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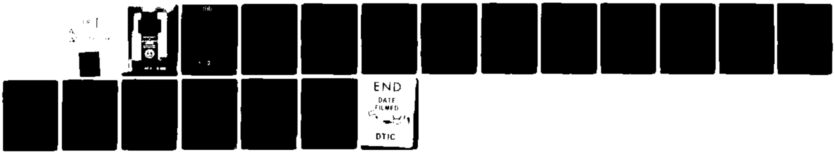

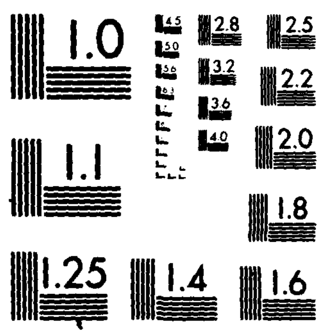

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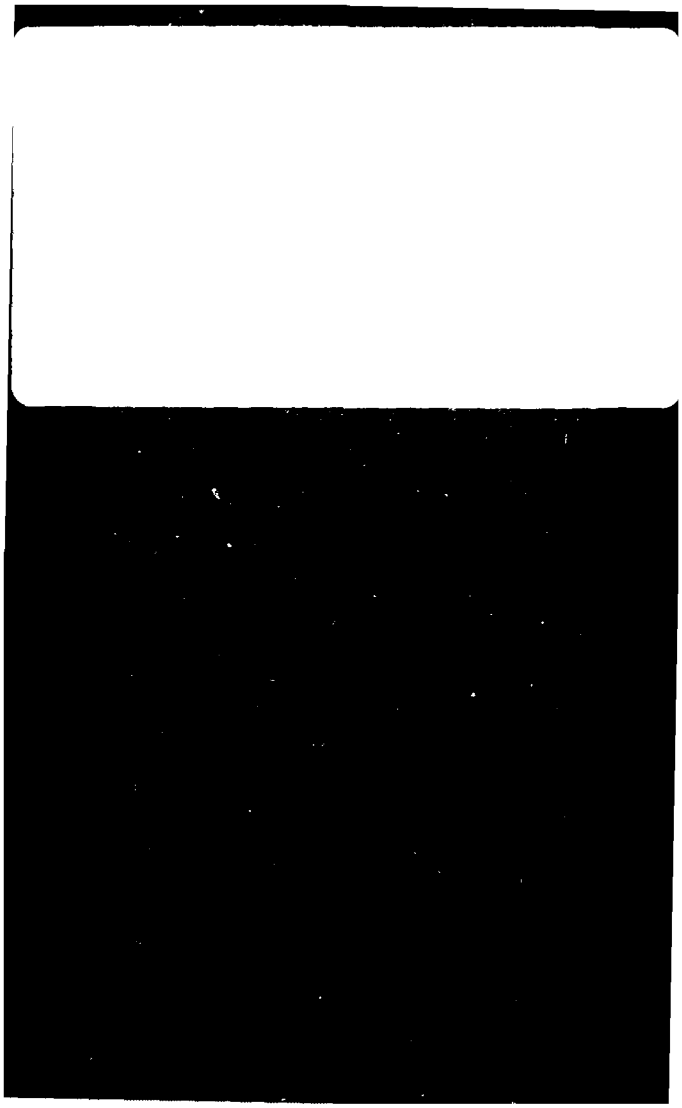

LEVEL II

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6 SOME PROPOSITIONS ON COST FUNCTIONS

10 Norman K. Womer

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Industrial Management Department

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SOME PROPOSITIONS

ON COST FUNCTIONS

INTRODUCTION

A usual intermediate product of neoclassical production theory is the cost function. The cost function relates cost of production to the quantity of output produced per unit of time, and in some cases, prices of resources used in the production process. The form of the cost function depends on the underlying production function and the competitive conditions in each of the resource markets. Its form also depends on constraints imposed on the firm's production decision. Cost functions derived from circumstances where one or more resources are fixed in quantity are labeled short run functions while those applicable to situations when all resources are variable are referred to as long run functions.

