EXECUTIVE SUMMARY

of the attached paper entitled
"Exact A Priori Matching of Mixed Boundary Conditions for Second Order Elliptic Problems"

by

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The research results reported in the present paper represent a substantial advance in the direction of providing more efficient, cost-saving techniques for solving a wide class of commonly occurring two-dimensional boundary value problems. In previous papers ([5], [6]), it has been shown that it is possible to dramatically reduce the cost of solving two-dimensional problems by amalgating three formerly disparate problem-solving tools, namely:

1. Computer graphics (visual feedback)
2. Numerical analysis (scientific computing software)
3. Qualitative information (the analyst's experience and insight, and "weak" mathematical theorems).

More specifically, in [5], Gordon and Hall pointed out (via examples) the practical utility of such an amalgamation. The problems considered therein were, however, restricted to elliptic boundary value problems subject to Dirichlet boundary conditions, i.e., problems in which the function values are specified on the perimeter of the domain. That paper, as well
as [6], focused on the issue of contrasting the usual way of initializing an iterative numerical solution method with the proposed new technique which uses the so-called "blending-function methods" of interpolation to a priori exactly match the boundary conditions.

As one would intuitively expect, starting with what literally "looks like" (computer graphics) a good approximation reduces the computation (numerical analysis) time very substantially. If, in addition, an analyst is provided a mechanism for quantifying his experience-based knowledge (qualitative information) of the particular class of problems under study, the "exact" solution is almost in hand.

The Gordon/Kelly paper [6] extends these early results, involving only Dirichlet boundary conditions, to the rather general problem of satisfying "mixed linear boundary conditions," i.e., boundary conditions of the form: $aF + \beta \frac{\partial F}{\partial n} = g$. The boundary conditions are, however, assumed to be "consistent." By this is meant that, at the corners of the region, the boundary conditions from either side "match."

The attached paper addresses the problem of inconsistently specified boundary conditions. In the simplest instance of Dirichlet conditions, this means that the function values do not match at the corners. Herein, we show how to actually construct bivariate functions which exactly match the above type-mixed linear boundary conditions, even when they are inconsistently specified, cf. Section III. Moreover, software has been developed which numerically performs the necessary operator multiplications and constructs the singular functions needed to accommodate such inconsistent boundary conditions. This software will soon be available for general distribution.
EXACT A PRIORI MATCHING OF MIXED BOUNDARY CONDITIONS
FOR SECOND ORDER ELLIPTIC PROBLEMS

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ABSTRACT

In this paper we consider the problem of con-
structing bivariate functions which exactly match
boundary conditions of the general form $a + \varepsilon F(\alpha\gamma)$
along the perimeter of the unit square. The
reason for wishing to do this is that substantial
savings in computation time can be realized in the
subsequent solution of the discretized boundary value
problem: If an iterative method is used to solve the
discretized problem, beginning with a good initial
approximation can dramatically reduce the number of
iterations required to achieve convergence; if a di-
rect solution method is used, a specified accuracy
and precision comes with a far fewer algebraic unknowns.
Although attention herein is restricted to rectangular
regions, the techniques developed can be
straightforwardly extended to any rectangular
polygons. The interpolation techniques which we de-
velop for exact boundary matching are illustrated by
several examples which are accompanied by perspective
views of their graphs and by contour plots.

I. Background and Introduction

To set the stage for the main ideas of this
paper, we begin by displaying the familiar bilinearly
blended interpolant to Dirichlet boundary conditions
on the perimeter of the unit square $S = [0,1]^2$. Let $F(x,y)$ be a supposed primitive function from
which the boundary conditions are extracted. Then,
the synthetic function $U(x,y)$ which we construct via
transfinite ("blending function") interpolation is
given by the following (cf., [1], [2], [5]):

$$U(x,y) = (1-x)F(x,0) + xF(x,1)
+ (1-y)F(0,y) + yF(x,0)
- (1-x)(1-y)F(0,0) - (1-x)yF(0,1)
- x(1-y)F(1,0) - xy(1,1).$$

(1.1)

It is easy to verify that $U = F$ on $S$.

