nu(h;p,0) = \int_0^\infty [\Phi(hs) - \Phi(-hs)]^p g_\nu(s)ds. \qquad (2.4)$$

Since $\rho = 0$ we see that Hahn and Hendrickson's Table 1 is applicable. In this case the statistic $M(p,\nu) = \max_{1 \leqq m \leqq p} |\hat{\theta}_m - \theta_m| \Big/ \sqrt{\sum_{i=1}^{p} c_{mi}^2 S^2/N}$ is known as the <u>Studentized maximum modulus</u>, and $P\{M(p,\nu) \leqq h\} = H_\nu(h;p,0)$. Pillai and Ramachandran (1954) had earlier tabulated h for $\rho = 0$ to two decimal places for $p = 2(1)8$; $\alpha = 0.05$; $\nu = 5(5)20,24,30,60,120,\infty$. Stoline and Ury (1979) have tabulated h to three decimal places for $p = k(k-1)/2$, $k = 3(1)20$; $\alpha = 0.2,0.1,0.05,0.01$; $\nu = 5,7,10,12(4)24,30,40,60,120,\infty$; in a later paper (Ury, Stoline and Mitchell (1980)) these tables were extended to cover $k = 20(2)50(5)80,90,100$; $\alpha = 0.2,0.1,0.05,0.01$; $\nu = 20(1)40(2)60(5)120,240,480,\infty$. (The constants tabulated by Stoline et al. were to be used for <u>joint</u> <u>two-sided</u> interval estimates for the $k(k-1)/2$ pairwise contrasts $\mu_{i_1} - \mu_{i_2}$ between the k population means. Such contrasts are <u>not</u> <u>orthogonal</u>, but as a consequence of an inequality of

Šidák (1967), the use of constants h determined for the case $\rho = 0$ results in intervals which are <u>conservative</u>, i.e., they achieve a joint confidence coefficient which <u>exceeds</u> the nominal $1-\alpha$ at the cost of having intervals which are somewhat broader than they need be.) Earlier, Games (1977), employing Šidák's (1967) multiplicative inequality, computed <u>conservative</u> constants, specifically for the problem of multiple comparisons for <u>non-orthogonal</u> contrasts, his tables give upper bounds to h to three decimal places for $p = 2(1)10(5)50$; $\alpha = 0.20, 0.10, 0.05, 0.01$; $\nu = 2(1)30, 40, 60, 120, \infty$. (We mention that Chen (1979) tabulated percentage points associated with random variables arising from a multivariate Student t-distribution (2.1) with zero correlations. However, his tables are not related to ours; they are used, for example, to find an interval estimate of $\max\{\mu_1,\ldots,\mu_k\}$ when the estimates $\hat{\mu}_i$ $(1 \leq i \leq k)$ are based on sample sizes which are not necessarily all equal.)

In applications involving multiple comparisons for <u>orthogonal</u> contrasts, our tables provide the constants needed to obtain <u>two-sided</u> interval estimates of p contrasts with joint confidence coefficient <u>exactly equal to</u> the nominal value of $1-\alpha$. The tables give $h = h_\nu(p,\rho,\alpha)$ to <u>five</u> significant figures for $\rho = 0$; $p = 2(1)31$; $\alpha = 0.2, 0.1, 0.05, 0.01$, $\nu = 2(1)30(5)60, 120, 240, \infty$.

<u>2.2 Construction of the tables</u>

In order to construct the tables it was necessary to obtain the solution in h to $H_\nu(h;p,0) = 1-\alpha$ where $H_\nu(h;p,0)$ is given by (2.4). To evaluate the infinite integral for finite values of ν, a 96-point Legendre quadrature formula was used (see Abromowitz and Stegun (1964), p, 919); for $\nu = \infty$ we have $s = 1$ and the infinite integral disappears. This method of calculation is not appropriate for $\nu = 1$ (since the pdf

$g_\nu(s)$ is then infinite at $s = 0$), and no h-values were computed for $\nu = 1$. The general computer program which was written for arbitrary ρ and h was first used for $\rho = 1/2$, $p = 2$, and the probabilities obtained were then checked against those given in Table 1 of Dunnett and Sobel (1954) which had been computed using an exact series expansion; agreement was found to the five decimal places given in each. The h-values were also checked against those given in Hahn and Hendrickson's Table 1 and in Stoline and Ury (1979), and agreed to the three decimal places given in these tables. Our h-values are believed to be correct to the four decimal places given.

3. ILLUSTRATIONS OF USES OF THE TABLES

In this section we consider several types of planned experiments in which the pertinent inferences are made on the basis of orthogonal contrasts among the treatment means. We shall point out some of the issues involved, and indicate how the multiple comparisons procedures used with the appropriate constants in our tables can control the experimentwise error rates for such inferences.

3.1 Experiments involving a single qualitative factor

Example 1: A 5-level experiment

Bennett and Franklin (1954), Section 7.34, consider an experiment involving five different methods of analyzing the concentration of iron in a standard solution. Two methods included agitation and three methods did not; four analyses were made with each method. The orthogonal contrasts under consideration (see their Table 7.9) were $(c_{m1}, c_{m2}, \ldots, c_{m5})$ = $(3,3,-2,-2,-2)$, $(1,-1,0,0,0)$, $(0,0,2,-1,-1)$, $(0,0,0,1,-1)$ for $m = 1,2,3,4$, respectively. Here $p = 4$, $\nu = 15$ and from our Tables 1 and 2 we find

Table 1

STUDENTIZED MAXIMUM MODULUS FOR 1.-ALPHA = .99

D.F.	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	12.7266	14.4366	15.6538	16.5916	17.3507	17.9860	18.5307	19.0067	19.4286	19.8070	20.1497	20.4626	20.7503	21.0164	21.2637
3	7.1267	7.9142	8.4796	8.9187	9.2764	9.5774	9.8367	10.0641	10.2663	10.4483	10.6134	10.7646	10.9038	11.0328	11.1529
4	5.4619	5.9854	6.3624	6.6562	6.8964	7.0992	7.2744	7.4284	7.5657	7.6895	7.8021	7.9052	8.0004	8.0887	8.1710
5	4.7003	5.1056	5.3974	5.6252	5.8117	5.9695	6.1060	6.2262	6.3335	6.4303	6.5185	6.5994	6.6742	6.7436	6.8083
6	4.2711	4.6109	4.8551	5.0457	5.2019	5.3341	5.4486	5.5496	5.6398	5.7212	5.7954	5.8636	5.9266	5.9851	6.0398
7	3.9978	4.2963	4.5104	4.6774	4.8142	4.9301	5.0305	5.1191	5.1982	5.2697	5.3349	5.3948	5.4502	5.5017	5.5498
8	3.8092	4.0795	4.2730	4.4237	4.5472	4.6517	4.7423	4.8223	4.8937	4.9583	5.0172	5.0713	5.1214	5.1679	5.2114
9	3.6716	3.9215	4.0999	4.2388	4.3526	4.4488	4.5322	4.6058	4.6716	4.7310	4.7853	4.8351	4.8813	4.9241	4.9642
10	3.5668	3.8013	3.9684	4.0984	4.2047	4.2946	4.3725	4.4412	4.5026	4.5582	4.6088	4.6554	4.6985	4.7386	4.7760
11	3.4845	3.7070	3.8652	3.9881	4.0886	4.1735	4.2471	4.3120	4.3700	4.4224	4.4703	4.5142	4.5549	4.5927	4.6281
12	3.4182	3.6310	3.7821	3.8993	3.9951	4.0761	4.1461	4.2079	4.2632	4.3131	4.3586	4.4005	4.4392	4.4752	4.5088
13	3.3636	3.5685	3.7138	3.8264	3.9183	3.9959	4.0631	4.1224	4.1753	4.2232	4.2668	4.3069	4.3440	4.3785	4.4107
14	3.3178	3.5152	3.6566	3.7654	3.8541	3.9289	3.9937	4.0508	4.1018	4.1479	4.1900	4.2286	4.2644	4.2976	4.3286
15	3.2790	3.4719	3.6082	3.7136	3.7996	3.8721	3.9349	3.9901	4.0395	4.0841	4.1248	4.1622	4.1968	4.2289	4.2590
16	3.2457	3.4338	3.5665	3.6692	3.7528	3.8233	3.8843	3.9380	3.9860	4.0293	4.0688	4.1051	4.1387	4.1699	4.1991
17	3.2167	3.4007	3.5304	3.6306	3.7122	3.7810	3.8404	3.8928	3.9395	3.9818	4.0203	4.0556	4.0883	4.1187	4.1471
18	3.1913	3.3717	3.4987	3.5968	3.6766	3.7439	3.8020	3.8532	3.8988	3.9401	3.9777	4.0122	4.0441	4.0738	4.1016
19	3.1688	3.3461	3.4708	3.5669	3.6452	3.7111	3.7681	3.8182	3.8629	3.9033	3.9401	3.9739	4.0052	4.0342	4.0613
20	3.1488	3.3233	3.4459	3.5404	3.6172	3.6820	3.7379	3.7871	3.8309	3.8706	3.9067	3.9398	3.9705	3.9990	4.0256
21	3.1310	3.3029	3.4237	3.5167	3.5923	3.6560	3.7109	3.7593	3.8024	3.8413	3.8768	3.9093	3.9394	3.9674	3.9936
22	3.1149	3.2846	3.4037	3.4953	3.5698	3.6325	3.6866	3.7342	3.7767	3.8150	3.8499	3.8819	3.9115	3.9390	3.9647
23	3.1003	3.2680	3.3855	3.4760	3.5495	3.6113	3.6646	3.7115	3.7534	3.7911	3.8255	3.8570	3.8862	3.9133	3.9386
24	3.0870	3.2528	3.3690	3.4584	3.5309	3.5920	3.6446	3.6909	3.7322	3.7694	3.8033	3.8345	3.8632	3.8899	3.9149
25	3.0749	3.2390	3.3540	3.4423	3.5140	3.5743	3.6264	3.6721	3.7128	3.7496	3.7831	3.8138	3.8422	3.8686	3.8932
26	3.0638	3.2263	3.3401	3.4276	3.4985	3.5582	3.6096	3.6548	3.6951	3.7314	3.7645	3.7949	3.8230	3.8490	3.8734
27	3.0535	3.2147	3.3274	3.4140	3.4842	3.5433	3.5942	3.6389	3.6788	3.7147	3.7475	3.7775	3.8052	3.8310	3.8551
28	3.0440	3.2039	3.3157	3.4015	3.4711	3.5296	3.5800	3.6243	3.6637	3.6993	3.7317	3.7614	3.7889	3.8144	3.8382
29	3.0352	3.1939	3.3048	3.3899	3.4589	3.5168	3.5668	3.6107	3.6498	3.6850	3.7171	3.7466	3.7737	3.7990	3.8226
30	3.0271	3.1846	3.2947	3.3791	3.4475	3.5050	3.5546	3.5981	3.6368	3.6718	3.7036	3.7327	3.7597	3.7847	3.8081
35	2.9937	3.1467	3.2533	3.3350	3.4012	3.4567	3.5045	3.5465	3.5838	3.6175	3.6482	3.6763	3.7022	3.7263	3.7488
40	2.9690	3.1187	3.2228	3.3025	3.3670	3.4211	3.4677	3.5085	3.5448	3.5776	3.6074	3.6347	3.6599	3.6833	3.7051
45	2.9501	3.0972	3.1994	3.2776	3.3408	3.3938	3.4394	3.4794	3.5149	3.5470	3.5761	3.6028	3.6274	3.6503	3.6716
50	2.9351	3.0802	3.1809	3.2579	3.3201	3.3722	3.4170	3.4563	3.4913	3.5228	3.5514	3.5776	3.6018	3.6242	3.6452
55	2.9230	3.0664	3.1659	3.2419	3.3033	3.3547	3.3989	3.4377	3.4721	3.5031	3.5313	3.5571	3.5810	3.6031	3.6237
60	2.9129	3.0549	3.1534	3.2286	3.2894	3.3402	3.3839	3.4222	3.4563	3.4869	3.5147	3.5403	3.5638	3.5856	3.6060
120	2.8586	2.9934	3.0866	3.1575	3.2146	3.2624	3.3034	3.3392	3.3711	3.3997	3.4257	3.4495	3.4714	3.4917	3.5107
240	2.8322	2.9635	3.0541	3.1229	3.1783	3.2246	3.2643	3.2989	3.3297	3.3574	3.3825	3.4055	3.4266	3.4462	3.4645
∞	2.8062	2.9342	3.0222	3.0890	3.1428	3.1876	3.2260	3.2595	3.2893	3.3160	3.3402	3.3624	3.3828	3.4017	3.4192

