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MORSE PROGRAMS: A TOPOLOGICAL API
TO SMOOTH CONSTRAINED OPTIMIZATION II.

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ABSTRACT

We consider nonlinear constrained optimization problems in which the objective function and constraint functions are sufficiently smooth. We focus on the programs which consist of both equality and inequality constraints, and we prove that the global optimum value function is twice continuously differentiable almost everywhere with respect to the parameters.

AMS(MOS) Subject Classification: 58A05, 90C30

Key Words: Morse functions, Nonlinear Programming, Sensitivity analysis, Second order conditions

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SIGNIFICANCE AND EXPLANATION

In the analysis of nonlinear optimization problems which arise in engineering, management science and economic theory, it is important not to assume that the relevant objective and constraint functions are convex. In this paper we give an analysis of such problems under the assumption that these functions are sufficiently smooth. We show that almost always one can expect that a nonlinear program will be "well-behaved" and that the global optimum value changes smoothly with changes in the data.

Morse Programs: A Topological Approach to Smooth Constrained Optimization II

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Introduction

In this paper, we continue the analysis of smooth nonlinear programming problems which we began in [4]. There we reduced the nonlinear programs to a finite family of "well-behaved" nonlinear programs, each of which consists of minimizing a Morse function on a manifold with boundary, by perturbing the objective function in a linear fashion and pertrubing the right hand side of constraints by adding a constant. We also gave the geometrical meaning of each "well-behaved" program.

Here we consider the nonlinear programs which consist of both equality and ineuality constraints and, in particular,

$$(\bar{P}): \qquad \underset{\|x\|^2 \leq c}{\text{minimize}} \{f(x) \quad \text{subject to} \quad g(x) = b\}$$

and its perturbation

$$(\bar{P}(u,v)): \quad \underset{\|x\|^2 \leq c}{\text{minimize}} \{f(x) - u^T x \quad \text{subject to} \quad g(x) = b + v\}$$

where $f: R^n \to R^1$, $g: R^n \to R^m$; $c > 0$; $u \in R^n$, $v \in R^m$; $n \geq m+1$. Our main results are: in the C^2 topology, Morse programs are open and dense in the family of $(\bar{P})$, where $f \in C^2$ and $g \in C^{n-m+1}$ (Theorem A), and if $f \in C^2$ and $g \in C^{n-m+1}$, then the global optimum value function $\bar{\omega}(u,v)$ for $(\bar{P}(u,v))$ is of class C^2 with respect to (u,v) on an open and dense set of $R^n \times R^m$ (Theorem B).

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1. Preliminaries and Notation.

A property that holds except on a subset of R^n whose Lebesgue measure is zero is said to hold at almost every $u \in R^n$. The complement of a measure zero set in R^n is said to have full measure in R^n.

The Jacobian matrix and the Hessian matrix of f at x are denoted by $Df(x)$ and $D^2f(x)$ respectively.

Let $f:M \to R^m$ be a C^γ map from a k-dimensional C^γ manifold M with boundary ∂M in R^n. Let (ϕ,U) be a local parametrization of M at x such that $x = \phi(u)$, $u \in U \subseteq H^k = \{x \in R^k | x_k \geq 0\}$. The tangent space T_xM of M at x is defined to be the image of $D\phi(u):R^k \to R^n$. A point $x \in M$ is a regular point of f if $D(f\phi)(u):R^k \to R^m$ is surjective, otherwise x is a critical point of f. A critical point x of $f:M \to R^1$ is nondegenerate if the $k \times k$ matrix $D^2(f\phi)(u)$ is nonsingular. It is easily shown that the above definitions do not depend on the choice of local parametrization. A point $y \in R^m$ is a regular value of f, denoted by $f \pitchfork y$, if every $x \in f^{-1}(y)$ is a regular point of f, otherwise y is a critical value of f. $f:M \to R^1$ is a Morse function if all critical points of f are nondegenerate.

Let $f:M \to N$ be a C^γ map, $A \subseteq N$ be a C^γ submanifold of N. f is transversal to A, denoted by $f \pitchfork A$, if for ever $x \in f^{-1}(A)$, $\text{Image}Df(x) + T_{f(x)}A = T_{f(x)}N$ holds, where $Df(x):T_xM \to T_{f(x)}N$ is the derivative of f. Two submanifolds A, B of M are transversal denoted by $A \pitchfork B$, if $i \pitchfork B$ where $i:A \to M$ is the inclusion map.

The proofs of the following theorems, which we will use in this paper, can be found in Gillemin and Pollack [5].

(1.1) Let $f:X \to Y$ be a C^{γ} map such that $f \pitchfork Z$ for a C^{γ} submanifold Z of Y, then $f^{-1}(Z)$ is a C^{γ} submanifold of X and $\dim f^{-1}(Z) = \dim X - \dim Y + \dim Z$.

(1.2) Let $f: X \to Y$ be a C^{γ} map of a C^{γ} manifold X with boundary ∂X onto a boundaryless C^{γ} manifold Y. If $f \pitchfork Z$ and $f|_{\partial X} \pitchfork Z$ for a boundaryless submanifold Z of Y, then $f^{-1}(Z)$ is a C^{γ} submanifold of X with boundary $\partial f^{-1}(Z) = f^{-1}(Z) \cap \partial X$ and $\dim f^{-1}(Z) = \dim X - \dim Y + \dim Z$.

(1.3) Let $f: X \to R^1$ be C^{γ} map such that $f \pitchfork c$ for some $c \in R^1$. Then $\{x | f(x) \leq c\}$ is a C^{γ} submanifold of X with boundary $f^{-1}(c)$.

(1.4) Let $f:X \to Y$ and $g: Y \to Z$ be C^{γ} maps. Suppose $g \pitchfork W$ for a C^{γ} submanifold W of Z. Then $g \circ f \pitchfork W$ if and only if $f \pitchfork g^{-1}(W)$.

