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AN ALTERNATIVE CONSIDERATION ON SINGULAR LINEAR STATE ESTIMATION

Y.F. Huang E. Fogel J.B. Thomas
Princeton Univ. Univ. of Notre Dame Princeton Univ.
Princeton, NJ 08544 Notre Dame, IN 46556 Princeton, NJ 08544

Abstract

The problem of designing an optimal state estimator for a linear, discrete-time system with a singular noise covariance matrix is considered. In this article, this problem is cast as a constrained optimization problem and the approach appears to be more direct. Solution to this optimization problem gives a reduced-order optimal state estimator.

I. INTRODUCTION:

In a linear stochastic system, the output measurement may be only partially noise corrupted. Although, in practice, one may argue that there exist no noise-free measurements, it is quite possible that some of the measurements are noise corrupted while the others are relatively accurate. Under the Gaussian assumption, this implies that the noise covariance matrix has both large and small eigenvalues, which easily leads to numerical difficulties in the implementation of the Kalman filter. It is convenient in this case to model these more accurate measurements as noise-free entities.


The main feature of the approach used in this paper is the following: after a proper similarity transformation, the state variables are decomposed into two parts, one to be estimated by a reduced-order filter and the other to be recovered exactly from the noise-free measurements. Then the dynamic equation of the latter part of the state equation is considered a constraint on the optimal estimation of the other part of the states. Hence the state estimation problem in this case is cast as a constrained optimization problem, which leads to a reduced-order optimal state estimator.

II. PROBLEM FORMULATION

A linear, discrete-time stochastic system can be described by the following equations:

\[ \begin{align*}
  x(k+1) &= A(k) x(k) + B(k) u(k), \quad k=0,1,2,\ldots \quad (1) \\
  y(k) &= C(k) x(k) + v(k), \quad k=1,2,\ldots \\
\end{align*} \]

where \( x(\cdot) \in \mathbb{R}^n \), \( u(\cdot) \in \mathbb{R}^p \), and \( y(\cdot) \in \mathbb{R}^m \).

To further specify the problem, the following assumptions are made:

1. \( x(0), u(0), u(1), \ldots, v(1), v(2), \ldots \) are independent random vectors with the following statistics:

\[ \begin{align*}
  \mathbb{E}[x(0)] &= x_0 \\
  \mathbb{E}[x(0)x^T(0)] &= \Sigma_x \\
  \mathbb{E}[u(k)] &= 0 \\
  \mathbb{E}[u(k)u^T(k-l)] &= \Sigma_u \\
  \mathbb{E}[v(k)] &= 0 \\
  \mathbb{E}[v(k)v^T(k-l)] &= \Sigma_v \\
\end{align*} \]

\[ \begin{align*} 
\mathbf{v}(k) & \equiv \mathbf{v}(k) \delta(i) \mathbf{v}(k, i) \\
\mathbf{E}[\mathbf{u}(k)\mathbf{v}^T(i)] & = 0_{p \times m} \mathbf{v}(k, i) \\
\mathbf{E}[\mathbf{u}(k)\mathbf{x}^T(0)] & = 0_{p \times m} \mathbf{v}(k) \\
\mathbf{E}[\mathbf{v}(k)\mathbf{x}^T(0)] & = 0_{m \times n} \mathbf{v}(k) 
\end{align*} \]

where \( \mathbf{u}^T(\cdot) \) and \( \mathbf{v}^T(\cdot) \) denote the transpose of vectors \( \mathbf{u}(\cdot) \) and \( \mathbf{v}(\cdot) \), respectively, and \( \delta(\cdot) \) denotes the Kronecker delta.

(ii) For any \( k \), the \( \mathbf{V}_v(k) \) is a non-negative definite matrix with rank \( m_1 \), where \( m_1 \leq m \). Under this assumption, implementation of the standard Kalman filter involves inversion of a matrix which may be singular. Tee and Athans [5] proposed an observer-estimator of order \( n - m_2 \) which performs as well as higher order estimators, where \( m_2 \leq m \).

(iii) For any \( k \), the \( \mathbf{C}(k) \) is of full rank, i.e. every element of the output measurement is independent of the others.

The objective here is to design an optimal state estimator of order \( n - m_2 \). Without loss of generality, one can assume that

\[ \mathbf{v}(k) = \begin{bmatrix} \mathbf{v}_1(k) \\ \mathbf{v}_2(k) \end{bmatrix} \]

where \( \mathbf{v}_1(k) \in \mathbb{R}^{m_1} \), and thus the covariance matrix of \( \mathbf{v}(k) \) can be written as

\[ \mathbf{V}_v(k) = \begin{bmatrix} \mathbf{V}_{v_1}(k) & 0_{m_1 \times m_2} \\ 0_{m_2 \times m_1} & \mathbf{V}_{v_2}(k) \end{bmatrix} \]

where \( \mathbf{V}_{v_1} \) is strictly positive definite.