While the length of the time period involved is frequently implicit, the cost function is a flow concept relating the time rate of cost to the time rate of output. As such, its relation to various stock concepts like the quantity of capital equipment, the cumulative output to be produced (volume), and work experience of the organization at any point in time is difficult to specify. These stock concepts are frequently ignored in classroom treatments of the subject and in many attempts at empirical estimation of the cost function. This simplification is sometimes justified if the firm is in a steady state situation where the demand rate is expected to continue indefinitely. Hirshleifer [6, 236-237] has argued that the simplification is consistent as well with the situation in which volume is expected to be proportionate to production rate. Nevertheless, explicit consideration of these stock concepts is of crucial importance in many industrial settings.

The stock concepts are so important that in some industries variations in costs are routinely explained only by stock changes, ignoring the flow concepts explicitly treated by economic theory. For example, the learning curves or progress functions developed to describe costs in the aircraft industry typically relate costs to the number of items produced without any explicit reference to production rate or rate of resource use.

Attempts to integrate these two methods of describing variations in costs of production, learning curves and cost functions, have met with only limited success. Alchian [1] and Hirshleifer [6] have discussed some of the issues involved and have provided some intuitive expectations about the form of the resulting relation. Alchian states nine propositions on the variations of program costs due to changes in production rate, volume, the time horizon and the firm's production experience. Hirshleifer extends this work and relates it to more traditional cost theory. However, both works stop short of integrating their cost theory with production theory. Preston and Keachie [8] also relate learning to cost functions, but their graphical analysis of the situation also lacks precision. Oi [7] goes somewhat further. He purports to derive Alchian's nine propositions from an intertemporal production function. His general treatment explains learning in terms of standard production theory, but he too fails to specify the form of the cost function. Rosen [9] also investigates learning and production, but he stops short of deriving a cost function.

Washburn [10] addresses these issues directly. He formulates a model which relates discounted cash flow to production rate and the cumulative number of items produced. His model draws heavily on (i) the characteristics of production on an assembly line and (ii) the efficiency of "crews" of labor at each position on the line. This rather specialized model points the way for the generalization below which is based on neoclassical production theory.

In this paper, a model is developed for a firm producing to an order which specifies a quantity and a delivery date for output. The order serves to constrain a production program which minimizes the discounted cost of producing at a constant output rate. The model is formed by augmenting a homogeneous production function with a learning hypothesis.

The model and the resulting cost function are developed below. The cost function is then compared to the results of Alchian, Hirshleifer, and Oi.

THE MODEL

The basic production model employed here uses a production function to relate output rate to two classes of inputs. The first class, termed labor services, is composed of resources whose use rate can be varied throughout the production program. The second class of resources is termed capital. Capital resources are acquired prior to the start of production and their quantity is not changed during the program. We assume that the relative prices of resources within the classes do not change so that each class may be represented as a single composite resource.

The variables of the model are described as follows:

q = rate of production on the program

$\ell(t)$ = rate of labor use at t

L = quantity of augmented or effective labor

K = quantity of capital

$Q(t) = \int_0^t q\, d\tau$ = cumulative production experience at t

δ = a parameter describing learning

γ = a returns to scale parameter

C = discounted program costs in units of labor

T = time horizon for the production program

V = volume of output to be produced by T

$P(t)$ = daily unit cost of capital in units of labor

ε = time elasticity of the cost of capital, a parameter

σ = elasticity of substitution

The production function relates the rate of labor use and the stock of capital to output rate at any point in time. It is assumed to be homogeneous of degree γ in the arguments K, capital, and L, effective (or augmented), labor. That is

$$q = L^{\gamma} h(K/L) \tag{1}$$

The production function is assumed to have positive marginal products and convex isoquants. This requires

$$h' > 0,\ h - h'K/L > 0, \text{ and } h'' < 0$$

Decreasing returns to scale are assumed by requiring $\gamma < 1$.