(1.5) Let X, Z be submanifolds of Y such that $X \pitchfork Z$. Then $X \cap Z$ is again a submanifold of Y, $\dim(X \cap Z) = \text{dix } X + \dim Z - \dim Y$ and $T_x(X \cap Z) = T_xX \cap T_xZ$ for any $x \in X \cap Z$.

(1.6) Let $f:X \to R^1$ be a C^2 map of a C^2 manifold X in R^n. Then for almost every $u \in R^n$, the function $f(x) - u^T x$ is a Morse function on X.

2. Morse Programs: Definition and Properties.

Let us consider a program

(R): $\quad$ minimize $\{f(x)$ subject to $g(x) \le b, \ h(x) = c\}$

and a perturbation

$(R(u,v,w))$: $\quad$ minimize $\{f(x) - u^T x$ subject to $g(x) \le b+v, \ h(x) = c+w\}$

where $f: R^n \to R^1$, $g: R^n \to R^m$ and $h: R^n \to R^p$ are of class C^2, $u \in R^n$, $v \in R^m$, $w \in R^p$, $n > p$.

Let $I = \{1, \cdots, m\}$ and let us denote

$$\bar{M}_{J,i} := \{x \mid g_J(x) = b_J, \ g_i(x) \le b_i \ , \ h(x) = c\}$$

$$\partial \bar{M}_{J,i} = \{x \mid g_J(x) = b_J, \ g_i(x) = b_i \ , \ h(x) = c\}$$

for all $J \subseteq I$ and $i \in I$. For notational convenience we denote $\bar{M}_J := \bar{M}_{J,i}$ if $i \in J$, and $\bar{X}_i := M_{j,i}$, $\partial \bar{X}_i := \partial \bar{M}_{J,i}$ if $J = \phi$. Note that if $i \in J$, then $\bar{M}_J = \bar{M}_{j,i} = \partial \bar{M}_{J,i}$. Let $X_i := \{x \mid g_i(x) \le b_i\}$ and $\partial X_i := \{x \mid g_i(x) = b_i\}$ for all $i \in I$.

Definition

A program (R) is a Morse program if (R) satisfies

$(\bar{M}1)$ $\quad g_i \pitchfork b_i$, $h|_{X_i} \pitchfork c$ and $h|_{\partial X_i} \pitchfork c$ for all $i \in I$, and $h \pitchfork c$.

$(\bar{M}2)$ $\quad g_J|_{\bar{X}_i} \pitchfork b_J$ and $g_J|_{\partial \bar{X}_i} \pitchfork b_j$ for all nonempty $J \subseteq I$ and $i \notin J$.

$(\bar{M}3)$ $\quad f$ is a Morse function on $h^{-1}(c)$, $\bar{M}_{J,i}$ and $\partial \bar{M}_{J,i}$ for all $J \subseteq I$ and $i \in I$

$(\bar{M}4)$ $\quad f|_{\bar{M}_{j,i}}$ has no critical points on $\partial \bar{M}_{J,i}$ for all $J \subseteq I$ and $i \notin J$.

Remark 1 With the absence of equality or inequality constraints, the above definition of a Morse program coincides with the one that I defined in [4].

Remark 2. $\bar{X}_i$ (or X_i) is a manifold with boundary $\partial\bar{X}_i$ (or ∂X_i) by ($\bar{M}1$) and (1.2) (or (1.3)); and $\bar{M}_{J,i}$ is a manifold with boundary $\partial\bar{M}_{j,i}$ by ($\bar{M}2$) and (1.2).

Definition x is a critical point of (R) if x is a feasible point of (R) (i.e. $g(x) \leq b$ and $h(x) = c$) and x is a critical point of $f|_{\bar{M}_{J(x)}}$ where $J(x) := \{j | g_j(x) = b_j\}$.

The following results are verified in essentially the same way as Theorem F and Theorem H in [4].

Proposition 1 If (R) is a Morse program and x is a critical point of (R) with $J := J(x)$, Then we have that

(a) $(|J|+p) \times n$ matrix $\begin{pmatrix} Dg_J(x) \\ Dh(x) \end{pmatrix}$ has full rank.

(b)[1] there exists a unique $(\lambda,\mu) \in R^m \times R^p$ such that

$Df(x)^T + Dg(x)^T\lambda + Dh(x)^T\mu = 0$, $\lambda_i \neq 0$ iff $i \in J$.

(c) $\mathcal{L}(x) := D^2f(x) + \sum_1^m \lambda_i D^2 g_i(x) + \sum_1^p \mu_j D^2 h_j(x)$ induces an isomorphism on $T_x\bar{M}_J$ [2]

(d) on $T_x\bar{M}_J$, $\mathcal{L}(x)$ is positive definite iff x is a local minimum; negative definite iff x is a local maximum; indefinite iff x is a saddle point of f on $\bar{M}_J$.

Proposition 2. If $f \in C^2$, $g \in C^n$ and $h \in C^{n-p+1}$, then for almost every fixed $(v,w) \in R^m \times R^p$, $(R(u,v,w))$ is a Morse program having at most one global solution for almost every $u \in R^n$.

---

1) $\lambda_j > 0$ ($\lambda_j < 0$) for all $j \in J$ if x is a local minimum (maximum) (see Luenberger [6], 10.6).

2) For $s \in T_x\bar{M}_J$, we project $\mathcal{L}(x)s$ orthogonally onto $T_x\bar{M}_J$. We call this linear homomorphism on $T_x\bar{M}_J$ by induced homomorphism of $\mathcal{L}(x)$ on $T_x\bar{M}_J$.

3. Equality Constraints and One Regular Inequality Constraint: Generic Property.

Let us consider a program

$(\bar{P})$: minimize $\{f(x)$ subject to $g(x) = b, \|x\|^2 \leq c\}$

and a perturbation of $(\bar{P})$

$(\bar{P}(u,v))$: minimize $\{f(x) - u^T x$ subject to $g(x) = b+v, \|x\|^2 \leq c\}$

where $f:R^n \to R^1$ and $g:R^n \to R^m$ are of class C^2 ; $u \in R^n$, $v \in R^m$; $c > 0$; $n \geq m+1$. Let $h(x) := \|x\|^2$, $D := \{x | \|x\|^2 \leq c\}$ and $S := \partial D :=$ $\{x | \|x\|^2 = c\}$, then $h \pitchfork c$ for all $c > 0$ i.e. h is a regular constraint.