Table 1 (continued)

STUDENTIZED MAXIMUM MODULUS FOR 1.-ALPHA = .99 (CONT.)

D.F.	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
2	21.4946	21.7112	21.9149	22.1072	22.2892	22.4620	22.6263	22.7830	22.9327	23.0759	23.2133	23.3451	23.4719	23.5940	23.7117
3	11.2652	11.3707	11.4700	11.5639	11.6528	11.7374	11.8178	11.8946	11.9680	12.0383	12.1058	12.1706	12.2329	12.2930	12.3509
4	8.2480	8.3204	8.3887	8.4533	8.5146	8.5728	8.6283	8.6813	8.7319	8.7805	8.8271	8.8720	8.9151	8.9567	8.9968
5	6.8690	6.9260	6.9798	7.0308	7.0791	7.1251	7.1690	7.2109	7.2510	7.2894	7.3264	7.3619	7.3961	7.4290	7.4609
6	6.0910	6.1392	6.1847	6.2278	6.2[illegible]	6.3077	6.3448	6.3804	6.4143	4469	6.4783	6.5084	6.5374	6.5654	6.5925
7	5.5949	5.6374	5.6775	5.7155	5.7515	5.7859	5.8187	5.8500	5.8800	5.9088	5.9365	5.9631	5.9887	6.0135	6.0374
8	5.2522	5.2906	5.3269	5.3613	5.3940	5.4251	5.4548	5.4832	5.5104	5.5365	5.5615	5.5857	5.6090	5.6314	5.6531
9	5.0018	5.0372	5.0707	5.1024	5.1326	5.1613	5.1887	5.2149	5.2400	5.2640	5.2872	5.3095	5.3310	5.3517	5.3718
10	4.8112	4.8443	4.8755	4.9052	4.9334	4.9602	4.9858	5.0103	5.0338	5.0563	5.0780	5.0989	5.1190	5.1384	5.1571
11	4.6613	4.6925	4.7221	4.7501	4.7767	4.8020	4.8262	4.8494	4.8716	4.8928	4.9133	4.9330	4.9520	4.9704	4.9881
12	4.5404	4.5702	4.5983	4.6250	4.6503	4.6744	4.6975	4.7195	4.7406	4.7609	4.7804	4.7992	4.8173	4.8348	4.8516
13	4.4410	4.4695	4.4965	4.5220	4.5463	4.5694	4.5915	4.6126	4.6328	4.6523	4.6709	4.6889	4.7063	4.7230	4.7392
14	4.3578	4.3853	4.4112	4.4358	4.4592	4.4814	4.5027	4.5230	4.5425	4.5613	4.5792	4.5966	4.6133	4.6294	4.6450
15	4.2872	4.3137	4.3388	4.3626	4.3852	4.4068	4.4273	4.4470	4.4658	4.4839	4.5014	4.5181	4.5343	4.5499	4.5649
16	4.2265	4.2522	4.2766	4.2997	4.3217	4.3426	4.3625	4.3816	4.3999	4.4175	4.4344	4.4506	4.4663	4.4815	4.4961
17	4.1738	4.1989	4.2226	4.2451	4.2664	4.2868	4.3062	4.3248	4.3426	4.3597	4.3762	4.3920	4.4073	4.4220	4.4363
18	4.1276	4.1521	4.1753	4.1972	4.2181	4.2379	4.2569	4.2750	4.2924	4.3091	4.3252	4.3406	4.3555	4.3699	4.3838
19	4.0868	4.1108	4.1334	4.1549	4.1753	4.1948	4.2133	4.2311	4.2481	4.2644	4.2801	4.2952	4.3098	4.3239	4.3375
20	4.0505	4.0740	4.0962	4.1173	4.1373	4.1563	4.1745	4.1919	4.2086	4.2246	4.2400	4.2548	4.2691	4.2829	4.2962
21	4.0181	4.0411	4.0630	4.0836	4.1032	4.1219	4.1398	4.1569	4.1732	4.1889	4.2040	4.2186	4.2326	4.2461	4.2592
22	3.9888	4.0115	4.0330	4.0533	4.0726	4.0910	4.1085	4.1253	4.1414	4.1568	4.1717	4.1860	4.1997	4.2130	4.2259
23	3.9624	3.9847	4.0058	4.0258	4.0448	4.0629	4.0802	4.0967	4.1126	4.1278	4.1424	4.1564	4.1700	4.1831	4.1957
24	3.9383	3.9603	3.9811	4.0009	4.0196	4.0374	4.0545	4.0707	4.0863	4.1013	4.1157	4.1296	4.1429	4.1558	4.1683
25	3.9163	3.9381	3.9586	3.9781	3.9965	4.0141	4.0309	4.0470	4.0624	4.0772	4.0914	4.1050	4.1182	4.1309	4.1432
26	3.8962	3.9176	3.9379	3.9572	3.9754	3.9928	4.0094	4.0252	4.0404	4.0550	4.0691	4.0826	4.0956	4.1081	4.1203
27	3.8776	3.8989	3.9189	3.9379	3.9560	3.9731	3.9895	4.0052	4.0203	4.0347	4.0485	4.0619	4.0747	4.0871	4.0991
28	3.8605	3.8815	3.9014	3.9202	3.9380	3.9550	3.9712	3.9868	4.0016	4.0159	4.0296	4.0428	4.0555	4.0678	4.0796
29	3.8447	3.8655	3.8851	3.9037	3.9214	3.9382	3.9543	3.9696	3.9844	3.9985	4.0120	4.0251	4.0377	4.0498	4.0616
30	3.8300	3.8506	3.8700	3.8885	3.9060	3.9226	3.9386	3.9538	3.9683	3.9823	3.9957	4.0087	4.0211	4.0332	4.0448
35	3.7698	3.7896	3.8084	3.8261	3.8429	3.8589	3.8742	3.8888	3.9028	3.9162	3.9291	3.9416	3.9535	3.9651	3.9762
40	3.7255	3.7448	3.7629	3.7801	3.7965	3.8120	3.8268	3.8410	3.8546	3.8676	3.8801	3.8921	3.9037	3.9149	3.9258
45	3.6916	3.7104	3.7281	3.7449	3.7609	3.7760	3.7905	3.8044	3.8176	3.8303	3.8425	3.8543	3.8656	3.8765	3.8871
50	3.6648	3.6832	3.7006	3.7171	3.7327	3.7476	3.7618	3.7754	3.7884	3.8008	3.8128	3.8243	3.8354	3.8461	3.8565
55	3.6430	3.6612	3.6783	3.6945	3.7099	3.7246	3.7385	3.7519	3.7647	3.7769	3.7887	3.8000	3.8109	3.8215	3.8317
60	3.6250	3.6430	3.6599	3.6759	3.6911	3.7055	3.7193	3.7325	3.7451	3.7572	3.7688	3.7800	3.7907	3.8011	3.8111
120	3.5284	3.5451	3.5608	3.5756	3.5897	3.6031	3.6159	3.6281	3.6398	3.6510	3.6618	3.6721	3.6821	3.6917	3.7010
240	3.4816	3.4976	3.5127	3.5271	3.5406	3.5535	3.5658	3.5776	3.5888	3.5995	3.6099	3.6198	3.6294	3.6386	3.6475
∞	3.4357	3.4512	3.4657	3.4795	3.4925	3.5049	3.5167	3.5280	3.5388	3.5491	3.5591	3.5686	3.5778	3.5866	3.5952