As an example of the use of (1.1), consider a
primitive function $F$ whose values along the four
gross edges of the unit square are:

$$F(0,y) = \sin(2\pi y) + 1.5
F(1,y) = y^2 (y-1) + .5$$

(1.2)

$$F(x,0) = 1.5 - x^2
F(x,1) = (x-1)^2 + .5.$$

The function $U$ which interpolates these boundary con-
ditions is given by (1.1) as:

$$U(x,y) = (1-x)\sin(2\pi y) + xy(y^2 - y - 2)
+ x^2 (2y-1) + 1.5.$$  

(1.3)

The graph of $U$ is shown in Fig. 1a, and a contour
plot of $U$ is given in Fig. 1b.

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II. Bivariate and Higher Dimensional Interpolation

Bivariate and higher dimensional interpolation is
most easily discussed in the formalism of projection
operators ([1], [2], [5], [6]). For instance, the
above expression (1.1) for $U$ can be more succinctly
expressed as the Boolean sum of the two elementary
projectors $P_x$ and $P_y$, given by:

$$P_x[F] = (1-x)F(0,y) + xF(1,y)$$

(1.4)

$$P_y[F] = (1-y)F(x,0) + yF(x,1).$$

By definition, the Boolean sum of two projectors
(idempotent linear operators) is: $P_x + P_y.$

$P_x - P_y.$ For the time being, we shall assume
that the primitive function $F$ is continuous at the
corners of the unit square, in which case the pro-
jectors commute:

$$P_x P_y[F] = P_y P_x[F].$$

(1.5)

(The main results of this paper are, as we shall soon
discuss, concerned with problems involving corner sin-
gularities; in those cases, the relevant projectors do
not commute.) In terms of these commutative pro-
jectors $P_x$ and $P_y$, expression (1.1) for $U$ simplifies:

$$U = (P_x \circ P_y)[F] = (P_y \circ P_x)[F].$$

(1.6)

Interpolation schemes of this type are known by
several aliases including Boolean sum interpolation,
transfinite interpolation, and blending function
interpolation. Of these, the term transfinite comes
closest to conveying the essence of this class of
techniques. These methods are distinguished from
classical finite dimensional interpolation schemes by
the fact that they incorporate a non-denumerable number
of scalar samples of $F$ into the interpolant. More
precisely, interpolation schemes of this class extract
from the bivariate primitive function $F$ uncountable
samples of $F$, not simply scalar samples. (Note that
$P_x[F]$ and $P_y[F]$ individually and $(P_x \circ P_y)[F]$ are
transfinite) Interpolants, the product $P_x P_y[F]$ =
$P_y P_x[F]$ is merely the standard four parameter bilinear
interpolant to the four corner values of $F$. (6)

Previous studies by Gordon and Hall [1], [3] and Gordon
and Kolly [6] have been aimed at demonstrating how, for
continuous boundary conditions, the transfinite, bil-
inearly blended interpolant (1.1) can be employed to
reduce the computational effort and obtain numerical
solutions to second order elliptic boundary value
problems. In these papers, the authors discuss
the following general approach: First, use (1.1) to
construct $U$, which exactly matches the given Dirichlet
boundary conditions and thus reduces the original
problem to one with homogeneous boundary conditions;
then, examine the original problem for any additional
information which may be inferred about the solution.
Such auxiliary knowledge, although perhaps merely
qualitative or heuristic, can often be used to advan-
tage in improving upon the first approximations $U$
obtained by simply matching the boundary conditions.

As an example, the solution to Laplace's equation
must satisfy a Maximum Principle. If the initial
estimate $U$ does not, then there are simple ways of con-
structing functions $V$ which vanish on $35$ and are such
that $U + V$ both match the given boundary conditions and satisfies the Maximum Principle, cf., [5] and [6].

For the Poisson equation $\nabla^2 U = 0$, the sign of $\sigma$ determines that the solution is (locally) either subharmonic or superharmonic [8], and this auxiliary information can be built into the exact boundary matching approximation $U + V$. In the actual testing of these ideas, we have found that interactive computer graphics is an almost indispensable aid.

