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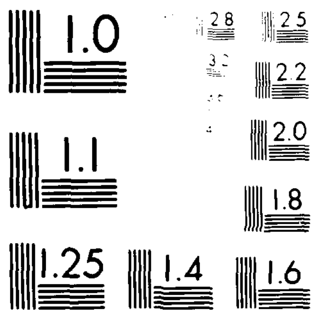
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MULTIVARIATE DISTRIBUTIONS IN RELIABILITY THEORY AND LIFE TESTING[1]

by

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ABSTRACT

Multivariate parametric distributions which are of interest in reliability theory and life testing are discussed. These include distributions with exponential, Weibull and gamma univariate marginal distributions. Other distributions of interest are the multivariate nonparametric distributions whose marginals have increasing *failure rates* (IFR), increasing failure rate averages (IFRA), are new better than used (NBU) or new better than used in expectation (NBUE). Also mentioned are univariate and multivariate processes which have associated with them distributions in the various nonparametric classes mentioned above.

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KEY WORDS

Multivariate exponential distributions; multivariate Weibull distributions; multivariate gamma distributions; multivariate exponential extensions; shock models; threshold model; gestation model; characteristic function equation; multivariate IFR; multivariate IFRA; multivariate NBU; multivariate NBUE.

1. INTRODUCTION

In this paper we discuss the various multivariate parametric and nonparametric classes of distributions which are of interest in reliability theory and life testing.

In Section 2 we concentrate on parametric distributions whose univariate marginals are either exponential, Weibull or gamma. We also discuss multivariate exponential extensions which are distributions whose marginals are not generally exponential but which were formulated utilizing concepts based on univariate exponential distributions.

Multivariate nonparametric classes are discussed in Section 3. These classes contain multivariate distributions which have increasing failure rate or increasing failure rate average or are new better than used or new better than used in expectation. Many formulations have been given for each of these classes, so that in this section only the most recent or the most established formulations are discussed in detail.

The paper concludes with a discussion of univariate and multivariate stochastic processes which are related to the nonparametric classes to which Section 3 was devoted.

2. PARAMETRIC DISTRIBUTIONS

2.1 Introduction

The univariate parametric distributions which have been most useful in reliability theory have been the exponential and Weibull distributions. Others which have been of some importance are the lognormal and the gamma distributions. A history of the use of these distributions is given in Chapter 1 of Barlow and Proschan (1965). An introduction to these distributions, models from which they arise, their properties and their use in reliability theory are contained in Mann, Shafer and Singpurwalla (1974), and in Barlow and Proschan (1965, 1975). See also Bain (1978). For more detailed expositions on these distributions and comprehensive bibliographies see Johnson and Kotz (1972).

Multivariate parametric distributions which are analogs of the univariate distributions previously mentioned are still being developed. Unlike the multivariate normal distribution, for most other multivariate distributions having marginals of one type there are many possible dependence structures and consequently many multivariate versions. Several multivariate exponential and related distributions, for example, have been developed. Many of these, along with their properties, are given in Basu and Block (1975) and in Block (1975). See also Johnson and Kotz (1972).

2.2 Bivariate Exponential and Related Distributions

Most of the distributions treated in this section have multivariate analogs, but for the purposes of clarity, and exposition we will treat the bivariate case first. As with all the distributions discussed we will say a distribution is a multivariate "   " if all of its univariate marginals are "   ". Therefore a multivariate exponential distribution will be one whose univariate marginals are exponential. We will say a distribution is a <u>multivariate exponential extension</u> if it is derived from properties of a univariate exponential. The univariate marginals of such a distribution will not necessarily be exponential.

Freund Distribution

Freund (1961) introduced one of the first bivariate distributions based on a model involving the exponential distribution. An interpretation of this model, given in Block (1975), is the following. Consider a two component system where the failure of one component affects the lifetime of the other component. Let Y_1 and Y_2 have independent exponential distributions with means α_1^{-1} and α_2^{-1} respectively (the initial distributions of the unaffected component lifetimes). Let Y_1' and Y_2' be independent of Y_1 and Y_2 and have independent exponential distributions with means $(\alpha_1')^{-1}$ and $(\alpha_2')^{-1}$ (the distributions of the affected components).