Table 2

STUDENTIZED MAXIMUM MODULUS FOR 1.-ALPHA = .95

D.F.	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	5.5714	6.3405	6.8863	7.3061	7.6454	7.9291	8.1723	8.3846	8.5727	8.7414	8.8941	9.0335	9.1616	9.2801	9.3902
3	3.9602	4.4297	4.7644	5.0232	5.2334	5.4098	5.5615	5.6944	5.8124	5.9185	6.0147	6.1027	6.1836	6.2586	6.3284
4	3.3820	3.7446	4.0030	4.2032	4.3661	4.5031	4.6212	4.7247	4.8168	4.8997	4.9750	5.0439	5.1074	5.1662	5.2210
5	3.0905	3.3992	3.6189	3.7890	3.9275	4.0442	4.1447	4.2330	4.3116	4.3824	4.4468	4.5057	4.5601	4.6105	4.6574
6	2.9161	3.1926	3.3888	3.5407	3.6644	3.7685	3.8584	3.9373	4.0076	4.0709	4.1285	4.1813	4.2300	4.2751	4.3172
7	2.8004	3.0555	3.2361	3.3757	3.4894	3.5851	3.6677	3.7402	3.8048	3.8631	3.9160	3.9646	4.0094	4.0510	4.0898
8	2.7181	2.9580	3.1275	3.2583	3.3648	3.4544	3.5317	3.5996	3.6601	3.7146	3.7643	3.8098	3.8517	3.8907	3.9270
9	2.6567	2.8852	3.0463	3.1706	3.2716	3.3566	3.4299	3.4943	3.5517	3.6034	3.6504	3.6936	3.7334	3.7704	3.8048
10	2.6091	2.8289	2.9834	3.1025	3.1993	3.2807	3.3508	3.4125	3.4674	3.5169	3.5619	3.6032	3.6413	3.6767	3.7097
11	2.5712	2.7839	2.9333	3.0483	3.1416	3.2201	3.2877	3.3471	3.4001	3.4478	3.4912	3.5310	3.5677	3.6018	3.6336
12	2.5402	2.7472	2.8924	3.0040	3.0945	3.1706	3.2362	3.2937	3.3450	3.3912	3.4333	3.4718	3.5074	3.5404	3.5712
13	2.5145	2.7168	2.8584	2.9671	3.0553	3.1294	3.1933	3.2493	3.2992	3.3442	3.3851	3.4226	3.4572	3.4893	3.5192
14	2.4928	2.6910	2.8296	2.9360	3.0222	3.0946	3.1570	3.2117	3.2605	3.3044	3.3443	3.3809	3.4147	3.4460	3.4753
15	2.4742	2.6690	2.8051	2.9094	2.9939	3.0649	3.1260	3.1796	3.2273	3.2703	3.3093	3.3452	3.3782	3.4089	3.4375
16	2.4581	2.6500	2.7838	2.8864	2.9694	3.0391	3.0991	3.1517	3.1985	3.2407	3.2791	3.3142	3.3467	3.3768	3.4048
17	2.4441	2.6334	2.7653	2.8663	2.9480	3.0166	3.0756	3.1273	3.1734	3.2149	3.2526	3.2872	3.3190	3.3486	3.3762
18	2.4317	2.6187	2.7489	2.8486	2.9292	2.9967	3.0549	3.1059	3.1512	3.1921	3.2292	3.2633	3.2947	3.3238	3.3510
19	2.4207	2.6057	2.7344	2.8328	2.9124	2.9791	3.0365	3.0868	3.1316	3.1719	3.2085	3.2421	3.2730	3.3017	3.3285
20	2.4109	2.5941	2.7214	2.8188	2.8974	2.9634	3.0201	3.0698	3.1140	3.1538	3.1899	3.2231	3.2536	3.2820	3.3084
21	2.4021	2.5837	2.7098	2.8061	2.8840	2.9492	3.0053	3.0544	3.0981	3.1375	3.1732	3.2060	3.2362	3.2642	3.2904
22	2.3941	2.5743	2.6992	2.7947	2.8718	2.9364	2.9919	3.0406	3.0838	3.1227	3.1581	3.1905	3.2204	3.2481	3.2740
23	2.3869	2.5657	2.6897	2.7843	2.8608	2.9248	2.9798	3.0280	3.0708	3.1094	3.1444	3.1765	3.2061	3.2335	3.2591
24	2.3803	2.5579	2.6809	2.7749	2.8507	2.9142	2.9687	3.0165	3.0589	3.0972	3.1319	3.1637	3.1930	3.2202	3.2456
25	2.3743	2.5507	2.6729	2.7662	2.8414	2.9044	2.9585	3.0059	3.0481	3.0860	3.1204	3.1519	3.1810	3.2080	3.2331
26	2.3687	2.5441	2.6656	2.7582	2.8330	2.8955	2.9492	2.9962	3.0381	3.0757	3.1098	3.1411	3.1700	3.1967	3.2217
27	2.3636	2.5381	2.6588	2.7509	2.8251	2.8872	2.9406	2.9873	3.0288	3.0662	3.1001	3.1311	3.1598	3.1863	3.2111
28	2.3588	2.5324	2.6525	2.7441	2.8179	2.8796	2.9326	2.9790	3.0203	3.0574	3.0911	3.1219	3.1504	3.1767	3.2013
29	2.3544	2.5272	2.6467	2.7378	2.8111	2.8725	2.9252	2.9714	3.0123	3.0492	3.0827	3.1133	3.1416	3.1678	3.1922
30	2.3503	2.5224	2.6413	2.7319	2.8049	2.8659	2.9183	2.9642	3.0050	3.0416	3.0749	3.1054	3.1335	3.1595	3.1838
35	2.3334	2.5024	2.6190	2.7077	2.7791	2.8388	2.8900	2.9348	2.9746	3.0104	3.0428	3.0725	3.0999	3.1253	3.1490
40	2.3209	2.4876	2.6025	2.6898	2.7600	2.8187	2.8690	2.9130	2.9520	2.9871	3.0190	3.0482	3.0750	3.0999	3.1231
45	2.3113	2.4762	2.5897	2.6759	2.7453	2.8031	2.8528	2.8961	2.9346	2.9692	3.0006	3.0293	3.0558	3.0803	3.1031
50	2.3036	2.4671	2.5796	2.6649	2.7335	2.7908	2.8399	2.8827	2.9208	2.9550	2.9860	3.0144	3.0405	3.0647	3.0872
55	2.2973	2.4597	2.5713	2.6560	2.7240	2.7807	2.8294	2.8718	2.9095	2.9434	2.9741	3.0022	3.0280	3.0520	3.0743
60	2.2922	2.4536	2.5644	2.6485	2.7161	2.7724	2.8207	2.8628	2.9002	2.9338	2.9642	2.9920	3.0177	3.0414	3.0635
120	2.2640	2.4203	2.5273	2.6082	2.6731	2.7271	2.7733	2.8136	2.8494	2.8814	2.9105	2.9370	2.9614	2.9841	3.0051
240	2.2502	2.4039	2.5090	2.5884	2.6519	2.7048	2.7500	2.7895	2.8244	2.8557	2.8840	2.9099	2.9337	2.9558	2.9763
∞	2.2365	2.3877	2.4909	2.5688	2.6310	2.6828	2.7270	2.7655	2.7996	2.8302	2.8578	2.8831	2.9063	2.9278	2.9478

Table 2 (continued)

STUDENTIZED MAXIMUM MODULUS FOR 1.-ALPHA = .95 (CONT.)