Learning enters the situation as labor augmenting technological change. So that

$$L = Q^{\delta}(t)\ell(t) \qquad (2)$$

The elasticity of effective labor with respect to cumulative experience, δ, is assumed to be positive and less than one.

One additional constraint is imposed on the production program: q is constrained to be independent of t so that

$$q = V/T \qquad (3)$$

and

$$Q(t) = qt \qquad (4)$$

This constraint is implicit in Alchian [1] and explicitly stated in Hirshleifer [6:238] and Oi [7:587]. The implications of relaxing the constraint are explored in some special cases by Womer [11].

We also assume that K cannot be varied during the production program. But if K and q are both constant during the program, then from (1) it is clear that L is not a function of t. Dividing (2) by q and rewriting yields

$$\ell(t)/q = (L/q)Q^{-\delta}(t) \qquad (5)$$

In an environment where K and q are fixed, this log-linear relation between labor required per unit of output and cumulative output is indistinguishable from a unit learning curve. Substituting from (4) shows

$$\ell(t) = L\, q^{-\delta}t^{-\delta} \qquad (6)$$

As a direct consequence of learning and a constant output rate, the rate of labor use falls with time.

In explaining his propositions Alchian [1] emphasizes the relation between planned volume and the technique of production chosen. He argues that ". . . increased expenditures on more durable dies (capital) should result in (a) more than proportional increase of output potential. . ." This, he implies, results in large volume programs tending to be more capital intensive than small volume programs.

Furthermore, Alchian's discussion makes it clear that he expects planned volume to affect the choice of production technique even in the absence of learning.

We suppose the mechanism which relates volume to production technique to operate as follows. With no learning ($\delta = 0$) a production technique is specified by the capital-labor ratio. If the production rate is fixed and if the technique of production does not change, the length of the program, T, limits V. Suppose that the unit cost of capital increases at a decreasing rate with T. Under these circumstances, expanding volume by increasing T requires a less than proportionate increase in expenditure on capital. The same quantity of capital is used however. Larger volume also requires more labor, which increases the present value of the wage bill. If the percentage increase in the expenditure on capital is less than the percentage increase in the present value of the wage bill, cost minimization requires that capital be substituted for labor by shifting to a more capital intense technique of production.

For convenience the daily unit cost of capital, P, is assumed to be a log-linear function of t.

$$P(t) = pt^{-\varepsilon} \tag{7}$$

where $0 \leq \varepsilon \leq 1$ and p is the unit capital on day one. So even if the discount rate were zero, the unit capital cost increases at a decreasing rate as required, i.e.

$$\int_0^T P(t)\,dt = pT^{1-\varepsilon}/(1-\varepsilon)$$

The unit cost of capital and program costs are measured in units of labor, so the wage rate is implicit. Also, time is measured in units large enough so that the discount rate is equal to one. The present value of the cost stream is, therefore,

$$\begin{aligned} C &= \int_0^T [\ell(t) + Kpt^{-\varepsilon}]e^{-t}dt \qquad (8) \\ &= Lq^{-\delta}\int_0^T t^{-\delta}e^{-t}dt + Kp\int_0^T t^{-\varepsilon}e^{-t}dt \end{aligned}$$

Let

$$\int_0^T t^{-\delta}e^{-t}dt = \Gamma(1-\delta,T) \text{ and}$$

$$\int_0^T t^{-\varepsilon}e^{-t}dt = \Gamma(1-\varepsilon,T)$$ denote the incomplete gamma function.

Then (8) may be written as

$$C = Lq^{-\delta}\Gamma(1-\delta,T) + Kp\Gamma(1-\varepsilon,T) \qquad (9)$$

Notice that $q^{-\delta}\Gamma(1-\delta,T)$ can be thought of as the discounted price of effective labor while $p\Gamma(1-\varepsilon,T)$ can be thought of as the discounted effective price of capital.

The firm's problem is to

$$\begin{aligned} &\text{Min } C = Lq^{-\delta}\Gamma(1-\delta,T) + Kp\Gamma(1-\varepsilon,T) \qquad (10) \\ &\text{s.t. } q = L^{\gamma}h(K/L) \end{aligned}$$

q is determined by the program ending requirements V and T in (3) so production rate is not a decision variable in (10).