Then it is easily shown that the distribution of the component lifetimes

$$(X_1, X_2) = \begin{cases} (Y_1, Y_1+Y_2') & \text{if } Y_1 < Y_2, \\ (Y_2+Y_1', Y_2) & \text{if } Y_2 < Y_1. \end{cases}$$

has the survival function

$$\bar{F}(x_1,x_2)=\begin{cases}\left(\dfrac{\alpha_1}{\alpha_1+\alpha_2-\alpha_2'}\right)\exp(-(\alpha_1+\alpha_2-\alpha_2')x_1-\alpha_2'x_2) \\ \quad+\left(\dfrac{\alpha_2-\alpha_2'}{\alpha_1+\alpha_2-\alpha_2'}\right)\exp(-(\alpha_1+\alpha_2)x_2) \quad \text{if } x_1<x_2\ , \\ \left(\dfrac{\alpha_2}{\alpha_1+\alpha_2-\alpha_1'}\right)\exp(-\alpha_1'x_1-(\alpha_1+\alpha_2-\alpha_1')x_2) \\ \quad+\left(\dfrac{\alpha_1-\alpha_1'}{\alpha_1+\alpha_2-\alpha_1'}\right)\exp(-(\alpha_1+\alpha_2)x_1) \quad \text{if } x_1>x_2\ . \end{cases} \tag{1}$$

Properties of this distribution are given in Johnson and Kotz (1972).

Marshall and Olkin and a Related Distribution

The distribution of Marshall and Olkin (1967a) has survival function

$$\bar{F}(t_1,t_2)=P(T_1>t_1,\ T_2>t_2)=\exp(-\lambda_1t_1-\lambda_2t_2-\lambda_{12}\max(t_1,t_2))$$
$$\text{for } t_1>0,\ t_2>0,$$

where λ_1, λ_2, λ_{12} are nonnegative. This distribution is derivable from 1) a fatal shock model, 2) a nonfatal shock model, and 3) a loss of memory model. For details on these models and for properties of this important distribution see Johnson and Kotz (1972) and Barlow and Proschan (1975). Estimation and testing for this model have been carried out in Bennis, Bain and Higgins (1972), Bhattacharya and Johnson (1973) and Proschan and Sullo (1975). A more general version of this distribution is given by Marshall and Olkin (1967b).

A recent characterization of this distribution due to Block (1977a) is that (T_1,T_2) has this distribution if and only if T_1 and T_2 are marginally exponential and $\min(T_1,T_2)$ is exponential and independent of T_1-T_2. Other characterizations of this distribution are given in Galambos and Kotz (1978) and in Basu and Block (1975).

A distribution which is closely related to the Marshall and Olkin distribution has been studied by Block and Basu (1974). It is also closely related to the Freund distribution and can be obtained from the interpretation of Freund's model given previously where the affect of one component on the other is a strain, i.e., $\alpha_1 < \alpha_1'$, $\alpha_2 < \alpha_2'$. For the choice $\alpha_1 = \lambda_1+\lambda_{12}(\frac{\lambda_1}{\lambda_1+\lambda_2})$, $\alpha_1' = \lambda_1+\lambda_{12}$, $\alpha_2 = \lambda_2+\lambda_{12}(\frac{\lambda_2}{\lambda_1+\lambda_2})$ and $\alpha_2' = \lambda_2+\lambda_{12}$ in (1) where λ_1, λ_2, λ_{12} are nonnegative and $\lambda_1+\lambda_2 > 0$, the distribution is obtained. It turns out that this distribution is also derivable from a loss of memory model similar to that of Marshall and Olkin. Furthermore this distribution is the absolutely continuous part of the Marshall and Olkin distribution. It should also be noted that the lifetime of the two organ system of Gross, Clark and Liu (1971) and also of the two organ subsystem of Gross (1973) is a special case of the maximum lifetime of this distribution. See the original paper for details, estimation and other properties. Estimation and testing have also been carried out by Mehrotra and Michalek (1976) and by Gross and Lam (1979). The latter authors consider an application of this distribution to bivariate relapse times for patients receiving different treatments.