D.F.	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
2	9.4930	9.5893	9.6800	9.7635	9.8465	9.9233	9.9964	10.0661	10.1326	10.1963	10.2574	10.3160	10.3724	10.4266	10.4789
3	6.3936	6.4548	6.5124	6.5669	6.6184	6.6674	6.7140	6.7585	6.8010	6.8417	6.8807	6.9182	6.9543	6.9890	7.0225
4	5.2723	5.3204	5.3658	5.4087	5.4493	5.4879	5.5247	5.5598	5.5934	5.6255	5.6564	5.6861	5.7146	5.7421	5.7686
5	4.7014	4.7427	4.7817	4.8185	4.8534	4.8866	4.9182	4.9484	4.9773	5.0049	5.0315	5.0570	5.0816	5.1053	5.1282
6	4.3567	4.3937	4.4287	4.4617	4.4931	4.5229	4.5513	4.5784	4.6044	4.6293	4.6532	4.6761	4.6982	4.7196	4.7401
7	4.1261	4.1602	4.1924	4.2229	4.2518	4.2793	4.3055	4.3305	4.3545	4.3775	4.3995	4.4207	4.4411	4.4608	4.4798
8	3.9611	3.9931	4.0233	4.0519	4.0790	4.1048	4.1293	4.1528	4.1753	4.1968	4.2175	4.2374	4.2566	4.2751	4.2929
9	3.8371	3.8675	3.8962	3.9233	3.9490	3.9735	3.9968	4.0191	4.0405	4.0609	4.0806	4.0995	4.1177	4.1352	4.1522
10	3.7406	3.7697	3.7971	3.8231	3.8477	3.8712	3.8935	3.9149	3.9353	3.9549	3.9737	3.9918	4.0093	4.0261	4.0423
11	3.6634	3.6914	3.7178	3.7428	3.7666	3.7892	3.8107	3.8313	3.8510	3.8699	3.8880	3.9055	3.9223	3.9385	3.9542
12	3.6001	3.6272	3.6528	3.6771	3.7001	3.7219	3.7428	3.7628	3.7818	3.8002	3.8177	3.8346	3.8509	3.8667	3.8818
13	3.5473	3.5737	3.5986	3.6222	3.6446	3.6659	3.6862	3.7056	3.7241	3.7419	3.7590	3.7755	3.7913	3.8066	3.8214
14	3.5026	3.5284	3.5527	3.5757	3.5976	3.6183	3.6381	3.6571	3.6752	3.6926	3.7093	3.7253	3.7408	3.7557	3.7701
15	3.4643	3.4896	3.5133	3.5359	3.5572	3.5776	3.5970	3.6155	3.6332	3.6502	3.6666	3.6823	3.6974	3.7120	3.7261
16	3.4311	3.4559	3.4792	3.5013	3.5222	3.5422	3.5612	3.5794	3.5968	3.6135	3.6295	3.6449	3.6597	3.6741	3.6879
17	3.4021	3.4264	3.4493	3.4710	3.4916	3.5112	3.5299	3.5478	3.5649	3.5812	3.5970	3.6121	3.6267	3.6408	3.6544
18	3.3764	3.4004	3.4229	3.4443	3.4646	3.4839	3.5023	3.5198	3.5367	3.5528	3.5683	3.5832	3.5976	3.6114	3.6248
19	3.3536	3.3772	3.3995	3.4205	3.4405	3.4595	3.4777	3.4950	3.5116	3.5275	3.5428	3.5575	3.5716	3.5853	3.5985
20	3.3332	3.3565	3.3785	3.3993	3.4190	3.4378	3.4556	3.4727	3.4891	3.5048	3.5199	3.5344	3.5484	3.5618	3.5749
21	3.3148	3.3378	3.3595	3.3801	3.3996	3.4181	3.4358	3.4527	3.4689	3.4844	3.4993	3.5136	3.5274	3.5407	3.5536
22	3.2982	3.3210	3.3424	3.3628	3.3820	3.4004	3.4179	3.4346	3.4505	3.4659	3.4806	3.4948	3.5084	3.5216	3.5343
23	3.2831	3.3056	3.3269	3.3470	3.3661	3.3842	3.4015	3.4181	3.4339	3.4491	3.4636	3.4777	3.4912	3.5042	3.5168
24	3.2693	3.2916	3.3127	3.3326	3.3515	3.3695	3.3866	3.4030	3.4187	3.4337	3.4481	3.4620	3.4754	3.4883	3.5007
25	3.2566	3.2788	3.2996	3.3194	3.3381	3.3559	3.3729	3.3892	3.4047	3.4196	3.4339	3.4477	3.4609	3.4737	3.4861
26	3.2450	3.2670	3.2877	3.3072	3.3258	3.3435	3.3603	3.3764	3.3919	3.4066	3.4208	3.4345	3.4476	3.4603	3.4725
27	3.2343	3.2560	3.2766	3.2960	3.3145	3.3320	3.3487	3.3647	3.3800	3.3947	3.4087	3.4223	3.4353	3.4479	3.4600
28	3.2243	3.2459	3.2663	3.2856	3.3040	3.3214	3.3380	3.3538	3.3690	3.3836	3.3975	3.4110	3.4239	3.4364	3.4485
29	3.2151	3.2366	3.2568	3.2760	3.2942	3.3115	3.3280	3.3437	3.3588	3.3733	3.3871	3.4005	3.4134	3.4258	3.4377
30	3.2065	3.2278	3.2480	3.2670	3.2851	3.3023	3.3187	3.3343	3.3493	3.3637	3.3775	3.3907	3.4035	3.4158	3.4277
35	3.1711	3.1919	3.2115	3.2301	3.2477	3.2644	3.2804	3.2956	3.3102	3.3241	3.3376	3.3505	3.3629	3.3749	3.3865
40	3.1448	3.1652	3.1844	3.2026	3.2198	3.2362	3.2519	3.2668	3.2811	3.2947	3.3079	3.3205	3.3327	3.3444	3.3558
45	3.1245	3.1446	3.1635	3.1814	3.1983	3.2145	3.2298	3.2445	3.2586	3.2720	3.2849	3.2974	3.3093	3.3209	3.3320
50	3.1083	3.1281	3.1468	3.1645	3.1812	3.1971	3.2123	3.2268	3.2406	3.2539	3.2667	3.2789	3.2907	3.3021	3.3131
55	3.0952	3.1148	3.1332	3.1507	3.1673	3.1830	3.1980	3.2123	3.2260	3.2392	3.2518	3.2639	3.2756	3.2868	3.2977
60	3.0842	3.1037	3.1220	3.1393	3.1557	3.1713	3.1861	3.2003	3.2139	3.2269	3.2394	3.2514	3.2630	3.2741	3.2849
120	3.0248	3.0432	3.0606	3.0771	3.0927	3.1075	3.1216	3.1350	3.1479	3.1602	3.1720	3.1834	3.1943	3.2049	3.2151
240	2.9955	3.0135	3.0304	3.0464	3.0616	3.0760	3.0897	3.1028	3.1153	3.1273	3.1388	3.1498	3.1604	3.1707	3.1806
∞	2.9664	2.9839	3.0004	3.0160	3.0307	3.0447	3.0581	3.0708	3.0829	3.0946	3.1058	3.1165	3.1268	3.1368	3.1463

Table 3

STUDENTIZED MAXIMUM MODULUS FOR 1.-ALPHA = .90

D.F.	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	3.8310	4.3787	4.7659	5.0630	5.3028	5.5031	5.6746	5.8242	5.9567	6.0754	6.1829	6.2809	6.3710	6.4543	6.5317
3	2.9894	3.3685	3.6371	3.8439	4.0113	4.1516	4.2720	4.3773	4.4708	4.5547	4.6307	4.7002	4.7641	4.8232	4.8782
4	2.6624	2.9755	3.1959	3.3676	3.5059	3.6220	3.7217	3.8090	3.8866	3.9563	4.0196	4.0774	4.1306	4.1799	4.2258
5	2.4905	2.7686	2.9549	3.1160	3.2386	3.3414	3.4299	3.5074	3.5762	3.6382	3.6944	3.7458	3.7932	3.8370	3.8779
6	2.3850	2.6415	2.8220	2.9609	3.0735	3.1680	3.2493	3.3205	3.3838	3.4408	3.4925	3.5398	3.5834	3.6238	3.6615
7	2.3137	2.5555	2.7253	2.8557	2.9615	3.0502	3.1265	3.1934	3.2528	3.3063	3.3549	3.3994	3.4403	3.4783	3.5137
8	2.2623	2.4936	2.6555	2.7798	2.8805	2.9650	3.0376	3.1013	3.1579	3.2088	3.2550	3.2974	3.3364	3.3726	3.4063
9	2.2236	2.4468	2.6028	2.7225	2.8193	2.9005	2.9703	3.0315	3.0859	3.1348	3.1792	3.2199	3.2574	3.2922	3.3246
10	2.1934	2.4103	2.5616	2.6776	2.7713	2.8499	2.9175	2.9767	3.0294	3.0767	3.1197	3.1591	3.1954	3.2290	3.2603
11	2.1692	2.3810	2.5286	2.6415	2.7328	2.8093	2.8750	2.9326	2.9838	3.0299	3.0717	3.1100	3.1453	3.1780	3.2085
12	2.1493	2.3570	2.5014	2.6119	2.7011	2.7759	2.8401	2.8964	2.9464	2.9913	3.0322	3.0695	3.1040	3.1359	3.1657
13	2.1327	2.3369	2.4788	2.5871	2.6747	2.7479	2.8109	2.8660	2.9150	2.9590	2.9990	3.0357	3.0694	3.1007	3.1298
14	2.1186	2.3199	2.4595	2.5661	2.6522	2.7242	2.7861	2.8402	2.8883	2.9316	2.9709	3.0068	3.0400	3.0707	3.0993
15	2.1066	2.3053	2.4430	2.5481	2.6329	2.7038	2.7647	2.8180	2.8654	2.9080	2.9466	2.9820	3.0146	3.0448	3.0730
16	2.0961	2.2926	2.4287	2.5324	2.6161	2.6861	2.7462	2.7987	2.8454	2.8874	2.9255	2.9604	2.9926	3.0223	3.0501
17	2.0870	2.2815	2.4161	2.5187	2.6014	2.6706	2.7299	2.7818	2.8279	2.8694	2.9070	2.9415	2.9732	3.0026	3.0300
18	2.0789	2.2717	2.4051	2.5066	2.5884	2.6568	2.7155	2.7669	2.8124	2.8534	2.8906	2.9247	2.9560	2.9851	3.0122
19	2.0717	2.2630	2.3952	2.4958	2.5769	2.6446	2.7027	2.7535	2.7987	2.8392	2.8760	2.9097	2.9407	2.9695	2.9963
20	2.0652	2.2552	2.3864	2.4862	2.5665	2.6336	2.6912	2.7416	2.7863	2.8265	2.8629	2.8963	2.9270	2.9555	2.9820
21	2.0594	2.2482	2.3784	2.4775	2.5572	2.6238	2.6809	2.7308	2.7751	2.8150	2.8511	2.8842	2.9147	2.9429	2.9692
22	2.0542	2.2418	2.3712	2.4696	2.5487	2.6148	2.6715	2.7211	2.7650	2.8046	2.8404	2.8732	2.9035	2.9314	2.9575
23	2.0494	2.2360	2.3647	2.4624	2.5410	2.6067	2.6630	2.7122	2.7559	2.7951	2.8307	2.8632	2.8932	2.9210	2.9469
24	2.0450	2.2308	2.3587	2.4559	2.5340	2.5993	2.6552	2.7041	2.7474	2.7864	2.8218	2.8541	2.8839	2.9115	2.9372
25	2.0410	2.2259	2.3532	2.4498	2.5276	2.5924	2.6480	2.6966	2.7397	2.7785	2.8136	2.8457	2.8753	2.9027	2.9283
26	2.0374	2.2215	2.3481	2.4443	2.5216	2.5861	2.6414	2.6897	2.7326	2.7711	2.8060	2.8380	2.8674	2.8946	2.9200
27	2.0340	2.2173	2.3435	2.4392	2.5161	2.5803	2.6353	2.6834	2.7260	2.7643	2.7991	2.8308	2.8601	2.8872	2.9124
28	2.0308	2.2135	2.3391	2.4345	2.5111	2.5750	2.6297	2.6775	2.7200	2.7581	2.7926	2.8242	2.8533	2.8803	2.9054
29	2.0279	2.2100	2.3351	2.4301	2.5063	2.5700	2.6245	2.6721	2.7143	2.7522	2.7866	2.8180	2.8470	2.8738	2.8988
30	2.0252	2.2067	2.3314	2.4260	2.5019	2.5653	2.6196	2.6670	2.7090	2.7468	2.7810	2.8123	2.8411	2.8678	2.8927
35	2.0140	2.1931	2.3160	2.4091	2.4838	2.5461	2.5994	2.6460	2.6872	2.7243	2.7579	2.7886	2.8169	2.8430	2.8674
40	2.0056	2.1830	2.3045	2.3965	2.4703	2.5318	2.5844	2.6303	2.6710	2.7075	2.7406	2.7709	2.7987	2.8245	2.8485
45	1.9992	2.1752	2.2956	2.3858	2.4599	2.5207	2.5728	2.6182	2.6584	2.6946	2.7273	2.7572	2.7847	2.8102	2.8339
50	1.9940	2.1689	2.2886	2.3790	2.4515	2.5119	2.5635	2.6085	2.6484	2.6842	2.7166	2.7462	2.7735	2.7987	2.8222
55	1.9899	2.1639	2.2828	2.3727	2.4448	2.5047	2.5559	2.6006	2.6402	2.6758	2.7079	2.7373	2.7644	2.7894	2.8127
60	1.9854	2.1596	2.2780	2.3675	2.4391	2.4987	2.5497	2.5941	2.6334	2.6687	2.7007	2.7299	2.7567	2.7816	2.8047
120	1.9674	2.1367	2.2519	2.3389	2.4083	2.4661	2.5154	2.5583	2.5963	2.6303	2.6611	2.6893	2.7151	2.7391	2.7613
240	1.9581	2.1253	2.2391	2.3247	2.3931	2.4499	2.4984	2.5406	2.5779	2.6113	2.6415	2.6691	2.6945	2.7179	2.7397
∞	1.9488	2.1141	2.2263	2.3107	2.3780	2.4339	2.4815	2.5229	2.5596	2.5923	2.6220	2.6490	2.6739	2.6969	2.7182