The first order conditions for cost minimization written in ratio form are

$$(\gamma L^{\gamma-1}h - L^{\gamma-1}h'K/L)/(L^{\gamma-1}h') = q^{-\delta}\Gamma(1-\delta,T)/[p\Gamma(1-\varepsilon,T)]$$

Let

$$x = q^{-\delta}\Gamma(1-\delta,T)/[p\Gamma(1-\varepsilon,T)] \tag{11}$$

be the effective labor to capital price ratio. Then

$$\gamma h/h' - K/L = x \tag{12}$$

Let g represent the solution to this condition, i.e.

$$K/L = g(x) \tag{13}$$

Substituting (13) into the objective function and using the constraint (1) yields

$$\begin{aligned} C &= q^{1/\gamma}h^{-1}[g(x)][q^{-\delta}\Gamma(1-\delta,T) + g(x)p\Gamma(1-\varepsilon,T)] \qquad (14) \\ &= q^{1/\gamma}h^{-1}[g(x)]p\Gamma(1-\varepsilon,T)[x + g(x)] \\ &= \gamma q^{1/\gamma}p\Gamma(1-\varepsilon,T)/h'[g(x)] \end{aligned}$$

the cost function.

With T fixed, the cost function shows that production rate can affect program costs in two ways.

First, increasing production rate causes costs to increase at an increasing rate because of diminishing returns to scale. But second, increasing production rate causes the price of effective labor to fall, decreasing the effective capital-labor ratio, increasing h′ and causing costs to fall. The relative magnitudes of these effects determine the impact of production rate on program costs.

Increasing volume with a constant production rate affects costs by increasing T. There are two effects here as well. First, the effective price of capital is increased, tending to increase costs. Second, the effective price ratio may be increased or decreased affecting costs by changing h′. Again, the magnitude of these effects and the sign of the second effect determine the net impact of volume on program costs.

The solution at (13) can also be used to investigate the impact of volume on the choice of the production technique.

We take the program capital-labor ratio to be the number of "days" of capital service divided by the total labor required for the program. Or

$$R = T\,K\,[\int_0^T \ell(t)\,dt]^{-1} \qquad (15)$$

Substituting from (6) and (13) and performing the indicated integration yields

$$R = (1-\delta)V^{\delta}g(x) \qquad (16)$$

Again there are two effects, a direct effect which is positive and an indirect effect which may be positive or negative. To evaluate the net effect of increasing volume at a constant production rate consider the partial derivative of (16) with q fixed

$$\partial R/\partial V|_q = (1-\delta)V^{\delta-1}g(x) + (1-\delta)V^{\delta}g'(x)[\partial x/\partial V|_q] \qquad (17)$$

Evaluating g´ as

$$g'(x) = g(x)\sigma/x \qquad (18)$$

where σ is the production function's elasticity of substitution yields

$$\partial R/\partial V|_q = (1-\delta)V^{\delta-1}g\{\delta+(V\sigma/x)[\partial x/\partial V|_q]\}$$

Performing the indicated differentiation using (12) and (3) yields

$$\partial R/\partial V|_q = (1-\delta)V^{\delta-1}g\{\delta+\sigma[T^{1-\delta}e^{-T}\Gamma^{-1}(1-\delta,T)-T^{1-\varepsilon}e^{-T}\,^{-1}(1-\varepsilon,T]\} \qquad (19)$$

Artin [2:10] shows that the incomplete gamma function, $\Gamma(\alpha,T)$, is log-convex in α. Furthermore,

$$T^{\alpha-1}e^{-T}\Gamma^{-1}(\alpha,T) = d\ln[\Gamma(\alpha,T)]/d\alpha \qquad (20)$$

Therefore, $T^{\alpha}e^{-T}\Gamma^{-1}(\alpha,T)$ increases with α. Using this fact, shows that the expression at (19) is positive if $\varepsilon \geq \delta$. If $\delta > \varepsilon$ the expression may be positive or negative depending on the sizes of δ and σ. If there is no learning, so that $\delta = 0$, cost minimization is clearly sufficient for R to increase as volume expands at a constant production rate. Despite this fact this model contradicts several of Alchian's and Hirshleifer's results.