In this section, we study a family of nonlinear programs with some equality constraints and one fixed regular constraint $h(x) \leq c$ ($c > 0$ is fixed). We will show that in C^2 topology Morse programs are open and dense in this family (Theorem A).

Let us recall the definition of a Morse program.

$(\bar{P})$ is a Morse program iff $(\bar{P})$ satisfies

(MP1) $g|_D \pitchfork b$, $g|_S \pitchfork b$

(MP2) f is a Morse function on $\bar{M} := g^{-1}(b) \cap D$ and $\partial\bar{M} := g^{-1}(b) \cap S$.

(MP3) $f|_{\bar{M}}$ has no critical points on $\partial\bar{M}$.

Remark

$(\bar{M}1)$ corresponds to (MP1) since $h \pitchfork c$. We do not have $(\bar{M}2)$ since we have only one inequality. $(\bar{M}3)$ corresponds to (MP2), and $(\bar{M}4)$ corresponds to (MP3).

Firstly, by Proposition 2, we have:

<u>Corollary 3.</u> <u>If</u> $f \in C^2$ <u>and</u> $g \in C^{n-m+1}$, <u>then for almost every fixed</u> $v \in R^m$, $(\bar{P}(u,v))$ <u>is a Morse program with a unique global solution for almost every</u> $u \in R^n$.

<u>Definition</u> Let $C^2(D,R^k)$ be a set of all C^2 functions from D to R^k for some $k \geq 1$. C^2 norm topology $\|\cdot\|_2$ on $C^2(D,R^k)$ is defined by $\|\phi\|_2 := \max_{x \in D} \{\|\phi(x)\|, \|D\phi(x)\|, \|D^2\phi(x)\|\}$ for $\phi \in C^2(D,R^k)$ where $\|\cdot\|$ is the Euclidean norm (all $n \times k$ matrices are considered to be in $R^{n \times k}$).

<u>Lemma 4.</u> <u>If</u> $g|_D \pitchfork b$, $g|_S \pitchfork b$, $\|g^n - g\|_2 \to 0$ <u>then</u> $g^n|_D \pitchfork b$, $g^n|_S \pitchfork b$ <u>for sufficiently large</u> n.

<u>Proof.</u> First of all we will show that if $g|_D \pitchfork b$, $\|g^n - g\|_2 \to 0$ then $g^n|_D \pitchfork b$ for sufficiently large n. Suppose it is not true. Then there exists $x^n \in g^{n^{-1}}(b) \cap D$ such that $Dg^n(x^n)$ is not of full rank for infinitely many n's.

Since D is compact, there exists a subsequence $\{x^{n_j}\}$ of $\{x^n\}$ such that $x^{n_j} \to x^0$ for some $x^0 \in D$. However $\|g^n - g\|_2 \to 0$ implies $g^{n_j}(x^{n_j}) \to g(x^0)$ and $Dg^{n_j}(x^{n_j}) \to Dg(x^0)$. Then $g(x^0) = b$ and since $g|_D \pitchfork b$, $Dg(x^0)$ is of full rank. Hence $Dg^{n_j}(x^{n_j})$ is of full rank for sufficiently large n_j which contradicts the assumption. Similarly we can show that $(g,h) \pitchfork (b,c)$ and $\|g^n - g\|_2 \to 0$ imply $(g^n,h) \pitchfork (b,c)$ for sufficiently large n. However by [4] Lemma 14(b), we have $(g,h) \pitchfork (b,c)$ iff $g|_S \pitchfork b$ hence we complete the proof. Q.E.D.

Under the same assumption of Lemma 4, by (1.2) we can claim that $\bar{M}^\ell := g^{\ell^{-1}}(b) \cap D$ is (n-m)-dimensional manifold with boundary $\partial\bar{M}^\ell := g^{\ell^{-1}}(b) \cap S$ for sufficiently large ℓ. Then we have

<u>Lemma 5</u> <u>If</u> $f|_{\bar{M}}$, $f|_{\partial\bar{M}}$ <u>are Morse functions and if</u> $\|f^n - f\|_2 \to 0$, <u>then</u> $f^n|_{\bar{M}^n}$, $f^n|_{\partial\bar{M}^n}$ <u>are Morse functions for sufficiently large</u> n.

Proof.

Let us define $F_b : R^n \times R^m \to R^n \times R^m$ and $G_b : R^n \times R^m \times R^1 \to R^n \times R^m \times R^1$ by, respectively,

$$F_b(x,\lambda) := (Df(x)^T + Dg(x)^T\lambda,\ g(x) - b) \ ,$$

and

$$G_b(x,\lambda,\mu) := (Df(x)^T + Dg(x)^T\lambda + Dh(x)^T\mu \ , \quad g(x) - b \ , \quad h(x) - c).$$

By the same argument as in [4] Theorem B, we have that

$$f|_{\bar{M}} \ \underline{\text{is a Morse function iff}} \ F_b|_{D\times R^m} \pitchfork 0$$

and

$$f|_{\partial\bar{M}} \ \underline{\text{is a Morse function iff}} \ G_b \pitchfork 0 \ .$$