Table 3 (continued)

STUDENTIZED MAXIMUM MODULUS FOR 1.-ALPHA = .90 (CONT.)

D.F.	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
2	6.6039	6.6716	6.7353	6.7954	6.8522	6.9062	6.9575	7.0064	7.0531	7.0978	7.1406	7.1818	7.2213	7.2594	7.2960
3	4.9296	4.9778	5.0232	5.0660	5.1066	5.1451	5.1818	5.2167	5.2501	5.2821	5.3128	5.3422	5.3705	5.3978	5.4241
4	4.2687	4.3090	4.3469	4.3827	4.4166	4.4488	4.4795	4.5088	4.5368	4.5636	4.5893	4.6140	4.6377	4.6606	4.6827
5	3.9161	3.9520	3.9858	4.0177	4.0480	4.0767	4.1041	4.1302	4.1552	4.1792	4.2021	4.2242	4.2454	4.2659	4.2856
6	3.6967	3.7297	3.7609	3.7904	3.8183	3.8448	3.8701	3.8942	3.9172	3.9393	3.9605	3.9809	4.0005	4.0194	4.0376
7	3.5468	3.5779	3.6072	3.6349	3.6611	3.6861	3.7099	3.7326	3.7543	3.7751	3.7950	3.8142	3.8327	3.8505	3.8676
8	3.4378	3.4674	3.4953	3.5217	3.5467	3.5705	3.5932	3.6148	3.6355	3.6553	3.6744	3.6927	3.7103	3.7273	3.7436
9	3.3549	3.3833	3.4102	3.4356	3.4596	3.4825	3.5043	3.5251	3.5450	3.5641	3.5824	3.6000	3.6169	3.6333	3.6490
10	3.2897	3.3172	3.3432	3.3677	3.3910	3.4132	3.4343	3.4544	3.4737	3.4921	3.5099	3.5269	3.5433	3.5591	3.5744
11	3.2370	3.2638	3.2890	3.3129	3.3356	3.3571	3.3776	3.3972	3.4160	3.4339	3.4512	3.4678	3.4837	3.4991	3.5140
12	3.1935	3.2197	3.2444	3.2677	3.2898	3.3109	3.3309	3.3500	3.3683	3.3859	3.4027	3.4189	3.4345	3.4495	3.4640
13	3.1571	3.1827	3.2069	3.2297	3.2514	3.2720	3.2916	3.3104	3.3283	3.3455	3.3620	3.3778	3.3931	3.4078	3.4220
14	3.1261	3.1512	3.1750	3.1974	3.2187	3.2389	3.2581	3.2765	3.2941	3.3110	3.3272	3.3428	3.3578	3.3722	3.3862
15	3.0993	3.1241	3.1474	3.1695	3.1904	3.2103	3.2293	3.2474	3.2647	3.2813	3.2972	3.3126	3.3273	3.3415	3.3552
16	3.0761	3.1005	3.1235	3.1452	3.1658	3.1854	3.2041	3.2220	3.2390	3.2554	3.2711	3.2862	3.3007	3.3147	3.3283
17	3.0556	3.0797	3.1024	3.1239	3.1442	3.1636	3.1820	3.1996	3.2165	3.2326	3.2481	3.2630	3.2774	3.2912	3.3045
18	3.0375	3.0613	3.0837	3.1050	3.1251	3.1442	3.1624	3.1798	3.1965	3.2124	3.2277	3.2424	3.2566	3.2703	3.2835
19	3.0214	3.0449	3.0671	3.0881	3.1080	3.1269	3.1449	3.1621	3.1786	3.1944	3.2095	3.2241	3.2381	3.2516	3.2647
20	3.0069	3.0302	3.0521	3.0729	3.0926	3.1114	3.1292	3.1462	3.1625	3.1782	3.1932	3.2076	3.2215	3.2348	3.2478
21	2.9938	3.0169	3.0387	3.0593	3.0788	3.0973	3.1150	3.1319	3.1481	3.1636	3.1784	3.1927	3.2065	3.2197	3.2325
22	2.9819	3.0048	3.0264	3.0469	3.0662	3.0846	3.1022	3.1189	3.1349	3.1503	3.1650	3.1792	3.1928	3.2060	3.2187
23	2.9711	2.9939	3.0153	3.0356	3.0548	3.0730	3.0904	3.1070	3.1229	3.1382	3.1528	3.1668	3.1804	3.1934	3.2060
24	2.9612	2.9838	3.0051	3.0252	3.0443	3.0624	3.0797	3.0962	3.1120	3.1271	3.1416	3.1556	3.1690	3.1819	3.1944
25	2.9521	2.9746	2.9957	3.0157	3.0346	3.0527	3.0698	3.0862	3.1019	3.1169	3.1313	3.1452	3.1585	3.1714	3.1838
26	2.9438	2.9661	2.9871	3.0069	3.0258	3.0437	3.0607	3.0770	3.0926	3.1075	3.1218	3.1356	3.1488	3.1616	3.1740
27	2.9360	2.9582	2.9791	2.9988	3.0175	3.0353	3.0523	3.0685	3.0840	3.0988	3.1130	3.1267	3.1399	3.1526	3.1649
28	2.9288	2.9509	2.9717	2.9913	3.0099	3.0276	3.0445	3.0606	3.0760	3.0907	3.1049	3.1185	3.1316	3.1442	3.1564
29	2.9222	2.9441	2.9648	2.9843	3.0028	3.0204	3.0372	3.0532	3.0685	3.0832	3.0973	3.1108	3.1239	3.1364	3.1486
30	2.9159	2.9378	2.9583	2.9778	2.9962	3.0137	3.0304	3.0464	3.0616	3.0762	3.0902	3.1037	3.1167	3.1292	3.1412
35	2.8902	2.9116	2.9317	2.9508	2.9689	2.9860	3.0024	3.0180	3.0329	3.0472	3.0609	3.0741	3.0868	3.0991	3.1109
40	2.8710	2.8920	2.9119	2.9306	2.9484	2.9653	2.9814	2.9968	3.0114	3.0255	3.0390	3.0520	3.0645	3.0765	3.0882
45	2.8561	2.8769	2.8965	2.9150	2.9325	2.9492	2.9651	2.9803	2.9948	3.0087	3.0220	3.0348	3.0471	3.0590	3.0705
50	2.8442	2.8648	2.8842	2.9025	2.9199	2.9364	2.9521	2.9671	2.9815	2.9952	3.0084	3.0211	3.0333	3.0450	3.0564
55	2.8344	2.8549	2.8741	2.8923	2.9095	2.9259	2.9415	2.9564	2.9706	2.9842	2.9973	3.0098	3.0219	3.0336	3.0448
60	2.8264	2.8466	2.8658	2.8838	2.9009	2.9172	2.9326	2.9474	2.9615	2.9750	2.9880	3.0005	3.0125	3.0241	3.0352
120	2.7821	2.8016	2.8199	2.8373	2.8537	2.8693	2.8841	2.8983	2.9118	2.9248	2.9372	2.9491	2.9606	2.9717	2.9824
240	2.7601	2.7792	2.7971	2.8141	2.8302	2.8454	2.8599	2.8738	2.8870	2.8997	2.9119	2.9235	2.9348	2.9456	2.9560
∞	2.7382	2.7568	2.7744	2.7910	2.8067	2.8217	2.8358	2.8494	2.8623	2.8747	2.8866	2.8980	2.9089	2.9195	2.9297