A COMPARISON TO PREVIOUS WORK

Alchian [1] and Oi [7] state nine propositions concerning the signs of first and second partial derivatives of total discounted program costs. Of these nine, two (propositions seven and eight) concern the effect of starting production at a date later than $t = 0$. They are not considered here. The learning hypothesis, Alchian's proposition nine, is explicitly included in the model. In addition, Hirshleifer [6] states two propositions. Each of these eight propositions, six due to Alchian and two due to Hirshleifer, is examined below.

By using (3), the cost function at (14) can be regarded as a function of any two of the three variables q, V, and T. The two forms of the function used below are distinguished as C (T,q) and C (V,q) to make clear which variables are explicitly involved.

The propositions to be examined are listed in Table 1. Of the eight propositions, our assumptions are sufficiently strong to satisfy propositions one, three through five, and Hirshleifer's first proposition. The other three propositions fail to be satisfied under at least some conditions. These three propositions each concern the behavior of the change in cost with respect to production rate.

Table 1

Proposition and Model Results

	Propositions	Sign Asserted	Model Results
		Alchian and Oi	
1.	$\partial C(V,q)/\partial q$	+	+
2.	$\partial^2 C(V,q)/\partial q^2$	+	+ if γ is small and σ is small
3.	$\partial C(V,q)/\partial V$	+	+
4.	$\partial^2 C(V,q)/\partial V^2$	-	-
5.	$\partial[C(V,q)/V]/\partial V$	-	-
6.	$\partial^2 C(V,q)/\partial V\partial q$	-	- if σ is large and $\delta>\varepsilon$
		Hirshleifer	
H1.	$\partial C(T,q)/\partial q$	+	+
H2.	$\partial^2 C(T,q)/\partial q^2$	+ if q is large	+ if γ is small and σ is small

In particular, Alchian's [1:25] proposition two is, "The increment in C (the discounted cost stream) is an increasing function of the output rate." In the context of our model however,

$$\partial^2 C(V,q)/\partial q^2 = q^{1/\gamma-2}h^{-1}(1/\gamma+\varepsilon-1)(1/\gamma+\varepsilon-2)p\Gamma(1-\varepsilon,T)g \qquad (21)$$
$$+ 2(1/\Gamma+\varepsilon-2)p\Gamma(2-\varepsilon,T)g + p\Gamma(3-\varepsilon,T)g$$
$$+ (1/\gamma-1)(1/\gamma-2)q^{-\delta}\Gamma(1-\delta,T) + 2(1/\gamma-2)q^{1-\delta}\Gamma(2-\delta,T)$$
$$+ q^{-\delta}\Gamma(3-\delta) - \sigma q^{-\delta}\Gamma(1-\delta,T)\Psi^2 g/(g+x)$$

where

$$\Psi = [\Gamma(2-\delta,T)/\Gamma(1-\delta,T)] - [\Gamma(2-\varepsilon,T)/\Gamma(1-\varepsilon,T)] - \varepsilon \qquad (22)$$

The fact that

$$\alpha\Gamma(\alpha,T) - \Gamma(\alpha+1,T) = T^{\alpha-1}e^{-T} > 0 \qquad (23)$$

can be used to show that the first six terms of (21) are positive if $\gamma<1/(1+\delta)$. So if $\sigma = 0$, which requires effective labor and capital to be used in fixed proportions, then $\gamma < 1/(1+\delta)$ is sufficient for Alchian's proposition two to hold. However, if σ is somewhat greater than zero and γ is large (close to 1) then (21) will be negative and proposition two will not hold.