Let $F_b^\ell(x,\lambda) := (Df^\ell(x)^T + Dg^\ell(x)^T\lambda,\ g^\ell(x) - b)$, then $f^\ell|_{\bar{M}^\ell}$ is a Morse function iff $F_b^\ell|_{D\times R^m} \pitchfork 0$. Now we will show that if $f|_{\bar{M}}$ is a Morse function and $\|f^\ell - f\|_2 \to 0$, then $f^\ell|_{\bar{M}^\ell}$ is a Morse function for sufficiently large ℓ . Suppose, to the contrary, there exists $\{x^\ell\}$ such that $F_b^\ell(x^\ell,\lambda^\ell) = 0$ and $DF_b^\ell(x^\ell,\lambda^\ell) = \begin{pmatrix} \mathcal{L}^\ell(x^\ell) & Dg^\ell(x^\ell)^T \\ Dg^\ell(x^\ell) & 0 \end{pmatrix}$ is singular, where

$$\lambda^\ell := -(Dg^\ell(x^\ell)Dg^\ell(x^\ell)^T)^{-1}Dg^\ell(x^\ell)Df^\ell(x^\ell)^T \tag{3.1}$$

and

$$\mathcal{L}^\ell(x^\ell) := D^2f^\ell(x^\ell) + \sum_1^m \lambda_j^\ell \, D^2g_j(x^\ell) \ .$$

Since $x^\ell \in D$ and D is compact, there exists a converging subsequence of $\{x^\ell\}$. For notational convenience, let $x^\ell \to x^*$ for some $x^* \in D$. Since $g^\ell(x^\ell) = b$ and $\|g^\ell - g\|_2 \to 0$ we have $g(x^*) = b$, and since $g|_D \pitchfork b$ we have that $Dg(x^*)$ has full rank.

Moreover $x^{\ell} \to x^{*}$, $\|f^{\ell} - f\|_2 \to 0$ and $\|g^{\ell} - g\|_2 \to 0$, so that we have $\lambda^{\ell} \to \lambda^{*} := -(Dg(x^{*})Dg(x^{*})^T)^{-1}Dg(x^{*})Df(x^{*})^T$ by (3.1). Hence, we obtain $F_b(x^{*},\lambda^{*}) = 0$ and because $f|_{\bar{M}}$ is a Morse function, $DF_b(x^{*},\lambda^{*})$ is nonsingular. However we have $DF_b^{\ell}(x^{\ell},\lambda^{\ell}) \to DF_b(x^{*},\lambda^{*})$. Hence $DF_b^{\ell}(x^{\ell},\lambda^{\ell})$ is nonsingular for sufficiently large ℓ and this contradicts the choice of $\{x^{\ell}\}$. By a similar argument, we prove that if $f|_{\partial\bar{M}}$ is a Morse function and if $\|f^{\ell} - f\|_2 \to 0$, then $f^{\ell}|_{\partial\bar{M}^{\ell}}$ is a Morse function for sufficiently large ℓ. Q.E.D.

<u>Lemma 6.</u> <u>Under the same assumptions of Lemma 5, if</u> $f|_{\bar{M}}$ <u>has no critical point on</u> $\partial\bar{M}$ <u>then</u> $f^n|_{\bar{M}^n}$ <u>has no critical point on</u> $\partial\bar{M}^n$ <u>for sufficiently large</u> n.

<u>Proof.</u> Suppose it is not true, then there exists $x^n \in \partial\bar{M}^n$ such that x^n is a critical point of $f^n|_{\bar{M}^n}$ for infinitely many n's. Then there exists a unique $\lambda^n \in R^m$ such that

$$Df^n(x^n)^T + Dg^n(x^n)^T\lambda^n = 0 .$$

Since D is compact, there exists a subsequence $\{x^{n_j}\}$ of $\{x^n\}$ such that $x^{n_j} \to x^0$ for some $x^0 \in D$. Since $\|f^n - f\|_2 \to 0$, $\|g^n - g\|_2 \to 0$ and $\|x^n\| = c$, we have $\|x^0\| = c$, $g(x^0) = b$ and $Df(x^0)^T + Dg(x^0)^T\lambda^0 = 0$ where $\lambda := \lambda(x^0)$ (see (3.1)). This shows $x^0 \in \partial\bar{M}$ is a critical point of $f|_{\bar{M}}$ which contradicts our assumption. Q.E.D.

Combining Corollary 3, Lemmas 4,5,6 we obtain

<u>Theorem A</u>

<u>In the</u> C^2 <u>topology, Morse programs are open and dense in the family of programs</u>

$$\underset{\|x\|^2 \leq c}{\text{minimize}} \{f(x) \ \underline{\text{subject to}}\ g(x) = b\}$$

<u>where</u> $f:R^n \to R^1$, $g:R^n \to R^m$; $f \in C^2$, $g \in C^{n-m+1}$; $n > m+1$.

4. Equality Constraints and One Regular Constraint: Sensitivity Analysis

Now we will discuss the global optimum value function

$$\bar{\omega}(u,v) := \underset{\|x\|^2 \leq c}{\text{minimum}} \{f(x) - u^T x \quad \text{subject to} \quad g(x) = b + v\}$$

for $(\bar{P}(u,v))$. The basic ideas are essentailly the same as those in Theorem E of Fujiwara [4], where I discussed the optimum value function

$$\omega(u,v) := \underset{x \in R^n}{\text{minimum}} \{f(x) - u^T x \quad \text{subject to} \quad g(x) = b + u\}$$

and I assumed $g : R^n \to R^m$ is proper (i.e. if $\|x\| \to \infty$, then $\|g(x)\| \to \infty$). Here we do <u>not</u> assume that g is a proper function, and the argument is more delicate.

First, let us denote

$$\bar{Z} := \{(u,v) \in R^n \times R^m | (\bar{P}(u,v)) \text{ is a Morse program}\}$$

and

$$\bar{Z}^! := \left\{(u,v) \in R^n \times R^m \middle| \begin{array}{l}(\bar{P}(u,v)) \text{ is a Morse program} \\ \text{with a unique global solution}\end{array}\right\}$$

<u>Lemma 7</u> $\bar{Z}$ <u>is an open set of</u> $R^n \times R^m$.