Numerical experiments using the techniques suggested in [5] and [6] to obtain good first approximations with which to enter standard iterative linear system solvers demonstrated that substantial reductions in total computational cost can be realized using such preprocessing methods. Inasmuch as, lacking any previously computed results, the standard initialization of an iterative scheme for solving large linear systems is to set all unknowns equal to zero (or some constant), it is not surprising that an exact boundary matching function which also incorporates readily available auxiliary information should produce a more rapidly convergent numerical solution.

What may be more surprising are the results reported by Mitchell, Marshall and Wait [19], [20], pp. 174-175], and by Rice [11]. Namely, that merely by reducing an elliptic problem with inhomogeneous boundary conditions to a problem with homogeneous conditions one is able to achieve a numerical solution of specified accuracy using far fewer algebraic unknowns. (Exact a priori matching conditions of boundary conditions, tantamount to reducing the original problem to a problem whose solution — the "residual" — must satisfy homogeneous boundary conditions.) Rice has observed this empirically for the collocation codes in the ELLPACK suite, and Marshall and Mitchell have reported this to be true in experiments contrasting standard bilinear finite elements with "exact boundary elements". For the potential flow problem $\nabla^2 U = 0$ with a source at a point $(r, \theta)$, the exact solution of which is $U = \log r$ where $r^2 = (x-0.437)^2 + (y+0.15)^2$, Marshall and Mitchell obtained results indicating that for a "weak" singularity at $(0.437, -0.15)$, more than 256 standard bilinear elements are required with inhomogeneous boundary conditions to achieve the same four-figure accuracy as can be obtained with 16 elements if the boundary conditions are first homogenized. If the singularity is located at $(0.437, -0.15)$, the comparison is roughly 256 elements to achieve three-figure accuracy with inhomogeneous conditions versus 64 with homogeneous, cf. Table 4, p. 175 of [16].

In brief, the development of methodologies and associated software preprocessors to a priori exactly match rather general boundary conditions, and thus permit their homogenization prior to discretization and numerical solution, promises considerable savings in total computational cost, whether the discrete linear system is solved by iterative or direct methods.

II. Transfinite Interpolation to Mixed (Consistent) Boundary Conditions

In this section, we consider rather general boundary conditions of the form $\sigma f + \delta (\partial F/\partial n) = g$ on the perimeter of the unit square. These boundary conditions are to be thought of as being associated with some second order elliptic boundary value problem and, without further mention, we shall assume that they are as such to guarantee that the problem is well-posed.

In particular, this means that the solution must be "matched" along at least one of the four edges, i.e., on at least one of the edges the functional value itself must be specified.

In [6], Gordon and Kelly considered mixed linear boundary conditions of the form:

$$ L_0[F] = \phi_0 F(0,y) + \phi_1 F(y,0) + \sigma_0 F(0,y) + \sigma_1 F(y,0) $$

along $x = 0$

$$ L_1[F] = \phi_0 F(1,y) + \phi_1 F(1,y) = \sigma_0 F(1,y) + \sigma_1 F(1,y) $$

along $y = 0$

$$ L_2[F] = \phi_0 F(x,0) + \phi_1 F(x,0) = \sigma_0 F(x,0) + \sigma_1 F(x,0) $$

along $x = 1$

$$ L_3[F] = \phi_0 F(x,1) + \phi_1 F(x,1) = \sigma_0 F(x,1) + \sigma_1 F(x,1) $$

along $y = 1$

In which the $\phi_0$ and $\phi_1$, $\sigma_0$ and $\sigma_1$, are constants, and the boundary conditions are consistent.

$$ L_{ij}[F] = \phi_{ij} F(y) \quad (i,j = 0,1). $$

In the case of Dirichlet conditions, the linear operators $L_i$ and $M_i$ are just:

$$ L_0[F] = F(0,y) = \phi_0(y), \quad L_1[F] = F(y,0) = \phi_1(y) $$

$$ M_0[F] = F(x,0) = \phi_0(x), \quad M_1[F] = F(x,1) = \phi_1(x) $$

and the consistency requirement simply means that the boundary conditions are continuous at the four corners:

$$ M_{ij}[\phi(y)] = L_{ij}[\phi(x)] \quad (i,j = 0,1). $$

**Theorem (Gordon/Kelly):** Let the $L_i$ and $M_i$ be as in (2.1) and define two projectors $P_x$ and $P_y$ as follows:

$$ P_x[F] = \phi_0(x) L_0[F] + \phi_1(x) L_1[F] $$

$$ P_y[F] = \phi_0(y) M_0[F] + \phi_1(y) M_1[F], $$

where the functions $\phi_0$ and $\phi_1$ satisfy the cardinality conditions:

$$ L_i[\delta_{ik}] = \delta_{ik}, \quad M_i[\delta_{jk}] = \delta_{jk} $$

for $i, k = 0, 1$ (Kronecker Delta).

Then, the function $U$ obtained from the Boolean sum of $P_x$ and $P_y$ exactly satisfies all of the specified boundary conditions.

**Proof:** The function $U$ is given by

$$ U = (P_x + P_y)[F] $$

in which we have used the cardinality conditions (2.6). To show that $M_i[U] = \delta_i[0] = h_i(x)$, we also use the consistency hypothesis (2.2).

As an illustration of this result, consider a function $F$ such that

$$ L_0[F] = F(0,y) = 1.2 $$

Then, $U$ will be the function $F(x,y)$ subject to the条件
\[ L_1[F] = 6F(x,y) - F_x(1,y) = 6\left(1n(-2)^2 + (y - 9)^2\right) + 1.5 \]
\[ + \left(\frac{2}{(x-2)^2 + (y - 9)^2}\right) \]
\[ M_0[F] = F(x,0) + 2F_y(x,0) = 1n(-x-1.2)^2 + (-9)^2 \]
\[ - \left(\frac{1.8}{(x-2)^2 + (y-9)^2} + 1.5 \right) \]
\[ M_1[F] = F(x,1) = 1n(-x-1.2)^2 + (-1)^2 + 1.5 \]

It may be easily verified that the four functions:
\[ \phi_0(x) = x - \frac{5}{6}, \quad \phi_1(x) = \frac{1}{6}, \quad \psi_0(y) = y - 1, \quad \psi_1(y) = 2 - y \]

satisfy the necessary cardinality conditions. Simply apply the formulas for the \( L_1 \) and \( M_0 \) to these univariate "blending functions" and confirm the Kronecker delta properties. The corner values are \( a_0 \) in the four functions, and \( a_1 \) in the two functions. In particular, we show that for any choice of the eight parameters \( a_1 \), \( a_8 \), and \( a_9 \), in (2.1), we can always find polynomials of degree three or less that satisfy (2.6).

**Lemma:** Let \( L^1 \) and \( M^1 \) be defined as in the above theorem. Then, there exist polynomials of maximal degree three such that (2.6) holds.

**Proof:** We need carry out the proof for only the \( \phi_0(x) \), since the \( \psi(y) \) are constructed independently and analogously. To this end, suppose that \( \phi_0 \) is cubic in \( x \):
\[ \phi_0(x) = a_0x + b_2x^2 + c_0x^3 + d_2x^3 \]  

By applying the linear operators \( L_0 \) and \( L_1 \) to \( \phi_0 \) and collecting terms, we obtain the linear system:
\[ L_0[\phi_0] = a_0a_0 + b_0b_0 = 0 \]  
\[ L_1[\phi_0] = a_1a_0 + (a_1 + b_1) + (a_1 + b_1) + (a_1 + b_1) + (a_1 + b_1) = 0 \]

for the determination of the polynomial coefficients \( a_1 \), \( b_0 \), and \( b_1 \). Since the constants \( a_0 \) is fixed and \( b_1 \) is determined. Clearly, since there are four unknowns and only two equations, this system is underdetermined.

The key to the proof of existence is the recognition that if \( a_0 = 0 \), then \( \phi_0 \) cannot be zero, and vice versa (\( a_1 \) and \( a_1 \)). With this mind, it is easy to show that the five equations of (2.6) cannot all hold simultaneously. Q.E.D.