Table 4

STUDENTIZED MAXIMUM MODULUS FOR 1.-ALPHA = .80

D.F.	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2.5493	2.9419	3.2173	3.4277	3.5970	3.7380	3.8585	3.9635	4.0564	4.1396	4.2147	4.2833	4.3462	4.4044	4.4584
3	2.1666	2.4757	2.6925	2.8582	2.9918	3.1033	3.1988	3.2821	3.3558	3.4219	3.4818	3.5364	3.5865	3.6329	3.6760
4	2.0057	2.2791	2.4703	2.6165	2.7344	2.8329	2.9172	2.9908	3.0561	3.1146	3.1676	3.2160	3.2605	3.3016	3.3399
5	1.9175	2.1710	2.3479	2.4830	2.5919	2.6829	2.7609	2.8290	2.8894	2.9435	2.9926	3.0374	3.0786	3.1167	3.1521
6	1.8620	2.1028	2.2704	2.3983	2.5014	2.5876	2.6613	2.7258	2.7829	2.8342	2.8807	2.9231	2.9621	2.9982	3.0318
7	1.8238	2.0558	2.2169	2.3398	2.4388	2.5214	2.5923	2.6541	2.7089	2.7582	2.8027	2.8435	2.8810	2.9156	2.9479
8	1.7959	2.0215	2.1778	2.2969	2.3928	2.4729	2.5415	2.6014	2.6545	2.7021	2.7453	2.7848	2.8211	2.8546	2.8859
9	1.7747	1.9953	2.1480	2.2642	2.3576	2.4357	2.5025	2.5609	2.6126	2.6591	2.7011	2.7396	2.7750	2.8077	2.8381
10	1.7580	1.9747	2.1244	2.2383	2.3298	2.4062	2.4717	2.5288	2.5795	2.6249	2.6661	2.7037	2.7384	2.7704	2.8002
11	1.7446	1.9580	2.1054	2.2173	2.3073	2.3824	2.4467	2.5028	2.5525	2.5972	2.6376	2.6746	2.7086	2.7400	2.7693
12	1.7335	1.9443	2.0896	2.2000	2.2887	2.3626	2.4259	2.4812	2.5302	2.5741	2.6140	2.6504	2.6838	2.7148	2.7436
13	1.7242	1.9328	2.0764	2.1855	2.2730	2.3460	2.4085	2.4630	2.5114	2.5547	2.5940	2.6299	2.6630	2.6935	2.7219
14	1.7163	1.9230	2.0652	2.1731	2.2597	2.3319	2.3936	2.4475	2.4953	2.5382	2.5770	2.6125	2.6451	2.6753	2.7034
15	1.7094	1.9145	2.0555	2.1624	2.2481	2.3196	2.3808	2.4341	2.4814	2.5238	2.5623	2.5974	2.6296	2.6595	2.6873
16	1.7035	1.9072	2.0471	2.1531	2.2381	2.3090	2.3696	2.4224	2.4693	2.5113	2.5494	2.5842	2.6161	2.6457	2.6733
17	1.6983	1.9007	2.0396	2.1449	2.2293	2.2996	2.3597	2.4121	2.4586	2.5003	2.5380	2.5725	2.6042	2.6336	2.6609
18	1.6937	1.8950	2.0331	2.1376	2.2214	2.2912	2.3509	2.4030	2.4491	2.4905	2.5280	2.5622	2.5937	2.6228	2.6499
19	1.6896	1.8899	2.0272	2.1311	2.2145	2.2838	2.3431	2.3948	2.4407	2.4817	2.5189	2.5529	2.5842	2.6131	2.6400
20	1.6859	1.8853	2.0220	2.1253	2.2082	2.2771	2.3361	2.3875	2.4330	2.4739	2.5108	2.5446	2.5757	2.6044	2.6311
21	1.6826	1.8812	2.0172	2.1201	2.2025	2.2711	2.3298	2.3809	2.4262	2.4668	2.5035	2.5371	2.5680	2.5966	2.6231
22	1.6796	1.8775	2.0129	2.1153	2.1974	2.2656	2.3240	2.3749	2.4199	2.4603	2.4969	2.5303	2.5610	2.5894	2.6159
23	1.6769	1.8741	2.0090	2.1110	2.1927	2.2607	2.3187	2.3694	2.4142	2.4544	2.4908	2.5241	2.5546	2.5829	2.6092
24	1.6744	1.8710	2.0054	2.1070	2.1884	2.2561	2.3139	2.3643	2.4090	2.4490	2.4852	2.5183	2.5488	2.5769	2.6031
25	1.6721	1.8681	2.0021	2.1034	2.1845	2.2519	2.3095	2.3597	2.4042	2.4440	2.4801	2.5131	2.5434	2.5714	2.5975
26	1.6700	1.8655	1.9991	2.1000	2.1808	2.2480	2.3054	2.3555	2.3998	2.4395	2.4754	2.5082	2.5384	2.5663	2.5923
27	1.6680	1.8630	1.9963	2.0969	2.1775	2.2444	2.3016	2.3515	2.3956	2.4352	2.4710	2.5037	2.5338	2.5616	2.5875
28	1.6662	1.8608	1.9937	2.0940	2.1744	2.2411	2.2981	2.3478	2.3918	2.4313	2.4670	2.4995	2.5295	2.5572	2.5830
29	1.6645	1.8587	1.9913	2.0914	2.1715	2.2380	2.2949	2.3444	2.3883	2.4276	2.4632	2.4956	2.5255	2.5531	2.5788
30	1.6629	1.8567	1.9890	2.0889	2.1687	2.2351	2.2918	2.3412	2.3850	2.4242	2.4596	2.4920	2.5218	2.5493	2.5749
35	1.6565	1.8486	1.9797	2.0785	2.1576	2.2232	2.2793	2.3281	2.3713	2.4100	2.4450	2.4770	2.5063	2.5335	2.5588
40	1.6516	1.8426	1.9728	2.0708	2.1492	2.2143	2.2698	2.3182	2.3610	2.3993	2.4340	2.4656	2.4947	2.5216	2.5467
45	1.6479	1.8379	1.9674	2.0648	2.1427	2.2074	2.2625	2.3105	2.3530	2.3910	2.4254	2.4569	2.4857	2.5124	2.5372
50	1.6449	1.8342	1.9631	2.0601	2.1375	2.2018	2.2567	2.3044	2.3466	2.3844	2.4186	2.4498	2.4785	2.5050	2.5296
55	1.6425	1.8312	1.9595	2.0562	2.1333	2.1973	2.2519	2.2994	2.3414	2.3790	2.4130	2.4440	2.4726	2.4989	2.5234
60	1.6404	1.8287	1.9566	2.0529	2.1298	2.1935	2.2479	2.2952	2.3370	2.3745	2.4083	2.4392	2.4676	2.4939	2.5183
120	1.6294	1.8148	1.9406	2.0351	2.1104	2.1729	2.2260	2.2723	2.3131	2.3496	2.3827	2.4128	2.4405	2.4660	2.4898
240	1.6239	1.8080	1.9327	2.0263	2.1008	2.1626	2.2151	2.2608	2.3012	2.3372	2.3698	2.3996	2.4268	2.4521	2.4755
∞	1.6184	1.8011	1.9248	2.0175	2.0912	2.1523	2.2042	2.2494	2.2892	2.3248	2.3570	2.3863	2.4132	2.4381	2.4611

Table 4 (continued)

STUDENTIZED MAXIMUM MODULUS FOR 1.-ALPHA = .80 (CONT.)