<u>Proof.</u> Suppose $\bar{Z}$ is not open at $(\bar{u},\bar{v}) \in \bar{Z}$. Then there exists a sequence $\{(u^\ell,v^\ell)\}$ such that $(u^\ell,v^\ell) \to (\bar{u},\bar{v})$ and $(\bar{P}(u^\ell,v^\ell))$ does not satisfy (MP1) or (MP2) or (MP3). Suppose $(\bar{P}(u^\ell,v^\ell))$ does not satisfy (MP1) infinitely often. Let $\bar{f}(x) = f(x) - \bar{u}^T x$, $\bar{f}^\ell(x) = f(x) - u^{\ell^T} x$; $\bar{g}(x) = g(x) - \bar{v}$, $\bar{g}^\ell(x) = g(x) - v^\ell$. Then $(u^\ell,v^\ell) \to (\bar{u},\bar{v})$ implies $\|\bar{f}^\ell - \bar{f}\|_2 \to 0$ and $\|\bar{g}^\ell - \bar{g}\|_2 \to 0$. Hence, by Lemma 4 , $(\bar{P}(u^\ell,v^\ell))$ satisfies (MP1) for sufficiently large ℓ and this contradicts the assumption. Similarly if $(\bar{P}(u^\ell,v^\ell))$ does not satisfy (MP2) (or (MP3)), then by Lemma 5 (or Lemma 6) we have a contradiction. Therefore $\bar{Z}$ is an open set of $R^n \times R^m$. Q.E.D.

Proposition 8. The number of critical points of $(\bar{F}(u,v))$ is finite for any $(u,v) \varepsilon \bar{Z}$, and it is locally constant on the open set $\bar{Z}$.

Proof.

Let us define $\hat{F} : R^n \times R^m \times R^n \times R^m \to R^n \times R^m$ and $\hat{G} : R^n \times R^m \times R^1 \times R^n \times R^m \to R^n \times R^m \times R^1$ by

$$\hat{F}(x,\lambda,u,v) := (Df(x)^T - u + Dg(x)^T\lambda \ , \quad g(x) - b - v)$$

and

$$\hat{G}(x,\lambda,\nu,u,v) := (Df(x)^T - u + Dg(x)^T\lambda + Dh(x)^T\nu \ , \quad g(x) - b - v, \quad h(x) - c).$$

Let $(\bar{u},\bar{v}) \varepsilon \bar{Z}$ and let $\bar{M} := g^{-1}(b + \bar{v}) \cap D$ and $\partial\bar{M} := g^{-1}(b + \bar{v}) \cap S$. Then $\bar{x}$ is a critical point of $(\bar{P}(\bar{u},\bar{v}))$ if and only if $\bar{x}$ is a critical point of either $f(x) - u^Tx|_{\bar{M}}$ or $f(x) - u^Tx|_{\partial\bar{M}}$. By (MP3), we have that no critical points of $f(x) - \bar{u}^Tx|_{\bar{M}}$ are on $\partial\bar{M}$. Therefore, $\bar{x}$ is a critical point of $(\bar{P}(\bar{u},\bar{v}))$ if and only if $\bar{x}$ satisfies either

(4.1) $\qquad \hat{F}(\bar{x},\bar{\lambda},\bar{u},\bar{v}) = 0$ for some $\bar{\lambda} \varepsilon R^m$ and $h(\bar{x}) < c$

or

(4.2) $\qquad \hat{G}(\bar{x},\bar{\lambda},\bar{\nu},\bar{u},\bar{v}) = 0$ for some $\bar{\lambda} \varepsilon R^m$ and $\bar{\nu} > 0.$

By (MP2) $f(x) - \bar{u}^Tx$ is a Morse function on $\bar{M}$ and on $\partial\bar{M}$, hence critical points of $f(x) - \bar{u}^Tx|_{\bar{M}}$ and critical points of $f(x) - \bar{u}^Tx|_{\partial\bar{M}}$ are isolated. Since $\bar{M}$ and $\partial\bar{M}$ are compact, the number of critical points of $(\bar{P}(\bar{u},\bar{v}))$ is finite. Let $\bar{x}^1,\cdots,\bar{x}^k$ be distinct critical points of $f(x) - \bar{u}^Tx|_{\bar{M}}$ and let $\bar{x}^{k+1},\cdots,\bar{x}^{k+\ell}$ be distinct critical points of $f(x) - \bar{u}^Tx|_{\partial\bar{M}}$. Then we have that $\{\bar{x}^1,\cdots,\bar{x}^k\} \cap \{\bar{x}^{k+1},\cdots,\bar{x}^{k+\ell}\} = \phi$ and the number of critical points of $(\bar{P}(\bar{u},\bar{v}))$ is $k+\ell$. Let $\bar{\lambda}^1,\cdots\bar{\lambda}^k$ be the associated Lagrange multipliers of $\bar{x}^1,\cdots\bar{x}^k$. Let $(\bar{\lambda}^{k+1},\bar{\nu}^{k+1}),\cdots,(\bar{\lambda}^{k+\ell},\bar{\nu}^{k+\ell})$ be the associated Lagrange multipliers of $\bar{x}^{k+1},\cdots,\bar{x}^{k+\ell}$.

By (MP2) we have that $\hat{F}_{(\bar{u},\bar{v})}|_{D\times R^m} \pitchfork 0$ and $\hat{G}_{(\bar{u},\bar{v})}|_{S\times R^m\times R^1} \pitchfork 0$, where $\hat{F}_{(u,v)}(x,\lambda) = \hat{F}(x,\lambda,u,v)$ and $\hat{G}_{(u,v)}(x,\lambda,\nu) = G(x,\lambda,\nu,u,v)$ (see the beginning of the proof of Lemma 5). In particular, therefore we have that $D\hat{F}_{(\bar{u},\bar{v})}(\bar{x}^i,\bar{\lambda}^i)$ and $D\hat{G}_{(\bar{u},\bar{v})}(\bar{x}^j,\bar{\lambda}^j,\bar{\nu}^j)$ are nonsingular for $i = 1,\cdots,k$ and for $j = k+1,\cdots,k+\ell$. Hence, by the implicit function theorem (Edwards [1], p. 417), there exist neighborhoods $\bar{U}^i(\bar{u})$, $\bar{V}^i(\bar{v})$, $\bar{X}^i(\bar{x}^i)$, $\bar{\Lambda}^i(\bar{\lambda}^i)$, and C^1 functions $x^i(\cdot,\cdot)$ and $\lambda^i(\cdot,\cdot)$ from $\bar{U}^i \times \bar{V}^i$ to, respectively, $\bar{X}^i$ and $\bar{\Lambda}^i$ such that $x^i(\bar{u},\bar{v}) = \bar{x}^i$, $\lambda^i(\bar{u},\bar{v}) = \bar{\lambda}^i$