Friday and Patil Distribution

Proschan and Sullo (1974) briefly suggest a model which incorporates both the Marshall and Olkin and the Freund distributions. Friday and Patil (1977) pursue the idea of a distribution containing these two distributions still further. They develop a similar although more general distribution than that of Proschan and Sullo which is derivable from (1) a threshold model, (2) a gestation model and (3) a warmup model. The survival function can be written as

$$\bar{F}(x_1,x_2) = \alpha_0 \bar{F}_A(x_1,x_2) + (1-\alpha_0)\,\bar{F}_S(x_1,x_2) \qquad (2)$$

where $\bar{F}_A(x_1,x_2)$ is given by (1), i.e. the Freund distribution, and

$$\bar{F}_S(x_1,x_2) = \exp(-(\alpha_1+\alpha_2)\max(x_1,x_2)) \text{ if } x_1>0,\ x_2>0$$

It is clear that for $\alpha_0 = 1$ equation (2) gives the Freund distribution. It should be noticed that this equation is of the form of the Marshall and Olkin distribution (see Theorem 1.5, Chapter 5 of Barlow and Proschan (1975)). The choice of parameters $\alpha_0 = (\lambda_1+\lambda_2)(\lambda_1+\lambda_2+\lambda_{12})^{-1}$ where $\lambda_1>0$, $\lambda_2>0$ and $\lambda_{12}>0$ and α_1, α_1', α_2, α_2' as chosen for the Block and Basu distribution in a previous section yields the Marshall and Olkin distribution.

Also contained in the paper of Friday and Patil is a nice summary of the relations among the various distributions just discussed. Also discussed are transformations from independence for the distribution and computer generation and efficiency for this distribution.

Downton Distribution

This distribution is a special case of a classical bivariate gamma distribution due to Wicksell and to Kibble. See Krishnaiah and Rao (1961) for a discussion of this gamma distribution and references. Downton (1970) developed a model which gave rise to this bivariate exponential distribution and proposed its use in the setting of reliability theory.

An interpretation of this model, which is due to Arnold (1975b) is presented here. Consider a two component system where the components are each subjected to nonfatal shocks which occur according to two independent Poisson processes. Assume the processes have rates $(1-\rho)\mu_1^{-1}$ and $(1-\rho)\mu_2^{-1}$ respectively where $0<\rho<1$, $0<\mu_1$, $0<\mu_2$. Let $\{X_{ij}\}_{j=1}^{\infty}$, $i=1,2$ represent interarrival times for each of the two processes. Assume that each component fails after N_1 and N_2 shocks respectively where $N_1 \equiv N_2 \equiv N$ is a geometric random variable with parameter $1-\rho$ (i.e. $P\{N=n\} = \rho^{n-1}(1-\rho)$, $n=1,2,\ldots$). The times to failure of the two components are then given by

$$(Y_1, Y_2) = \left(\sum_{i=1}^{N_1} X_{i1}, \sum_{i=1}^{N_2} X_{i2}\right). \tag{3}$$

By conditioning on N and using characteristic functions it is easily seen that (Y_1,Y_2) has density

$$f(y_1,y_2) = \frac{\mu_1\mu_2}{1-\rho} e^{-\frac{\mu_1 y_1+\mu_2 y_2}{1-\rho}} I_0\left(\frac{2\sqrt{\rho\mu_1\mu_2 y_1 y_2}}{1-\rho}\right) \tag{4}$$

for $y_1>0$, $y_2>0$

where I_0 is the modified Bessel function of the first kind of order 0. This is the bivariate exponential distribution given by (2.10) of Downton (1970).