This result also contradicts Oi's claims concerning cost as a function of production rate. If the partial derivatives referred to in Oi's footnote [7:590] are actually taken, we find that Oi's own example ($\gamma = 1, \varepsilon = \delta = 0$) contradicts his claim that costs increase at an increasing rate with production rate.

The impact of this result is that independent of the discount rate, learning, or changes in the unit cost of capital with T, the proposition holds generally only if we have both substantial decreasing returns to scale and no input substitution possibilities.

The marginal cost curve is also the subject of Alchian's proposition six. He states as a "conjectural proposition" the "the marginal present value-cost with respect to increased rates of output decreases as the total contemplated output increases." Or $\partial^2C(V,q)/\partial q\partial V < 0$. Differentiating (14)

$$\begin{aligned}\partial^2C(V,q)/\partial q\partial V = {} & q^{1/\gamma-2}h^{-1}e^{-T}\{(1/\gamma+\varepsilon-1)pgT^{-\varepsilon} \qquad (24)\\ & + pgT^{1-\varepsilon} + (1/\gamma-1)q^{-\delta}T^{-\delta} + q^{-\delta}T^{1-\delta} \\ & - \sigma q^{-\delta}\Gamma(1-\delta,T)[T^{-\delta}\Gamma^{-1}(1-\delta,T) - T^{-\varepsilon}\Gamma^{-1}(1-\varepsilon,T)]\Psi g/(g+x)\}\end{aligned}$$

The first four terms of (24) are positive so Alchian's proposition six fails to hold if $\sigma = 0$. If $\delta > \varepsilon$ the last term is negative and large values for σ under these circumstances will cause the proposition to hold. This proposition also fails to hold in Oi's example [7].

Hirshleifer [6] asserts that $\partial^2C(T,q)/\partial q^2$ may be negative for low values of q but that as q increases, the expression will become positive. Performing this differentiation yields

$$\partial^2C(T,q)/\partial q^2 = q^{1/\gamma-2}p\Gamma(1-\varepsilon,T)h^{-1}\{(1/\gamma-1)(1/\gamma)g + (1/\gamma-\delta)(1/\gamma-\delta-1)x-\sigma\delta^2gx/(g+x)\} \tag{25}$$

If $\delta = 0$, decreasing returns to scale is sufficient for (25) to be positive as the proposition asserts. But if $\delta > 0$ and $\sigma > 0$, (25) will be negative for large γ, contradicting the proposition.

SUMMARY OF RESULTS

In this paper, the interaction between returns to scale and learning was investigated using a homogeneous production function. A cost function for the problem of production to customer order was derived. In this situation, several exceptions to previously derived propositions concerning the form of the resulting cost function were uncovered.

In particular, Alchian's assertion that with V fixed, marginal costs increase with production rate was found to require that the production function exhibit substantial decreasing returns to scale if there were possibilities for input substitution. Alchian's "conjectural proposition" that marginal costs decrease with volume, was found to hold only in the presence of input substitution and substantial learning. In the presence of learning, exceptions were also found to Hirshleifer's proposition that marginal costs rise with production rate when the length of the program is fixed.

Ultimately, the shape of the cost function is an empirical question. While no studies were uncovered which simultaneously addressed learning, input substitution and returns to scale, these have been studied separately. Conway and Shultz [4] report estimates for δ ranging from about 0.2 to 0.5 in various assembly operations. Dhrymes' [5] study of the CES production function yields estimates of γ in the range .997 to 1.218 and estimates of σ in the range 0.05 to 1.984 for manufacturing in the U.S. If these ranges of estimates are indicative of the true parameter space, it seems likely that empirical support may be available both for and against the three questioned propositions in different settings. Until this empirical work is completed, the shape of the program cost function in any particular circumstance must be regarded as an open question.

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In this paper, a model is developed for a firm producing to an order which specifies a quantity and a delivery date for output. The order serves to constrain a production program which minimizes the discounted cost of producing at a constant output rate. The model is formed by augmenting a homogenous production function with a learning hypothesis and deriving a cost function. The cost function is then compared to several results due to Alchian, Hirshleifer, and Oi.

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