(4.3) $$\bar{X}^i \subseteq D - S ,$$

and

(4.4) $$\hat{F}(x,\lambda,u,v) = 0 \iff x = x^i(u,v),\ \lambda = \lambda^i(u,v) \quad \text{on } \bar{X}^i \times \bar{\Lambda}^i \times \bar{U}^i \times \bar{V}^i$$

for $i = 1,\cdots,k;$ and there exist neighborhoods $\bar{U}^j(\bar{u})$, $\bar{V}^j(\bar{v})$, $\bar{X}^j(\bar{x}^j)$, $\bar{\Lambda}^j(\bar{\lambda}^j)$, $\bar{N}^j(\bar{\nu}^j)$ and C^1 functions $x^j(\cdot,\cdot)$, $\lambda^j(\cdot,\cdot)$, $\nu^j(\cdot,\cdot)$ from $\bar{U}^j \times \bar{V}^j$ to, respectively, $\bar{X}^j$, $\bar{\Lambda}^j$, $\bar{N}^j$ such that $x^j(\bar{u},\bar{v}) = \bar{x}^j$, $\lambda^j(\bar{u},\bar{v}) = \bar{\lambda}^j$, $\nu^j(\bar{u},\bar{v}) = \bar{\nu}^j$ and

(4.5) $$\hat{G}(x,\lambda,\nu,u,v) = 0 \iff (x,\lambda,\nu) = (x^j(u,v),\ \lambda^j(u,v),\ \nu^j(u,v))$$

$$\text{on } \bar{X}^j \times \bar{\Lambda}^j \times \bar{N}^j \times \bar{U}^j \times \bar{V}^j$$

for $j = k+1,\cdots,k+\ell$.

Since $\bar{Z}$ is open (Lemma 7), we can choose $\bar{U}(\bar{u})$ and $\bar{V}(\bar{v})$ such that $\bar{U} \subseteq \bigcap_{i=1}^{k+\ell} \bar{U}^i$, $\bar{V} \subseteq \bigcap_{i=1}^{k+\ell} \bar{V}^i$, and $\bar{U} \times \bar{V} \subseteq \bar{Z}$, and such that $x^1(\bar{U},\bar{V}),\cdots,$ $x^{k+\ell}(\bar{U},\bar{V})$ are pairwise disjoint. Now (4.1) - (4.5) imply that the number of critical points of $(\bar{P}(\bar{u},\bar{v}))$ is no less that $k + \ell$ for (u,v) $\epsilon\ \bar{U} \times \bar{V}$, because $x^1(u,v), \cdots, x^{k+\ell}(u,v)$ are distinct critical points of $(\bar{P}(u,v))$ for $(u,v)\ \epsilon\ \bar{U} \times \bar{V}$. We claim that, in actual fact, it is exactly $k + \ell$. Suppose, to the contrary, there exists $\{(u^\alpha,v^\alpha)\}$ such that $(u^\alpha,v^\alpha)\ \epsilon\ \bar{U} \times \bar{V}$, $(u^\alpha,v^\alpha) \to (\bar{u},\bar{v})$ and the number of critical points of

$(\bar{P}(u^\alpha,v^\alpha))$ is greater than $k+\ell$. Firstly, assume that there exist infinitely many $\{(x^\alpha,\lambda^\alpha)\}$ such that $x^\alpha \not\in \{x^1(u^\alpha,v^\alpha),\cdots,x^k(u^\alpha,v^\alpha)\}$, $\hat{F}(x^\alpha,\lambda^\alpha,u^\alpha,v^\alpha) = 0$, and $h(x^\alpha) < c$. Since $\{x^\alpha\} \subseteq \bar{M}$ and $\bar{M}$ is compact, there exists a converging subsequence of $\{x^\alpha\}$. For notational convenience, let us denote $x^\alpha \to x^*$ for some $x^* \in \bar{M}$. Since

$$\lambda^\alpha := \lambda(x^\alpha) := (Dg(x^\alpha)Dg(x^\alpha)^T)^{-1}Dg(x^\alpha)(Df(x^\alpha)^T - u^\alpha)$$

and $u^\alpha \to \bar{u}$, we obtain $\lambda^\alpha \to \lambda^* := \lambda(x^*)$. Then we have $(x^\alpha,\lambda^\alpha,u^\alpha,v^\alpha) \to (x^*,\lambda^*,\bar{u},\bar{v})$, hence by the continuity of $\hat{F}$, $\hat{F}(x^*,\lambda^*,\bar{u},\bar{v}) = 0$.