Downton also shows that if instead of $N_1 \equiv N_2 \equiv N$ in his derivation (and equivalently in the above derivation) we let (N_1,N_2) assume various other bivariate geometric distributions a distribution of the same form as (4) is obtained.

As mentioned initially the above distribution can be obtained as the special case of a particular bivariate gamma distribution. This distribution is obtained as follows. Let $(X_1,Y_1),(X_2,Y_2),\ldots,(X_n,Y_n)$ be iid bivariate standard normal distributions with correlation ρ. Then

$$\left(\sum_{i=1}^{n} X_i^2 \,,\; \sum_{i=1}^{n} Y_i^2\right)$$

has a correlated bivariate gamma (chi-square) distribution. For the case n=2, a distribution of the form (4) is obtained.

Hawkes Distribution

The bivariate exponential distribution of Hawkes (1972) is obtained from the same model as that of Downton. The only difference is in the choice of the bivariate geometric distribution (N_1,N_2). This bivariate geometric distribution which was derived by Hawkes was derived independently by Esary and Marshall (1973) and by Arnold (1975a). The model given here is from the Esary and Marshall paper.

Consider two devices which receive nonfatal shocks at discrete time periods labeled by the positive integers. The occurrence of these shocks is independent and at each cycle there is a probability P_{11} that both devices survive the shocks, probability P_{10} that the first survives but the second does not, probability P_{01} that the second survives and the first does not and P_{00} that both devices fail. Then letting N_i be the number of shocks to failure for devices i=1,2, and conditioning on the occurrence of the first cycle, as in Arnold (1975b), it is easily seen that the characteristic function $\phi(t_1,t_2)$ satisfies

$$\phi(t_1,t_2) = e^{it_1+it_2}(p_{00}+p_{01}\phi(0,t_2)+p_{10}\phi(t_1,0)+p_{11}\phi(t_1,t_2)). \quad (5)$$

This is essentially the characteristic function equation of Paulson and Uppuluri (1972b) and is easily solved (i.e. take $t_1=0$ and solve for $\phi(0,t_2)$, then for $\phi(t_1,0)$ similarly, then for $\phi(t_1,t_2)$). The bivariate distribution which has this characteristic function is given by

$$P(N_1>n_1,N_2>n_2) = \begin{cases} p_{11}^{n_1}(p_{01}+p_{11})^{n_2-n_1} & \text{if } n_1 \le n_2, \\ p_{11}^{n_2}(p_{10}+p_{11})^{n_1-n_2} & \text{if } n_2 \le n_1. \end{cases} \quad (6)$$

Using this distribution in the model of Downton, Hawkes in slightly different notation (i.e. $p_{ij}=p_{ji}$ for all $i=0,1,\ j=0,1$) obtains his distribution. By taking $p_{01}=p_{10}=0$ it can be seen that $N_1 \equiv N_2$ and so the *Downton* distribution is a special case of Hawkes. The resulting transform is given in Hawkes (1972) along with some properties.

Paulson Distribution

Paulson (1973) derives a bivariate exponential distribution through a characteristic function equation. This equation is the generalization of a one dimensional characteristic function equation which arises from a compartment model (see Paulson and Uppuluri (1972a)). A generalization of the compartment model also leads to the bivariate equation.

The bivariate equation is given by

$$\phi(t_1,t_2) = \psi(t_1,t_2)\left[p_{00}+p_{01}\phi(0,t_2)+p_{10}\phi(t_1,0)+p_{11}\phi(t_1,t_2)\right]$$

where $p_{00}+p_{01}+p_{10}+p_{11} = 1$, $p_{10}+p_{11}<1$, $p_{01}+p_{11}<1$ and $\psi(t_1,t_2) = \left[(1-i\theta_1 t_1)(1-i\theta_2 t_2)\right]^{-1}$. Then solving for $\phi(t_1,t_2)$ in the above equation leads to the bivariate characteristic function

$$\phi(t_1,t_2) = \left[(1-i\theta_1 t_1)(1-i\theta_2 t_2)-p_{11}\right]^{-1}\left[p_{00}+p_{10}(1-i\mu_1 t_1)^{-1} + p_{01}(1-i\mu_2 t_2)^{-1}\right]$$

where $\mu_1 = \theta_1(p_{00}+p_{01})^{-1}$ and $\mu_2 = \theta_2(p_{00}+p_{10})^{-1}$.