As an example, consider the following consistent boundary conditions:

\[ L_0[F] = F(0,y) + F_x(0,y) = 1.5e^{-y} + .2 \]
\[ L_1[F] = F(1,y) = .25e^{-(y + 1)} \]
\[ M_0[F] = 2F(x,0) + F_y(x,0) = .25e^{-x(1 + 1n6x)} + .4 \]
\[ M_1[F] = 2F(x,1) - F_y(x,1) = .75e^{-(1 + 1n6x)} + .4 \]

Here, \( a_0 = 1, b_0 = 1, c_0 = 1, d_0 = 2, e_0 = 2, \) and \( f_0 = 1 \). So that, by following the above algorithm, we obtain the blending functions:
\[ \phi_0(x) = 1 - x^2 \]
\[ \phi_1(x) = x^2 \]
\[ \phi_0(y) = .5 + y \]
\[ \phi_1(y) = -y^3 \]
which yield the function:

\[
U(x, y) = 1.5x(1-x^2)e^{-y} + 25x^2e^{-1(1+y)}cosy + 0.25x^2e^{-1(1+y)}cosy - 0.75e^{-1(1-y)}(1-e^{-1(1-y)})^3 - 0.5x(1-x^2)(e^{-1(1-x^2)})^3
\]

(2.19)

By applying the four linear operators \( L_k \) and \( M_k \) to this last expression, it can be confirmed that:

\[
L_k[U] = L_k[F] \quad \text{and} \quad M_k[U] = M_k[F] \quad (i,j = 0,1)
\]

i.e., \( U \) does satisfy the required boundary conditions.

The graph and contour plot of \( U \) are shown in Figs. 3a and 3b.

III. Transfinite Interpolation to Inconsistent Mixed Boundary Conditions

As a practical matter, boundary conditions for elliptic problems are quite frequently not consistently specified. By this we mean that, although the solution must be smooth (analytic) inside the problem domain, it may and does have singularities (discontinuities) on the boundary. An elementary example is this: this is the textbook heat conduction problem of determining the equilibrium temperature conditions are such that the boundary value problem will undoubtedly be aware of these two conditions be consistent, i.e.:

\[
\lim_{y \to 0} f(x, y) = g(x) \quad \text{and} \quad \lim_{y \to 0} h(x, y) = 0
\]

(3.1)

In words, for fixed \( i \) and \( j \), the function \( U_{ij} \) vanishes under operation by any of the six linear functionals \( L_k \) and \( M_k \) of \( (k=1, i, j) \). When operated on by \( L_k \) of \( M_k \), the result is \( L_k[U] \) or \( M_k[U] \), respectively.

The case of pure Dirichlet boundary conditions, \( \alpha_j = \alpha_0 = \beta_0 = \beta_j = 0 \) is the simplest to interpret. Suppose the boundary conditions are such that:

\[
\begin{align*}
F_1(x, j) &= 1 \text{m} F_1(x, j) = L_1 U_1[F] \\
F_2(x, j) &= 1 \text{m} F_2(x, j) = M_1 U_1[F]
\end{align*}
\]

(3.2)

and \( L_1 U_1[F] \neq M_1 U_1[F] \); cf., for example, Fig. 4a.

The function \( U_{ij} \) which we shall construct will satisfy, for the case of Dirichlet conditions:

\[
\begin{align*}
1 \text{m} U_{ij}(x, j) &= L_1 U_{ij}[F] \\
1 \text{m} U_{ij}(x, j) &= M_1 U_{ij}[F]
\end{align*}
\]

(3.3)

and at the three corners other than \((i,j), U_{ij} \) will vanish.

In the general case, suppose for the moment that we have the required functions \( U_{ij}(x, y) \) which satisfy conditions (3.2). Let \( W \) be equal to the sum of these four corner functions:

\[
W(x, y) = U_{00}(x, y) + U_{01}(x, y) + U_{10}(x, y) + U_{11}(x, y)
\]

(3.4)

Clearly, \( W \) satisfies the eight conditions:

\[
\begin{align*}
L_k W &= L_k U_{ij}[F] \\
M_k W &= M_k U_{ij}[F]
\end{align*}
\]

(3.5)