It is easy to see that there exists a non-singular matrix \( \mathbf{Q}(k) \) such that the transformation

\[ \mathbf{z}(k) = \mathbf{Q}(k) \mathbf{x}(k) \]  

yields the following state and measurement equations

\[ \begin{align*} 
\mathbf{z}_1(k+1) & = \mathbf{A}_1 \mathbf{z}_1(k) + \mathbf{B}_1 \mathbf{v}_1(k) \\
\mathbf{z}_2(k+1) & = \mathbf{A}_2 \mathbf{z}_1(k) + \mathbf{B}_2 \mathbf{v}_2(k) 
\end{align*} \]

and

\[ \begin{align*} 
\mathbf{z}_1(k) & = \begin{bmatrix} \mathbf{z}_{11}(k) \\ \mathbf{z}_{12}(k) \end{bmatrix} \\
\mathbf{z}_2(k) & = \begin{bmatrix} \mathbf{z}_{21}(k) \\ \mathbf{z}_{22}(k) \end{bmatrix} \\
\mathbf{u}(k) & = \begin{bmatrix} \mathbf{u}_1(k) \\ \mathbf{u}_2(k) \end{bmatrix} \\
\mathbf{v}(k) & = \begin{bmatrix} \mathbf{v}_1(k) \\ \mathbf{v}_2(k) \end{bmatrix} 
\end{align*} \]

It is obvious from (7) and (8) that \( \mathbf{z}_1(k) \) and \( \mathbf{z}_2(k) \) are mutually dependent; therefore the estimation of \( \mathbf{z}_1(k) \) does depend on the dynamic behavior of \( \mathbf{z}_2(k) \). Thus the filtering problem becomes that of finding an optimal \( \hat{\mathbf{z}}_1(k+1|k+1) \) subject to (7) and constrained by (8), where \( \hat{\mathbf{z}}_1(k+1|k+1) \) denotes the estimate of \( \mathbf{z}_1(k+1) \) given measurements up to time \( k+1 \). Note, from (6) and (7), that the state \( \mathbf{z}_2(k) \) can be regarded as a deterministic input in the Kalman filtering problem.

8. THE REDUCED-ORDER OPTIMAL STATE ESTIMATOR

In this section, the optimal estimator for \( \mathbf{z}_1(k) \) is derived where the performance measure is the trace of the error covariance matrix. Defining the vector \( \mathbf{s}(k) \) as

\[ \mathbf{s}(k) \triangleq \mathbf{z}_1(k) - \mathbf{P}(k) \mathbf{z}_2(k) : \mathbf{P}(k) \in \mathbb{R}^{n_1 \times m_2} \]

from (5), (7) and (8), one obtains
state estimator is formulated by the following equations

\[ \dot{\hat{x}}(k+1|k+1) = [I - K^*(k+1)H(k+1)]F^*(k)\hat{x}(k) + K^*(k+1)(z(k+1) - H(k+1)\hat{x}(k)) \]

\[ z_2(k+1) = H(k+1)G\xi(k)z_2(k) \]  

\[ z_2(k+1) = \tilde{z}_2(k+1) + \tilde{P}^*(k+1)z_2(k) \]  

\[ z_2(k+1) = \tilde{z}_2(k+1) + P^*(k+1)z_2(k) \]  

\[ z_2(k+1) = \tilde{z}_2(k+1) + P^*(k+1)z_2(k) \]  

\[ \tilde{z}_2(k+1) = \tilde{z}_2(k+1) + P^*(k+1)z_2(k) \]

where

\[ V^{*}(k) = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \]

and the following special cases are of interest:

**Case 1:** \( \lambda_1(k) = 0_{m_1x_m2} \)

**Case 2:** \( \lambda_2(k) = 0_{m_2x_m2} \), i.e., \( \lambda_1(k) \) is a singular non-zero matrix. In this case, only some components of \( z_2(k) \) contain information about \( (u(k), z_1(k)) \). Thus the similarity transformation discussed in Section 2 can be redefined so as to isolate only those elements of \( z_2(k) \) which constitute a constraint on \( (u(k), z_1(k)) \).

Hence the Lagrange multiplier \( P(k+1) \) that should be considered is an element in \( \mathbb{R}^{(n+1)x_r} \), where \( r < m_2 \). Alternatively, any member in \( \Sigma^*(k+1) \) can be used in the filter realization.

**Case 3:** \( \lambda_1(k) \) is positive-definite. This condition can be fulfilled when \( V_1(k) \) is positive-definite for any \( k = 0, 1, 2, \ldots \).

In this case, \( \Sigma^*(k+1) \) contains one and
only one element \( P^*(k+1) \), which is given by (19.d).

When \( P^*(-) \) is uniquely specified (Case 3), one can compare the error covariance matrix given by (20.g) with that obtained for arbitrary \( P(-) \) and observe the same expression for \( V_z(k+1|k+1) \).

The difference is that \( \Gamma^*(k+1) \) of (20.f) has the following property

\[
\text{Tr}[\Gamma^*(k+1)] < \text{Tr}[\Gamma(k+1)]
\]

where \( \Gamma(-) \) is obtained from non-optimal \( P(-) \).

Finally, the implementation of this estimator should be initiated as follows:

\[
\hat{x}(0) = \mathbb{E}[x(0)] = x_0
\]

i.e.

\[
\hat{x}_1(0) = \mathbb{E}[z_1(0)]
\]

therefore

\[
\begin{bmatrix}
v_z(0) \\
z_1(0)
\end{bmatrix} =
\begin{bmatrix}
Q_0 & V_0^T(0) \\
0 & Q_1 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]

and

\[
P^*(0) = \begin{bmatrix} 0 & a & b & 0 \end{bmatrix}
\]

5. CONCLUSION

A reduced-order optimal state estimator for a linear, discrete-time system associated with a singular noise covariance matrix has been derived in this paper. The main idea in this derivation is to cast this singular state estimation problem as a constrained optimization problem.

The estimator derived here is fundamentally the same as that derived by Fairman [8]. The major differences are: the approach here is more straightforward, the optimality of the estimator is more explicitly exposed and, furthermore, the possibility of nonuniqueness of \( P^*(-) \) is discussed here.

It is worth mentioning that the estimator given here requires lower order matrix inversion than the standard full-order Kalman filter does in the singular case; thus the computational efficiency is improved. This estimation procedure can be applied similarly to smoothing and predicting problems or systems with colored noise.

REFERENCES


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