Note that $h(x^*) < c$, because if $h(x^*) = c$, then we obtain $\hat{G}(x^*,\lambda^*,0,\bar{u},\bar{v}) = 0$ and this implies that x^* is a critical point of $f(x) - \bar{u}^T x|_{\partial\bar{M}}$ with $\nu = 0$, which contradicts the fact that $(\bar{u},\bar{v}) \in \bar{Z}$. Therefore x^* is a critical point of $f(x) - \bar{u}^T x|_{\bar{M}}$, and hence $(x^*,\lambda^*) = (\bar{x}^i,\bar{\lambda}^i)$ *for some* $i = 1,\cdots,k$. But this contradicts (4.4), because then for sufficiently large α, we have that

$$(x^\alpha,\lambda^\alpha,u^\alpha,v^\alpha) \in \bar{X}^i \times \bar{\Lambda}^i \times \bar{U}^i \times \bar{V}^i$$

$$\hat{F}(x^\alpha,\lambda^\alpha,u^\alpha,v^\alpha) = 0$$

$$\text{and} \quad x^\alpha \not\in \{x^1(u^\alpha,v^\alpha),\cdots,x^k(u^\alpha,v^\alpha)\}\ .$$

Similarly if we assume that there exist infinitely many $\{(x^\alpha,\lambda^\alpha,\nu^\alpha)\}$ such that $x^\alpha \not\in \{x^{k+1}(u^\alpha,v^\alpha),\cdots,x^{k+\ell}(u^\alpha,v^\alpha)\}$ and $\hat{G}(x^\alpha,\lambda^\alpha,\nu^\alpha,u^\alpha,v^\alpha) = 0$, then we can arrive at a contradiction. Hence, the number of critical points of $(\bar{P}(u,v))$ is $k+\ell$ in a small neighborhood of $(\bar{u},\bar{v})$. Q.E.D.

<u>Corollary 9.</u> $\bar{Z}^!$ <u>is an open set of</u> $R^n \times R^m$.

If $f \in C^2$ and $g \in C^{n-m+1}$, then by Corollary 3, $\bar{Z}$ and $\bar{Z}^!$ are dense sets of $R^n \times R^m$, hence we obtain

Theorem B. *If* $f \in C^2$ *and* $g \in C^{n-m+1}$, *then the global optimum value function* $\bar{\omega}(u,v)$ *for* $(\bar{P}(u,v))$ *is of class* C^2 *with respect to* (u,v) *on the open and dense set* $\bar{Z}'$ *of* $R^n \times R^m$.

Proof. Using the same notation as in Proposition 8, we have

$$\bar{\omega}(\bar{u},\bar{v}) = \min_{1 \leq i \leq k+\ell} \{f(x^i(\bar{u},\bar{v})) - \bar{u}^T x^i(\bar{u},\bar{v})\} .$$

Hence $\bar{\omega}(\bar{u},\bar{v}) = f(x^i(\bar{u},\bar{v})) - \bar{u}^T x^i(\bar{u},\bar{v})$ for some i. It is easily shown (see, for example, Luenberger [6], 10.5), that

$$D\bar{\omega}(\bar{u},\bar{v}) = -(x^i(\bar{u},\bar{v}),\ \lambda^i(u,v))$$

and

$$D^2\bar{\omega}(\bar{u},\bar{v}) = -\begin{pmatrix} Dx^i(\bar{u},\bar{v}) \\ D\lambda^i(\bar{u},\bar{v}) \end{pmatrix} .$$

Therefore $\bar{\omega}$ is in C^2. The rest of the proof is derived by Lemma 7, Proposition 8, and Corollary 9. Q.E.D.

Remark

The differentiability of the *local* optimum value function was given by Fiacco/McCormick [2] and Fiacco [3], using the implicit function theorem. Our result presented here is *not* for the *local*, *but* the *global* optimum value function. (See also [4] Proposition 6, and Theorem E).

5. Fixed and Variable Constraints.

As a natural extension of Section 3, we consider a smooth nonlinear program defined by a set of variable constraints (which we are allowed to perturb) and fixed constraints (which we are not allowed to perturb). Namely we consider a program

$$(S): \quad \underset{\substack{G(x)\leq 0 \\ H(x)=0}}{\text{minimize}} \{f(x) \text{ subject to } g(x) \leq b,\ h(x) = c\}$$

and a perturbation of (S)

$$(S(u,v,w)): \quad \underset{\substack{G(x)\leq 0 \\ H(x)=0}}{\text{minimize}} \{f(x) - u^T x \text{ subject to } g(x) \leq b+v,\ h(x) = c+w\}$$

where f,g,h,G,H are of class C^2 from R^n respectively to R^1, R^m, R^p, R^r, R^s; $u \in R^n$, $v \in R^m$, $w \in R^p$.

We impose a condition (c0) to G and H

(c0) $\qquad (G_\alpha, H) \pitchfork (0_\alpha, 0)$ for every $\alpha \subseteq \{1,2,\cdots,r\}$.

For example, if we take $G_k(x) = -x_k$ for $k = 1, \cdots, n$, then G satisfies (c0), and (S) becomes

$$\underset{x \geq 0}{\text{minimize}} \{f(x) \text{ subject to } g(x) \leq b,\ h(x) = c\}.$$

Moreover, if $G \in C^n$, $H \in C^{n-s+1}$, then we can assume that, generically, (c0) is satisfied (cf. [4], Lemma 11).

Spingarn ([7],[8],[9]) considered a more general fixed constraint set, named "cyrtohedron" which contains degenerate points and he showed that the problem is reduced to solving at most a countable number of programs of type (S) ([9], (3.7)). In our framework, we consider a program (S) and we impose the condition (c0) on G and H so that we do not have degenerate points. The basic idea is the same as that shown in [4] Theorem H; namely, we perturb the right hand side so that the feasible region becomes a union of a finite family of manifolds with boundary. We then perturb the objective function so that it becomes a Morse function on each manifold and it has no critical points on the boundary (hence strict complementarity holds). Then, we derive the necessary conditions for the optimality of this type of problem, which is a special case of Spingarn ([9],(3.9)).

Let $N_\alpha := \{x|G_\alpha(x) = 0,\ H(x) = 0\}$ for $\alpha \subseteq \{1,\cdots,r\}$, then N_α is a manifold of dimension $n - |\alpha| - s$ by (c0) and (1.1). Let us consider all $(b,c) \in R^m \times R^p$ that satisfy the following conditions (c1)-(c3);

(c1): $\quad (g_J,h) \pitchfork (b_J,c)$ for all $J \subseteq \{1,\cdots,m\}$

(c2): $\quad (g_J,h)|_{N_\alpha} \pitchfork (b_J,c)$ for all J and α.