It can be shown that this is exactly the form of the Hawkes distribution and henceforth we will refer to this distribution as the Hawkes-Paulson distribution. For properties see Paulson (1973) and Hawkes (1972).

Arnold Classes

In describing these classes Arnold uses what he calls a generalized multivariate geometric distribution. It is easily seen that this is a reparametrized version of (6) in the bivariate case. Thus we let (N_1, N_2) be the bivariate distribution given by (6). Then Arnold's bivariate classes $\varepsilon_n^{(2)}$ consist of the random variables

$$(Y_1, Y_2) = \left(\sum_{i=1}^{N_1} X_{i1}, \sum_{i=1}^{N_2} X_{i2} \right)$$

where (X_{i1}, X_{i2}) for $i=1,2,\ldots$ are bivariate iid rvs with distributions in $\varepsilon_{n-1}^{(2)}$ for $n>1$ and where $\varepsilon_0^{(2)}$ consists of (X,X) where X is exponential. Clearly the marginals are exponential for all the classes. It also is not hard to show that $\varepsilon_1^{(2)}$ contains the pair of independent exponentials (see Arnold (1975a)) and also contains the Marshall and Olkin distribution (see the nonfatal shock model discussed in Barlow and Proschan (1975)). Furthermore, it follows from the derivations given here of the Downton and the Hawkes distributions that these are contained in $\varepsilon_2^{(2)}$ since $\varepsilon_1^{(2)}$ contains the independent exponentials.

The Arnold classes of distributions have been described using the characteristic function equation approach of Paulson and Uppuluri (1972b) and Paulson (1973) in Block, Paulson and Kohberger (1975). In this latter paper the characteristic function equation approach has been used to derive properties of the distributions in this class, including descriptions of the standard distributions in the class, infinite divisibility of the distributions, moment properties and asymptotic properties. These results are summarized, without proof, in Block (1977b), in which it is also shown how the distributions in the class lead to multivariate shock models of the type studied in the univariate case by Esary, Marshall, and Proschan (1973).

2.3. Multivariate Exponential and Related Distributions

Most of the bivariate models in the preceding section have multivariate $(n>3)$ analogs. In general the ideas are similar to the bivariate case, but the notational complexity is greatly increased. Without giving many details, we will briefly discuss the multivariate situation.

The Freund distribution has been generalized to the multivariate case by Weinman (1966) but only for identically distributed marginals. See Johnson and Kotz (1972) for details concerning this distribution. Block (1975) has considered a generalization of the Freund distribution for the case when the marginals need not be identically distributed as well as generalizing the Block and Basu (1974) and the Proschan and Sullo (1974) models in the same paper.

Generalization of the Downton (1970), Hawkes (1972) and Paulson (1973) distributions implicitly exist within the framework of the general multivariate gamma distribution of Krishnamoorthy and Parthasarathy (1951) (see also Krishnaiah and Rao (1961) and Krishnaiah (1977)) and also within the framework of the Arnold classes. A specific parametric form has been given in Hsu, Shaw and Tyan (1977).

Recently a multivariate exponential distribution has been proposed by Bryant (1979) which arises in the context of certain cycling systems.

2.4. Multivariate Gamma Distributions

Unlike multivariate exponential distributions multivariate gamma distributions have a long history and many of their distributional properties have been discussed in the literature. A good reference for these distributions and their properties is Johnson and Kotz (1972).

Krishnaiah (1977) has specifically discussed multivariate gamma distributions in a reliability setting. In that paper, Krishnaiah discusses the distributions of order statistics and linear combinations of variables from various multivariate gamma distributions and their use in the area of simultaneous test procedures. Furthermore he discusses multivariate distributions arising from mixtures of distributions of type similar to those encountered in the models of Downton (1970), Hawkes (1972), Arnold (1975) and Block (1977).