From this, we draw the important conclusion that, by virtue of the linearity of the operators \( L_k \) and \( M_k \):

\[
\begin{align*}
L_k W &= 0 \\
M_k W &= 0
\end{align*}
\]

(3.6)

If the solution to the original interpolation problem is again denoted by \( U \), we want to represent \( U \) as the sum of \( W \) and a yet to be determined function \( V \).
Now, since \( U \) is to satisfy the boundary conditions
\[
 L_i[U] = L_i[F] \quad \text{and} \quad M_j[U] = M_j[F] \quad (i,j = 0,1),
\]
we have from (3.7) that:
\[
 L_i M_j[V] = L_i M_j[F] = 0 \quad (i,j = 0,1),
\]
which is to say that the function \( V \) satisfies consistent boundary conditions, as defined in relation (2.2). Therefore, we can actually construct \( V \) using the techniques presented in the previous section. Referring back to (2.7), we have that:
\[
 V(x,y) = \Phi_x (F-W) + \Phi_y (F-W),
\]
the last because of (3.7).

In summary, we first construct the function \( U \), for each corner \((1,1)\). Then, we compute the derived boundary conditions, \( L_i[F-W] \) and \( M_j[F-W] \), and use these in expression (3.10) for \( V \). The function \( U = W + V \) will then exactly satisfy the original, inconsistent boundary conditions:
\[
 L_i[U] = L_i[F] \quad \text{and} \quad M_j[U] = M_j[F] \quad (i,j = 0,1).
\]

We shall now without derivation, display the equations (3.13a) and (3.13b) for \((i,j = 0,1)\).

\[\begin{align*}
 U_{10}(x,y) &= \Phi_0(x) \Phi_0(y) [T(\theta_{00}) L_{00} M_{00}[F]] \\
 &+ (1-T(\theta_{00})) M_{00}[F]
 U_{11}(x,y) &= \Phi_0(x) \Phi_1(y) [T(\theta_{01}) L_{01} M_{01}[F]] \\
 &+ (1-T(\theta_{01})) M_{01}[F]
\end{align*}\]

\[\begin{align*}
 U_{10}(x,y) &= \Phi_1(x) \Phi_0(y) [T(\theta_{10}) L_{10} M_{10}[F]] \\
 &+ (1-T(\theta_{10})) M_{10}[F]
 U_{11}(x,y) &= \Phi_1(x) \Phi_1(y) [T(\theta_{11}) L_{11} M_{11}[F]] \\
 &+ (1-T(\theta_{11})) M_{11}[F].
\end{align*}\]

The functions \( \Phi_0(x) \) and \( \Phi_1(y) \) are the same as in (2.6), and the \( T(\theta) \) must satisfy the following conditions:
\[
 T(\theta_{10}) = 1 \quad \text{at} \quad 0 \quad \text{and} \quad \theta_{10} = \frac{\pi}{2}
 T(\theta_{01}) = 1 \quad \text{at} \quad 0 \quad \text{and} \quad \theta_{11} = \frac{\pi}{2}
 T(\theta_{11}) = 0 \quad \text{at} \quad \theta_{11} = \frac{\pi}{4}
 T(\theta_{01}) = 0 \quad \text{at} \quad \theta_{01} = \frac{\pi}{4}
\]

The selection of \( T(\theta) \) is to satisfy the following conditions:
\[
 T(\theta_{10}) = 1 \quad \text{at} \quad 0 \quad \text{and} \quad \theta_{10} = \frac{\pi}{2}
 T(\theta_{01}) = 1 \quad \text{at} \quad 0 \quad \text{and} \quad \theta_{01} = \frac{\pi}{2}
\]

In the case of Dirichlet boundary conditions, only equations (3.13a) must hold, and they are quite simply satisfied by taking:
\[
 (1 - \frac{\pi}{4}) \quad \theta_{10} = 0, \quad \theta_{01} = \frac{\pi}{2}
\]