D.F.	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
2	4.5088	4.5560	4.6003	4.6422	4.6818	4.7194	4.7551	4.7891	4.8216	4.8527	4.8825	4.9111	4.9386	4.9650	4.9905
3	3.7162	3.7540	3.7894	3.8229	3.8546	3.8847	3.9133	3.9405	3.9666	3.9915	4.0154	4.0383	4.0604	4.0816	4.1021
4	3.3756	3.4091	3.4406	3.4703	3.4985	3.5252	3.5507	3.5749	3.5981	3.6203	3.6415	3.6620	3.6816	3.7005	3.7187
5	3.1852	3.2163	3.2455	3.2731	3.2992	3.3240	3.3476	3.3701	3.3917	3.4122	3.4320	3.4509	3.4692	3.4867	3.5037
6	3.0632	3.0926	3.1203	3.1465	3.1713	3.1948	3.2172	3.2385	3.2590	3.2785	3.2972	3.3152	3.3325	3.3492	3.3653
7	2.9780	3.0053	3.0329	3.0581	3.0819	3.1045	3.1260	3.1465	3.1661	3.1849	3.2029	3.2202	3.2369	3.2529	3.2683
8	2.9151	2.9425	2.9683	2.9926	3.0157	3.0376	3.0585	3.0784	3.0974	3.1156	3.1330	3.1498	3.1659	3.1815	3.1965
9	2.8666	2.8933	2.9184	2.9422	2.9647	2.9860	3.0064	3.0257	3.0443	3.0620	3.0791	3.0954	3.1111	3.1263	3.1409
10	2.8281	2.8542	2.8788	2.9020	2.9241	2.9450	2.9649	2.9838	3.0020	3.0194	3.0360	3.0521	3.0675	3.0823	3.0966
11	2.7966	2.8223	2.8465	2.8693	2.8909	2.9115	2.9310	2.9497	2.9675	2.9846	3.0009	3.0167	3.0318	3.0464	3.0604
12	2.7706	2.7958	2.8196	2.8421	2.8634	2.8836	2.9028	2.9212	2.9388	2.9556	2.9717	2.9872	3.0021	3.0164	3.0303
13	2.7485	2.7734	2.7969	2.8191	2.8401	2.8600	2.8790	2.8971	2.9145	2.9310	2.9469	2.9622	2.9769	2.9911	3.0048
14	2.7296	2.7543	2.7775	2.7994	2.8201	2.8398	2.8586	2.8765	2.8936	2.9100	2.9257	2.9408	2.9554	2.9693	2.9828
15	2.7133	2.7377	2.7606	2.7823	2.8028	2.8223	2.8409	2.8586	2.8755	2.8918	2.9073	2.9223	2.9366	2.9505	2.9638
16	2.6990	2.7231	2.7459	2.7674	2.7877	2.8070	2.8254	2.8429	2.8597	2.8758	2.8912	2.9060	2.9202	2.9339	2.9472
17	2.6864	2.7103	2.7329	2.7542	2.7743	2.7935	2.8117	2.8291	2.8457	2.8617	2.8769	2.8916	2.9057	2.9193	2.9324
18	2.6752	2.6989	2.7213	2.7424	2.7624	2.7814	2.7995	2.8168	2.8333	2.8491	2.8642	2.8788	2.8928	2.9063	2.9193
19	2.6652	2.6887	2.7109	2.7319	2.7518	2.7707	2.7886	2.8058	2.8221	2.8378	2.8529	2.8673	2.8812	2.8946	2.9076
20	2.6561	2.6796	2.7016	2.7225	2.7422	2.7609	2.7788	2.7958	2.8121	2.8277	2.8426	2.8570	2.8708	2.8841	2.8970
21	2.6480	2.6713	2.6932	2.7139	2.7336	2.7522	2.7699	2.7869	2.8030	2.8185	2.8334	2.8477	2.8614	2.8746	2.8874
22	2.6406	2.6637	2.6855	2.7062	2.7257	2.7442	2.7619	2.7787	2.7948	2.8102	2.8250	2.8392	2.8528	2.8660	2.8787
23	2.6338	2.6568	2.6785	2.6990	2.7185	2.7369	2.7545	2.7712	2.7872	2.8026	2.8173	2.8314	2.8450	2.8581	2.8707
24	2.6276	2.6505	2.6721	2.6925	2.7118	2.7302	2.7477	2.7643	2.7803	2.7955	2.8102	2.8242	2.8378	2.8508	2.8634
25	2.6218	2.6447	2.6662	2.6865	2.7057	2.7240	2.7414	2.7580	2.7739	2.7891	2.8036	2.8176	2.8311	2.8441	2.8566
26	2.6165	2.6393	2.6607	2.6809	2.7001	2.7183	2.7356	2.7522	2.7680	2.7831	2.7976	2.8115	2.8250	2.8379	2.8503
27	2.6116	2.6343	2.6556	2.6758	2.6949	2.7130	2.7303	2.7467	2.7625	2.7775	2.7920	2.8059	2.8192	2.8321	2.8445
28	2.6071	2.6297	2.6509	2.6710	2.6900	2.7081	2.7253	2.7417	2.7574	2.7724	2.7868	2.8006	2.8139	2.8267	2.8391
29	2.6028	2.6253	2.6465	2.6665	2.6855	2.7035	2.7206	2.7370	2.7526	2.7676	2.7819	2.7957	2.8090	2.8217	2.8341
30	2.5989	2.6213	2.6424	2.6624	2.6813	2.6992	2.7163	2.7326	2.7481	2.7631	2.7774	2.7911	2.8043	2.8171	2.8293
35	2.5824	2.6045	2.6254	2.6450	2.6637	2.6814	2.6982	2.7143	2.7296	2.7443	2.7584	2.7720	2.7850	2.7976	2.8097
40	2.5700	2.5919	2.6125	2.6320	2.6504	2.6680	2.6846	2.7005	2.7157	2.7302	2.7442	2.7576	2.7705	2.7829	2.7949
45	2.5604	2.5821	2.6025	2.6218	2.6401	2.6575	2.6740	2.6898	2.7048	2.7192	2.7330	2.7463	2.7591	2.7714	2.7833
50	2.5526	2.5742	2.5945	2.6137	2.6318	2.6491	2.6655	2.6811	2.6961	2.7104	2.7241	2.7373	2.7500	2.7622	2.7740
55	2.5463	2.5678	2.5880	2.6070	2.6251	2.6422	2.6585	2.6740	2.6889	2.7031	2.7168	2.7299	2.7425	2.7546	2.7663
60	2.5410	2.5624	2.5825	2.6014	2.6194	2.6364	2.6527	2.6681	2.6829	2.6971	2.7106	2.7237	2.7362	2.7483	2.7599
120	2.5119	2.5327	2.5522	2.5707	2.5881	2.6047	2.6204	2.6355	2.6498	2.6636	2.6767	2.6894	2.7015	2.7133	2.7246
240	2.4973	2.5178	2.5370	2.5552	2.5724	2.5887	2.6042	2.6190	2.6332	2.6467	2.6596	2.6721	2.6841	2.6956	2.7067
∞	2.4827	2.5028	2.5218	2.5397	2.5566	2.5727	2.5879	2.6025	2.6164	2.6297	2.6425	2.6547	2.6665	2.6778	2.6887

that $h_{15}(4,0,\alpha) = 3.6082$ and 2.8051 for $\alpha = 0.01$ and 0.05, respectively. The joint confidence intervals with an experimentwise error rate of α are thus given by

$$\left\{ \begin{array}{l} \hat{\theta}_1 \pm h_{15}(4,0,\alpha)\sqrt{30s^2/4} \\ \hat{\theta}_2 \pm h_{15}(4,0,\alpha)\sqrt{2s^2/4} \\ \hat{\theta}_3 \pm h_{15}(4,0,\alpha)\sqrt{6s^2/4} \\ \hat{\theta}_4 \pm h_{15}(4,0,\alpha)\sqrt{2s^2/4} \end{array} \right\} . \qquad (3.1)$$

As pointed out in the text, the breakdown into single degrees of freedom in any particular situation will be dictated by the characteristics of the experiment. Two cases frequently used for 4-level experiments are $(c_{m1},c_{m2},c_{m3},c_{m4}) = (1,1,-1,-1), (1,-1,1,-1), (1,-1,-1,1)$ and $(c_{m1},c_{m2},c_{m3},c_{m4}) = (1,-1,0,0), (1,1,-2,0), (1,1,1,-3)$ for $m = 1,2,3$, respectively.

If the number of analyses made with each method were not all equal, some (and possibly all) of the contrasts under consideration would <u>not be orthogonal</u>. Then the use of our h-values yields a confidence coefficient greater than $1-\alpha$. (See Šidák (1967), equation (8).)

<u>3.2 2^n factorial experiments</u>

<u>Example 2: A 2^4 experiment</u>

Cochran and Cox (1957), Section 5.24a, analyze the yields obtained in a 2^4 experiment with four fertilizers (m = manure, n = nitrogen, p = phosphorus, k = potassium), each at two levels, conducted in four randomized complete blocks: the experiment was carried out to study the effects of these fertilizers on the yield of grass. The 64 yields per plot (total over 6 harvests, km. per 3-meter row) were used to compute the effect means which were given as $M = 18.0$, $N = 21.3$, $P = 5.5$, $k = 24.1$, $MN = 3.2$, $MP = -0.3$, $MK = -7.5$, $NP = 3.5$, $NK = 10.9$, $PK = 3.2$, $MNP = -1.4$, $MNK = -8.5$, $MPK = 0.8$, $NPK = 0.5$, $MNPK = -1.6$. The estimated standard error of an effect mean was computed to be $\sqrt{s^2/2^{n-2}r} = \sqrt{(90.5)/4(4)} = 2.38$ where $s^2 = 90.5$ is the error mean square based on $\nu = 45$ d.f., $r = 4$ is the number of replications of each treatment combination, and $n = 4$ is the number of factors. For $\nu = 45$ the Student t-values are 2.690 and 2.014 for $\alpha = 0.01$ and $\alpha = 0.05$, respectively. The effect means $\pm t_{45}^{\alpha}(2.38)$ are exhibited in Columns 2 and 3 of Table 5 for $\alpha = 0.01$ and $\alpha = 0.05$, respectively. Cochran and Cox give analogous information in the lower half of their Table 5.1a where they indicate effects that are statistically significant at the 1% (**) and 5%(*) levels, namely (M,N,K,MK,NK,MNK) and (P), respectively.

To control the <u>experimentwise</u> error rate, the corresponding h-values from our Tables 1 and 2 are $h_{45}(15,0,0.01) = 3.6503$ and $h_{45}(15,0,0.05) = 3.0803$ for $\alpha = 0.01$ and $\alpha = 0.05$, respectively. The effect means $\pm h_{45}(15,0,\alpha)$ (2.38) are exhibited in Columns 4 and 5 of Table 5 for $\alpha = 0.01$ and $\alpha = 0.05$, respectively. We note from Columns 2 and 4 that MK and MNK are now "significant" at the 5% (rather than at the 1% level) and that P is not significant at the 5% level. Thus for this experiment, controlling the <u>experimentwise</u> error rate has reduced the number of statistically significant results.

The experimenter must decide whether the per contrast or the experimentwise error rate is more pertinent in his particular experiment. In the extreme case of a single significant contrast, the experimenter using per contrast error rates would be left feeling unsure whether the effect

Table 5

A 2^4 complete factorial experiment[1/]

($p = 15$, $\nu = 45$)

Effect	Confidence interval on effect mean[2/]: Error rate α per contrast, $\alpha = 0.01$, $t_{45}^{\alpha} = 2.690$	Error rate α per contrast, $\alpha = 0.05$, $t_{45}^{\alpha} = 2.014$	Error rate α per experiment, $\alpha = 0.01$, $h_{45}^{\alpha} = 3.6503$	Error rate α per experiment, $\alpha = 0.05$, $h_{45}^{\alpha} = 3.0803$
M	18.0±6.4(**)	18.0±4.8(*)	18.0±8.7(**)	18.0±7.3(*)
N	21.3±6.4(**)	21.3±4.8(*)	21.3±8.7(**)	21.3±7.3(*)
P	5.5±6.4	5.5±4.8(*)	5.5±8.7	5.5±7.3
K	24.1±6.4(**)	24.1±4.8(*)	24.1±8.7(**)	24.1±7.3(*)
MN	3.2±6.4	3.2±4.8	3.2±8.7	3.2±7.3
MP	- 0.3±6.4	- 0.3±4.8	- 0.3±8.7	- 0.3±7.3
MK	- 7.5±6.4(**)	- 7.5±4.8(*)	- 7.5±8.7	- 7.5±7.3(*)
NP	3.5±6.4	3.5±4.8	3.5±8.7	3.5±7.3
NK	10.9±6.4(**)	10.9±4.8(*)	10.9±8.7(**)	10.9±7.3(*)
PK	3.2±6.4	3.2±4.8	3.2±8.7	3.2±7.3
MNP	- 1.4±6.4	- 1.4±4.8	- 1.4±8.7	- 1.4±7.3
MNK	- 8.5±6.4(**)	- 8.5±4.8(*)	- 8.5±8.7	- 8.5±7.3(*)
MPK	0.8±6.4	0.8±4.8	0.8±8.7	0.8±7.3
NPK	0.5±6.4	0.5±4.8	0.5±8.7	0.5±7.3
MNPK	- 1.6±6.4	- 1.6±4.8	- 1.6±8.7	- 1.6±7.3

[1/] Cochran and Cox (1957), Table 5.1a.