2.5. Multivariate Weibull Distributions

Multivariate Weibull distributions were discussed by Marshall and Olkin (1967a) in the context of their discussion on multivariate exponential distributions. Specifically they define a multivariate Weibull distribution by assuming $(T_1,\dots,T_n)$ has their multivariate exponential distribution and then considering

$$(T_1',\dots,T_n') = (T_1^{1/\alpha_1},\ T_2^{1/\alpha_2},\dots,T_n^{1/\alpha_n}) \qquad (7)$$

where $\alpha_i > 0$ for $i=1,\dots,n$ which then has univariate Weibull marginal distributions. This procedure certainly could be extended for any multivariate exponential distribution. Moeschberger (1974) has studied bivariate Weibull distributions of this form, deriving properties and discussing maximum likelihood expectation.

David (1974) and Lee and Thompson (1974) have introduced multivariate Weibull distributions of the form $(T_1,\dots,T_n)$ where $T_i = \min(U_J\colon i \in J)$, $\phi \neq J \subset \{1,\dots,n\}$, $P\{U_J > x\} = e^{-\lambda_J x^{\alpha_J}}$, $x>0$ and U_J are independent. These distributions need not have Weibull marginals if the α_J are not all equal. Arnold (1967) has also considered Weibull distributions of a similar form, but his restriction that they belong to an additive family forces $\alpha_J = \alpha$ for all J. Thus these distributions are also of the form (7).

Recently Spurrier and Weier (1979) have modified the Freund model using Weibull instead of exponential distributions.

3. NONPARAMETRIC CLASSES OF DISTRIBUTIONS

Various classes of distributions which describe the way in which component lifetimes wear out have been discussed by many authors. The most important of these classes are 1) the increasing failure rate (IFR) class, 2) the increasing failure rate average (IFRA) class, 3) the new better than used (NBU) class, and 4) the new better than used in expectation (NBUE) class. These have been extensively discussed in the literature. The case where the lifetimes are independent (which we call the univariate case) are discussed in the book of Barlow and Proschan (1975) and also in the expository paper of Block and Savits (1980). The case where the components are dependent (called the multivariate case) has also been discussed in the latter paper. but since the development in the field is so rapid many new results have appeared since this last mentioned paper. We give a brief introduction to the univariate case, some background in the multivariate case and then outline the most recent developments.

3.1. Univariate Classes

The most prominent of the nonparametric classes used in reliability theory is the class of distributions which have increasing failure rate. See Block and Savits (1980) for background and motivation for this class In the following we let T be a random variable with distribution function $F(x)$ such that $F(0)=0$ (the usual assumption is $F(0^-)=0$, but for the purpose of exposition we use the above) having density $f(x)$ (if it exists). We say F has increasing failure rate (IFR) if the survival function $\bar{F}(x)=P\{T>x\}$ satisfies

$$\frac{\bar{F}(x+t)}{\bar{F}(x)} \quad \text{decreases in } x>0 \text{ for } t>0 \tag{8}$$

or equivalently (if the density exists)

$$r(x) = \frac{f(x)}{\bar{F}(x)} \quad \text{increases in } x>0.$$

A distribution F has <u>increasing failure rate average</u> (IFRA) if

$$\bar{F}(\alpha t) \geq \bar{F}^{\alpha}(t) \quad \text{for all } 0<\alpha\leq 1 \text{ and all } t\geq 0$$

or equivalently (if the density exists)

$$t^{-1}(\int_0^t r(x)dx) \quad \text{increases in } t>0.$$

Another equivalent formulation (see Block and Savits (1976)) is

$$\int_0^{\infty} h^{\alpha}(x/\alpha)dF(x) \geq \{\int_0^{\infty} h(x)dF(x)\}^{\alpha} \quad \text{for all } 0<\alpha\leq 1 \tag{9}$$

and all nonnegative increasing functions h.