(c3) $\quad g_i|_{\bar{M}_J \cap N_\alpha} \pitchfork b_i$ for all J, α, and $i \notin J$,

where $\quad \bar{M}_J := \{x|g_J(x) = b_J,\ h(x) = c\}$.

Note that if $g \in C^n$, $h \in C^{n-p+1}$ then the set of all (b,c) satisfying (c1) - (c3) has full measure in $R^m \times R^p$ by Sard's theorem and Fubini's theorem (cf. [4], Lemma 11). By (c1) and (1.1), $\bar{M}_J$ is $(n - |J| - p)$-dimensional manifold for all J; by (c2), (1.1), (1.4), and (1.5), $\bar{M}_J \cap N_\alpha$ is $(n - |J| - p - |\alpha| - s)$-dimensional manifold; by (c3) and (1.2), $\bar{M}_J \cap N_\alpha \cap g_i^{-1}(-\infty,b_i]$ is $(n - |J| - p - |\alpha| - s)$-dimensional manifold with boundary $\bar{M}_J \cap N_\alpha \cap g_i^{-1}(b_i)$. Then by (1.6), for almost every $u \in R^n$, $f(x) - u^Tx|_{\bar{M}_J \cap N_\alpha}$ is a Morse function for all J and α. By [4] Proposition 13, for almost every $u \in R^n$, $f(x) - u^Tx|_{\bar{M}_J \cap N_\alpha \cap g_i^{-1}(-\infty,b_i]}$ has no critical points on $\bar{M}_J \cap N_\alpha \cap g_i^{-1}(b_i)$ for all J, α, and $i \notin J$.

Now, let us fix $u \in R^n$ and $(b + v,\ c + w) \in R^m \times R^p$ satisfying the above conditions. Let $x^* \in R^n$ be a feasible point of $(S(u,v,w))$ and a critical point of $f(x) - u^Tx|_{\bar{M}'_J \cap M_\alpha}$, where $J = \{i|g_i(x^*) = b_i + v_i\}$, $\alpha = \{k|G_k(x^*) = 0\}$ and $\bar{M}'_J = \{x|g_J(x) = b_J + v_J,\ h(x) = c + w\}$. Then by (c2), (1.1), (1.4) and (1.5), we have $T_{x^*}(\bar{M}'_J \cap N_\alpha) = T_{x^*}\bar{M}'_J \cap T_{x^*}N_\alpha$ and

$T_{x^*}(\bar{M}'_J \cap N_\alpha) = \mathrm{Ker}\, Dg_J(x^*) \cap \mathrm{Ker}\, Dh(x^*) \cap \mathrm{Ker}\, DG_\alpha(x^*) \cap \mathrm{Ker}\, DH(x^*)$.
Hence, by [4] Lemma 1, there exist unique $\lambda^* \in R^m$, $\mu^* \in R^p$, $\xi^* \in R^r$ and $\eta^* \in R^s$, such that

$$Df(x^*)^T - u + Dg(x^*)^T\lambda^* + Dh(x^*)^T\mu^* = -(DG(x^*)^T\xi^* + DH(x^*)^T\eta^*) \in T_{x^*}N_\alpha ,$$

$$\lambda^*_{J^c} = 0, \quad \xi^*_{\alpha^c} = 0$$

where $J^c = \{1, \cdots, m\} - J$ and $\alpha^c = \{1, \cdots, r\} - \alpha$. Moreover, using the same argument as that of the proof of [4] Theorem G, we have

$$\lambda^*_i \neq 0 \text{ iff } i \in J .$$

Hence, we obtain a special case of Spingarn ([9],(3.9)),

<u>Proposition 10</u>

Suppose $g \in C^n$ and $h \in C^{n-p+1}$. Then for almost every fixed $(v,w) \in R^m \times R^p$, $(S(u,v,w))$ has the following properties for almost every $u \in R^n$.

<u>If</u> x <u>is a feasible point of</u> $(S(u,v,w))$ <u>and a critical point of</u> $f(x) - u^Tx$ <u>on</u> $\bar{M}'_J \cap N_\alpha$ <u>where</u> $J = \{i|g_i(x) = b_i + v_i\}$, $\alpha = \{k|G_k(x) = 0\}$, $\bar{M}'_J = g_J^{-1}(b + v) \cap h^{-1}(c + w)$, $N_\alpha = G_\alpha^{-1}(0_\alpha) \cap H^{-1}(0)$, <u>then</u>

(a) $(Dg_J(x)^T, Dh(x)^T, DG_\alpha(x)^T, DH(x)^T)$ <u>has full rank.</u>

(b) <u>there exist unique</u> $\lambda \in R^m$, $\mu \in R^p$, $\xi \in R^r$, $\eta \in R^s$ <u>such that</u>

$$Df(x)^T - u + Dg(x)^T\lambda + Dh(x)^T\mu = -(DG(x)^T\xi + DH(x)^T\eta) \in T_xN_\alpha;$$

$\lambda_i \neq 0$ <u>iff</u> $i \in J$; $\xi_k = 0$ <u>for</u> $k \notin \alpha$.

(c) $$\mathcal{L}(x) = D^2f(x) + \sum_i \lambda_i D^2g_i(x) + \sum_j \mu_j D^2h_j(x) + \sum_k \xi_k D^2G_k(x) + \sum_\ell \eta_\ell D^2H_\ell(x)$$ <u>induces an isomorphism on</u> $T_x(\bar{M}'_J \cap N_\alpha)$.

(d) <u>on</u> $T_x(\bar{M}'_J \cap N_\alpha)$, $\mathcal{L}(x)$ <u>is positive definite if</u> x <u>is a local minimum; negative definite if</u> x <u>is a local maximum, indefinite iff</u> x <u>is a saddle point on</u> $\bar{M}'_J \cap N_\alpha$.

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We consider nonlinear constrained optimization problems in which the objective function and constraint functions are sufficiently smooth. We focus on the programs which consist of both equality and inequality constraints, and we prove that the global optimum value function is twice continuously differentiable almost everywhere with respect to the parameters.

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