[2/] Standard error of effect mean is $\sqrt{s^2/2^{n-2}r} = \sqrt{(90.5)/4(4)} = 2.38$. The entry in each cell in the body of the table is either (effect mean) ± $t_{45}^{\alpha}(2.38)$ or (effect mean) ± $h_{45}^{\alpha}(2.38)$. The intervals indicated by (**) or (*) do not cover zero.

was a real one, given that the experiment has provided multiple opportunities for one of them to be significant. On the other hand, in an experiment with several significant contrasts as in the present example, strict use of the experimentwise error rate procedure makes it more difficult for the smaller effects to be declared significant after the larger ones have been identified.

We have assumed in the above analysis that à priori the experimenter was interested in controlling the experimentwise error for all 15 contrasts. If, à priori, he had been interested in only the 4 main effects and the 6 two-factor interactions (and he had made that decision without being influenced by the data) then $p = 10$, $\nu = 45$, and the appropriate h-value for $\alpha = 0.01$ and $\alpha = 0.05$ would be 3.5149 and 2.9346, respectively; now MK is still significant at the 5% level and P′ is still not significant at the 5% level, while the status of MNK (as well as MNP, MPK, NPK and MNPK) in terms of possible significance would be unknown. If, after looking at the data, the experimenter decided that only M,N,K,NK and MNK were of interest, then he still must use the original factor $h_{45}(15,0,\alpha)$ in reporting his final results. Effectively, what he has done here is "data snooping" in the sense of Scheffé (1959), p. 80, and he must pay for that privilege by using the larger h-value if he desires to make statistically legitimate confidence statements with experimentwise control over the error rate. In this situation the inference must be limited to a particular set of orthogonal contrasts specified in advance.

It thus is clear that in 2^n complete factorial experiments where n is "large," it is to the experimenter's advantage if he can specify à priori which contrasts are and/or are not of interest; analogous considerations hold for fractional factorial experiments. In certain types of experiments it is easy to identify certain contrasts which are not of interest. We consider such a problem in Example 3.

Example 3: A 2^3 experiment with two classification factors

Consider a 3-factor experiment, each factor at two levels, where the factors are diet (Diet 1 vs. Diet 2), sex (male vs. female), and age (old vs. young). The purpose of the experiment is to study the effect of change in diet on gain in weight. Here the treatment factor of interest is diet while sex and age are classification variables. Thus, denoting the main effect of diet, sex and age by A, B and C, respectively, and analogously for their interactions, the experimenter would be interested in the $p = 4$ orthogonal contrasts associated with A, AB, AC and ABC rather than in all 7 orthogonal contrasts. Similarly, in a 2^4 experiment two of which are classification factors, the experimenter would be interested in at most the $p = 12$ orthogonal contrasts A, B, AB, AC, AD, ACD, BC, BD, BCD, ABC, ABD, ABCD. See Cox (1958), Examples 6.3 and 6.4, for a discussion of treatment factors and classification factors.

3.3 3^n factorial experiments, all factors quantitative

Example 4: A 3^3 experiment

Davies (1978), pp. 332-336, reports the results of a 3^3 experiment, each factor quantitative and equally spaced, all treatment combinations replicated twice. The variable under study is the yield of a chemical process, the three factors being: i) C, the concentration of an inorganic material (A) in the free water present in the reaction mixture, ii) V, the volume of free water present in the reaction mixture, and iii) N, the amount of a second inorganic material (B) in the reaction mixture. Each factor was studied at three equally spaced levels, and two replications of each of the 27 treatment combinations was obtained. A quadratic response surface was fit to the data using orthogonal polynomials, and the total d.f. for treatments was partitioned into 18 individual d.f. associated with the 6 main effects $(C_L, C_Q, V_L, V_Q, N_L, N_Q)$ and the 12 two-factor interactions $(C_L \times V_L,\ C_Q \times V_L,\ C_L \times V_Q,\ C_Q \times V_Q,\ C_L \times N_L,\ C_Q \times N_L,\ C_L \times N_Q,\ C_Q \times N_Q,\ V_L \times N_L,\ V_Q \times N_L,\ V_L \times N_Q,\ V_Q \times N_Q)$: the remaining 8 degrees of freedom representing the three-factor interactions were pooled. Here the subscripts L and Q represent linear and quadratic, respectively. There were 27 d.f. associated with the error mean square.

For $\nu = 27$ the Student t-values are 2.771 and 2.052, for $\alpha = 0.01$ and $\alpha = 0.05$, respectively. Using these values (actually the corresponding $F^1_{27} = (t_{27})^2$ values were used) Davies reported the 8 effects $(C_L,\ C_Q,\ V_L,\ V_Q,\ C_L \times V_L,\ C_Q \times V_L,\ C_L \times V_Q,\ C_Q \times N_L)$ as being statistically significant at the 1% level and the 2 effects $(N_Q,\ C_L \times N_L)$ significant at the 10% level. (Note: We are reporting the results here as tests of significance to conform with Davies, but we would have preferred to present our results as interval estimates as in our Table 5.)

To control the experimentwise error rate the corresponding h-values from our Tables 1, 2 and 3 are $h_{27}(18,0,0.01) = 3.8989$, $h_{27}(18,0,0.05) = 3.2560$, and $h_{27}(18,0,0.10) = 2.9582$ for $\alpha = 0.01$, $\alpha = 0.05$ and $\alpha = 0.10$, respectively. Thus, if the experimenter wished to control the experimentwise error rate for the 18 orthogonal contrasts of interest he would assert that the 3 effects $(C_L, C_Q, C_L{\times}V_L)$ are significant at the 1% level, the 3 effects $(V_L, V_Q, C_Q{\times}N_L)$ are significant at the 5% level, the one effect $(C_Q{\times}V_L)$ is significant at the 10% level, and the remaining 11 effects are not significant at the 10% level. (The experimenter had decided à priori that the 8 effects associated with the three-factor interactions were not of interest, and hence this total sum of squares based on 8 d.f. was not partitioned into the 8 individual one d.f. sums of squares associated with each of the remaining relevant orthogonal contrasts.)

Thus the same considerations arise in the analysis of this experiment as arose in the analysis of the 2^4 experiment of Example 2. And the same caveats hold here as well.

3.4 An application to biological assay

The purpose of biological assay is to estimate the potency ratio, ρ, of two biological preparations which have dose-response curves which can be represented by the same form of regression function and which differ only in the factor ρ in the dose scale. (We use the symbol ρ here as in Finney (1978), p. 41.) A common situation is one in which the response scale is linear in log dose. In this case parallel straight lines can be fit to the two sets of data; an estimate of $\log \rho$ is then given by the horizontal distance between them. A useful experimental design for such situations is the so-called symmetric (k,k)-point design in which k dose levels equally spaced on a log scale are used for each preparation (the k levels being different for the two preparations) with n observations being taken at each of the $2k$ design points;

the usual values for k are 2, 3 or 4. See Finney (1978), p. 105.

In analyzing the data from a (k,k) bioassay, the sum of squares between treatments has $2k-1$ d.f. which can be separated into $2k-1$ meaningful orthogonal components. See Finney (1978). pp. 105-109, for an example with $k = 3$ where the 5 orthogonal contrasts are denoted by L_p (preparations), L_1 (average linear regression), L_1' (parallelism), L_2 (average quadratic regression), L_2' (difference between quadratics). The first two enter into the calculation of the estimated relative potency while the remaining three are used to test the validity of the assay. However, significance tests at level $\alpha = 0.05$ for each of these three will result in an error rate for the assay approaching $1 - (1-\alpha)^3 = 0.14$. If it is desired to control the experimentwise error rate at a specified value α, then the constants h tabulated in our paper can be used.

To apply our constants to Finney's example, the largest of the three mean squares for the validity contrasts, which here is L_2 having a value of 0.001606 (see Finney's table 5.2.2), is expressed as a ratio to the error mean square based on 30 d.f.; this ratio then is compared for $\alpha = 0.05$ with $(h_{30}(3,0,0.05))^2 = (2.5224)^2 = 6.36$ instead of referring it to tables of $F^1_{30} = (t^1_{30})^2 = (2.042)^2 = 4.17$. In this experiment the ratio is only 0.52, a clearly non-significant result.

In Finney's description, the contrast L_p is also considered as a test of a type of assay validity, and as such it could be included with the other three to form a set of four simultaneous tests involving orthogonal contrasts. Actually, L_p provides a measure of how successful the experimenter has been in choosing comparable dose levels of the two preparations, and a significant value provides a signal to the experimenter that he might be comparing the two preparations at different portions of the dose-response curve rather than necessarily invalidating the assay. Thus it might be preferable to consider it separately from the other three validity contrasts, as illustrated in the preceding paragraph.

Similar considerations apply with more than three dose levels of each preparation. For example, in a (4,4) bioassay there would be 5 orthogonal contrasts in addition to L_p and L_1. If linearity is assumed then these provide 5 separate tests for assay validity which can be tested by using $h_\nu(5,0,\alpha)$ from our tables in order to achieve an experimentwise error α for the validity tests. On the other hand, if a quadratic dose response curve is assumed, two of them enter into the calculation of the estimated potency, as described by Finney (1978), p. 122, leaving a set of 3 orthogonal contrasts to test the validity of the bioassay.

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ABSTRACT

In many experimental situations the pertinent inferences are made on the basis of orthogonal contrasts among the treatment means (as in 2^n factorial experiments). In this setting a particularly useful form of inference is one involving multiple comparisons. The present paper describes situations in which such inferences are meaningful, gives examples of their use, and provides an extensive set of tables of constants needed to implement such multiple comparison procedures. The procedures can also be used for statistically legitimate "data snooping" (in the sense of Scheffé (1959), p. 80) to help decide which contrasts within a specified set warrant further study.

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