A distribution F is <u>new better than used</u> (NBU) if

$$\bar{F}(x+t) \leq \bar{F}(x)\bar{F}(t) \quad \text{for all } x\geq 0,\ t\geq 0. \tag{10}$$

A distribution F is <u>new better than used in expectation</u> (NBUE) if

$$\int_t^{\infty} \bar{F}(x)dx \leq \mu\bar{F}(t) \quad \text{for all } t>0$$

where $\mu = \int_0^{\infty} \bar{F}(x)dx$ is finite.

Dual versions for all the above definitions exist by reversing the monotonicity or the inequality. Since the treatment of these concepts is similar we shall omit it.

3.2. Multivariate Classes

As in the parametric case many multivariate extensious are possible. Many versions of multivariate IFR, IFRA, NBU and NBUE have been proposed. For various IFR and IFRA extensions see Marshall (1974) and Esary and Marshall (1979) respectively. In the following we discuss the particular multivariate IFR and IFRA notations which at this time appear to be the most important ones. Various multivariate concepts of NBU and NBUE are given in Block and Savits (1980). As of yet, no clear favorites have emerged but there have been several recent papers on this subject. We shall attempt to describe some of this development. In the following we let F be the multivariate distribution of the random variable $\underset{\sim}{T} = (T_1, \ldots, T_n)$ which is assumed to satisfy $\bar{F}(\underset{\sim}{0}) = \bar{F}(0, \ldots, 0) = 1$ where we let $\bar{F}(\underset{\sim}{t}) = \bar{F}(t_1, \ldots, t_n) = P(T_1 > t_1, \ldots, T_n > t_n)$.

The distribution F is said to be <u>MIFR</u> if

$$\frac{\bar{F}(\underset{\sim}{x} + t\underset{\sim}{1})}{\bar{F}(\underset{\sim}{x})} \text{ decreases in } \underset{\sim}{x} \geq \underset{\sim}{0} \text{ for all } t > 0 \tag{11}$$

where $\underset{\sim}{1} = (1, \ldots, 1)$.

This generalizes (8) and has many important properties (see Block and Savits (1980)). Various variants of this condition are possible (see Marshall (1979)) but the above version best captures the intuitive idea of the model that components in the same environment run for the same time (i.e. $t_1 = t_2 = \ldots = t_n = t$) but may be of different ages (i.e. $\underset{\sim}{x} = (x_1, \ldots, x_n)$). This concept also satisfies many of the important basic properties that one would expect of such a multivariate generalization. See Chapter 5 of Barlow and Proschan (1975) for the statements and proofs of these properties.

The concept of multivariate IFRA which we now discuss is a generalization of (9). We say F is <u>MIFRA</u> if

$$E^{\alpha}\left[h(\underset{\sim}{T})\right] \leq E\left[h^{\alpha}(T/\alpha)\right] \quad \text{for all } 0<\alpha\leq 1$$

and for all continuous nonnegative increasing functions h. Recall that F is the df of $\underset{\sim}{T}$. A distribution satisfying this condition has all of the properties one would expect of a generalization of the univariate IFRA concept. See Block and Savits (1979b).

Esary and Marshall (1979) have proposed various other concepts of multivariate IFRA, many of them having intuitive appeal. Unfortunately all of them fail to satisfy at least one of the basic properties which the MIFRA distributions possess. This is demonstrated in Block and Savits (1978b).

Block and Savits (1980) describe a wide variety of possible definitions for both the concepts of multivariate NBU and multivariate NBUE. Some of these were definitions given by Buchanan and Singpurwalla (1977), others were based on the multivariate IFRA concepts of Esary and Marshall and still others were based on Laplace transform characterizations of NBU and NBUE which appeared in Block and Savits (1979a) and on other characterizations of NBU and NBUE which appeared in Block and Savits (1978a). At the time the paper of Block and Savits (1980) was written the only thing that was clear was that many multivariate NBU and NBUE concepts were possible. Since then some order has begun to appear in this field.