For the more general operators \( L_i \) and \( M_j \), the cubic function
\[
 T(\theta_{ij}) = \left( \frac{\theta_{ij}}{\pi} - 1 \right)^2 \left( \frac{\theta_{ij}}{\pi} + 1 \right) \quad (i,j = 0,1)
\]

satisfies (3.13a), (3.13b) and (3.14) as required. Figure 5a shows a perspective view and 5b the contour plot of the interpolant to the mixed inconsistent boundary conditions:

\[\begin{align*}
 L_0[F] &= F(0,y) = \cosh(\frac{\pi}{4}(1-y)) + 1
 L_{10}[F] &= F(x,0) = 0.5x^2 + 1
 L_{01}[F] &= F_y(x,1) = 0
\end{align*}\]

where:
\[
 \Phi_0(x) = 1 - 0.5x \quad \Phi_1(x) = 0.5x \quad (3.18)
\]

and
\[
 L_0 M_0[F] = 1 \quad L_0 M_1[F] = 2.5
 M_0 L_0[F] = 3.509 \quad M_0 L_1[F] = 0
 M_1 L_0[F] = 0 \quad M_1 L_1[F] = 0.
\]

Figure 6a shows a perspective and 6b the contour plot of the interpolant to the mixed inconsistent boundary conditions:
\[
 L_0[F] = F(0,y) = .5 \sin(x(4y + .5)) + .7
 L_1[F] = F(1,y) = 2y(1-y)\cos(x(2y-.25)) + .2
\]

(3.20)


$$M_0[F] = F_y(x,0) = 5(1-x)$$

$$M_1[F] = F_y(x,1) = 0$$

where:

$$\phi_0(x) = 1 - x$$

$$\phi_1(x) = x$$

$$\psi_0(y) = x - 0.5x^2$$

$$\psi_1(y) = 0.5x^2$$

and

$$L_0^M[F] = 5$$

$$L_1^M[F] = 0$$

$$M_0^L[F] = 1.414$$

$$M_1^L[F] = 0$$

$$L_0^L[F] = 0$$

$$L_1^L[F] = 0$$

$$M_1^L[F] = -1.414.$$  (3.21)

Note that although the boundary conditions are inconsistent at the three corners (0, 0), (1, 0), and (1, 1), the function value of the interpolant is inconsistent only at the corner (1, 1).

To illustrate the construction of $U(x,y)$ from $W(x,y)$ and $V(x,y)$ we will consider a very simple problem with Dirichlet boundary conditions and a discontinuity at (1, 1):

$$L_0^M[F] = F(0,y) = y^2$$

$$L_1^M[F] = F(1,y) = 0$$

$$M_0^L[F] = F(x,0) = 0$$

$$M_1^L[F] = F(x,1) = 0.$$  (3.22)

Obviously, $L_i^M[F] = M_i^L[F]$ for $i = 0$ and $j = 1, 0$. But, $L_i^M[F] = 0$ and $M_i^L[F] = 1$. The blending functions are:

$$\phi_0(x) = 1 - x$$

$$\phi_1(x) = x$$

$$\psi_0(y) = 1 - y$$

$$\psi_1(y) = y.$$  (3.23)

which yield

$$U(x,y) = (1-x)[y(1 - \frac{2\theta_{01}}{\pi}) - y + y^2]$$

$$= (1-x)y(\frac{2\theta_{01}}{\pi})$$

where $\theta_{01}$ is defined in (3.11).

References


Figure 2a.

Figure 2b.

Figure 3a.

Figure 3b.

Figure 4a.

Figure 4b.
EXACT A PRIORI MATCHING OF MIXED BOUNDARY CONDITIONS
FOR SECOND ORDER ELLIPTIC PROBLEMS

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ERRATA

1. p. 2, equation (2.7), \((P_x + P_y)[F]\) should be \((P_x + P_y)[F]\).

2. p. 3, equation (2.11), \[
(y-1) \ln\left(\frac{(x-1.2)^2 + .81}{(x-1.2)^2 + .81}\right)
\]
should be \(\[
(y-1) \ln\left(\frac{(x-1.2)^2 + .81}{(x-1.2)^2 + .81}\right)
\]

3. p. 5, equation (3.20), \(L_1[F] = F(1,y) = 2y(1-y)\cos(\pi(2y-.25)) + .2\)

should be \(L_1[F] = F(1,y) = 2y(1-y)\cos(\pi(2y-.25)) + .2\)
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-8