Marshall and Shaked (1979) introduce a compelling concept of NBU. This definition is that a random vector $\underset{\sim}{T}=(T_1,\ldots,T_n)$ is multivariate NBU if

$$P\{\underset{\sim}{T} \in (\alpha+\beta)A\} \leq P\{\underset{\sim}{T} \in \alpha A\}P\{\underset{\sim}{T} \in \beta A\} \text{ for all } \alpha>0,\ \beta>0 \quad (12)$$

and all upper (or increasing) sets A in R^n. (A is an upper set if $\underset{\sim}{x} \in A$ and $\underset{\sim}{x} \leq \underset{\sim}{y}$ imply $\underset{\sim}{y} \in A$). It is clear that this is a general version of the type of definition studied by Buchanan and Singpurwalla (1977) and very recently by Ghosh and Ebrahimi (1980), i.e.

$$\bar{F}(\underset{\sim}{x}+\underset{\sim}{y}) \leq \bar{F}(\underset{\sim}{x})\ \bar{F}(\underset{\sim}{y}) \quad \text{for all } \underset{\sim}{x}>\underset{\sim}{0} \text{ and } \underset{\sim}{y}>\underset{\sim}{0} \quad (13)$$

where $\underset{\sim}{x}$ and $\underset{\sim}{y}$ are perhaps further constrained. Furthermore Marshall and Shaked have four alternate characterizations of (12). Several of these involve the concept that $\underset{\sim}{T}$ is multivariate NBU if and only if $g(\underset{\sim}{T})$ is univariate NBU for all g of a certain type. Many other properties are proven.

The classes of multivariate NBU and NBUE introduced by Buchanan and Singpurwalla (1977), where the NBU distributions are defined by properties which are cases of (13) and the NBUE distributions are integrated versions of these, have been further studied by Ghosh and Ebrahimi (1980). These authors study the relationships among these definitions (and some variants of them), their properties and also demonstrate how some multivariate shock models give rise to them. They also make connections with some of the IFRA concepts. Griffith (1979) has also considered multivariate shock models leading to some of these concepts.

A recent paper by El-Neweihi, Proschan and Sethuraman (1980) discuss the multivariate class which arises as minimums of independent univariate NBU random variables. These distributions arise in the same way as the Marshall and Olkin distribution and also in the same way as one of the definitions of multivariate IFRA (i.e. Condition C) of Esary and Marshall (1979). These have properties similar to those of distributions with exponential minimums studied by Esary and Marshall (1979). Various relationships and properties are given. One of them is given that $\underset{\sim}{T}=(T_1,\ldots,T_n)$ has this NBU property then it has the Marshall and Olkin distribution if and only if, for example, $\min_{1\le i\le n} T_i$ is exponential. Relationships are given between the present definition and various other definitions.

3.3 Processes

The concept of a one dimensional stochastic process being of a type described by one of the four classes has been proposed by Ross (1979). Essentially he has discussed processes which are decreasing (or increasing) and whose first entry times into a state are all IFRA. For these processes, which he calls IFRA processes, he proves a closure theorem. He also studies NBU processes. NBU processes are also considered by El-Neweihi, Proschan and Sethuraman (1978) who also prove a closure theorem.

Extensions of Ross's ideas to multivariate processes have been accomplished by Block and Savits (1979c). These authors study several types of multivariate processes having IFRA type properties.

Recently Arjas (1979) has considered IFR processes. He has discussed both the univariate and multivariate cases.

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Multivariate parametric distributions which are of interest in reliability theory and life testing are discussed. These include distributions with exponential, Weibull and gamma univariate marginal distributions. Other distributions of interest are the multivariate nonparametric distributions whose marginals have increasing failure rates (IFR), increasing failure rate averages (IFRA), are new better than used (NBU) or new better than used in expectation (NBUE). Also mentioned are univariate and multivariate processes

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processes which have associated with them distributions in the various nonparametric classes mentioned above.

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